Argument filterings and usable rules in higher-order rewrite systems

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Argument Filterings and Usable Rules in Higher-Order Rewrite Systems

SUZUKI Sho† KUSAKARI Keiichirou†
Frédéric BLANQUI‡

† Graduate School of Information Science, Nagoya University
‡ INRIA, France

Abstract: The static dependency pair method is a method for proving the termination of higher-order rewrite systems *à la* Nipkow. It combines the dependency pair method introduced for first-order rewrite systems with the notion of strong computability introduced for typed λ-calculi. Argument filterings and usable rules are two important methods of the dependency pair framework used by current state-of-the-art first-order automated termination provers. In this paper, we extend the class of higher-order systems on which the static dependency pair method can be applied. Then, we extend argument filterings and usable rules to higher-order rewriting, hence providing the basis for a powerful automated termination prover for higher-order rewrite systems.

1 Introduction

Various extensions of term rewriting systems (TRSs) [22] for handling functional variables and abstractions have been proposed [11, 19, 9, 20, 12]. In this paper, we consider higher-order rewrite systems (HRSs) [19], that is, rewriting on β-normal η-long simply-typed λ-terms using higher-order matching. For example, the typical higher-order function *foldl* can be defined by the following HRS R_{foldl}:

$$\begin{cases} \text{foldl}(\lambda xy.F(x,y), X, \text{nil}) \to X \\ \text{foldl}(\lambda xy.F(x,y), X, \text{cons}(Y,L)) \\ \qquad\qquad \to \text{foldl}(\lambda xy.F(x,y), F(X,Y), L) \end{cases}$$

Then the function sum computing the sum of the elements can be defined by the HRS R_{sum}, which is the union of R_{foldl} and the following rules:

$$\begin{cases} \text{add}(0,Y) \to Y, \quad \text{add}(\text{s}(X),Y) \to \text{s}(\text{add}(X,Y)) \\ \text{sum}(L) \to \text{foldl}(\lambda xy.\text{add}(x,y), 0, L) \end{cases}$$

Also, the function len can be defined by the HRS R_{len}, which is the union of R_{foldl} and the following rule:

$$\text{len}(L) \to \text{foldl}(\lambda xy.\text{s}(x), 0, L)$$

The static dependency pair method is a method for proving the termination of higher-order rewrite systems. It combines the dependency pair method introduced for first-order rewrite systems [1] with Tait and Girard's notion of strong computability introduced for typed λ-calculi [7]. It was first introduced for simply-typed term rewriting systems (STRSs) [14] and then extended to HRSs [16]. The static dependency pair method consists in showing the non-loopingness of each static recursion component independently, the set of static recursion components being computed through some static analysis of the possible sequences of function calls.

This method applies only to plain function-passing (PFP) systems. In this paper, we provide a new definition of PFP that significantly enlarges the class of systems on which the method can be applied. It is based on the notion of accessibility introduced in [3] and extended to HRSs in [2].

For the HRS $R_{\mathrm{sum}} \cup R_{\mathrm{len}}$, the static dependency pair method returns the following two components:

$$\left\{ \begin{array}{l} \mathrm{foldl}^\sharp(\lambda xy.F(x,y), X, \mathrm{cons}(Y,L)) \\ \qquad\qquad \to \mathrm{foldl}^\sharp(\lambda xy.F(x,y), F(X,Y), L) \end{array} \right\}$$
$$\left\{ \ \mathrm{add}^\sharp(\mathrm{s}(X), Y) \to \mathrm{add}^\sharp(X,Y) \ \right\}$$

The static dependency pair method proves the termination of the HRS $R_{\mathrm{sum}} \cup R_{\mathrm{len}}$ by showing the non-loopingness of each component.

In order to show the non-loopingness of a component, the notion of reduction pair is often used. Roughly speaking, it consists in finding a well-founded quasi-ordering in which the component rules are strictly decreasing and all the original rules are non-increasing.

Argument filterings, which consist in removing some arguments of some functions, provide a way to generate reduction pairs. First introduced for TRSs [1], it has been extended to STRSs [12, 15]. In this paper, we extend it to HRSs.

In order to reduce the number of constraints required for showing the non-loopingness of a component, the notion of usable rules is also very important. Indeed, a finer analysis of sequences of function calls show that not all original rules need to be taken into account when trying to prove the termination of a component. This analysis was first conducted for TRSs [5, 8] and has been extended to STRSs [21, 15]. In this paper, we extend it to HRSs.

All together, this paper provides a strong theoretical basis for the development of an automated termination prover for HRSs, by extending to HRSs some successful techniques used by modern state-of-the-art first-order termination provers like for instance [6, 8].

