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Abstract: The aim of this paper is to present a new class of mixed finite elements on quadrilaterals and hexahedra where the approximation is polynomial on each element K. The degrees of freedom are the same as those of classical mixed finite elements. However, in general, with this kind of finite elements, the resolution of second order elliptic problems leads to non conforming approximations. In the particular case when the finite elements are parallelograms or parallelepipeds, we can notice that our method is conform and coincides with the classical mixed finite elements on structured meshes.
First, a motivation for the study of the Pseudo-conforming polynomial mixed finite elements method is given, and the convergence of the method established. Then, numerical results that confirm the error estimates, predicted by the theory, are presented.

Key-words: mixed finite elements, polynomial approximation, non conforming approximation, quadrilateral meshes, hexahedral meshes

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Eléments finis polynomiaux dans Hdiv pour des maillages en quadrilatères et hexaèdres

Résumé : Le but de ce travail est de présenter une nouvelle classe d'éléments finis mixtes pour des maillages en quadrilatères et en hexaèdres pour lesquels l'approximation est polynômiale sur chaque élément K. Les degrés de liberté sont les même que ceux des éléments finis mixtes classiques. Cependant, avec ce nouveau type d'élément fini, la résolution de problèmes elliptiques du second ordre ne fournit pas, en général,une approximation conforme. Mais dans le cas particulier où les éléments sont des parallélogrammes ou des parallélépipèdes, on peut remarquer que notre méthode est conforme est coincide avec les éléments finis mixtes classiques sur des maillages structurés.
Dans une première section on présente les motivations de cette étude. Dans la section suivante, on présente et étudie des 'eléments finis mixtes pseudo-conforme. Et dans la derniŕe section on présente quelques tests numériques confirmant les résultats théoriques annoncés.

Mots-clés : Eléments finis mixtes, approximation polynômiale, approximation non conforme, maillage en quadrilatéres et hexaédres

1 Introduction

Quadrilaterals and hexahedra are often used in meshers particularly in geophysical applications and in fluids mechanics. When the geometry and the medium are structured, regular rectangular meshes are used. Otherwise general convex quadrilaterals or hexahedra are used. Then, with mixed finite elements ([31],[9]), we must construct finite elements on the mesh by using multilinear mappings noted F to a reference rectangle or rectangular solid.
The jacobian of these mappings leads to non polynomial basis functions on the elements of the mesh and introduces non polynomial matrices in the partial differential operators and the use of the Piola transform to work on the reference element is effective only when the mapping is linear otherwise a loss of order of convergence is observed ([3]).
In this paper, we are interested in quadrilateral and hexahedral meshes. One way for obtaining polynomial basis functions is to cut the quadrilaterals into triangles (or hexahedra into tetrahedra) and work with macro-elements ([21], [22], [23]). It is not our process. We choose to build finite elements by considering quadrilaterals and hexahedra as distortions of parallelograms and parallelepipeds. It is important to note here that the reserved vocabulary is the one of mathematicians; therefore an hexahedron is an example of polyhedron and its faces are plane. In the literature of the mechanics, usually an hexahedron denotes the image of a cube by a Q_1 transformation; commonly, the faces of a "trilinear hexahedral element" (for instance, see [19]) are not plane; they are nappes of hyperbolic paraboloids. Note that the inversibility of the transformation of a biunit cube into an hexahedron is still open ([20], [36]). Clearly, the fact of considering flat faces is restrictive but we can notice that non structured meshes in hexahedra (i.e. with plane faces) can be obtained and used in non academic meshers (see [29]). The generalization of the forthcoming analysis to "trilinear hexahedra" shall be not tackled in this paper.

In the presented method, the basis functions are built under conditions of weak-continuity of the unknowns between the elements. In the general case, the resulting mixed finite element is not conforming but the conditions of weak-continuity are sufficient to ensure the expected order of convergence. In the particular case of a parallelotope, the resulting mixed finite element is conforming and coincides with the t classical mixed finite element on a parallelotope. Returning to the general case, we call pseudo-conforming such a finite element.
The rest of the paper is organized as follow. The section 2 of the paper is devoted to the finite elements geometry. The choosen approach allows us to describe jointly quadrilaterals and hexahedra. In the section 3 we explain why the extension of the Raviart-Thomas finite elements to general quadrilaterals and hexehedra is not suitable. The section 4 deals with our pseudo-conforming finite elements; from a model problem we look at the conditions that our finite element must satisfy to obtain the expected *a priori* error estimates. And then we give solutions to built the finite element basis. Finally, in the last section, some numerical simulations are presented. In this paper we use the following notations:
For a vector $\mathbf{v} \in \mathbb{R}^{\mathbf{n}}$, $|\mathbf{v}|$ is the length of the vector $\mathbf{v}$; in matrix notation $\mathbf{v}$ is represented by the column vector $(v_1, ..., v_n)^T$ and then $|\mathbf{v}| = \left\{\sum_{1\leq j\leq n} |v_j|^2\right\}^{1/2}$, the Euclidean norm of the associated column vector. And for a square matrix B, $\|B\|$ is the spectral norm.

For a triangle or a quadrilateral K, $|K|$ is the area of K and if γ is a side of K, $|\gamma|$ is the lenght of γ; for a tetrahedron or an hexahedron K, $|K|$ is the volume of K and if γ is a face of K, $|\gamma|$ is the area of γ.

For a polyhedral domain K, we note

$$H^m(K) = \left\{v \in L^2(K); \partial^\alpha v \in L^2(K), \text{ for all } \alpha \text{ with } |\alpha| \leq m\right\}$$

equipped with the norm and the semi-norm

$$\|v\|_{m,K} = \left(\sum_{|\alpha|\leq m} \int_K |\partial^\alpha v|^2 dx\right)^{1/2}, \quad |v|_{m,K} = \left(\sum_{|\alpha|=m} \int_K |\partial^\alpha v|^2 dx\right)^{1/2}.$$

We consider also the following norm and semi-norm

$$\|v\|_{m,\infty,K} = \max_{|\alpha|\leq m} \left\{\operatorname{ess\,sup}_{x\in K} |\partial^\alpha v|\right\}, \quad |v|_{m,\infty,K} = \max_{|\alpha|=m} \left\{\operatorname{ess\,sup}_{x\in K} |\partial^\alpha v|\right\}.$$

$P(K)$ is the vectorial space $\{\mathbf{x} \in K \mapsto p(\mathbf{x});\ p \in P\}$, where P is a N variables polynomial space and K is a domain in $\mathbb{R}^N$. For any integer k, P_k is the space of polynomial functions of degree $\leq k$, while Q_k is the space of polynomial functions of degree $\leq k$ in each variable.

For each polyhedral K, h_K denotes the diameter of K and ρ_K denotes the diameter of the largest ball contained in K.

2 The geometry

2.1 Vertex and face numbering.

In $\mathbb{R}^N$ with $N = 2$ or 3, let K be a convex non-degenerated quadrilateral when $N = 2$, a convex non-degenerated hexahedron when $N = 3$. Let $\left\{\mathbf{a}_i \in \mathbb{R}^N,\ 1 \leq i \leq 2^N\right\}$ be the vertices of K. We use hereafter the word "face" for 2D and 3D geometries with the following vocabulary convention: for $N = 2$, a face of a quadrilateral K designates a side of K. We designate by "edge" of K a side of K when $N = 2$, the intersection of two adjacent quadrangular faces of K when $N = 3$.

Two vertices which do not belong to a same face of K are said opposite vertices. The numbering of the vertices is shown on Figure 1 and Figure 2. Note that this vertex numbering is such that

$$\begin{array}{ll} \forall\ i = 2, ..., 2^{N-1}, & [\mathbf{a}_1, \mathbf{a}_i] \text{ is an edge of } K; \\ \forall\ i = 1, ..., 2^{N-1}, & \mathbf{a}_i \text{ and } \mathbf{a}_{2^N+1-i} \text{ are opposite vertices of } K. \end{array}$$

The center of a polyhedral is the isobarycenter of its vertices; we note $\mathbf{a}_0$ the center of K:

$$\mathbf{a}_0 = \frac{1}{2^N} \sum_{1\leq i\leq 2^N} \mathbf{a}_i.$$

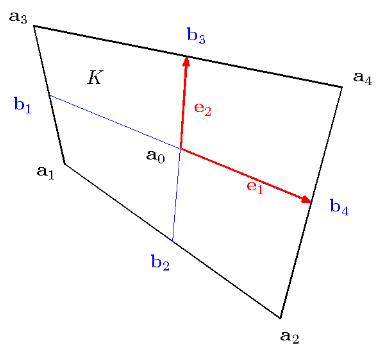


Figure 1: Numerotation $(N = 2)$

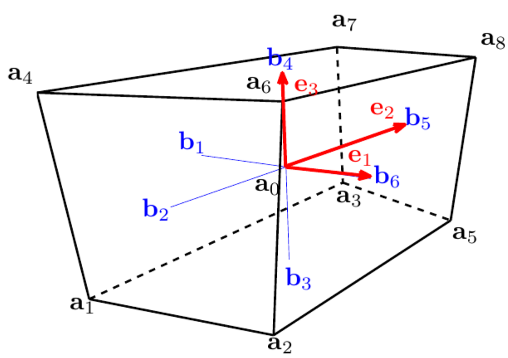


Figure 2: Numerotation $(N = 3)$

Let now $\{\gamma_m \subset \mathbb{R}^N,\ 1 \leq m \leq 2N\}$ be the set of the faces of K. Two faces without common vertex are said opposite faces. The face numbering is shown on Figure 1 and Figure 2. This numbering is such that

$$\begin{array}{l} \displaystyle\bigcap_{1 \leq i \leq N} \gamma_i = \mathbf{a}_1; \\ \forall\ m = 1, ..., N-1, \quad \mathbf{a}_{m+1} \notin \gamma_m; \\ \forall\ m = 1, ..., N, \quad \gamma_m \text{ and } \gamma_{2N+1-m} \text{ are opposite faces of } K. \end{array}$$

Last, let $\mathbf{b}_m$ be the center of the face γ_m, for $m = 1, ..., 2N$, and let us introduce the vectors $\mathbf{e}_m \in \mathbb{R}^N$ defined by

$$\forall\ m = 1, ..., N, \quad \mathbf{e}_m = \mathbf{a}_0 - \mathbf{b}_m\ \ (= \mathbf{b}_{2N+1-m} - \mathbf{a}_0)\,.$$

Since K is assumed to be a nondegenerated polyhedron, $(\mathbf{e}_1, ..., \mathbf{e}_N)$ is a basis of $\mathbb{R}^N$.

2.2 Affine-equivalent elements.

Let $\widehat{K} = [-1, +1]^N$ be the reference square when $N = 2$, the reference cube when $N = 3$. The vertices of $\widehat{K}$ are denoted by $\widehat{\mathbf{a}}_i,\ 1 \le i \le 2^N$ and the faces are denoted by $\widehat{\gamma}_m,\ 1 \le m \le 2N$.

We choose the following vertex numbering

$$\begin{array}{ll} \text{for } N=2: & \widehat{\mathbf{a}}_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix},\ \widehat{\mathbf{a}}_2 = \begin{pmatrix} +1 \\ -1 \end{pmatrix} \\ \text{for } N=3: & \widehat{\mathbf{a}}_1 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix},\ \widehat{\mathbf{a}}_2 = \begin{pmatrix} +1 \\ -1 \\ -1 \end{pmatrix},\ \widehat{\mathbf{a}}_3 = \begin{pmatrix} -1 \\ +1 \\ -1 \end{pmatrix},\ \widehat{\mathbf{a}}_4 = \begin{pmatrix} -1 \\ -1 \\ +1 \end{pmatrix} \end{array}$$

and

$$\text{for } N=2 \text{ and } N=3: \quad \widehat{\mathbf{a}}_i = -\widehat{\mathbf{a}}_{1+2^N-i}, \qquad 1 + 2^{N-1} \le i \le 2^N.$$

The face numbering is defined by

$$\begin{array}{ll} \forall\ m = 1,\ ...,\ N, & \widehat{\gamma}_m = \left\{ \widehat{\mathbf{x}} = (\widehat{x}_1, ..., \widehat{x}_N)^T \in \widehat{K};\ \widehat{x}_m = -1 \right\}; \\ \forall\ m = 1,\ ...,\ N, & \widehat{\gamma}_{2N+1-m} = \widehat{\gamma}_m. \end{array}$$

Let $\widehat{\mathbf{b}}_m$ be the center of the face $\widehat{\gamma}_m$, for $m = 1, ..., 2N$. The canonical basis $(\widehat{\mathbf{e}}_1, ..., \widehat{\mathbf{e}}_N)$ of $\mathbf{R}^N$ can be simply express with the vectors $\widehat{\mathbf{b}}_m$

$$\widehat{\mathbf{e}}_m = -\widehat{\mathbf{b}}_m \ \left(= \widehat{\mathbf{b}}_{2N+1-m}\right), \quad 1 \le m \le N.$$

Let B_K be the change of basis matrix given by

$$B_K \widehat{\mathbf{e}}_m = \mathbf{e}_m, \quad 1 \le m \le N..$$

and $F_K^\sharp$ be the invertible affine mapping

$$F_K^\sharp: \quad \widehat{\mathbf{x}} \in \mathbf{R}^N \to F_K^\sharp\left(\widehat{\mathbf{x}}\right) = \mathbf{a}_0 + B_K \widehat{\mathbf{x}}$$

This mapping $F_K^\sharp$ is the unique affine mapping such that

$$F_K^\sharp(\widehat{\mathbf{b}}_m) = \mathbf{b}_m, \quad 1 \le m \le N.$$

It is a bijection between $\widehat{K}$ and its image

$$K^\sharp = F_K^\sharp\left(\widehat{K}\right).$$

As image of the reference parallelotope by an invertible affine mapping, $K^\sharp$ is a parallelotope. The associated parallelotope of K being by definition the parallelotope which has the same face centers than K. We see that $K^\sharp$ is the associated parallelotope of K and we have $K^\sharp$ $=K$ if and only if K is a parallelotope. Let

$$K^\vee = (F_K^\sharp)^{-1}\,(K)\,.$$

The parallelotope associated to the polyhedron $K^\vee$ is the reference parallelotope $\widehat{K}$. For the analysis of quadrangular and hexahedral finite element, it is useful to precise how the element K is distorted.

