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Intriguing Patterns in the Roots of the Derivatives of some Random Polynomials

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ABSTRACT

Our observations show that the sets of real (respectively complex) roots of the derivatives of some classical families of random polynomials admit a rich variety of patterns looking like discretized curves. To bring out the shapes of the suggested curves, we introduce an original use of fractional derivatives. Then we set several conjectures and outline a strategy to explain the presented phenomena. This strategy is based on asymptotic geometric properties of the corresponding complex critical points sets.

Key words: random polynomials, random matrices, roots of polynomials, fractional derivatives, critical points, patterns.

(AMS classification 60G99, 60H25, 60B20, 15B52, 65H04, 26A33, 35B38, 30C15)

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1. INTRODUCTION

Computer algebra systems are powerful tools for performing experiments and simulations in Mathematics. They serve to illustrate known properties, already rigorously proved, or conjectures; to find examples, to show that a bound is sharp, to estimate some values or behaviors. Once in a blue moon, experiments reveal unexpected patterns or phenomena. After the surprise, the repetition of experiments and variations to test robustness, comes the time to share the observations and the quest for explanations.

This paper relates my experiments, relying on the computer algebra system Maple, on the roots sets of univariate polynomials of medium degrees, and of their iterated derivatives. Generations of mathematicians studied these basic objects, so it seemed unlikely that simple graphics should uncover any surprising feature. The originality of the presented approach was to choose random polynomials, compute roots of a great number of derivatives and consider averaged objects and phenomena. I started observing the real roots and then looked at the complex ones, as they are more amenable to algebraic interpretations.

Random matrices are matrix-valued random variables. Their study have encountered a great deal of interest in the last decades, since many important properties of disordered physical systems can be represented mathematically using eigenvectors and eigenvalues of matrices with elements drawn randomly from statistical distributions. Their characteristic polynomials form a special class of random polynomials. See [14], or for a first insight the article Random matrix in Wikipedia . Random polynomials is a classical field of interest in Mathematics and Statistics; several families of random polynomials have been described in great details, see [8]. The number and distribution of real and complex roots of random polynomial present regular structures (see section 2 below) which are statistical consequences of the properties of their coefficients distributions. This is also the case for eigenvalues of random matrices, see [5].

For a fixed degree n, we consider several bases $g_i(x)$ of polynomials of degree at most n. We form the polynomial $f := \sum_{i=0}^{n} a_i g_i(x)$, the set of coefficients a_i being instances of $n+1$ independent normal centered standard distributions. In our experiments, we also consider characteristic polynomials of matrices whose entries are independent normal centered standard distributions.

A critical point of a polynomial $f(x)$ is a root of its derivative $f'(x)$. As random polynomials are generic, they admit only critical points that are not root of f; in the sequel we will only be interested by these critical points. By Rolle theorem, between two roots of f there is at least a root of f' while in the complex plane, by Gauss-Lucas theorem, the critical points of f are contained in the convex hull of the roots of f. There are several improved versions of this theorem, see the excellent book [16] which contains many fine results and enlightening historical notes.

Our general project is to concentrate on some families of random polynomials and set new conjectures, on the set of their critical points, suggested by experiments, observations and numerical evidences. It extends our previous works [11], [7], [9], [10]. Our conjectures, hopefully transformed into theorems, could then act as an oracle and indicate the estimated number and locations of the roots of a random polynomial and of its derivatives. This information could be used to derive better average complexity bounds for roots isolation algorithms.

The paper is organized as follows. In section 2, different families of random polynomials are examinated; some of their properties will be recalled and illustrated. Section 3 is devoted to our experiments on the sets of real roots of these polynomials and their derivatives; we organize them in a Variation diagram. Then we point out intriguing patterns and set a conjecture to try to express formally a part of the observed phenomena. Section 4 introduces our definition of a polynomial bivariate factor P of the fractional derivatives of a polynomial f, P induces a continuation between the roots sets of f and xf'. With this tool, we define an algebraic spline curve we call the "stem" of the polynomial f; it is of particular interest for random polynomials, In addition, section 4 describes experimental results on the stems, points out another intriguing phenomenon and poses some conjectures which leads to analyze the influence of the complex roots of f. Section 5 concentrates on the relative locations of the complex roots of f and f'. For some random polynomials, an interesting pairing is observed and its consequences explored. Finally we conclude discussing a tentative analysis and synthesis of our observations based on the symmetries of the limit distribution of the complex roots of f.