2 Preliminaries

We assume that the reader is familiar with notions for HRSs [18], and notions related with static dependency pair methods [16].

A *preterm* is generated from an infinite set of typed variables $\mathcal{V}$ and a set of typed function symbols Σ by λ-abstraction and λ-application. We denote by $\mathcal{T}$ the set of *(simply-typed) terms*, which is an η-long β-normal form. We denote by $t{\downarrow}$ the η-long β-normal form of t. In general, a term t is of the form $\lambda x_1 \ldots x_m.at_1 \ldots t_n$ where $a \in \Sigma \cup \mathcal{V}$. We abbreviate this by $\lambda\overline{x_m}.a(\overline{t_n})$. For a term $t \equiv \lambda\overline{x_m}.a(\overline{t_n})$, we define $top(t) = a$ and $args(t) = \{\overline{t_n}\}$. A *higher-order rewrite rule* is a pair (l, r) of terms, denoted by $l \to r$, such that $top(l) \in \Sigma$, $type(l) = type(r) \in \mathcal{B}$ and $FV(l) \supseteq FV(r)$. An HRS is a set of higher-order rewrite rules. The *reduction relation* $\underset{R}{\to}$ of an HRS R is defined by $s \underset{R}{\to} t$ iff $s \equiv C[l\theta{\downarrow}]$ and $t \equiv C[r\theta{\downarrow}]$ for some rewrite rule $l \to r \in R$, context $C[\,]$ and substitution θ. An HRS R is said to be *finitely branching* if $\{t' \mid t \underset{R}{\to} t'\}$ is a finite set for any term t.

A term t is said to be *terminating* or *strongly normalizing* for an HRS R, denoted by $SN(R,t)$, if there is no infinite rewrite sequence of R starting from t. We write $SN(R)$ if $\forall t.SN(R,t)$. A well-founded relation $>$ on terms is a *reduction order* if $>$ is closed under substitution and context. An HRS R is terminating iff $R \subseteq >$ for some reduction order $>$.

A term t is said to be *strongly computable* in an HRS R if $SC(R,t)$ holds, which is inductively defined on simple types as follows: $SN(R,t)$ if $type(t) \in \mathcal{B}$, and $\forall u \in \mathcal{T}_\alpha.(SC(R,u) \Rightarrow SC(R,(tu){\downarrow}))$ if $type(t) = \alpha \to \beta$. We also define the set $\mathcal{T}_{SC}^{args}(R) = \{t \mid \forall u \in args(t).SC(R,u)\}$.

3 Improved Static Dependency Pair Method

In this section, we introduce the static dependency pair method for plain function-passing (PFP) HRSs [16] but extend the class of PFP systems by redefining the notion of safe subterms by using the notion of accessibility [2].

Definition 3.1 The *stable subterms* of t are $SSub(t) = SSub_{FV(t)}(t)$ where $SSub_X(t) = \{t\} \cup SSub'_X(t)$, $SSub'_X(\lambda x.s) = SSub_X(s)$, $SSub'_X(a(\overline{t_n})) = \bigcup_{i=1}^n SSub_X(t_i)$ if $a \notin X$, and $SSub'_X(t) = \emptyset$ otherwise.

Definition 3.2 (Safe subterms - New definition) The set of *safe subterms* of a term l is $safe(l) = \bigcup_{l' \in args(l)} \{t{\downarrow} \mid t \in Acc(l'), FV(t) \subseteq FV(l')\}$ where $t \in Acc(l')$ (t is *accessible in* l') if either:

1. $t = l'$ or $t \in SSub(l')$, $type(t) \in \mathcal{B}$ and $FV(t) \subseteq FV(l')$,
2. $\lambda x.t \in Acc(l')$ and $x \notin FV(l')$,
3. $t(x\downarrow) \in Acc(l')$ and $x \notin FV(t) \cup FV(l')$,
4. $f(\overline{t_n}) \in Acc(l')$, $t_i = \lambda\overline{x_k}.t$, $type(t) \in \mathcal{B}$ and $\{\overline{x_k}\} \cap FV(t) = \emptyset$,
5. $x(\overline{t_n}) \in Acc(l')$, $t_i = t$ and $x \notin FV(\overline{t_n}) \cup FV(l')$.

Strictly speaking, $safe(l)$ may not be included in $Sub(l)$ and, because of (3), accessible terms are β-normal preterms not necessarily in η-long form.

Definition 3.3 (Plain Function-Passing [16]) An HRS R is *plain function-passing* (PFP) if for any $l \to r \in R$ and $Z(\overline{r_n}) \in Sub(r)$ such that $Z \in FV(r)$, there exists $k \leq n$ such that $Z(\overline{r_k})\downarrow \in safe(l)$.