2.3 Distortion parameters

When $N = 2$, let $\mathbf{d}$ be the vector of $\mathbb{R}^2$ given by

$$\mathbf{d} = \frac{1}{4}\,(\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3 + \mathbf{a}_4)\,. \tag{1}$$

We can interpret $2\mathbf{d}$ as the vector whose extremities are the mid-points of the diagonals of the quadrilateral K. This means that the quadrilateral K is a parallelogram if and only if $\mathbf{d} = 0$. It is easy to see that the vertices of $K^\sharp$ (the parallelogram associated to the quadrilateral K), are given by

$$\mathbf{a}_i^\sharp = \mathbf{a}_i - s_i\,\mathbf{d}, \quad 1 \le i \le 4$$

where

$$s_1 = s_4 = +1, \qquad s_2 = s_3 = -1. \tag{2}$$

When $N = 3$, let $\mathbf{d}_0,\ \mathbf{d}_1,\ \mathbf{d}_2$ and $\mathbf{d}_3$ be the vectors of $\mathbb{R}^3$ given by

$$\begin{cases} \mathbf{d}_0 = \dfrac{1}{8}\,(\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3 - \mathbf{a}_4 + \mathbf{a}_5 + \mathbf{a}_6 + \mathbf{a}_7 - \mathbf{a}_8)\,, \\ \mathbf{d}_1 = \dfrac{1}{8}\,(\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 - \mathbf{a}_4 - \mathbf{a}_5 - \mathbf{a}_6 + \mathbf{a}_7 + \mathbf{a}_8)\,, \\ \mathbf{d}_2 = \dfrac{1}{8}\,(\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 - \mathbf{a}_4 - \mathbf{a}_5 + \mathbf{a}_6 - \mathbf{a}_7 + \mathbf{a}_8)\,, \\ \mathbf{d}_3 = \dfrac{1}{8}\,(\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 - \mathbf{a}_6 - \mathbf{a}_7 + \mathbf{a}_8)\,. \end{cases} \tag{3}$$

These four vectors $\mathbf{d}_m$ are chosen for the hexahedron K to be a parallelepiped if and only if $\mathbf{d}_0 = \mathbf{d}_1 = \mathbf{d}_2 = \mathbf{d}_3 = 0$. The vertices of $K^\sharp$ (the parallelepiped associated to the hexahedron K) are

$$\mathbf{a}_i^\sharp = \mathbf{a}_i - \sum_{0 \le m \le 3} s_{i,m}\,\mathbf{d}_m, \quad \text{for } 1 \le i \le 8$$

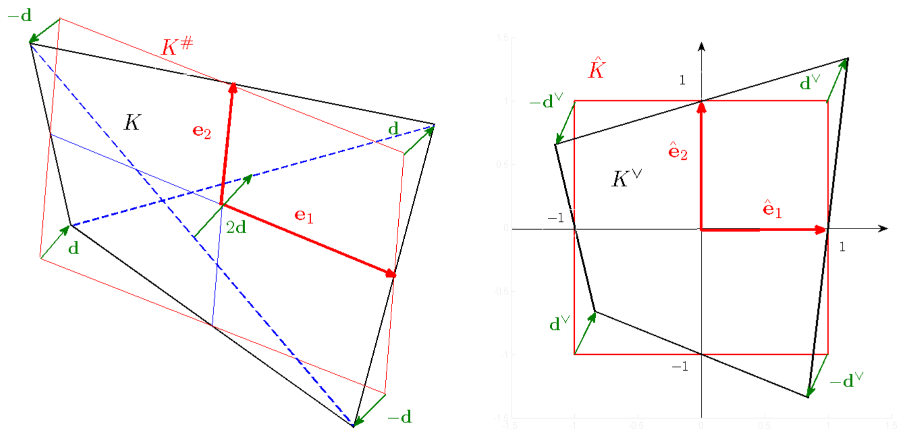


Figure 3: Distortion parameters of quadrilateralss

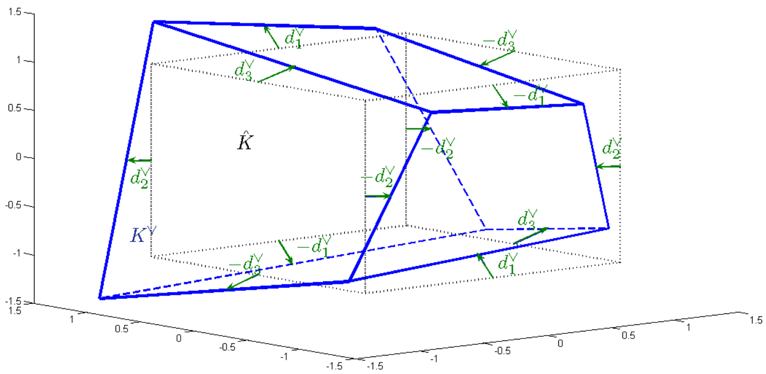


Figure 4: Distortion parameters of hexahedra

where

$$(s_{i,m}) = \begin{pmatrix} +1 & +1 & +1 & +1 \\ -1 & +1 & -1 & -1 \\ -1 & -1 & +1 & -1 \\ -1 & -1 & -1 & +1 \\ +1 & -1 & -1 & +1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \\ -1 & +1 & +1 & +1 \end{pmatrix} \tag{4}$$

From the equality

$$\sum_{1\le i\le 8} |\sum_{0\le m\le 3} s_{i,m}\mathbf{d}_m|^2 = 8 \sum_{0\le m\le 3} |\mathbf{d}_m|^2 \ ,$$

we deduce that:

$$\sum_{0\le m\le 3} s_{i,m}\mathbf{d}_m = 0 \text{ for } 1\le i\le 8 \Longrightarrow \mathbf{d}_m = 0 \text{ for } 0\le m\le 3.$$

Thus as announced, the hexadron K is a parallelepiped if and only if $\mathbf{d}_m = 0$ for $0 \le m \le 3$. We resume the results for 2D and 3D geometries in the following proposition:

Proposition 2.1 *Let $N^* = \left(2^N - N - 1\right) N$; there exist a vector $\mathbf{d} \in \mathbb{R}^{N^*}$ and matrices S_i, $1 \le i \le 2^N$, with N rows and N^* columns and which entries are ± 1, such that the vertex $\mathbf{a}_i$ of K and the vertex $\mathbf{a}_i^\sharp$ of $K^\sharp$ are linked by the relation*

$$\mathbf{a}_i = \mathbf{a}_i^\sharp + S_i\ \mathbf{d}, \quad 1\le i\le 2^N. \tag{5}$$

More precisely, for $N = 2$, $\mathbf{d} \in \mathbb{R}^2$ is given by (1) and the matrices S_i are square matrices of order 2 satisfying $S_i = s_i I$, where s_i are scalars given by (2). For $N = 3$, $\mathbf{d} = \left(\mathbf{d}_0^T,\ \mathbf{d}_1^T,\ \mathbf{d}_2^T,\ \mathbf{d}_3^T\right)^T$ is identified to a vector of $\mathbb{R}^{12}$, its coordinates $\mathbf{d}_m$ are given by (3) and the matrices $S_i := (s_{0,i}I,\ s_{1,i}I,\ s_{2,i}I,\ s_{3,i}I)$ are 3 rows and 12 columns matrices. The scalars $s_{i,m}$ are given by (4).

From (5) we deduce

$$\mathbf{d} = \frac{1}{4^{N-2}}\ {S_i}^T(\mathbf{a}_i - \mathbf{a}_i^\sharp),\ \ 1\le i\le 2^N.$$

We have $K = K^\sharp$ if and only if $\mathbf{d} = 0$ in $\mathbb{R}^{N^*}$.

Definition 2.2 *The vector $\mathbf{d}$ is named the distortion vector of K.*

Let us now introduce the N^* numbers δ_m defined by:

$$\begin{array}{lll} \mathbf{d} = \sum\limits_{1\le m\le 2} \delta_m\ \mathbf{e}_m & & \text{when } N=2, \\ \mathbf{d}_l = \sum\limits_{1\le m\le 3} \delta_{m+3l}\ \mathbf{e}_m, & 0\le l\le 3 & \text{when } N=3. \end{array} \tag{6}$$

where we recall that $\mathbf{e}_m$ is given by $\mathbf{e}_m = \mathbf{a}_0 - \mathbf{b}_m$. These parameters are invariant by affine mapping; in particular for the distortion vector $\mathbf{d}^\vee$ of $K^\vee$ we have if $N = 2$

$$\mathbf{d}^\vee = \sum_{1\le m\le 2} \delta_m\ \widehat{\mathbf{e}}_m$$

and if $N = 3$

$$\mathbf{d}_l^\vee = \sum_{1\le m\le 3} \delta_{m+3l}\ \widehat{\mathbf{e}}_m, \quad 0\le l\le 3$$

Definition 2.3 *The numbers* $(\delta_m)_{1\leq m\leq N^*}$ *are said the distortion parameters of* K.

Since the mapping $F_K^\sharp$ is invertible affine, K is a convex polyhedron if and only if $K^\vee$ is a convex polyhedron. So we see that the convexity of K and the face planarity when $N=3$ can be expressed by a set of constraints on the distortion parameters only. For $N=2$, it is easy to show that K is a convex quadrilateral if and only if we have

$$|\delta_1| + |\delta_2| < 1.$$

For $N = 3$, we can write a set of 6 equations and 18 inequations on the 12 distortion parameters which means that K is a convex hexahedron; but we cannot use this set of non-linear constraints.

From now on, we shall assume for $N=3$ as for $N=2$ that

$$\sum_{1\leq m\leq N^*} |\delta_m| < 1 \tag{7}$$

holds . Then $K^\vee$ contains $B(\mathbf{0}, 1/\sqrt{N})$ the ball centered at the origin and of radius $1/\sqrt{N}$ and $K^\vee$ is contained in the cube $[-2,+2]^N$. The polyhedron K is contained in the parallelotope

$$K^{2\sharp} = F_K^\sharp\left([-2,+2]^N\right)$$

This element $K^{2\sharp}$ is homothetic to $K^\sharp$ with a ratio equal to 2. Then, we have the inequality

$$h_K \leq 2h_{K^\sharp} \ .$$

Last, we note that the Euclidean norm of the distortion vector of K satisfies

$$\frac{1}{2N}\left(\sum_{1\leq m\leq N^\sharp} |\delta_m|\right)\rho_{K^\sharp} \leq |\mathbf{d}| \leq \frac{1}{2}\left(\sum_{1\leq m\leq N^\sharp} |\delta_m|\right) h_{K^\sharp}.$$

3 R.T. FE extension to quadrilaterals and hexahedra

The convergence results in $H(div)$ of RT, BDM or BDFM finite elements on meshes with parallelepipeds are well known ([32],[31], [9],...). But it is only enough recently that the loss of convergence order on general quadrilaterals or hexahedra was revealing ([2],[3]).
Let us look at the origin of the problem. Let S_h be a space of approximation supposed included in $H(div,\Omega)$. each $\mathbf{p}\in S_h$ verifies the fluxes reciprocity. (i.e. $\mathbf{p}\cdot\mathbf{n}$) on each internal face of the elements ([31], [9], [27], [26]). As the reciprocal image by F_K of the normal $\mathbf{n}$ at a face of K is not normal to the corresponding face of $\widehat{K}$, we must use the piola transform ([31]) on the vectorial variables to obtain the fluxes reciprocity:

$$\mathbf{p}(\mathbf{x}) = \frac{1}{J_{F_K}}\mathbf{DF}(\widehat{\mathbf{x}})\widehat{\mathbf{p}}(\widehat{\mathbf{x}}).$$

where $\mathbf{DF}$ is the jacobian matrix of the transformation F_K and J_{F_K} is determinant (supposed to be positive). And we have the following relations:

$$\int_{\partial K} \mathbf{p} \cdot \mathbf{n}\, u\, ds = \int_{\partial \widehat{K}} \widehat{\mathbf{p}} \cdot \widehat{\mathbf{n}}\, \widehat{u}\, d\widehat{s} \int_K div \mathbf{p}\, u dx = \int_{\widehat{K}} div \widehat{\mathbf{p}}\, \widehat{u}\, d\widehat{x}$$

We remark that: K is a parallelepiped if and only if $\mathbf{DF}$ is constant.
Let k be an integer, we note $S_h = \left\{ \mathbf{p}_h \in H(div, \Omega)/\widehat{\mathbf{p}}|_{\widehat{K}} \in \mathbf{RT}_k(\widehat{K}) \right\}$ and $\Pi_h \mathbf{p}$ an interpolation operator in S_h and we have the well-known results

