Viscosity Solution

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Related Concepts

– Partial differential equation
– Weak solution
– Shape from shading
– Distance function
– Optimal control

Definition

Viscosity solution is a notion of weak solution for a class of partial differential equations of Hamilton-Jacobi type.

Background

A first order partial differential equation of the type

$$H(x, u(x), Du(x)) = 0 \tag{1}$$

is called an Hamilton-Jacobi equation. A function u is said to be a classical solution of(1) over a domain if u is continuous and differentiable over the entire domain and x, $u(x)$ and $Du(x)$ (the gradient of u at x) satisfy the above equation at every point of the domain. Consider the boundary value problem

$$|u'(x)| - 1 = 0 \text{ for } x \in (-1, 1),\ u(\pm 1) = 0. \tag{2}$$

By Rolle's theorem it is easily seen that classical solutions of the previous problem do not exist, whereas there exist infinite many weak solutions, i.e. continuous functions which satisfy the equation at almost every point (the saw-tooth solutions, see Figure 1-a).

At first this situation can seem rather atypical, but in fact it is nothing of the sort. For example, it concerns the distance functions which are widely used in computer vision. In particular, it is easy to see that the distance function of a closed curve in a plan, which typically has strong edges (see Figure 1), is almost everywhere a solution of the Eikonal equation

$$|Du(x)| - F(x) = 0 \tag{3}$$

with $F(x) = 1$ and $u(x) = 0$ on the curve. Also, in general, as in the case of equation (2), this last equation has no classical solution. Another typical example concerns the shape from shading problem which naturally yields in Hamilton-Jacobi equations having the same behavior. In particular, by modelling the problem with an orthographic camera, a directional front lighting and a lambertian surface, the problem consists then in solving an Eikonal equation in which the function F depends on the considered image.

It is therefore very important to have a theory which allows merely continuous functions to be solutions of Hamilton-Jacobi equation and to provide at the same time a way to select the relevant solution among the weak solutions of the problem.

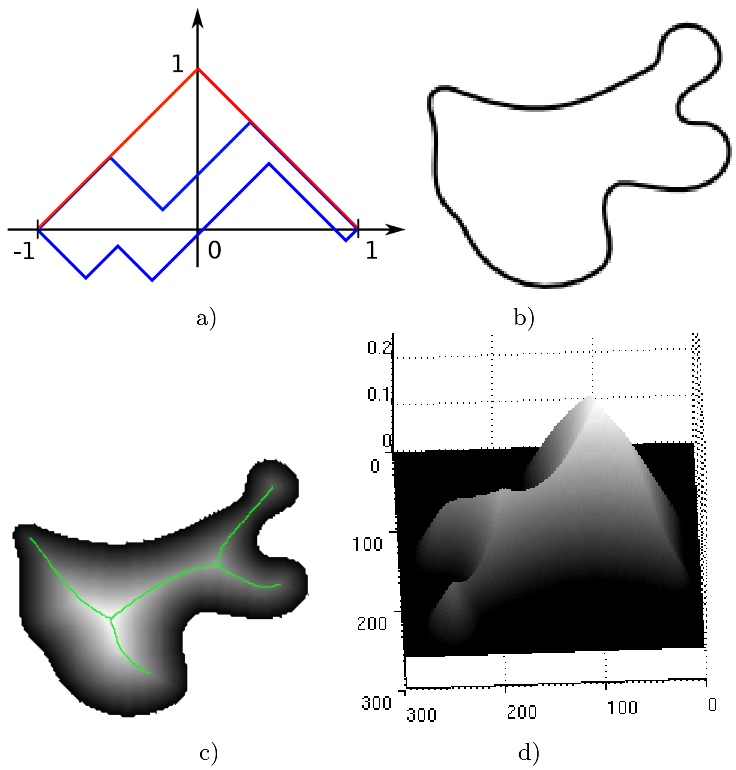

Fig. 1. *Distance functions in 1D and 2D:* a) three examples of weak solutions of equation (2); The red curve corresponds to the viscosity solution. b) Example of a closed curve in the plan. c) Distance function (represented as a color map) to the curve displayed in b). To improve the visibility, the distance function is only displayed inside the curve. As usually, one can distinguish strong edges we have partly highlighted by green curves. This gives the skeleton of the shape. d) 3D representation of the function distance c). The distance function is the viscosity solution of the Eikonal equation (3), with $F(x) = 1$ and $u(x) = 0$ on the curve.

Theory

The notion of viscosity solution was introduced at the beginning of the '80s by M.G.Crandall and P.L.Lions [11] and it is related to Kruzkov's theory of entropy solutions for scalar conservation laws.

The basic idea is to replace the differential $Du(x)$ at a point x where it does not exist (for example because of a kink in u) with the differential $D\phi(x)$ of a smooth function ϕ touching the graph of u, from above for the subsolution condition and from below for the supersolution one, at the point x.