2. RANDOM POLYNOMIALS

The study of random polynomials is a classical and very active subject in Mathematics and Statistics It is at the core of extensive recent research and has also many applications in Physics and Economics; two books [2] and [8] are dedicated to it. Already in 1943, Mark Kac [13] gave an explicit formula for the expectation of the number of roots of a polynomial in a class that now bears his name (see below). The subject is naturally related to the study of eigenvalues of random matrices with its applications in Physics, see [5].

For a fixed degree n, we consider several bases $g_i(x)$ of polynomial of degree at most n, then we form the polynomial

$$f := \sum_{i=0}^{n} a_i g_i(x).$$

The coefficients a_i being instances of $n+1$ independent normal centered standard distributions $N(0,1)$. We are concerned with averaged asymptotic behaviors when n tends to infinity, but in our experiments we chose n between 32 and 128. We also considered characteristic polynomials of matrices with various shapes whose entries are independent normal centered standard distributions.

Let's start with the following methodological point. In statistics, averaged properties are generally observed through a series of realizations forming a sample. However in our setting, some families of large degree random polynomials, the uniformity of a distribution of roots, a symmetry or an intriguing regular shape shows up in almost each experiment. A single large object is enough to represent the whole ensemble, in other words a significant sample contains only one element. This convenient behavior can be related to a property of some disordered systems called "self-averaging". However, our situation is more complicated, since we did not fix in advance any feature of the observed shapes; they are extracted from the pictures.

We now list some classes of random polynomials, we specify their names and the corresponding bases $g_i(x)$ (cf. the above formula).

- Kac polynomials: the basis is $g_i(x) = x^i$.
- $SO(2)$-polynomials: $g_i(x) = \sqrt{\binom{n}{i}}x^i$.
- Weyl-polynomials: $g_i(x) = \sqrt{\frac{1}{i!}}x^i$.

Then, the following less commonly studied families; in this paper we give them the following names (NC stands for normal combination):

- NC-Bernstein : $g_i(x) = \binom{n}{i}(1+x)^i(1-x)^{n-i}$.
- NC-Chebyshev: the basis is made by the Chebyshev polynomials of degree i, for i between 0 and n.

Then the characteristic polynomials of several classes of random matrices

- matrices whose entries are instances of independent standard normal distribution,
- symmetric matrices whose entries are instances of independent standard normal distribution,
- random unitary matrices obtained by taking the eigenvectors of a matrice of the previous class.

Sparse analog of these classes and other distributions of their coefficients (or entries) are also very interesting; but the listed classes are already rich enough to express our observations and conjectures.

Number of real roots and distribution of complex roots

- The asymptotic number of real roots of a Kac polynomial is about $\frac{2}{\pi}\ln n$, the distribution of the complex roots tends to a uniform distribution on the unit circle; Figure 1 shows the roots of a Kac polynomial of degree 128.

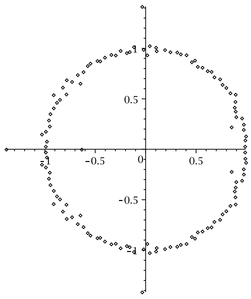

Figure 1: Complex roots of a Kac polynomial

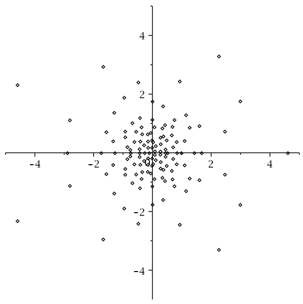

Figure 2: Complex roots of a $SO(2)$ polynomial

- The asymptotic number of real roots of a $SO(2)$- polynomial is about $\sqrt{n}$, the distribution of the complex roots tends to a uniform distribution on the Riemann sphere. Figure 2 shows their location on the complex plane for a polynomial of degree 128.

- The asymptotic number of real roots of a Weyl polynomial is about $\frac{2}{\pi}\sqrt{n}$, the distribution of the complex roots tends to a uniform distribution on the disc centered at the origin and of radius $\sqrt{n}$.