Proposition 3.1 *if K is a parallelepiped then we have the following interpolation errors:*

$$\|\mathbf{p} - \mathbf{p}_h\|_{0,\Omega} \leq Ch^{k+1} |\mathbf{p}|_{k+1,\Omega}$$

$$\|div(\mathbf{p} - \mathbf{p}_h)\|_{0,\Omega} \leq Ch^{k+1} |div(\mathbf{p})|_{k+1,\Omega}$$

Let us consider the case of a finite element of lowest degree (i.e. $k = 0$), and examine what are the interpolation erros when K is not a parallelepiped. The Piola transform give us:

$$\widehat{\mathbf{p}}(\widehat{\mathbf{x}}) = \mathbf{M}(\widehat{\mathbf{x}})\mathbf{p}(\mathbf{x}) \tag{8}$$

$\mathbf{M}$ is the transposed cofactor matrix of $\mathbf{DF}$.
We note $\widehat{\Pi}\widehat{\mathbf{p}}$ the interpolate of $\widehat{\mathbf{p}}$ in $\mathbf{RT}_0(\widehat{K})$.
First let us consider the case $N = 2$.
The transformation F_K is bilinear and we have:

$$F_K : \quad \widehat{\mathbf{x}} \in \mathbb{R}^2 \to F_K(\widehat{\mathbf{x}}) = \mathbf{a}_0 + B_K \widehat{\mathbf{x}} + \frac{1}{4} \widehat{\mathbf{x}}_1 \widehat{\mathbf{x}}_2 \mathbf{d},$$

$$\mathbf{DF}(\widehat{\mathbf{x}}) = \frac{1}{2} \left(\mathbf{m}_1 + \frac{1}{2} \widehat{\mathbf{x}}_2 \mathbf{d} | \mathbf{m}_2 + \frac{1}{2} \widehat{\mathbf{x}}_1 \mathbf{d} \right)$$

and

$$J_{F_K}(\widehat{\mathbf{x}}) = \frac{1}{4} \left(\mathbf{m}_1 \cdot \mathbf{m}_2^{\perp} + \frac{1}{4} \mathbf{m}_1 \cdot \mathbf{d}^{\perp} \widehat{\mathbf{x}}_1 + \frac{1}{4} \mathbf{d} \cdot \mathbf{m}_1^{\perp} \widehat{\mathbf{x}}_2 \right).$$

One can remark that the lines of the matrix $\mathbf{M}$ belong to $\mathbf{RT}_0(\widehat{K})$. The usual technic to estimate the interpolation error is to remark that $(P_0)^2 \subset \mathbf{RT}_0(\widehat{K})$, then we have

$$\|\widehat{\mathbf{p}} - \widehat{\Pi}\widehat{\mathbf{p}}\|_{0,\widehat{K}} \leq C |\widehat{\mathbf{p}}|_{1,\widehat{K}}. \tag{9}$$

To come back to the element K we use the relation (8) and for instance we have

$$\frac{\partial \widehat{\mathbf{p}}_1}{\partial \widehat{\mathbf{x}}_1} = m_{11} \frac{\partial \mathbf{p}_1}{\partial \widehat{\mathbf{x}}_1} + m_{12} \frac{\partial \mathbf{p}_2}{\partial \widehat{\mathbf{x}}_1} + \frac{\partial m_{11}}{\partial \widehat{\mathbf{x}}_1} \mathbf{p}_1 + \frac{\partial m_{12}}{\partial \widehat{\mathbf{x}}_1} \mathbf{p}_2.$$

When the transformation F_K is not affine, the partial derivatives of m_{ij} are *a priori* different of 0 and from (9) we are not able to obtain better than:

$$\|\mathbf{p} - \Pi_h \mathbf{p}\|_{0,\Omega} \leq C\|\mathbf{p}\|_{1,\Omega}$$

but by noticing that the space :

$$W = \left\{ \widehat{\mathbf{p}} \in (P_1(K))^2; \frac{\partial \widehat{\mathbf{p}}_1}{\partial \widehat{\mathbf{x}}_2} = 0, \frac{\partial \widehat{\mathbf{p}}_2}{\partial \widehat{\mathbf{x}}_1} = 0, \frac{\partial \widehat{\mathbf{p}}_1}{\partial \widehat{\mathbf{x}}_1} + \frac{\partial \widehat{\mathbf{p}}_2}{\partial \widehat{\mathbf{x}}_2} = 0 \right\} \subset \mathbf{RT}_0(\widehat{K}),$$

we have more precisely

$$\|\widehat{\mathbf{p}} - \widehat{\Pi}\widehat{\mathbf{p}}\|_{0,\widehat{K}} \leq C \left(\| \frac{\partial \widehat{\mathbf{p}}_1}{\partial \widehat{\mathbf{x}}_2} \|_{0,\widehat{K}} + \| \frac{\partial \widehat{\mathbf{p}}_2}{\partial \widehat{\mathbf{x}}_1} \|_{0,\widehat{K}} + \| \frac{\partial \widehat{\mathbf{p}}_1}{\partial \widehat{\mathbf{x}}_1} + \frac{\partial \widehat{\mathbf{p}}_2}{\partial \widehat{\mathbf{x}}_2} \|_{0,\widehat{K}} \right).$$

And by using the properties of the lines of $\mathbf{M}$ one have:

$$\|\mathbf{p} - \Pi_h \mathbf{p}\|_{0,\Omega} \leq Ch|\mathbf{p}|_{1,\Omega}$$

Furthermore, the Piola transform gives us the following relation on the divergence:

$$\frac{1}{J_{F_K}} div\widehat{\mathbf{p}}(\widehat{\mathbf{x}}) = div\mathbf{p}(\mathbf{x})$$

We have $div(\mathbf{RT}_0(\widehat{K})) = P_0(\widehat{K})$ and if $J_F(\widehat{\mathbf{x}})$ is a constant function, we deduce that

$$div(\Pi_h \mathbf{p}) = \Pi_K^{(0)}(div(\mathbf{p})$$

where $\Pi_K^{(0)}$ is the L^2- projection operator on K ([31]). but, in the case of some quadrilateral, J_{F_K} is affine and this property is lost. On the other hand, it is always true on $\widehat{K}$

$$\|div(\widehat{\mathbf{p}} - \widehat{\Pi}\widehat{\mathbf{p}})\|_{0,\widehat{K}} = \|div(\widehat{\mathbf{p}}) - \Pi_{\widehat{K}}^{(0)} div(\widehat{\mathbf{p}})\|_{0,\widehat{K}} \leq C|div(\widehat{\mathbf{p}})|_{1,\widehat{K}}$$

and

$$\widehat{\nabla} div(\widehat{\mathbf{p}}) = J_{F_K} \widehat{\nabla} div(\mathbf{p}) + div(\mathbf{p}) \widehat{\nabla} J_{F_K} \tag{10}$$

then

$$|div(\widehat{\mathbf{p}})|_{1,\widehat{K}} \leq C \left(h_K |div(\mathbf{p})|_{1,K} + \|div(\mathbf{p})\|_{0,K} \right),$$

and finaly we are not able to obtain better than

$$\|div(\mathbf{p} - \Pi_h \mathbf{p})\|_{0,\Omega} \leq C\|div(\mathbf{p})\|_{1,\Omega}$$

In the 3D case, the transformation is trilinear:

$$F_K : \quad \widehat{\mathbf{x}} \in \mathbb{R}^3 \to F_K(\widehat{\mathbf{x}}) = \mathbf{a}_0 + B_K \widehat{\mathbf{x}} + \widehat{\mathbf{x}}_1 \widehat{\mathbf{x}}_2 \mathbf{d}_3 + \widehat{\mathbf{x}}_1 \widehat{\mathbf{x}}_3 \mathbf{d}_2 + \widehat{\mathbf{x}}_2 \widehat{\mathbf{x}}_3 \mathbf{d}_1 + \widehat{\mathbf{x}}_1 \widehat{\mathbf{x}}_2 \widehat{\mathbf{x}}_3 \mathbf{d}_0,$$

and

$$\mathbf{DF}(\widehat{\mathbf{x}}) = B_K + (\widehat{\mathbf{x}}_2\mathbf{d}_3 + \widehat{\mathbf{x}}_3\mathbf{d}_2 - \widehat{\mathbf{x}}_2\widehat{\mathbf{x}}_3\mathbf{d}_0|\widehat{\mathbf{x}}_1\mathbf{d}_3 + \widehat{\mathbf{x}}_3\mathbf{d}_1 - \widehat{\mathbf{x}}_1\widehat{\mathbf{x}}_3\mathbf{d}_0|\widehat{\mathbf{x}}_1\mathbf{d}_2 + \widehat{\mathbf{x}}_2\mathbf{d}_1 - \widehat{\mathbf{x}}_1\widehat{\mathbf{x}}_2\mathbf{d}_0)\,.$$

$J_{F_K} \in Q_2 \cap P_4$, and its expression is very complicate. Just as previously, the lines of $\mathbf{M}$ belong to $\mathbf{RT}_1$. As $\frac{\partial m_{ij}}{\partial \widehat{\mathbf{x}}_k}$ are of order 1 in h_K, we are able to obtain better than

$$\|\mathbf{p} - \Pi_h\mathbf{p}\|_{0,\Omega} \leq C\|\mathbf{p}\|_{1,\Omega}$$

We also have $\widehat{\nabla} J_{F_K}$ of order 1 in h_K then

$$\|div(\mathbf{p} - \Pi_h\mathbf{p})\|_{0,\Omega} \leq C\|div\mathbf{p}\|_{1,\Omega}$$

And these results can be confirmed by numerical tests (see section 5).

4 Pseudo-conforming finite element in $H(div)$

There are many studies on non conforming finite elements in H^1 ([37], [8],[34],[39], ...). But in our knowledge, there are very few works on non conforming finite elements in $H(div)$. Let us recall the conformity requirements of the space $H(div,\Omega)$ ([27]): Flux reciprocity on each face.
If a function $\mathbf{p}_h : \Omega \to \mathbb{R}^N$ satisfies

1. $\mathbf{p}_h|_K \in H_1(K)^N$ for each $K \in \mathcal{T}_h$,
2. for each face γ jointly to 2 elements of $\mathcal{T}_h$, i.e. $\gamma = \overline{K}_1 \cap \overline{K}_2$ the normal traces γ of $\mathbf{p}_h|_{K_1}$ and $\mathbf{p}_h|_{K_2}$ are the same, i.e. $\mathbf{p}_h|_{K_1} \cdot \mathbf{n} = \mathbf{p}_h|_{K_2} \cdot \mathbf{n}$ where $\mathbf{n}$ is a normal at the face.

then $\mathbf{p}_h \in H(div,\Omega)$. And reciprocally if $\mathbf{p}_h \in H(div,\Omega)$ and if 1. is satisfied then 2. is satisfied.
Remark: There is an important difference between the conformity in the space H^1 and $H(div)$. The trace of H^1 function belongs to $H^{1/2}$ while the normal trace of $H(div)$ function belongs to $H^{-1/2}$. this explains the definition previously given where we must suppose that the function is most regular than $H(div)$ (i.e. H^1) on each element. And it is going to have an important consequence on the "test patch"; to control the non conformity error we must not only impose the mean joins of the normal traces but also the mean joins of the momenta of order 1 of the normal traces .

4.1 The model

We consider the second order elliptic model problem:

$$\begin{cases} -div(A\,\mathbf{grad}u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{11}$$

where $A = (a_{i,j})$ is a symmetric matrix satisfying

$$\forall x \in \bar{\Omega}, \forall \xi \in R^2, \quad c\sum_{i=1}^{2} \xi_i^2 \leq \sum_{i,j=1}^{2} a_{i,j}(x)\xi_i\xi_j \leq c^{-1}\sum_{i=1}^{2} \xi_i^2,$$

and $\Gamma := \partial\Omega$ is the boundary of a polyhedral domain $\Omega \subset R^N$. Let $\mathcal{T}_h$ be a triangulation of Ω into quadrilaterals. Let $\partial\mathcal{T}_h$ denotes the set of the edges of the elements of $\mathcal{T}_h$ and $\partial\mathcal{T}_h \backslash \partial\Omega$ denotes the set of interior edges. For each element γ of $\partial\mathcal{T}_h \backslash \partial\Omega$, there exist K^+ and K^- in $\mathcal{T}_h$ such that $\bar{K}^+ \cap \bar{K}^- = \gamma$. The unitary outward normal of K^+ is noted $\mathbf{n}^+$ and the normal of a face is defined by $\mathbf{n} = \mathbf{n}^+$. For each subset γ of $\partial\Omega$, $\mathbf{n}$ denotes the unitary outward normal of Ω.
The classical mixed variational formulation ([31], [9]) for (11) is: find $u \in L^2(\Omega)$ and $\mathbf{p} \in H(div, \Omega)$ such that

$$\int_\Omega div\mathbf{p}\, v dx - \int_\Omega f v dx = 0 \qquad \forall v \in L^2(\Omega), \tag{12}$$

$$\int_\Omega A^{-1}\mathbf{p}.\mathbf{q} - \int_\Omega u\; div\,\mathbf{q}\; dx = 0 \qquad \forall \mathbf{q} \in H(div, \Omega). \tag{13}$$