For example, the HRS R_{foldl} displayed in the introduction is PFP, because $safe(\text{foldl}(\lambda xy.F(x,y), X, \text{cons}(Y, L))) = \{\lambda xy.F(x,y), X, \text{cons}(Y,L), Y, L\}$ and $F\downarrow \equiv \lambda xy.F(x,y) \in safe(\text{foldl}(\lambda xy.F(x,y), X, \text{cons}(Y,L)))$.

The definition of safeness given in [16] corresponds to case (1). This new definition therefore includes much more terms, mainly higher-order patterns [17]. This greatly increases the class of rules that can be handled and the applicability of the method since it reduces the number of dependency pairs.

For instance, the new definition allows us to handle the following rule:

$$D(\lambda x.\sin(Fx))y \to D(\lambda x.Fx)y \times \cos(Fy)$$

Indeed, $l' = \lambda x.\sin(Fx) \in Acc(l')$ by (1), $\sin(Fx) \in Acc(l')$ by (2), $Fx \in Acc(l')$ by (2) and $F \in Acc(l')$ by (3). Therefore, $safe(l) = \{l', \lambda x.Fx, y\}$. With the previous definition, we had $safe(l) = \{l', y\}$ only.

Lemma 3.4 Let R be an HRS, $l \to r \in R$, θ be a substitution. Then $l\theta\downarrow \in \mathcal{T}_{SC}^{args}(R)$ implies $SC(R, s\theta\downarrow)$ for any $s \in safe(l)$.

Proof. Refer to Lemma 4.3 in [16] and Lemma 10 in [2]. □

This definition of safeness can be further improved (in case 4) by using more complex interpretations for base types than just the set of strongly normalizing terms, but this requires to check more properties. We leave this for future work.

We now recall the definitions of static dependency pair, static recursion component and reduction pair, and the basic theorems concerning these notions, including the subterm criterion [16].

Definition 3.5 (Static dependency pair [16]) Let R be an HRS. All top symbols of the left-hand sides of rewrite rules, denoted by $\mathcal{D}_R$, are called *defined symbols*, whereas all other function symbols, denoted by $\mathcal{C}_R$, are *constructors*.

We define the *marked* term $t^\sharp$ by $f^\sharp(\overline{t_n})$ if t has the form $f(\overline{t_n})$ with $f \in \mathcal{D}_R$; otherwise $t^\sharp \equiv t$. Then, let $\mathcal{D}_R^\sharp = \{f^\sharp \mid f \in \mathcal{D}_R\}$.

We also define the set of *candidate subterms* as follows: $Cand(\lambda\overline{x_m}.a(\overline{t_n})) = \{\lambda\overline{x_m}.a(\overline{t_n})\} \cup \bigcup_{i=1}^{n} Cand(\lambda\overline{x_m}.t_i)$.

Now, a pair $\langle\, l^\sharp,\ a^\sharp(\overline{r_n})\,\rangle$, denoted by $l^\sharp \to a^\sharp(\overline{r_n})$, is said to be a *static dependency pair* in R if there exists $l \to r \in R$ such that $\lambda\overline{x_m}.a(\overline{r_n}) \in Cand(r)$, $a \in \mathcal{D}_R$, and $a(\overline{r_k})\downarrow \notin safe(l)$ for all $k \leq n$. We denote by $SDP(R)$ the set of static dependency pairs in R.

Example 3.6 Let PFP-HRS R_{ave} be the union of R_{sum}, R_{len} and the following rules:

$$\begin{cases} \text{sub}(X, 0) \to X, \quad \text{sub}(0, Y) \to 0 \\ \text{sub}(\text{s}(X), \text{s}(Y)) \to \text{sub}(X, Y) \\ \text{div}(0, \text{s}(Y)) \to 0 \\ \text{div}(\text{s}(X), \text{s}(Y)) \to \text{s}(\text{div}(\text{sub}(X, Y), \text{s}(Y))) \\ \text{ave}(L) \to \text{div}(\text{sum}(L), \text{len}(L)) \end{cases}$$