- The asymptotic number of real roots of the characteristic polynomials of a general random matrix (as above) is about $\sqrt{\frac{2}{\pi}}\sqrt{n}$, the distribution of the complex roots tends to a uniform distribution on the disc centered at the origin and of radius $\sqrt{n}$. Figure 3 shows them for a matrix of size 128.

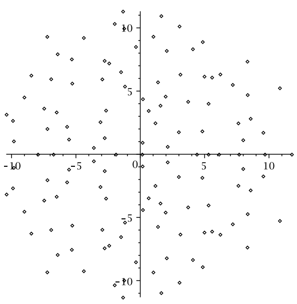

Figure 3: Complex eigenvalues of a random matrix

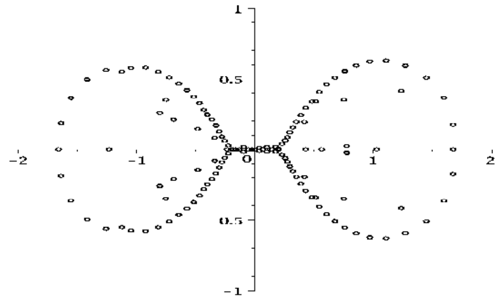

Figure 4: Complex roots of a NC Chebyshev

In the three previous pictures we notice that the limit distribution is almost uniform in angles around the origin, (rotational symmetric) this property is completed by an axial symmetry over the real axis due to complex conjugation. This observation can be quantified: [17] computed for Kac polynomials the density function $h_n(x, y)$ of the number of complex roots near a complex point $x+iy$, completing Kac's computation of the density function of the number of real roots near a real point x.

For the two other classes, NC-Bernstein and NC-Chebyshev, we have "only" a limit central symmetry.

- The asymptotic number of real roots of a polynomial in NC-Bernstein is about $\sqrt{2n}$, [7].

- Figures 4 shows, for a NC-Chebyshev polynomial of degree 128, the distribution of the complex roots, they concentrate along a segment of the real axis and two ovals centered at -1 and 1.

3. VARIATION DIAGRAM (VD)

We consider a polynomial $f(x)$ with real coefficients of degree n and its i-th derivative $F[i] = f^{(i)}(x)$ for $i = 0..n-1$. The sets of real roots of the n polynomials $F[i]$, appears in what, in French high schools is called "tableau de variations". In the 19-th century, the number of sign variations were used by Budan and by Fourier to estimate the number of roots of f in an interval, see [16], chapter 10. In the 20-th century, R. Thom relied on the signs of $f^{(i)}$ to distinguish and label the different roots of f, see [3].

We chose to organize all these roots with a 2D diagram, that we call Variation diagram (VD):

$$VD := \cup_i \ \{fsolve(F[i], x)\} \times \{n-i\}$$

Note that the second coordinate indicates the degree of the polynomial $F[i]$. As the iterated derivatives of a generic polynomial do not have multiple roots, they change sign at each root.

Example:

$f := (x-5).(x^2-x+4)$; $f' = 3(x-1).(x-3)$; $f'' = 6(x-2)$.

These polynomials have respectively 1, 2, 1 real roots:

$$VD = \{[5, 3], [1, 2], [3, 2], [2, 1]\}.$$

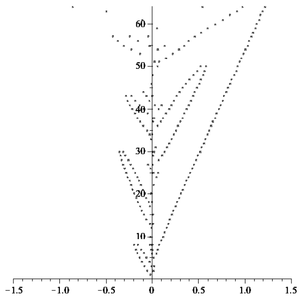

Figure 5: VD of a Kac polynomial of degree 64

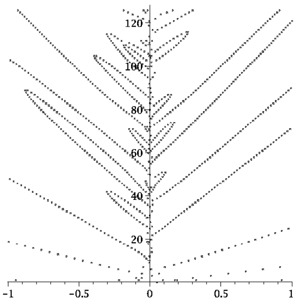

Figure 6: (Truncated) VD of a SO(2) polynomial

Our first experiments with Kac polynomials found that their VD admit unexpected structured patterns. The roots of the successive derivatives present almost curved alignments. To our best knowledge, this phenomenon has not been explored before.

We made more experiments with different instances of Kac polynomials and got very similar patterns, then we repeated the experiments with the different bases defined in the previous section. See Figures 5 to 8.