We consider

$$M_h = \left\{v_h \in L^2(\Omega);\, v_h|_K \in P_0 \forall K \in \mathcal{T}_h\right\},$$

$$L_{\mathcal{T}_h} = \left\{\mathbf{q} \in (L^2(\Omega))^N;\, \mathbf{q}|_K \in H(div, K) \forall K \in \mathcal{T}_h\right\}$$

and

$$L_h = \left\{\mathbf{q}_h \in (L^2(\Omega))^N;\, \mathbf{q}_h|_K \in P_K^N \forall K \in \mathcal{T}_h\right\},$$

where P_K is a polynomial space.
Where $\mathbf{p} \in H(div, \Omega) + L_{\mathcal{T}_h}$,we define

$$\|\mathbf{p}\|^2_{Hdiv(0,h)} = \sum_{K \in \mathcal{T}_h} \|\mathbf{p}\|^2_{H(div,K)}.$$

We can remark that M_h is defined as usual, and that $L_{\mathcal{T}_h} \not\subseteq H(div, \Omega)$.Then, we are only interesting by the non conformity approximation of $\mathbf{q}$.
For each $(u, \mathbf{p}, \mathbf{q}) \in L^2(\Omega) \times H(div, \Omega) \times H(div, \Omega) + L^2(\Omega) \times L_{\mathcal{T}_h} \times L_{\mathcal{T}_h}$ we define

$$c_h(u, \mathbf{p}, \mathbf{q}) = \sum_{K \in \mathcal{T}_h} \int_K A^{-1}\mathbf{p}.\mathbf{q} - \sum_{K \in \mathcal{T}_h} \int_K u\; div\,\mathbf{q}\; dx.$$

A non conforming mixed finite element method for problem (**??**) is: find $u_h \in M_h$ and $\mathbf{q}_h \in L_h$ such that

$$\sum_{K \in \tau_h} \int_K v_h\; div\,\mathbf{p}_h\; dx - \sum_{K \in \tau_h} \int_K f v_h dx = 0 \qquad \forall v_h \in M_h, \tag{14}$$

$$\sum_{K \in \tau_h} \int_K A^{-1}\mathbf{p}_h.\mathbf{q}_h - \sum_{K \in \tau_h} \int_K u_h\; div\,\mathbf{q}_h\; dx = 0 \qquad \forall \mathbf{q}_h \in L_h. \tag{15}$$

If the following $inf - sup$ condition is satisfied:

$$\inf_{\{v_h \in M_h; \|u_h\|_{0,\Omega}=1\}} \sup_{\{\mathbf{q}_h \in L_h, \|\mathbf{q}_h\|_{Hdiv(0,h)}=1\}} b(u_h, \mathbf{q}_h) \geq \beta > 0, \tag{16}$$

where $b(u_h, \mathbf{q}_h) = \sum_{K \in \mathcal{T}_h} \int_K u_h \; div\, \mathbf{q}_h \; dx$, then the problem (14),(15) admits a unique solution and we have the *a priori* error estimate

$$\|\mathbf{p} - \mathbf{p}_h\|_{Hdiv(0,h)} + \|u - u_h\|_{0,h} \leq$$
$$c\left(\inf_{v_h \in M_h} \|u - v_h\|_{0,h} + \inf_{\mathbf{q}_h \in L_h} \|\mathbf{p} - \mathbf{q}_h\|_{Hdiv(0,h)} + \sup_{0 \neq \mathbf{q}_h \in L_h} \frac{|c_h(u, \mathbf{p}, \mathbf{q}_h)|}{\|\mathbf{q}_h\|_{0,h}}\right). \tag{17}$$

Remark: In (17) we can use the norm of $\mathbf{q}_h$ in $H(div)$ rather than L^2 but this is not useful in this context.
Using the Green formula : $\int_K u\, div(\mathbf{q}_h) dx = -\int_K \mathbf{grad} u \cdot \mathbf{q}_h dx + \int_{\partial K} u\, \mathbf{q}_h \cdot \mathbf{n} d\sigma$ we prove that:

$$c_h(u, \mathbf{p}, \mathbf{q}_h) = \sum_{\gamma \in \partial \mathcal{T}_h} \int_\gamma u[\mathbf{q}_h . \mathbf{n}] d\sigma.$$

4.2 Local error estimates

4.2.1 Interpolation error in $H(div)$

let $\Psi_K = P_K^N$ a vectorial space of vectorial polynomial functions on K. One suppose that the set

$$\Sigma_K = \left\{ \mathbf{w} \to \frac{1}{|\gamma_m|} \int_{\gamma_m} \mathbf{w}.\mathbf{n}\, d\sigma; 1 \leq m \leq 2N \right\}$$

is Ψ_K-unisolvant. The basis functions of the mixed finite element $(K,\ \Psi_K,\ \Sigma_K)$ are noted by $\psi_{m,K}$ and the interpolation operator associated (i.e. Withney opetator) is denoted by Π_K^W. For each function $\mathbf{p}$ such that $\int_{\gamma_m} \mathbf{p}.\mathbf{n}\, d\sigma; 1 \leq m \leq 2N$ exists, we have :

$$\Pi_K^W \mathbf{p} = \sum_{1 \leq m \leq 2N} \frac{1}{|\gamma_m|} \int_{\gamma_m} \mathbf{p}.\mathbf{n}\, d\sigma\ \psi_{m,K}.$$

The basis functions $\psi_{m,K}$ and all the functions of Ψ_K can be extended to $K^{2\sharp}$.

Proposition 4.1 *Supposed that the parameter distortion of K satisfy (7) and that the inclusion $P_0(K)^N \subset \Psi_K$ holds then for each function $\mathbf{p} \in (H^1)^N(K)$ we have*

$$\left\|\mathbf{p} - \Pi_K^W \mathbf{p}\right\|_{0,K} \leq 4h_{K^\sharp} \left(\sum_{1 \leq m \leq 2N} \|\psi_{m,K}\|_{0,\infty,K^{2\sharp}} \right) |\mathbf{p}|_{1,K}\,. \tag{18}$$

Moreover, if we suppose that $div(\Psi_K) = P_0(K)$ *and* $div(\Pi_K^W \mathbf{p}) = \frac{1}{|K|}\int_K div(mathbf p)dx$ *, there exists a constant c independent of K such that for each function p satisfying* $\mathbf{p} \in (H^1)^3(K)$ *and* $div(\mathbf{p}) \in H^1(K)$ *we have*

$$\|div(\mathbf{p} - \Pi_K^W \mathbf{p})\|_{0,K} \leq c h_{K^\sharp} |div(\mathbf{p})|_{1,K}. \tag{19}$$

Proof. Let be $\xi \in \gamma_i$. Using the Taylor expansion with integral residue, for each $\mathbf{x} \in K$ we have

$$\mathbf{p}(\xi) = \mathbf{p}(\mathbf{x}) + \int_0^1 (1-\theta) D\mathbf{p}(\mathbf{x} + \theta(\xi - \mathbf{x}))(\xi - \mathbf{x}) d\theta.$$

consequently

$$\frac{1}{|\gamma_m|}\int_{\gamma_m} \mathbf{p}(\xi)\cdot \mathbf{n}_m d\sigma =$$
$$\mathbf{p}(\mathbf{x})\cdot \mathbf{n}_m + \frac{1}{|\gamma_m|}\int_{\gamma_m}\left(\int_0^1 (1-\theta) D\mathbf{p}(\mathbf{x} + \theta(\xi - \mathbf{x}))(\xi - \mathbf{x}) d\theta\right)\cdot \mathbf{n}_m d\sigma,$$

and

$$\Pi_K^W \mathbf{p}(\mathbf{x}) = \sum_{1\leq m\leq 2N} \mathbf{p}(\mathbf{x})\cdot \mathbf{n}_m \psi_{m,K}(\mathbf{x})$$
$$+ \sum_{1\leq m\leq 2N} \frac{1}{|\gamma_m|}\int_{\gamma_m}\left(\int_0^1 (1-\theta) D\mathbf{p}(\mathbf{x} + \theta(\xi - \mathbf{x}))(\xi - \mathbf{x}) d\theta\right)\cdot \mathbf{n}_m d\sigma \psi_{m,K}(\mathbf{x})$$

Using the assumption: $P_0(K)^N \subset \Psi_K$ we have:

$$\left(\mathbf{p} - \Pi_K^W \mathbf{p}\right)(\mathbf{x}) =$$
$$-\sum_{1\leq m\leq 2N} \frac{1}{|\gamma_m|}\int_{\gamma_m}\left(\int_0^1 (1-\theta) D\mathbf{p}(\mathbf{x} + \theta(\xi - \mathbf{x}))(\xi - \mathbf{x}) d\theta\right)\cdot \mathbf{n}_m d\sigma \psi_{m,K}(\mathbf{x})$$

and

$$|\left(\mathbf{p} - \Pi_K^W \mathbf{p}\right)(\mathbf{x})| \leq$$
$$h_K \sum_{1\leq m\leq 2N} \frac{1}{|\gamma_m|}\int_{\gamma_m}\left(\int_0^1 (1-\theta) \|D\mathbf{p}(\mathbf{x} + \theta(\xi - \mathbf{x}))\| d\theta\right) d\sigma |\psi_{m,K}(\mathbf{x})|$$

where $\|D\mathbf{p}\|$ is the spectral norm of the matrix $D\mathbf{p}$.
For $\mathbf{x}$ a.e. in K, we note $g(\mathbf{x},\xi) = \int_0^1 (1-\theta)\|D\mathbf{p}(\mathbf{x} + \theta(\xi - \mathbf{x}))\|$ and we have

$$\|\mathbf{p} - \Pi_K^W \mathbf{p}\|_{0,K}^2 \leq h_K^2 \sum_{1\leq m\leq 2N} \frac{1}{|\gamma_m|^2}\int_{\gamma_m}\int_K g(\mathbf{x},\xi)^2 d\mathbf{x} d\sigma \int_K |\psi_{m,K}(\mathbf{x})|^2 d\mathbf{x}.$$

Since K is a star domain towards each point ξ, we have $\int_K g(\mathbf{x},\xi)^2 d\mathbf{x} \leq \|D\mathbf{p}\|^2_{0,K}$. So one obtain

$$\|\mathbf{p} - \Pi^W_K \mathbf{p}\|_{0,K} \leq h_K \left(\sum_{1\leq m\leq 2N} \|\psi_{m,K}\|_{0,\infty,K} \right) |\mathbf{p}|_{1,K}.$$

and *a fortiori*

$$\|\mathbf{p} - \Pi^W_K \mathbf{p}\|_{0,K} \leq 4h_{K^\sharp} \left(\sum_{1\leq m\leq 2N} \|\psi_{m,K}\|_{0,\infty,K^{2\sharp}} \right) |\mathbf{p}|_{1,K}.$$

The interpolation error of the divergence does not raise problem because we supposed $div(\Psi_K) = P_0(K)$. This corresponds to the classical situation where the interpolate of the divergence is the orthogonal projection of $div(\mathbf{p})$ on $P_0(K)$ equipped with the scalar product of $L^2(K)$. Consequently, we have

$$\|div(\mathbf{p} - \Pi^W_K \mathbf{p})\|_{0,K} \leq ch_K |div(\mathbf{p})|_{1,K}.$$

and *a fortiori* there exists a constant c such that

$$\|div(\mathbf{p} - \Pi^W_K \mathbf{p})\|_{0,K} \leq ch_{K^\sharp} |div(\mathbf{p})|_{1,K}.$$

■

Proposition 4.2 *Suppose that there exists an integer r sufficiently big so that $\Psi_K \subseteq P^N_r(K)$ Then, for each $\mathbf{q} \in \Psi_K$ there exist a constant c_r that depends only on r such that*

$$|\mathbf{q}|_{1,K^{2\sharp}} \leq c_r \frac{1}{\rho_{K^\sharp}} \|\mathbf{q}\|_{0,K^{2\sharp}} \tag{20}$$

Proof. For each integer r there exists a constant $\hat{c}_r$ depending only on de r such that

$$\forall \mathbf{q} \in P^N_r(\hat{K}) \quad |\mathbf{q}|_{1,\hat{K}} \leq \hat{c}_r \|\mathbf{q}\|_{0,\hat{K}}$$

Using the invertible affine transformation from $\hat{K}$ on $K^{2\sharp}$, we obtain the following inequality

$$\forall \mathbf{q} \in P^N_r(K^{2\sharp}) \quad |\mathbf{q}|_{1,K^{2\sharp}} \leq \hat{c}_r \frac{2\sqrt{N}}{\rho_{K^\sharp}} \|\mathbf{q}\|_{0,K^{2\sharp}}.$$

The announced inequality (20) is obtained with $c_r = 2\sqrt{N}\hat{c}_r$. ■

4.2.2 error estimations on the faces

Proposition 4.3 *Suppose that the distortion parameters of K satisfy (7). Then, there exists a constant C, independent of the geometry of K, such that: $\forall u \in H^2(K)$ and $\forall m$ with $1 \leq m \leq 2N$, we have*

$$\left\| u - \pi^1_{\gamma_m} u \right\|_{0,\gamma_m} \leq C\ h^{3/2}_{K^\sharp} \left(\frac{h_{K^\sharp}}{\rho_{K^\sharp}} \right)^{1/2} |u|_{2,K} \tag{21}$$

Proof. let $u \in H^1(K)$ and γ_m be a face of K. The best approximation of the trace of u on γ_m in $L^2(\gamma_m)$ by a $P_1(\gamma_m)$-polynomial satisfies:

$$\left\|u - \pi^1_{\gamma_m}(u)\right\|_{0,\gamma_m} = \inf_{p \in P_1(\gamma_m)} \|u - p\|_{0,\gamma_m}.$$