Then, the set $SDP(R_{\text{ave}})$ consists of the following 11

pairs:

$$\left\{\begin{array}{l}
\mathrm{foldl}^\sharp(\lambda xy.F(x,y), X, \mathrm{cons}(Y,L)) \\
\qquad\qquad \to \mathrm{foldl}^\sharp(\lambda xy.F(x,y), F(X,Y), L) \\
\mathrm{add}^\sharp(\mathrm{s}(X), Y) \to \mathrm{add}^\sharp(X,Y) \\
\mathrm{sum}^\sharp(L) \to \mathrm{foldl}^\sharp(\lambda xy.\mathrm{add}(x,y), 0, L) \\
\mathrm{sum}^\sharp(L) \to \mathrm{add}^\sharp(x,y) \\
\mathrm{sub}^\sharp(\mathrm{s}(X), \mathrm{s}(Y)) \to \mathrm{sub}^\sharp(X,Y) \\
\mathrm{div}^\sharp(\mathrm{s}(X), \mathrm{s}(Y)) \to \mathrm{div}^\sharp(\mathrm{sub}(X,Y), \mathrm{s}(Y)) \\
\mathrm{div}^\sharp(\mathrm{s}(X), \mathrm{s}(Y)) \to \mathrm{sub}^\sharp(X,Y) \\
\mathrm{len}^\sharp(L) \to \mathrm{foldl}^\sharp(\lambda xy.\mathrm{s}(x), 0, L) \\
\mathrm{ave}^\sharp(L) \to \mathrm{div}^\sharp(\mathrm{sum}(L), \mathrm{len}(L)) \\
\mathrm{ave}^\sharp(L) \to \mathrm{sum}^\sharp(L), \quad \mathrm{ave}^\sharp(L) \to \mathrm{len}^\sharp(L)
\end{array}\right.$$

Definition 3.7 (Static dependency chain [16]) Let R be an HRS. A sequence $u_0^\sharp \to v_0^\sharp, u_1^\sharp \to v_1^\sharp, \ldots$ of static dependency pairs is a *static dependency chain in* R if there exist $\theta_0, \theta_1, \ldots$ such that $v_i^\sharp\theta_i{\downarrow} \xrightarrow[R]{*} u_{i+1}^\sharp\theta_{i+1}{\downarrow}$ and $u_i\theta_i{\downarrow}, v_i\theta_i{\downarrow} \in \mathcal{T}_{SC}^{args}(R)$ for all i.

Note that, for all i, $u_i^\sharp\theta_i$ and $v_i^\sharp\theta_i$ are terminating, since strong computability implies termination.

Proposition 3.8 [16] Let R be a PFP-HRS. If there exists no infinite static dependency chain then R is terminating.

Definition 3.9 (Static recursion component [16]) Let R be an HRS. The *static dependency graph of* R is the directed graph in which nodes are $SDP(R)$ and there exists an arc from $u^\sharp \to v^\sharp$ to $u'^\sharp \to v'^\sharp$ if the sequence $u^\sharp \to v^\sharp$, $u'^\sharp \to v'^\sharp$ is a static dependency chain.

A *static recursion component* is a set of nodes in a strongly connected subgraph of the static dependency graph of R. We denote by $SRC(R)$ the set of static recursion components of R.

A static recursion component C is *non-looping* if there exists no infinite static dependency chain in which only pairs in C occur and every $u^\sharp \to v^\sharp \in C$ occurs infinitely many times.

Proposition 3.10 [16] Let R be a PFP-HRS such that there exists no infinite path in the static dependency graph. If all static recursion components are non-looping, then R is terminating.

Example 3.11 For the PFP-HRS R_{ave} in Example 3.6, the set $SRC(R_{\mathrm{ave}})$ consists of the following four static recursion components:

$$\left\{\begin{array}{l}
\mathrm{foldl}^\sharp(\lambda xy.F(x,y), X, \mathrm{cons}(Y,L)) \\
\qquad\qquad \to \mathrm{foldl}^\sharp(\lambda xy.F(x,y), F(X,Y), L)
\end{array}\right\}$$
$$\left\{\mathrm{add}^\sharp(\mathrm{s}(X), Y) \to \mathrm{add}^\sharp(X,Y)\right\}$$
$$\left\{\mathrm{sub}^\sharp(\mathrm{s}(X), \mathrm{s}(Y)) \to \mathrm{sub}^\sharp(X,Y)\right\}$$
$$\left\{\mathrm{div}^\sharp(\mathrm{s}(X), \mathrm{s}(Y)) \to \mathrm{div}^\sharp(\mathrm{sub}(X,Y), \mathrm{s}(Y))\right\}$$

To prove the non-loopingness of components, the notions of subterm criterion and reduction pair have been proposed. The subterm criterion was introduced on TRSs [8], and then extended to STRSs [14] and HRSs [16]. Reduction pairs [13] are an abstraction of the notion of weak-reduction order [1].