Definition 1. i) *A continuous function u is said to be a viscosity subsolution of* (1) *if for any x and for any smooth function ϕ such that $u - \phi$ has a maximum point at x, then*

$$H(x, u(x), D\phi(x)) \leq 0;$$

ii) *A continuous function u is said to be a viscosity supersolution of* (1) *if for any x and for any smooth function ϕ such that $u - \phi$ has a minimum point at x, then*

$$H(x, u(x), D\phi(x)) \geq 0;$$

iii) *A continuous function u is said to be a viscosity solution of the Hamilton-Jacobi equation if it is a viscosity subsolution and supersolution.*

There is also an equivalent definition for viscosity solution which involves the notion of sub and super-differentials (see [5]).

It is straightforward to observe that solutions in the classical sense are viscosity solutions. Inversely, if a viscosity solution u is differentiable at x, then it solves the equation at this point in the classical sense. Hence, the notion of viscosity solution includes the one of classical solution.

By looking closely at the definition, one can understand rather intuitively how it allows a specific weak solution to be selected among the other ones and which of them is selected. In fact, this definition eliminates a certain type of edges. In particular for Hamilton-Jacobi equation (2), the definition allows upward edges, but not downward edges. Thus, in Figure 1, all the weak solutions of equation (2) which have downward edges are then excluded. Also, only the maximal weak solution (represented in red) is a viscosity solution.

In addition, the definition of viscosity solutions selects among the almost everywhere solutions the one which is consistent with the regularized problem. In fact, the name "viscosity" is motivated by the consistency of the notion with the method of "vanishing viscosity": the viscosity solution of equation (2) can be obtained as the limit for $\epsilon \to 0$ of the classical solutions of

$$-\epsilon u''(x) + |u'(x)| - 1 = 0 \text{ for } x \in (-1, 1),\ u(\pm 1) = 0. \tag{4}$$

(the term ϵ has the physical meaning of a viscosity coefficient).

The main characteristics of the notion of viscosity solution are

i) A very efficient and flexible way to prove uniqueness theorems and comparison principles;

ii) General existence results obtained via the adaptation of the classical Perron's method, by approximation arguments (such as the vanishing viscosity method in (3)), by means of representation formulas (via dynamic programming methods in optimal control theory), etc ;

iii) The stability of the notion of viscosity solution with respect to uniform convergence, and its generalizations, which allow to prove for example the convergence of numerical schemes;

iv) The correct formulation of various boundary conditions, including the classical Dirichlet, Neumann and oblique derivative conditions.

For good accounts of the viscosity solution theory we refer to [5], [7], [4], [12]. Finally, let us note that the notion and the theory of viscosity solutions have been extended to a large type of partial differential equations. In particular a number of results allow us to deal with second order equations [10], degenerate equations [9] and integro-differential equations [1,8]. The application to convergence of numerical schemes is also very important; see for example [6] and the appendix by Falcone in [5].

Applications

The range of applications of the notions of viscosity solution and Hamilton-Jacobi equations is enormous, including common class of partial differential equations such as evolutive problems and problems with boundary conditions, equations arising in optimal control theory (the Hamilton-Jacobi-Bellman equation), differential games (the Isaacs' equation), second-order equations arising in stochastic optimal control and stochastic differential games, geometric equations (mean curvature and Monge-Ampere equations), etc.

In computer vision, it has various applications. In particular, the distance functions and the Eikonal equations are widely used. Nowadays, thanks to the links between the viscosity solutions and the optimal control [5], one can easily prove that the distance functions correspond to the viscosity solutions of the Eikonal equations, which, moreover, provides various convenient tools for computing them. Also, all these notions have played an important role in shape representation [13,23], morphology [21,2], tractography [17,14,18] and in general, in image processing [22,3,18]. Furthermore, they are intensively used in the level set framework where the curves and the surfaces are represented by their signed distance functions [22,16]. The latter framework is used extensively for example in segmentation and 3D reconstruction.

Another main application of the notion of viscosity solution is to Shape From Shading problems, which give rise to first order differential and integro-differential equations of Hamilton-Jacobi type; see [15], [20] and the entry "*Shape from Shading*". A natural question in this context is why the viscosity solutions provide suitable solutions to this specific problem. In other words, why would the viscosity solutions have more sense than any other weak solution? In fact, here, the values of the viscosity solutions mainly come from its amazing stability combined with its consistency with the classical solution. To be more clear, let us consider the Shape From Shading problem with a continuous image I of a real scene. In such a case, to be physically plausible, the real surface u^* behind this image must be smooth (C^1); otherwise any infinitesimal displacement of the light direction would break this continuity property. Let I^* be the virtual image generated by the considered image formation model with surface u^*. I^* is necessary close to I, if not this would mean that the considered model is not appropriate. Then, thanks to the stability properties, the unique[1] viscosity solution u to the shape from shading Hamilton-Jacobi equation associated with the real image I is close to u^*, because u^* is the viscosity solution to the same equation in which we replace I by I^* (since it is also a solution in the classical sense). In other words, among all the weak solutions of the considered Shape From Shading equation (which has no solution in the classical sense with the real image I because of the modeling errors and the noise), the viscosity solution is necessarily close to the real surface which has been photographed.

Recommended Readings

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