We denote by $\gamma_m^\vee$ the reciprocal image of γ_m by the application $F_K^\sharp$, $u^\vee = u \circ F_K^\sharp$ and $p^\vee = p \circ F_K^\sharp$. We remark that $p^\vee \in P_1(\gamma_m^\vee)$. As for each $p \in P_1(\gamma_m)$ $(u-p)\circ F_K^\sharp = u^\vee - p^\vee$,we deduce that

$$\|u - p\|_{0,\gamma_m} \leq \left\|B_{K^\sharp}^{-1}\right\|^{(N-1)/2} \|u^\vee - p^\vee\|_{0,\gamma_m^\vee}.$$

each $p^\vee \in P_1(\gamma_m^\vee)$ can be considered as the restriction of a polynomial $p^\vee \in P_1(K^\vee)$. Then,using Lemma **??**,there exists a constant C such that

$$\|u^\vee - p^\vee\|_{0,\gamma_m^\vee} \leq C \|u^\vee - p^\vee\|_{1,K^\vee}.$$

and consequently we have

$$\inf_{p^\vee \in P_1(\gamma_m^\vee)} \|u^\vee - p^\vee\|_{0,\gamma_m^\vee} \leq C \inf_{p^\vee \in P_1(K^\vee)} \|u^\vee - p^\vee\|_{1,K^\vee}$$

and

$$\inf_{p^\vee \in P_1(\gamma_m^\vee)} \|u^\vee - p^\vee\|_{0,\gamma_m^\vee} \leq C \left\|u^\vee - \pi^1_{K^\vee}(u^\vee)\right\|_{1,K^\vee}$$

To continue, we introduce $\Pi_K^\vee$ a P_1-Lagrange P_1 interpolate on $K^\vee$. Using a Taylor expansion of $u^\vee$ with integral residue, we prove that there exists a constant C such that

$$\|u^\vee - \Pi_{K^\vee}(u^\vee)\|_{1,K^\vee} \leq C|u^\vee|_{2,K}.$$

consequently

$$\left\|u^\vee - \pi^1_{K^\vee}(u^\vee)\right\|_{1,K^\vee} \leq C|u^\vee|_{2,K}$$

and equally

$$|u^\vee|_{2,K^\vee} \leq \|B_K\|^{N/2+1} |u|_{2,K}.$$

From the previous inequality we deduce that

$$\left\|u - \pi^1_{\gamma_m}(u)\right\|_{0,\gamma_m} \leq C \left\|B_K^{-1}\right\|^{(N-1)/2} \|B_K\|^{N/2+1} |u|_{2,K^\vee}.$$

Then we use (7), it is sufficient to remark that

$$\|B_K\| \leq \frac{1}{2} h_{K^\sharp} \qquad \text{et} \qquad \left\|B_K^{-1}\right\| \leq \frac{2\sqrt{N}}{\rho_{K^\sharp}}$$

to conclude the demonstration of the proposition. ■

4.3 Convergence

Suppose that the solution u of (11) is regular (i.e $u \in H^2(\Omega)$ et $div(\mathbf{p} \in H^1(\Omega))$ and the mesh too.

Proposition 4.4 $\inf - \sup$ *condition*
Suppose that

1. *for each K , Σ_K is Ψ_K-unisolvent,*
2. *for each $\mathbf{q}_h \in L_h$, $div(\mathbf{q}_h)$ belongs to M_h.*
3. *the basis functions $\psi_{m,K}$ of Ψ_K satisfy $\exists C > 0$ $\|\psi_{m,K}\|_{0,\infty,K^{2\#}} < C$*

Under these three assumptions the $inf - sup$ condition (16) holds..

Proof. Since the domain Ω is regular, the $\inf - \sup$ condition on the continuous problem (cf [31], [9]) gives :
For each $u \in L^2(\Omega)$, there exists $\mathbf{p} \in H^1(\Omega)$ such that $div(\mathbf{p}) = u$ and the estimate $\|\mathbf{p}\|_{1,\Omega} \leq C\|u\|_{0,\Omega}$ holds with a constant C independent of the mesh.
Therefore, this property is true for each $u_h \in M_h$. Let $\mathbf{p}_h = \Pi_h^W \mathbf{p}$ be the Withney-interpolant of $\mathbf{p}$ in L_h and we want to prove that $\|\mathbf{p}_h\|_{Hdiv(0,h)} \leq C\|u_h\|_{0,\Omega}$. Using the assumption 2, we have $div(\mathbf{p}_h) = div(\mathbf{p})$ on each K.
Furthermore, for each $\mathbf{x}$ in K, we have

$$|\mathbf{p}_h(\mathbf{x})| \leq \sum_{1\leq m\leq 2N} \frac{1}{|\gamma_m|} |\int_{\gamma_m} \mathbf{p}.\mathbf{n}\, d\sigma|\ |\psi_{m,K}(\mathbf{x})|$$

then

$$\|\mathbf{p}_h\|_{0,K} \leq \sum_{1\leq m\leq 2N} \|\mathbf{p}.\mathbf{n}\|_{0,\gamma_m} \sum_{1\leq m\leq 2N} \|\psi_{m,K}\|_{0,\infty,K}$$

and using lemma **??** and the assumption 3 we obtain

$$\|\mathbf{p}_h\|_{0,K} \leq C\|\mathbf{p}\|_{1,K}.$$

Equally, for each $\mathbf{x}$ in K, we have

$$|div(\mathbf{p}_h(\mathbf{x}))| \leq \sum_{1\leq m\leq 2N} \frac{1}{|\gamma_m|} |\int_{\gamma_m} \mathbf{p}.\mathbf{n}\, d\sigma|\ |div(\psi_{m,K}(\mathbf{x}))|,$$

since $div(\psi_{m,K}(\mathbf{x}))$ is constant on K, we deduce

$$\|div\mathbf{p}_h\|_{0,K} \leq C\|\mathbf{p}\|_{1,K}.$$

Finally we have

$$\|\mathbf{p}_h\|_{Hdiv(0,h)} \leq C\|\mathbf{p}\|_{1,\Omega} \leq C\|u_h\|_{0,\Omega}.$$

We notice that $b(u_h, \mathbf{p}_h) = \|u_h\|^2_{0,\Omega}$ and conclude that

$$\inf_{\{v_h \in M_h; \|u_h\|_{0,\Omega}=1\}} \sup_{\{\mathbf{q}_h \in L_h, \mathbf{q}_h \neq 0\}} \frac{b(u_h, \mathbf{q}_h)}{\|\mathbf{q}_h\|_{Hdiv(0,h)}} \geq \inf_{\{v_h \in M_h; \|u_h\|_{0,\Omega}=1\}} \frac{b(u_h, \mathbf{p}_h)}{\|\mathbf{p}_h\|_{Hdiv(0,h)}}$$
$$\geq \frac{1}{C} > 0.$$

■

Proposition 4.5 *Approximation error*
Let be the following assumptions:

1. *K, Σ_K is Ψ_K-unisolvent,*
2. *$\exists r > 0\ \forall K\ P_0(K)^N \subset \Psi_K \subset (P_r(K))^N$,*
3. *$div(\Psi_K) = P_0$ and $div(\Pi^W_K \mathbf{p}) = \frac{1}{|K|}\int_K div(mathbffp)dx$*
4. *the basis functions $\psi_{m,K}$ of Ψ_K satisfy $\exists C > 0\ \|\psi_{m,K}\|_{0,\infty,K^{2\#}} < C$.*

Under these 4 assumptions we have

$$\inf_{\mathbf{q}_h \in L_h} \|\mathbf{p} - \mathbf{q}_h\|_{Hdiv(0,h)} \leq Ch\left(|u|_{2,\Omega} + |div(\mathbf{p})|_{1,\Omega}\right).$$

Proof. Since $\mathbf{p} = A\mathbf{grad}u$ and using Proposition 4.1 , we have immediately the result.

$$\left(\sum_{K \in \mathcal{T}_h} \left\|\mathbf{p} - \Pi^W_K \mathbf{p}\right\|^2_{H(div,K)}\right) \leq Ch^2\left(|p|^2_{1,\Omega} + |div(\mathbf{p})|^2_{1,\Omega}\right).$$

where C does not depend on the mesh. ■
Remark: The approximation error on u_h does not raise problem and is bounded by $Ch|u|_{1,\Omega}$.

Proposition 4.6 *Consistency error*
If the Patch Test conditions are satisfied, namely

$$\forall \mathbf{q}_h \in L_h, \quad \forall \gamma \in \partial\mathcal{T}_h, \quad \forall p \in P_1(\gamma) \quad \int_\gamma p\,[\mathbf{q}_h \cdot \mathbf{n}]d\sigma = 0,$$

then

$$|c_h(u, \mathbf{p}, \mathbf{q}_h)| \leq ch|u|_{2,\Omega}\|\mathbf{q}_h\|_{0,h}.$$

Proof. Let γ be in $\partial\mathcal{T}_h \backslash \partial\Omega$. From the Patch Test, for each polynomial $p \in P_1(\gamma)$ we have

$$\int_\gamma u[\mathbf{q}_h \cdot \mathbf{n}]d\sigma = \int_\gamma (u - p)[\mathbf{q}_h \cdot \mathbf{n}]d\sigma$$

consequently

$$|\int_\gamma u[\mathbf{q}_h\cdot\mathbf{n}]d\sigma| \leq \|u-\pi_\gamma^1 u\|_{0,\gamma}\|[\mathbf{q}_h\cdot\mathbf{n}]\|_{0,\gamma}$$

From Propositions **??**, 4.3 et 4.2 we deduce that

$$\begin{aligned}|\int_\gamma u\,\mathbf{q}_h^+\cdot\mathbf{n}^+d\sigma| &\leq ch^2\|\mathbf{q}_h^+\|_{1,K^+}|u|_{2,K^+}\\ &\leq ch\|\mathbf{q}_h^+\|_{0,K^{+2\sharp}}|u|_{2,K^+}\end{aligned}$$

and finally

$$|\int_\gamma u[\mathbf{q}_h\cdot\mathbf{n}]d\sigma| \leq ch\left(\|\mathbf{q}_h^+\|_{0,K^{+2\sharp}}|u|_{2,K^+}+\|\mathbf{q}_h^-\|_{0,K^{-2\sharp}}|u|_{2,K^-}\right).$$

Then we sum on all the face γ. In the right hand side of the inequality an element K appears at most $2N$ times, so we have

$$\begin{aligned}|\sum_{\gamma\in\partial\mathcal{T}_h}\int_\gamma u[\mathbf{q}_h.\mathbf{n}]d\sigma| &\leq ch\left(\sum_{K\in\tau_h}\|\mathbf{q}_h\|_{0,K^{2\sharp}}\right)|u|_{2,\Omega}\\ &\leq ch\|\mathbf{q}_h\|_{0,\Omega}|u|_{2,\Omega}.\end{aligned}$$

■

Consequently, the pseudo-conforming mixed finite element converges with order 1.

4.4 Polynomial mixed finite elements

To build a pseudo-conforming finite element, two approach can be considered. In the first approach, we use the finite element BDM_1 ([9])which is adapted to our problem in dimension 2 and 3. In the second approach, we use a hierarchical basis built from a pseudo-conforming finite element in H^1.

4.4.1 The quadrilateral case

Our goal is to build $\mathbf{W}_K$ a space of polynomial functions on K of dimension 8 in order to impose the mean value of the flux and to control its first momentum on each face. We suppose that $P_0^2\subset\mathbf{W}_K$ and $div(\mathbf{W}_K)=P_0$.
For each function $\mathbf{w}$ belonging to $P_0^2\oplus\mathbf{x}P_0$ we have

$$\forall p\in P_1,\ \int_{\gamma_m}p\mathbf{w}.\mathbf{n}\,d\sigma=\frac{1}{|\gamma_m|}\int_{\gamma_m}p\,d\sigma\int_{\gamma_m}\mathbf{w}.\mathbf{n}\,d\sigma.$$

The result is obvious for $\mathbf{w}$ belonging to P_0^2. Moreover, it still true for $\mathbf{w}=\mathbf{x}$ since $\mathbf{x}.\mathbf{n}$ is constant.