Definition 3.12 (Subterm criterion [16]) Let R be an HRS and $C \in SRC(R)$. We say that C satisfies the *subterm criterion* if there exists a function π from $\mathcal{D}_R^\sharp$ to non-empty sequences of positive integers such that:

- $u|_{\pi(top(u^\sharp))} \rhd_{sub} v|_{\pi(top(v^\sharp))}$ for some $u^\sharp \to v^\sharp \in C$,
- and the following conditions hold for any $u^\sharp \to v^\sharp \in C$:
  - $u|_{\pi(top(u^\sharp))} \trianglerighteq_{sub} v|_{\pi(top(v^\sharp))}$,
  - $\forall p \prec \pi(top(u^\sharp)).top(u|_p) \notin FV(u)$,
  - and $\forall q \prec \pi(top(v^\sharp)).q = \varepsilon \vee top(v|_q) \notin FV(v) \cup \mathcal{D}_R$.

Definition 3.13 (Reduction pair [1, 13]) A pair $(\gtrsim, >)$ of relations is a *reduction pair* if $\gtrsim$ and $>$ satisfy the following properties:

- $>$ is well-founded and closed under substitutions,
- $\gtrsim$ is closed under contexts and substitutions,
- and $\gtrsim \cdot > \subseteq >$ or $> \cdot \gtrsim \subseteq >$.

In particular, $\gtrsim$ is a *weak reduction order* if $(\gtrsim, \gtrsim \setminus \lesssim)$ is a reduction pair.

Proposition 3.14 [16] Let R be a PFP-HRS such that there exists no infinite path in the static dependency graph. Then, $C \in SRC(R)$ is non-looping if C satisfies one of the following properties:

- C satisfies the subterm criterion.
- There is a reduction pair $(\gtrsim, >)$ such that $R \subseteq \gtrsim$, $C \subseteq \gtrsim \cup >$ and $C \cap > \neq \emptyset$.

Example 3.15 Let $\pi(\mathrm{foldl}^\sharp) = 3$ and $\pi(\mathrm{add}^\sharp) = \pi(\mathrm{sub}^\sharp) = 1$. Then, every static recursion component C except the one for div (cf. Example 3.11) satisfies the subterm criterion. Hence, these static recursion components are non-looping.

4 Argument Filterings

An argument filtering generates a weak reduction order from an arbitrary reduction order. The method was first proposed on TRSs [1], and then extended to STRSs [12, 15]. In this section, we expand this technique to HRSs.

Definition 4.1 An *argument filtering function* is a function π such that, for every $f \in \Sigma$ of type $\alpha_1 \to \cdots \to \alpha_n \to \beta$ with $\beta \in \mathcal{B}$, $\pi(f)$ is either a positive integer $i \leq n$ if $\alpha_i = \beta$, or a list of positive integers $[i_1, \ldots, i_k]$ with $i_1, \ldots, i_k \leq n$. Then we also define $\pi(\lambda\overline{x_m}.a(\overline{t_n}))$ as follows:

$$\begin{cases} \lambda\overline{x_m}.\pi(t_i) & \text{if } a \in \Sigma \text{ and } \pi(a) = i \\ \lambda\overline{x_m}.a(\pi(t_{i_1}), \ldots, \pi(t_{i_k})) & \text{if } a \in \Sigma \text{ and } \pi(a) = [\overline{i_k}] \\ \lambda\overline{x_m}.a(\pi(t_1), \ldots, \pi(t_n)) & \text{if } a \in \mathcal{V} \end{cases}$$

Given an argument filtering π and a binary relation $>$, we define $s \gtrsim_\pi t$ by $\pi(s) > \pi(t)$ or $\pi(s) \equiv \pi(t)$, and $s >_\pi t$ by $\pi(s) > \pi(t)$. We also define the substitution θ_π by $\theta_\pi(x) \equiv \pi(\theta(x))$. Finally, we define the typing function $type_\pi$ after argument filtering as $type_\pi(a) = \alpha_{i_1} \to \cdots \to \alpha_{i_k} \to \beta$ if $a \in \Sigma$, $\pi(a) = [i_1, \ldots, i_k]$, $type(a) = \alpha_1 \to \cdots \alpha_n \to \beta$ and $\beta \in \mathcal{B}$; otherwise $type_\pi(a) = type(a)$.

In the examples, except stated otherwise, $\pi(f) = [1, \ldots, n]$ if $type(f) = \alpha_1 \to \cdots \to \alpha_n \to \beta$ and $\beta \in \mathcal{B}$. For instance, if $\pi(\mathrm{sub}) = [1]$ then $\pi(\mathrm{div}^\sharp(\mathrm{sub}(X, Y), \mathrm{s}(Y))) \equiv \mathrm{div}^\sharp(\mathrm{sub}(X), \mathrm{s}(Y))$.

Note that our argument filtering method never destroys the well-typedness, which is easily proved by induction on terms.

Theorem 4.2 For any argument filtering π and term $t \in \mathcal{T}$, $\pi(t)$ is well-typed under the typing function $type_\pi$ and $type_\pi(\pi(t)) = type(t)$.

In the following, we prove the soundness of the argument filtering method as a generating method of weak reduction orders. To this end, we first prove a lemma required for showing that $>_\pi$ and $\gtrsim_\pi$ are closed under substitution.

Lemma 4.3 $\pi(t\theta\downarrow) \equiv \pi(t)\theta_\pi\downarrow$.

Proof. By induction on preterm $t\theta$ ordered with $\underset{\beta}{\to} \cup \rhd_{sub}$. □

Theorem 4.4 For any reduction order $>$ and argument filtering function π, $\gtrsim_\pi$ is a weak reduction order.

Proof. It is easily shown that $s \gtrsim_\pi t \Rightarrow C[s] \gtrsim_\pi C[t]$ by induction on $C[\,]$. From Lemma 4.3, we have $s \gtrsim_\pi t \Rightarrow s\theta\downarrow \gtrsim_\pi t\theta\downarrow$, and $s >_\pi t \Rightarrow s\theta\downarrow >_\pi t\theta\downarrow$. Remaining properties are routine. □

Example 4.5 Consider the PFP-HRS R_{ave} in Example 3.6. Every static recursion component except $\{\mathrm{div}^\sharp(\mathrm{s}(X), \mathrm{s}(Y)) \to \mathrm{div}^\sharp(\mathrm{sub}(X, Y), \mathrm{s}(Y))\}$ is non-looping (cf. Example 3.15). We can prove its non-loopingness with the argument filtering method, by taking $\pi(\mathrm{sub}) = \pi(\mathrm{div}^\sharp) = [1]$, and the normal higher-order reduction ordering $>^n_{rhorpo}$, written $(>_{rhorpo})_n$ in [10]. From Proposition 3.14, the static recursion component for div is non-looping, and R_{div} is terminating.

5 Usable Rules

In order to reduce the number of constraints required for showing the non-loopingness of a component, the notion of usable rules is widely used. This notion

was introduced on TRSs [5, 8] and then extended to STRSs [21, 15]. In this section, we extend it to HRSs.

To illustrate the interest of this notion, we start with some example.

Example 5.1 We consider the data type heap ::= lf | nd(nat, heap, heap) and the PFP-HRS R_{heap} defined by the following rules:

$$\left\{\begin{array}{l}
\mathrm{add}(0,Y)\to Y,\quad \mathrm{add}(\mathrm{s}(X),Y)\to \mathrm{s}(\mathrm{add}(X,Y))\\
\mathrm{map}(\lambda x.F(x),\mathrm{nil})\to \mathrm{nil}\\
\mathrm{map}(\lambda x.F(x),\mathrm{cons}(X,L))\\
\qquad\to \mathrm{cons}(F(X),\mathrm{map}(\lambda x.F(x).L))\\
\mathrm{merge}(H,\mathrm{lf})\to H,\quad \mathrm{merge}(\mathrm{lf},H)\to H\\
\mathrm{merge}(\mathrm{nd}(X_1,H_{11},H_{12}),\mathrm{nd}(X_2,H_{21},H_{22}))\\
\qquad\to \mathrm{nd}(X_1,H_{11},\mathrm{merge}(H_{12},\mathrm{nd}(X_2,H_{21},H_{22}))\\
\mathrm{merge}(\mathrm{nd}(X_1,H_{11},H_{12}),\mathrm{nd}(X_2,H_{21},H_{22}))\\
\qquad\to \mathrm{nd}(X_2,\mathrm{merge}(\mathrm{nd}(X_1,H_{11},H_{12}),H_{21}),H_{22})\\
\mathrm{fl}(\lambda xyz.F(x,y,z),X,\mathrm{lf})\to X\\
\mathrm{fl}(\lambda xyz.F(x,y,z),X,\mathrm{nd}(Y,H_1,H_2))\\
\qquad\to F(X,\mathrm{fl}(F{\downarrow},X,H_1),\mathrm{fl}(F{\downarrow},X,H_2))\\
\mathrm{sumT}(H)\to \mathrm{fl}(\lambda xyz.\mathrm{add}(x,\mathrm{add}(y,z)),0,H)\\
\mathrm{hd}(\mathrm{nil})\to \mathrm{lf},\quad \mathrm{hd}(\mathrm{cons}(X,L))\to X\\
\mathrm{l2t}(\mathrm{nil})\to \mathrm{nil},\quad \mathrm{l2t}(\mathrm{cons}(H,\mathrm{nil}))\to \mathrm{cons}(H,\mathrm{nil})\\
\mathrm{l2t}(\mathrm{cons}(H_1,\mathrm{cons}(H_2,L)))\\
\qquad\to \mathrm{l2t}(\mathrm{cons}(\mathrm{merge}(H_1,H_2),\mathrm{l2t}(L)))\\
\mathrm{list2heap}(L)\to \mathrm{hd}(\mathrm{l2t}(\mathrm{map}(\lambda x.\mathrm{nd}(x,\mathrm{lf},\mathrm{lf}),L)))
\end{array}\right.$$