Consider now the following vectorial space of polynomial functions:

$$\Psi_K = \{\mathbf{w} \in \mathbf{W}_K ; \text{for } 1 \leq m \leq 4\,, \forall p \in P_1,$$
$$\left.\int_{\gamma_m} p\mathbf{w}.\mathbf{n}\, d\sigma = \frac{1}{|\gamma_m|}\int_{\gamma_m} p\, d\sigma \int_{\gamma_m} \mathbf{w}.\mathbf{n}\, d\sigma \right\}.$$

with the following set of degrees of freedom:

$$\Sigma_K = \left\{\mathbf{w} \rightarrow \int_{\gamma_m} \mathbf{w}.\mathbf{n}\, d\sigma ; 1 \leq m \leq 4\right\}$$

First approach Let us recall the definition of the spaces $BDM_{[k]}$ for $k \geq 1$

$$BDM_{[k]} = \Big\{\mathbf{w}^\vee \mid \mathbf{w}^\vee = \mathbf{v}(x_1^\vee, x_2^\vee) + r\,\mathbf{curl}({x_1^\vee}^{k+1}{x_2^\vee}^{k})$$
$$+ s\,\mathbf{curl}({x_1^\vee}^{k}{x_2^\vee}^{k+1}), \mathbf{v}(x_1^\vee, x_2^\vee) \in (P_k)^2\Big\}$$

We denote br $BDM_{[1]}^K$ the space:

$$BDM_{[1]}^K = \left\{\mathcal{P}_{K^\vee} \circ \mathbf{w}^\vee \circ \left(F_K^\sharp\right)^{-1} ; \mathbf{w}^\vee \in BDM_{[1]}\right\}$$

where $\mathcal{P}_{K^\vee}$ is the Piola transform defined by:

$$\mathbf{w}^\vee \longrightarrow \frac{1}{\det B_K} B_K \mathbf{w}^\vee .$$

Obviously we have $P_0^2 \subseteq \Psi_K$ et $div(\Psi_K) = P_0$. So, we have the following Proposition:

Proposition 4.7 *For each quadrilateral K convex, (K, Ψ_K, Σ_K) is a finite element of Raviart-Thomas type.*

Proof. For each polynomial $\mathbf{w}$ we set :

$$\widehat{I}_m(\mathbf{w}) = \int_{\widehat{\gamma}_m} \widehat{\mathbf{w}}.\mathbf{n}\, d\sigma \quad 1 \leq m \leq 4, \qquad \widehat{I}_m(\mathbf{w}) = \int_{\widehat{\gamma}_{m-4}} \sigma\, \widehat{\mathbf{w}}.\mathbf{n}\, d\sigma \quad 5 \leq m \leq 8,$$

and

$$I_m^\vee(\mathbf{w}) = \int_{\gamma_m^\vee} \mathbf{w}^\vee.\mathbf{n}\, d\sigma \quad 1 \leq m \leq 4, \qquad I_m^\vee(\mathbf{w}) = \int_{\gamma_{m-4}^\vee} \sigma\, \mathbf{w}^\vee.\mathbf{n}\, d\sigma \quad 5 \leq m \leq 8.$$

Next, we introduce the polynomials $r_j \in BDM_{[1]}$ satisfying:

$$\widehat{I}_m(r_j) = \delta_{m,j},\ 1 \leq m, j \leq 8.$$

The polynomials r_j exist since they correspond to the basis functions of the finite element $BDM_{[1]}$. Consider now the square matrix T of order 8 satisfying $T_{m,j} = I_m^\vee(r_j)$. A symbolic (or direct) calculus gives

$$detT = \left(1-{\delta_1}^2\right)\left(1-{\delta_2}^2\right)\left(1-(\delta_1+\delta_2)^2\right)\left(1-(\delta_1-\delta_2)^2\right).$$

Since $|\delta_1|+|\delta_2|<1$ we have $detR > 0$. That concludes the demonstration. ■

Remarks

- if $d=0$ (i.e. K is a parallelogram) then $(F_K^\sharp)^{-1}(\Psi_K) = \mathbf{RT}_{[0]}^K = \left\{\mathbf{q}^\vee \circ (F_K^\sharp)^{-1}; \mathbf{q}^\vee \in \mathbf{RT}_{[0]}\right\}$.
- The basis function on $K^\vee$ can be calculated explicitly even if it is certainly more efficient to calculate then numerically. We note

$$\phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \phi_3 = \begin{pmatrix} x_1^\vee \\ x_2^\vee \end{pmatrix}, \phi_4 = curl(\omega_K),$$

where

$$\begin{aligned}\omega_K\left(x_1^\vee, x_2^\vee\right) = \; & \frac{3\delta_1\delta_2}{\left(1-(\delta_1+\delta_2)^2\right)\left(1-(\delta_1-\delta_2)^2\right)}\left(1-(x_1^\vee)^2-(x_2^\vee)^2\right) \\ & -\frac{\left(1-2\delta_1^2-2\delta_2^2+\delta_1^4+7\delta_1^2\delta_2^2+\delta_2^4-3\delta_1^2\delta_2^4-3\delta_1^4\delta_2^2\right)}{(1-\delta_1^2)(1-\delta_2^2)\left(1-(\delta_1+\delta_2)^2\right)\left(1-(\delta_1-\delta_2)^2\right)}x_1^\vee x_2^\vee \\ & +\frac{\delta_2\left(1+\delta_1^2-2\delta_2^2-2\delta_1^4+\delta_1^2\delta_2^2+\delta_2^4\right)}{(1-\delta_1^2)(1-\delta_2^2)\left(1-(\delta_1+\delta_2)^2\right)\left(1-(\delta_1-\delta_2)^2\right)}(x_1^\vee)^2 x_2^\vee \\ & +\frac{\delta_1\left(1-2\delta_1^2+\delta_2^2+\delta_1^4+\delta_1^2\delta_2^2-2\delta_2^4\right)}{(1-\delta_1^2)(1-\delta_2^2)\left(1-(\delta_1+\delta_2)^2\right)\left(1-(\delta_1-\delta_2)^2\right)}x_1^\vee (x_2^\vee)^2.\end{aligned}$$

With these notations the basis functions are

$$\begin{aligned}\psi_{1,K}^\vee\left(x_1^\vee, x_2^\vee\right) &= \frac{1}{8}(2-\delta_2)\phi_1 - \frac{1}{8}\delta_1\phi_2 + \frac{1}{8}\phi_3 + \frac{1}{8}(\delta_1^2-(1-\delta_2))^2\phi_4 \\ \psi_{2,K}^\vee\left(x_1^\vee, x_2^\vee\right) &= -\frac{1}{8}\delta_2\phi_1 + \frac{1}{8}(2-\delta_1)\phi_2 + \frac{1}{8}\phi_3 - \frac{1}{8}(\delta_2^2-(1-\delta_1))^2\phi_4 \\ \psi_{3,K}^\vee\left(x_1^\vee, x_2^\vee\right) &= \frac{1}{8}(2+\delta_2)\phi_1 - \frac{1}{8}\delta_1\phi_2 + \frac{1}{8}\phi_3 + \frac{1}{8}(\delta_1^2-(1+\delta_2))^2\phi_4 \\ \psi_{4,K}^\vee\left(x_1^\vee, x_2^\vee\right) &= -\frac{1}{8}\delta_2\phi_1 - \frac{1}{8}(2+\delta_1)\phi_2 + \frac{1}{8}\phi_3 - \frac{1}{8}(\delta_2^2-(1+\delta_1))^2\phi_4\end{aligned}$$

Proposition 4.8 *Suppose that there exists a constant $\alpha > 0$ such that for each $K \in \mathcal{T}_h$, $\quad |\delta_1|+|\delta_2| \le 1-\alpha$, then the assumptions of Propositions 4.5 and 4.6 are satisfied.*

Proof. The 3 first assumptions of 4.5 are clearly satisfied and since $|\delta_1|+|\delta_2| \le 1-\alpha$, $\dfrac{1}{|\det T|}$ are bounded, the $\psi_{i,K}$ are bounded on $K^{2\#}$. And finally, by construction, the Patch Test is satisfied. ■

Second approach We search the space $\mathbf{W}_K^\vee$ as $\mathbf{W}_K^\vee = \mathbf{RT}_0 \oplus \mathbf{Z}_K^\vee$ where $\mathbf{Z}_K^\vee$ is a vectorial polynomial space of dimension 4 whose the functions basis are noted $(\mathbf{r}_j^\vee; \quad 5 \leq j \leq 8)$. Since $div(\mathbf{RT}_0) = P_0$, we can impose $div(\mathbf{r}_j^\vee) = 0$ for $j = 5, ..., 8$. As we want a hierarchical basis the $\mathbf{r}_j^\vee$ must verify

$$\int_{\gamma_m^\vee} \mathbf{r}_{j+4}^\vee \cdot \mathbf{n}_m^\vee d\sigma = 0 \text{ pour } j, m = 1, ...4.$$

To built these basis functions, we introduce

$$\begin{aligned}
\lambda_1^\vee(\mathbf{x}_1^\vee, \mathbf{x}_2^\vee) &= \delta_2 \mathbf{x}_1^\vee - (\delta_1 - 1)\mathbf{x}_2^\vee - (\delta_1 - 1) \\
\lambda_2^\vee(\mathbf{x}_1^\vee, \mathbf{x}_2^\vee) &= (\delta_2 - 1)\mathbf{x}_1^\vee - \delta_1 \mathbf{x}_2^\vee + (\delta_2 - 1) \\
\lambda_3^\vee(\mathbf{x}_1^\vee, \mathbf{x}_2^\vee) &= (\delta_2 + 1)\mathbf{x}_1^\vee - \delta_1 \mathbf{x}_2^\vee - (\delta_2 + 1) \\
\lambda_4^\vee(\mathbf{x}_1^\vee, \mathbf{x}_2^\vee) &= \delta_2 \mathbf{x}_1^\vee - (\delta_1 + 1)\mathbf{x}_2^\vee + (\delta_1 - 1)
\end{aligned}$$

When $\lambda_m^\vee$ vanishes, it corresponds to the equation of the straight lines passing by the two vertices of the edge m. So the functions $r_{4+i}^\vee = \prod_{j=1, j\neq i}^4 \lambda_i^\vee, 1 \leq i \leq 4$ vanish on 3 edges and we are going to prove that $\mathbf{r}_{j+4}^\vee = \mathbf{curl}(r_{j+4}^\vee); \quad 1 \leq j \leq 4$ satisfy the previous conditions. On the face $\gamma_m^\vee$, for $m \neq j$, an explicit calculus gives:

$$\mathbf{r}_{j+4}^\vee \cdot \mathbf{n}_m^\vee = \left(\prod_{k=1, k\neq j, m}^4 \lambda_k^\vee \right) \mathbf{curl}(\lambda_m^\vee) \cdot \mathbf{n}_m^\vee + \lambda_m^\vee \mathbf{curl} \left(\prod_{k=1, k\neq j, m}^4 \lambda_k^\vee \right) \cdot \mathbf{n}_m^\vee = 0 \text{ on } \gamma_m^\vee$$

because $\mathbf{curl}(\lambda_m^\vee)$ is perpendicular at $\mathbf{n}_m^\vee$ and $\lambda_m^\vee = 0$.
As $div(\mathbf{r}_{j+4}^\vee) = 0$ we have

$$\int_{\partial K^\vee} \mathbf{r}_{j+4}^\vee \cdot \mathbf{n}^\vee d\sigma = 0 \text{ et donc } \int_{\gamma_j} \mathbf{r}_{j+4}^\vee \cdot \mathbf{n}_j^\vee d\sigma = 0.$$

We have to show that the $\mathbf{W}_K^\vee$-unisolvency can be obtained from $\mathbf{RT}_0$-unisolvency.

$$\left\{ \mathbf{w}^\vee \to \frac{1}{|\gamma_m^\vee|} \int_{\gamma_m^\vee} \mathbf{w}^\vee \cdot \mathbf{n}^\vee \, d\sigma, \mathbf{w}^\vee \to \frac{1}{|\gamma_m^\vee|} \int_{\gamma_m^\vee} \sigma \mathbf{w}^\vee \cdot \mathbf{n}^\vee \, d\sigma; 1 \leq m \leq 4 \right\}$$

Each element of $\mathbf{w}^\vee$ can be written $\mathbf{w}^\vee = \mathbf{w}_1^\vee + \mathbf{w}_2^\vee$ with $\mathbf{w}_1^\vee \in \mathbf{RT}_0$ et $\mathbf{w}_2^\vee \in \mathbf{Z}_K^\vee$. From the construction $\mathbf{Z}_K^\vee$, $\mathbf{w}_2^\vee$ is the *curl* of a function $q_2^\vee$ that vanishes at the quadrilateral's vertices. So the tangential gradient of q_2 vanishes at least at one point on each face. Let us remark that $\nabla q_2^\vee \cdot \mathbf{t}_m = \mathbf{w}_2^\vee \cdot \mathbf{n}_m$. If we suppose that $\mathbf{w}_1^\vee = 0$, then $\mathbf{w}_2^\vee$ verifies on each face $\gamma_m^\vee$:

$$\begin{aligned}
&\int_{\gamma_m^\vee} \mathbf{w}_2^\vee \cdot \mathbf{n}_m^\vee d\sigma = 0, \\
&\int_{\gamma_m^\vee} \sigma \mathbf{w}_2^\vee \cdot \mathbf{n}_m^\vee d\sigma = 0, \\
&\exists \sigma_0 \in \gamma_m^\vee; \mathbf{w}_2^\vee(\sigma_0) \cdot \mathbf{n}_m^\vee = 0.
\end{aligned} \tag{22}$$

As the restriction of $\mathbf{w}_2^\vee \cdot \mathbf{n}_m$ at the face $\gamma_m^\vee$ belongs to P_2, it is obvious that (22) implies $\mathbf{w}_2^\vee \cdot \mathbf{n}_m = 0$ on $\gamma_m^\vee$. On the other hand, for each $q^\vee$ we have the following green formula:

$$\int_{K^\vee} div(\mathbf{w_2^\vee}) q^\vee dx - \int_{K^\vee} \mathbf{w_2^\vee} \cdot \nabla q^\vee dx = \int_{\partial K^\vee} \mathbf{w_2^\vee} \cdot \mathbf{n} d\sigma$$

then

$$\int_{K^\vee} \mathbf{w_2^\vee} \cdot \nabla q^\vee dx = 0.$$

and finally $\mathbf{w_2^\vee} = 0$.

4.4.2 Hexahedron case

We consider now $\mathbf{W}_K$ a space of polynomial functions on K of dimension 18 in order to impose the mean value of the flux and to control its 2 first momenta on each face. We suppose that $P_0^3 \subset \mathbf{W}_K$ and $div(\mathbf{W}_K) = P_0$.