The static recursion components for fl consists of

$$\left\{\begin{array}{l}\mathrm{fl}^\sharp(\lambda xyz.F(x,y,z),X,\mathrm{nd}(Y,H_1,H_2))\\ \qquad\to \mathrm{fl}(\lambda xyz.F(x,y,z),X,H_i)\end{array}\right\}$$

for $i = 1, 2$, and their union. By taking $\pi(\mathrm{fl}) = 3$, these components satisfy the subterm criterion. The static recursion components for add, map and merge also satisfy the subterm criterion. Hence it suffices to show that the following three static recursion components for l2t are non-looping:

$$\left\{\begin{array}{l}\mathrm{l2t}^\sharp(\mathrm{cons}(H_1,\mathrm{cons}(H_2,L)))\\ \qquad\to \mathrm{l2t}^\sharp(\mathrm{cons}(\mathrm{merge}(H_1,H_2),\mathrm{l2t}(L)))\ \cdots(1)\end{array}\right\}$$
$$\left\{\ \mathrm{l2t}^\sharp(\mathrm{cons}(H_1,\mathrm{cons}(H_2,L)))\to \mathrm{l2t}^\sharp(L)\ \cdots(2)\ \right\}$$
$$\{(1),(2)\}$$

The component $\{(2)\}$ satisfies the subterm criterion. By taking $\pi(\mathrm{cons}) = [2]$ and $\pi(\mathrm{l2t}) = \pi(\mathrm{l2t}^\sharp) = 1$, we can orient the static dependency pairs (1) and (2) by using the normal higher-order recursive path ordering [10]: $\pi(\mathrm{l2t}^\sharp(\mathrm{cons}(H_1,\mathrm{cons}(H_2,L)))) \equiv \mathrm{cons}(\mathrm{cons}(L)) >^n_{rhorpo} \mathrm{cons}(L) \equiv \pi(\mathrm{l2t}^\sharp(\mathrm{cons}(\mathrm{merge}(H_1,H_2),\mathrm{l2t}(L))))$ and $\pi(\mathrm{l2t}^\sharp(\mathrm{cons}(H_1,\mathrm{cons}(H_2,L)))) \equiv \mathrm{cons}(\mathrm{cons}(L)) >^n_{rhorpo} L \equiv \pi(\mathrm{l2t}^\sharp(L))$. However, in contrast to Example 4.5, the non-loopingness of $\{(1)\}$ and $\{(1),(2)\}$ cannot be shown with the previous techniques. Indeed, we cannot solve the constraint $R_{\text{heap}} \subseteq \gtrsim$. More precisely, we cannot orient the rule for hd, because $\pi(\mathrm{hd}(\mathrm{cons}(X,L))) \equiv \mathrm{hd}(\mathrm{cons}(L))$ does not contain the variable X occurring in the right-hand side.

The notion of usable rule solves this problem, that is, it allows us to ignore the rewrite rule for hd for showing the non-loopingness of l2t.

Definition 5.2 (Usable rules) We denote $f >_{\text{def}} g$ if g is a defined symbol and there is some $l \to r \in R$ such that $top(l) = f$ and g occurs in r.

We define the set $\mathcal{U}(t)$ of usable rules of a term t as follows. If, for every $X(\overline{t_n}) \in Sub(t)$, $\overline{t_n}$ are distinct bound variables, then $\mathcal{U}(t) = \{l \to r \in R \mid f >^*_{\text{def}} top(l)$ for some $f \in \mathcal{D}_R$ occurs in $t\}$. Otherwise, $\mathcal{U}(t) = R$. The usable rules of a static recursion component C is $\mathcal{U}(C) = \bigcup\{\mathcal{U}(v^\sharp) \mid u^\sharp \to v^\sharp \in C\}$.

For each $\alpha \in \mathcal{B}$, we associate the new function symbols $\bot_\alpha$ and c_α with $type(\bot_\alpha) = \alpha$ and $type(\mathrm{c}_\alpha) = \alpha \to \alpha \to \alpha$. We define the HRS C_e as $C_e = \{\mathrm{c}_\alpha(x_1,x_2) \to x_i \mid \alpha \in \mathcal{B}, i = 1, 2\}$.

We omit the index α whenever no confusion arises.