As in dimension 2, for each function $\mathbf{w}$ in $P_0^3 \oplus \mathbf{x} P_0$ we have

$$\forall p \in P_1, \int_{\gamma_m} p\mathbf{w}.\mathbf{n}\, d\sigma = \frac{1}{|\gamma_m|} \int_{\gamma_m} p\, d\sigma \int_{\gamma_m} \mathbf{w}.\mathbf{n}\, d\sigma.$$

Next we consider the following vectorial polynomial space:

$$\Psi_K = \left\{ \mathbf{w} \in \mathbf{W}_K;\ \text{pour}\ 1 \le m \le 8\,, \forall p \in P_1,\ \int_{\gamma_m} p\mathbf{w}.\mathbf{n}\, d\sigma = \frac{1}{|\gamma_m|} \int_{\gamma_m} p\, d\sigma \int_{\gamma_m} \mathbf{w}.\mathbf{n}\, d\sigma \right\}. \tag{23}$$

and the following set of degrees of freedom:

$$\Sigma_K = \left\{ \mathbf{w} \to \int_{\gamma_m} \mathbf{w}.\mathbf{n}\, d\sigma; 1 \le m \le 6 \right\}$$

.

First approach Let us recall the space BDM in dimension 3:

$$\begin{aligned} BDM_{[k]} = \{ & \mathbf{w} = \mathbf{v}(\mathbf{x}_1^\vee, \mathbf{x}_2^\vee, \mathbf{x}_3^\vee) \\ & + \sum_{i=0}^{k} r_i \mathbf{curl}(0, 0, \mathbf{x}_1^\vee \mathbf{x}_2^{\vee\, i+1} \mathbf{x}_3^{\vee\, k-i}) + \sum_{i=0}^{k} s_i \mathbf{curl}(\mathbf{x}_2^\vee \mathbf{x}_3^{\vee\, i+1} \mathbf{x}_1^{\vee\, k-i}, 0, 0) \\ & + \sum_{i=0}^{k} t_i \mathbf{curl}(\mathbf{x}_3^\vee \mathbf{x}_1^{\vee\, i+1} \mathbf{x}_2^{\vee\, k-i}, 0, 0),\ \mathbf{v}(\mathbf{x}_1^\vee, \mathbf{x}_2^\vee, \mathbf{x}_3^\vee) \in (P_k)^3, r_i, s_i, t_i \in \mathbb{R} \Big\} \end{aligned}$$

And we note $BDM_{[1]}^K$ the following space:

$$BDM_{[1]}^K = \left\{ \mathcal{P}_{K^\vee} \circ \mathbf{w}^\vee \circ \left(F_K^\sharp\right)^{-1}; \mathbf{w}^\vee \in BDM_{[1]} \right\}$$

. We have $dim(BDM_{[1]}^K) = 18$ and we choose $\mathbf{W}_K = BDM_{[1]}^K$.
Obviously we have $P_0^3 \subset \Psi_K$ and $div(\Psi_K) = P_0$. For each polynomial $\mathbf{w}$ we impose :

$$\widehat{I}_m(\mathbf{w}) = \int_{\widehat{\gamma}_m} \mathbf{w}.\mathbf{n}\, d\sigma \ , \quad \widehat{I}_{m+6}(\mathbf{w}) = \int_{\widehat{\gamma}_m} \sigma_1\, \mathbf{w}.\mathbf{n}\, d\sigma, \quad \widehat{I}_{m+12}(\mathbf{w}) = \int_{\widehat{\gamma}_m} \sigma_2\, \mathbf{w}.\mathbf{n}\, d\sigma \quad 1 \leq m \leq 6.$$

and

$$\check{I}_m(\mathbf{w}) = \int_{\check{\gamma}_m} \mathbf{w}.\mathbf{n}\, d\sigma, \quad \check{I}_{m+6}(\mathbf{w}) = \int_{\check{\gamma}_m} \sigma_1\, \mathbf{w}.\mathbf{n}\, d\sigma, \quad \check{I}_{m+12}(\mathbf{w}) = \int_{\check{\gamma}_m} \sigma_2\, \mathbf{w}.\mathbf{n}\, d\sigma \quad 1 \leq m \leq 6.$$

where σ_1 and σ_2 are the curvilinear abscissa of the face.
We consider the basis $\{\mathbf{r}_j\}_{j=1,\ldots,18}$ of the finite element $BDM_{[1]}$ (i.e. $\widehat{I}_m(\mathbf{r}_j) = \delta_{m,j},\ 1 \leq m, j \leq 18$ and the square matrix T of order 18 such that $T_{m,j} = \check{I}_m(\mathbf{r}_j)$. T can be written $T = I + B$ where B is matrix depending only on the distortion parameters. We deduce easily the following propositions:

Proposition 4.9 *For each hexahedron K such that $\|B\| \leq 1$, (K, Ψ_K, Σ_K) is a finite element of Raviart-Thomas type..*

Proposition 4.10 *Moreover if we suppose that there exists a real $\alpha > 0$ such that for each $K \in \tau_h$ we have $\|B\| \leq 1 - \alpha$ then the assumptions of Propositions 4.5 and 4.6 are verified.*

And finally we have the expected convergence in $O(h)$ of the solution obtained by using our pseudo-conform finite element.
Remark: It is difficult to characterise in another way what are the admissible hexahedra (i.e. hexahedra such that (K, Ψ_K, Σ_K) is a finite element). However numerically one can verify the class admissible hexahera is large, for instance the figure 5 shows some examples of admissible hexahedra obtained by deformations of the cube.

Second approach Here we decompose $\mathbf{W}_K^\vee = \mathbf{RT}_0 \oplus \mathbf{Z}_K^\vee$ where $\mathbf{Z}_K^\vee$ is a vectorial polynomial space of dimension 12. These functions are denoted $(\mathbf{r}_{i+6}^\vee; \quad 1 \leq i \leq 12)$.
To built the basis functions, we proceed as in 2 case by introducing the functions:

$$\lambda_m^\vee(\mathbf{x}_1^\vee, \mathbf{x}_2^\vee, \mathbf{x}_3^\vee) = det\left(\mathbf{x}^\vee - \mathbf{b}_m^\vee | \mathbf{b}_{m,3}^\vee - \mathbf{b}_{m,2}^\vee | \mathbf{b}_{m,4}^\vee - \mathbf{b}_{m,1}^\vee\right)$$

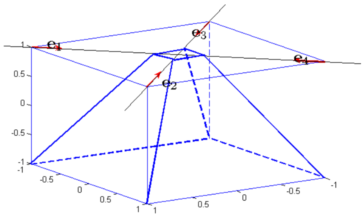
When $\lambda_m^\vee$ vanishes, it corresponds to the equation of the plan passing by the 4 vertices of the face m. And we consider

$$r_{i+8}^\vee = \prod_{j=1, j\neq i}^{6} \lambda_i^\vee, 1 \leq i \leq 6$$

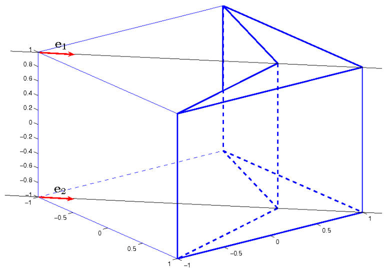
For each face $\gamma_m^\vee$, we note $\mathbf{t}_m^{(1),\vee}$ and $\mathbf{t}_m^{(2),\vee}$ a basis of tangent vectors at the face $\gamma_m^\vee$ (for instance we can take the medians of the face) and we define for $1 \leq m \leq 6$

$$\mathbf{r}_{2m-1+6}^\vee = \mathbf{curl}(r_{m+8}^\vee \mathbf{t}_m^{(1),\vee}) \text{ et } \mathbf{r}_{2m+6}^\vee = \mathbf{curl}(r_{m+8}^\vee \mathbf{t}_m^{(2),\vee}),$$

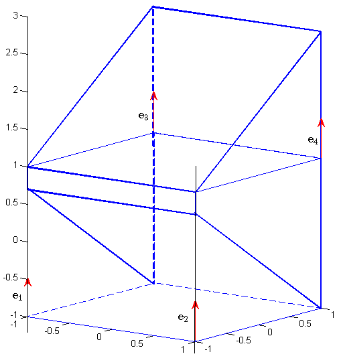
and we have the following proposition:



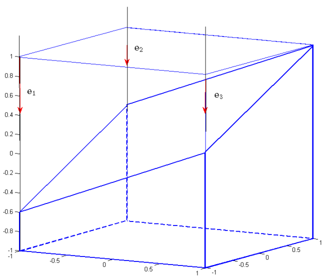
[example 1]



[example 2]



[example 3]



[example 4]

Figure 5: Examples of admissible hexahedra

Proposition 4.11

$$\int_{\gamma_m^\vee} \mathbf{r}_{i+6}^\vee \cdot \mathbf{n}_m^\vee d\sigma = 0 \quad 1 \le i \le 12, \quad 1 \le m \le 6.$$

Proof. We have $\mathbf{curl}(r_m^\vee \mathbf{t}_m^{(j),\vee}) = \nabla r_m^\vee \wedge \mathbf{t}_m^{(j),\vee}$ and for $k \neq m$ we obtain

$$\nabla r_m^\vee = \left(\prod_{j\neq i,k}^{6} \lambda_j\right) \nabla \lambda_k^\vee + \lambda_k^\vee \nabla \left(\prod_{j\neq i,k}^{6} \lambda_j^\vee\right)$$

and

$$\mathbf{curl}(r_m^\vee \mathbf{t}_m^{(j),\vee} \cdot \mathbf{n}_k^\vee = \left(\left(\prod_{j\neq i,k}^{6} \lambda_j^\vee\right) \nabla \lambda_k^\vee\right) \cdot \mathbf{n}_k^\vee + \left(\lambda_k^\vee \nabla \left(\prod_{j\neq i,k}^{6} \lambda_j^\vee\right)\right) \cdot \mathbf{n}_k^\vee = 0 \text{ sur } \gamma_k^\vee,$$

because $\nabla \lambda_k^\vee$ is perpendicular at the face $\gamma_k^\vee$ and $\lambda_k^\vee$ vanishes on $\gamma_k^\vee$.
As the divergence of the functions $\mathbf{r}_{i+6}^\vee$ are 0

$$\int_{\partial K^\vee} \mathbf{curl}(r_{m+8}^\vee \mathbf{t}_m^{(j),\vee}) \cdot \mathbf{n}^\vee d\sigma = 0$$

then

$$\int_{\partial \gamma_m^\vee} \mathbf{curl}(r_{m+8}^\vee \mathbf{t}_m^{(j),\vee}) \cdot \mathbf{n}_m^\vee d\sigma = 0$$

■

We have to show that the $\mathbf{W}_K^\vee$-unisolvency for the degrees of freedom

$$\left\{\mathbf{w}^\vee \to \frac{1}{|\gamma_m^\vee|} \int_{\gamma_m^\vee} \mathbf{w}^\vee \cdot \mathbf{n}^\vee \, d\sigma, \mathbf{w}^\vee \to \frac{1}{|\gamma_m^\vee|} \int_{\gamma_m^\vee} \sigma_1 \mathbf{w}^\vee \cdot \mathbf{n}^\vee \, d\sigma, \right.$$
$$\left. \mathbf{w}^\vee \to \frac{1}{|\gamma_m^\vee|} \int_{\gamma_m^\vee} \sigma_2 \mathbf{w}^\vee \cdot \mathbf{n}^\vee \, d\sigma; 1 \le m \le 6 \right\}$$

can be obtained from $\mathbf{RT}_0$-uni solvency. Each $\mathbf{w}^\vee$ can be decomposed $\mathbf{w}^\vee = \mathbf{w}_1^\vee + \mathbf{w}_2^\vee$ with $\mathbf{w}_1^\vee \in \mathbf{RT}_0$ and $\mathbf{w}_2^\vee \in \mathbf{Z}_K^\vee$. From the definition of $\mathbf{Z}_K^\vee$, we have $\mathbf{w}_2^\vee = \mathbf{curl}(q_2^\vee \mathbf{t}_m^{(j),\vee}) = \nabla q_2^\vee \wedge \mathbf{t}_m^{(j),\vee}$ where $q_2^\vee$ vanishes on the edges and is of constant sign on each face of the hexahedron. Suppose that $\mathbf{w}_1^\vee = 0$, then $\mathbf{w}_2^\vee$ verify on each face $\gamma_m^\vee$

$$\begin{aligned} \int_{\gamma_m^\vee} \mathbf{w}_2^\vee \cdot \mathbf{n}_m^\vee d\sigma = 0, \\ \int_{\gamma_m^\vee} \sigma_1 \mathbf{w}_2^\vee \cdot \mathbf{n}_m^\vee d\sigma = 0, \\ \int_{\gamma_m^\vee} \sigma_2 \mathbf{w}_2^\vee \cdot \mathbf{n}_m^\vee d\sigma = 0. \end{aligned} \tag{24}$$

As

$$\mathbf{w}_2^\vee \cdot \mathbf{n}_m^\vee = \nabla q_2^\vee \cdot \mathbf{t}_m^{(j),\vee} \wedge n_m^\vee = \nabla q_2^\vee \cdot (\alpha \mathbf{t}_m^{(1),\vee} + \beta \mathbf{t}_m^{(2),\vee}) \quad (\alpha, \beta \in \mathbb{R}).$$

we have

$$\int_{\gamma_m^\vee} (\alpha\sigma_1 + \beta\sigma_2) \nabla q_2^\vee \cdot (\alpha \mathbf{t}_m^{(1),\vee} + \beta \mathbf{t}_m^{(2),\vee}) d\sigma = 0 \quad (\alpha, \beta \in \mathbb{R}).$$

By integration by part and taking account that $q_2^\vee$ vanishes on $\partial\gamma_m^\vee$ we obtain

$$\int_{\gamma_m^\vee} q_2^\vee d\sigma = 0.$$

As $q_2^\vee$ is of constant sign, we deduce that it vanishes on $\gamma_m^\vee$ as well as $\mathbf{w}_2^\vee \cdot \mathbf{n}_m^\vee$. On the other hand, for each polynomial $q^\vee$ we have:

$$\int_{K^\vee} div(\mathbf{w_2}^\vee) q^\vee dx - \int_{K^\vee} \mathbf{w_2}^\vee \cdot \nabla q^\vee dx = \int_{\partial K^\vee} \mathbf{w_2}^\vee \cdot \mathbf{n} q^\vee \, d\sigma$$

then

$$\int_{K^\vee} \mathbf{w_2}^\vee \cdot \nabla q^\vee dx = 0.$$

and finally $\mathbf{w_2}^\vee = 0$.
Two propositions analogous at 4.9 et 4.10 can be expressed.