When we show the non-loopingness of a static recursion component using a reduction pair, Proposition 3.14 requires showing that $R \subseteq \gtrsim$. The non-loopingness is not guaranteed by simply replacing R with $\mathcal{U}(C)$. We can supplement the gap with the HRS C_e.

Theorem 5.3 Let R be a finitely-branching PFP-HRS. Then $C \in SRC(R)$ is non-looping if there exists a reduction pair $(\gtrsim, >)$ such that $\mathcal{U}(C) \cup C_e \subseteq \gtrsim$, $C \subseteq \gtrsim \cup >$, and $C \cap > \neq \emptyset$.

Proof. From Lemma 5.7, 5.9 and 5.10. These lemmas will be given at the end of this section. □

Example 5.4 We show the termination of the PFP-HRS R_{heap} in Example 5.1. We have to show the non-loopingness of the components $\{(1)\}$ and $\{(1),(2)\}$. To this end, it suffices to show that the constraint $\mathcal{U}(\{(1),(2)\}) \cup C_e \subseteq \gtrsim$ can be solved (instead of $R_{\mathrm{heap}} \subseteq \gtrsim$). The usable rules of $\{(1),(2)\}$ are seven rules for merge and l2t. The weak reduction order $(>^n_{rhorpo})_\pi$ orient the rules. Since $C_e \subseteq (>^n_{rhorpo})_\pi$, we conclude that R_{heap} is terminating.

In the rest of this section, we present a proof of Theorem 5.3. We assume that R is a finitely-branching PFP-HRS, C is a static recursion component of R, and $\Delta = \{top(l) \mid l \to r \in R \setminus \mathcal{U}(C)\}$.

The key idea of the proof is to use the following interpretation I.

Thanks to the Well-ordering theorem, we assume that every non-empty set of terms T has a least element least(T).

Definition 5.5 For a terminating term $t \in \mathcal{T}_\alpha$, $I(t)$ is defined as follows: $\lambda x.I(t')$ if $t \equiv \lambda x.t'$, $a(\overline{I(t_n)})$ if $t \equiv a(\overline{t_n})$ and $a \notin \Delta$, and $\mathrm{c}_\alpha(a(\overline{I(t_n)}), Red_\alpha(\{I(t') \mid t \xrightarrow[R\setminus\mathcal{U}(C)]{} t'\}))$ if $t \equiv a(\overline{t_n})$ and $a \in \Delta$. Here, for each $\alpha \in \mathcal{B}$, $Red_\alpha(T)$ is defined as $\bot_\alpha$ if $T = \emptyset$; otherwise $\mathrm{c}_\alpha(u, Red_\alpha(T \setminus \{u\}))$ where $u \equiv \mathrm{least}(T)$. We also define θ^I by $\theta^I(x) \equiv I(\theta(x))$ for a terminating substitution θ.

Theorem 5.6 For any terminating t, $I(t)$ is well-typed and $type(I(t)) = type(t)$.

Proof. By induction on t ordered by $\rhd_{sub} \cup \underset{R}{\to}$. □

Lemma 5.7 Let t be a term and θ be a substitution such that $t\theta\!\downarrow$ is terminating. Then, $I(t\theta\!\downarrow) \xrightarrow[C_e]{*} I(t)\theta^I\!\downarrow \xrightarrow[C_e]{*} t\theta^I\!\downarrow$.

Proof. By induction on $(\{type(x) \mid x \in dom(\theta)\}, t)$ ordered by the lexicographic combination of the multiset extension $\rhd_s^{mul}$ of $\rhd_s$, and $\rhd_{sub} \cup \underset{R}{\to}$. □

Lemma 5.8 Let t be a term and θ be a permutation such that $t\theta\!\downarrow$ is terminating. Then, $I(t\theta\!\downarrow) \equiv I(t)\theta^I\!\downarrow$.

Proof. By induction on t ordered by $\rhd_{sub} \cup \underset{R}{\to}$. □

Lemma 5.9 Let $l \to r \in C \cup \mathcal{U}(C)$ and θ be a substitution such that $r\theta\!\downarrow$ is terminating. Then, $I(r\theta\!\downarrow) \equiv r\theta^I\!\downarrow$.

Proof. By induction on t, we can show the stronger property $I(t\theta\!\downarrow) \equiv t\theta^I\!\downarrow$ for any $l \to r \in C \cup \mathcal{U}(C)$ and $t \in Sub(r)$. □

Lemma 5.10 If $s \underset{R}{\to} t$ and s is terminating, then $I(s) \xrightarrow[\mathcal{U}(C)\cup C_e]{+} I(t)$.

Proof. From Lemma 5.7 and 5.9. □

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