5 Some numerical tests

We take $\Omega = (]0,1[^N$and the exact solution is $u(x) = \prod_{i=1}^N sin(\pi x_i)$.
Our goal is to test our pseudo-conforming finite elements on general quadrilateral and hexahedra (i.e. non parallelepipedic or asymptotically parallelepipedic when the mesh is refined). Consequently, we chose meshes constituted by a pattern whose the shape is the same for all the refined meshes used. We considered two types of mesh, In 2D, the first mesh is a mesh in chevron given in [3] and the second is a mesh in honeycomb, In 3D the fisrt mesh is constituted of truncated pyramid and the second is a 3D-generalisation of mesh in chevron When we used the RT_0 finite element extended to general quadrilaterals or hexahera (i.e. with a non-linear Piola transform), the numerical tests confirm the lost of convergence of the solution in $Hdiv$ (see Figures 8 and 9)

Concerning our new pseudo-conforming mixed finite elements, the results are numerically the same by using the two different approaches to build the basis functions. The convergence curves on Figure 10 correspond to the choice of Ψ_K given by (22) and as expected u_h (resp. $\mathbf{p_h}$) converges in L^2 (resp. $Hdiv$) with the order 1.

5.0.3 Hexahedron case

The convergence curves on Figure 11 corresponds to the choice of Ψ_K given by (23) and as in the 2d case u_h (resp. $\mathbf{p_h}$) converges in L^2 (resp. $Hdiv$) with the order 1.

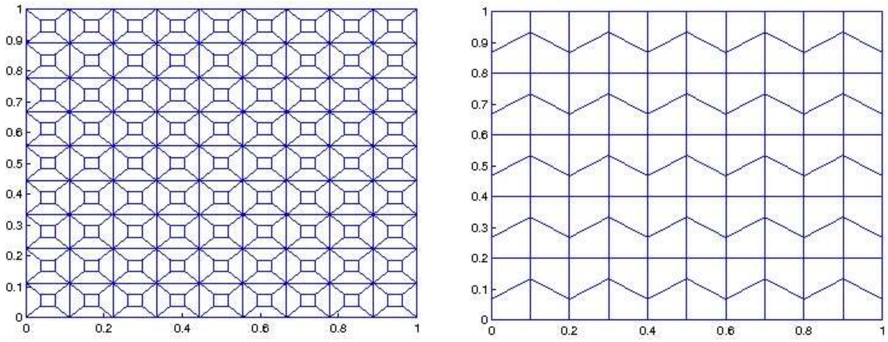


Figure 6: Meshes in honeycomb and chevron

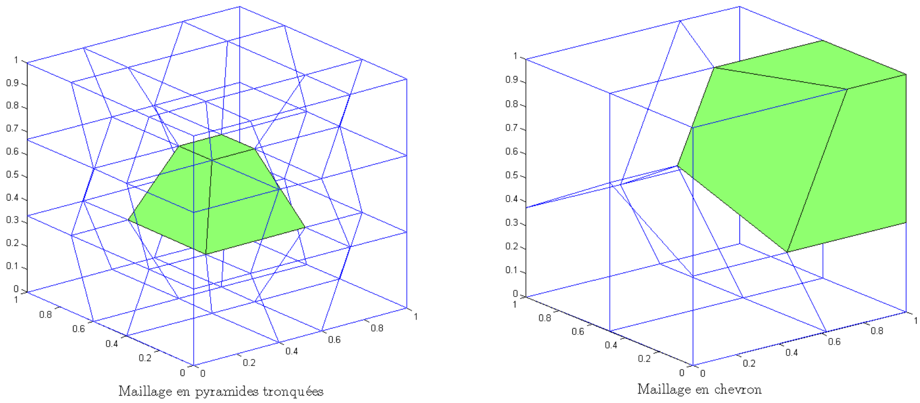


Figure 7: Meshes in truncated pyramid and chevrons

6 Conclusion

One of the motivations of this work refers to the loss of convergence problem when using classical mixed finite elements on quadrilaterals and hexahedra. The pseudo-conforming finite elements are a good answer to this problem and the theoretical part of this paper allows us to build pseudo-conforming finite elements of higher order without any particular difficulty. We can also notice that the Raviart-Thomas finite elements are often used in *a*

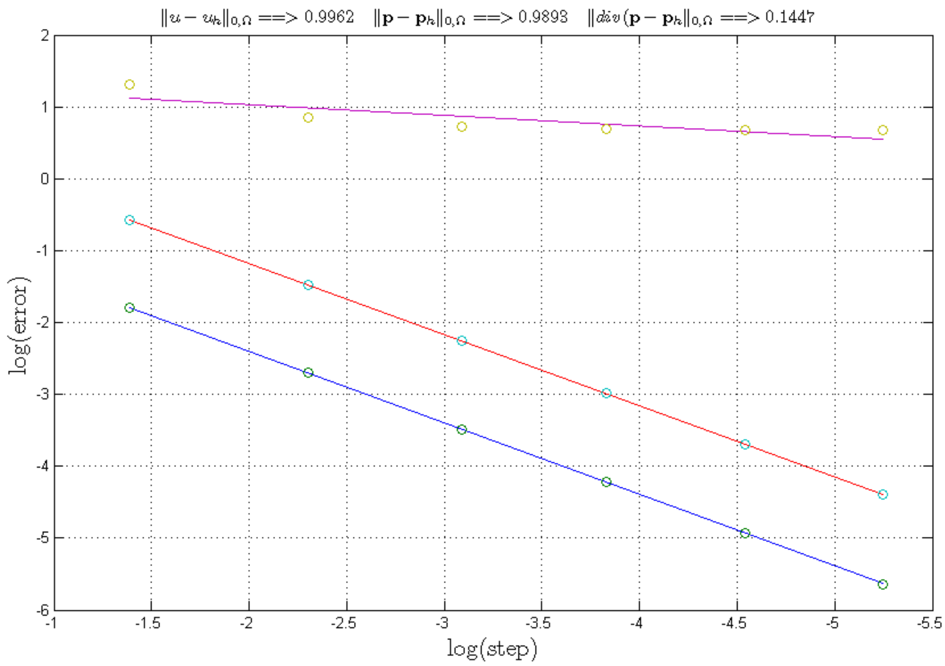


Figure 8: 2D convergence curves of extented RT_0-FE on meshes in chevron

posteriori error estimates (see for instance [25], [18], [1]) and our pseudo-conforming mixed finite can be used when the quadrilateral or hexahedron meshes are considered.

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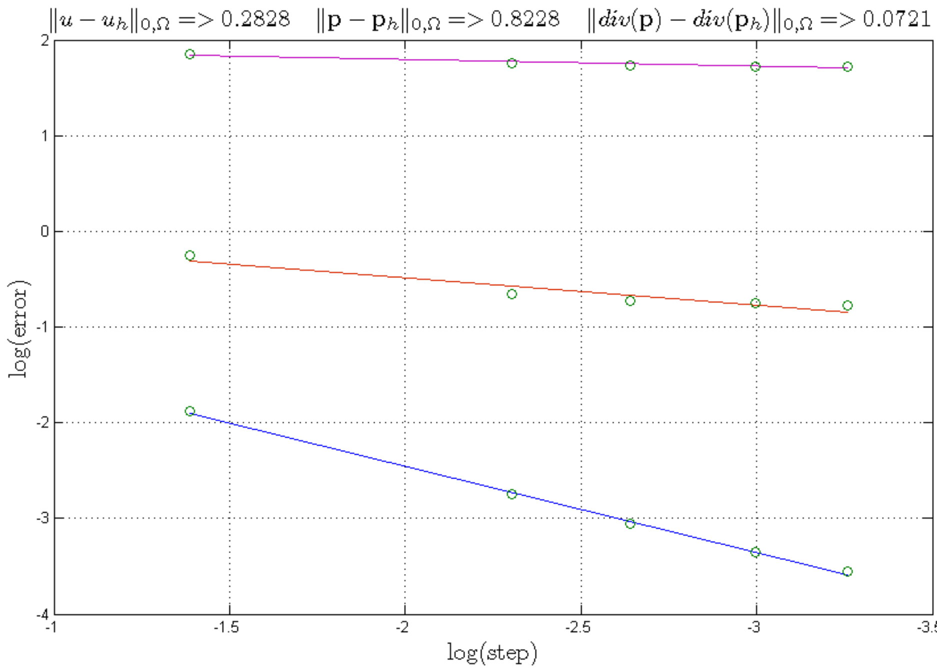

Figure 9: 3D convergence curves of extented RT_0-FE on meshes in chevron

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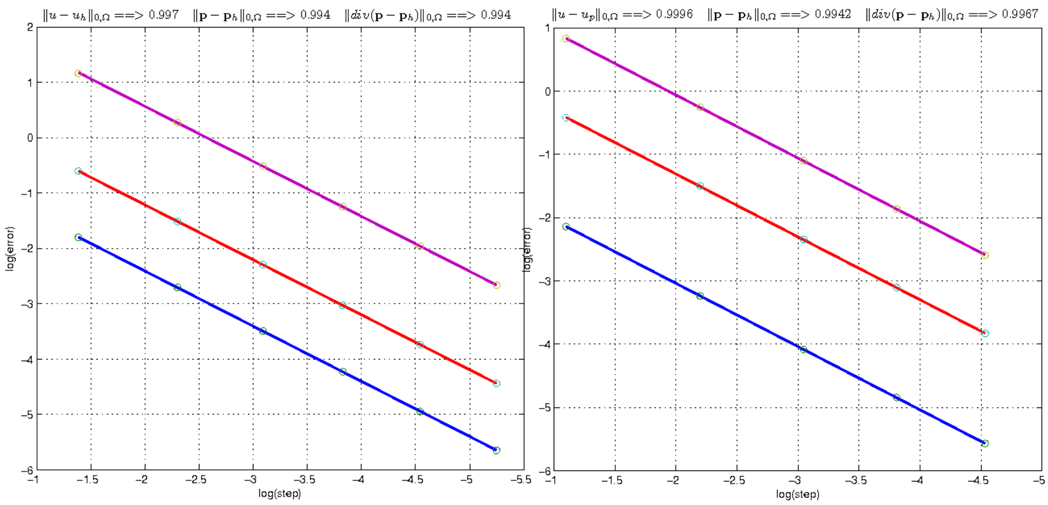

Figure 10: Convergence curves: Meshes in chevron (left), Meshes in honeycomb (right)

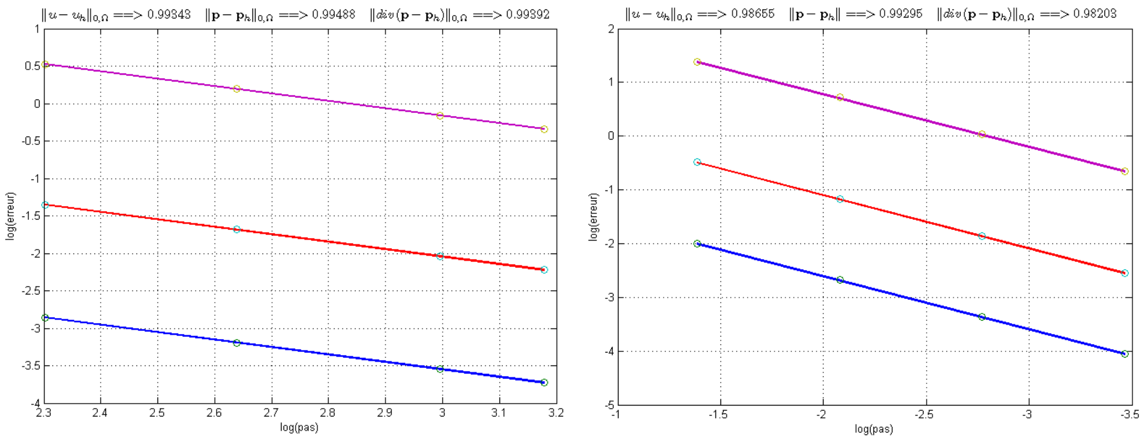

Figure 11: Convergence curves: Meshes in chevron (left), Meshes in truncated pyramid (right)

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