As illustrated by these pictures and many more, for the cited families of random polynomial the observed feature (almost alignments along lines or ovals) seems robust. In particular following our observation we conjecture:

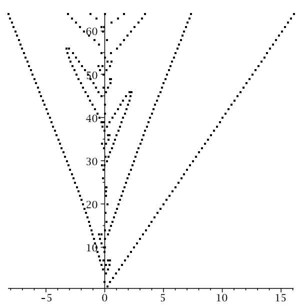

Figure 7: VD of a Weyl polynomial of degree 64

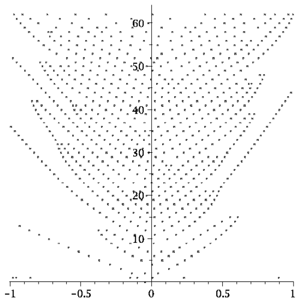

Figure 8: (Truncated) VD of a CN Bernstein

CONJECTURE 1. *For Kac, $SO(2)$ or Weyl random polynomials the external real roots (i.e. biggest or smallest) of all derivatives of f tend to be almost surely aligned along a line, when n tends to infinity.*

In order to strengthen these observations, we looked for a method to connect the points in coherence with our visual intuition. A natural strategy is to view the integer orders of derivation as discretized steps; hence to look for generalized derivatives with continuous orders.

4. FRACTIONAL DERIVATIVES

The attempt to introduce and compute with derivatives or antiderivatives of non-integer orders goes back to the 17-th century. In their fine book [15] dedicated to this subject, the authors relate that an integral equation, the tautochrone, was solved by Abel in 1823 using a semi derivative attached to the integral $\int_0^x \sqrt{x-t} f(t)dt$. In 1832 Liouville expanded functions in series of exponentials and defined q-th derivatives of such a series by operating term-by-term for q a real number. Riemann proposed another approach via a definite integral. The cited book provides in its introduction a nice presentation of the historical progression of the concept from 1695 to 1975 through a hundred citations.

An important property is that two such fractional derivation commute. However for non-integer orders of derivation, the fractional derivative at a point x of a function f does not only depend on the graph of f very near x; fractional derivation does not commute with the translations on the variable x. The traditional adjective "fractional", corresponding to the order of derivation, is misleading since it needs not be rational.

Let us emphasize that nowadays in Mathematics, fractional derivatives are mostly used for the study of PDE in Functional analysis. They are presented via Fourier or Laplace transforms. Fractional derivatives are seldom encountered in Polynomial algebra.

4.1 A polynomial factor

In order to interpolate the previous dotted curves, we consider a polynomial factor of the fractional derivatives of the polynomial f. We rely on Peacock's rule (1833) for monomials:

$$Diff_a(x^n, x) := \frac{n!}{(n-a)!} x^{n-a};\ for\, a > 0,\ n\, integer.$$

which immediately implies the following key fact.

LEMMA 1. *Let $f(x)$ be a polynomial of degree n, then*

$$x^a\Gamma(-a)Diff_a(f)$$

is a polynomial in x and a rational fraction in a with denominator $(n-a)(n-a-1)...(-a)$.

To interpolate the non vanishing roots of the successive derivatives of a polynomial f, only fractional derivatives with $0 < a < 1$, up to a power of x are needed. So we set the following definition and notation.

DEFINITION 1. *Let $f = \sum a_i x^i$ be a degree n polynomial. We call (monic) polynomial factor of a fractional derivative of order a of f,the polynomial $x^a\frac{(n-a)!}{n!}Diff_a(f,x))$. It is a polynomial of total degree n in x and a, which writes*

$$P_a(f) := a_n x^n \ + \sum_{i=0}^{n-1}(\prod_{j=i+1}^{n} 1-\frac{a}{j})a_i x^i.$$

4.2 Stem

DEFINITION 2. *We call Stem of a polynomial f of degree n, the union of the real curves formed by the roots of all the monic polynomial factors of the derivatives $f^{(i)}$ of f, for i from 0 to $n-1$ and $0 \leq a < 1$. A stem is a C^0 spline of algebraic curves.*

Here is a simple example to illustrate the regularity of the join between two successive curves forming the stem of f:

$$f = (x-1)(x-3) = x^2 - 4x + 3 \ ; \ f' = 2(x-2).$$

Hence,

$$P_a(f) = x^2 - 2x(2-a) + 3(2-a)(1-a)/2,$$

$$P_b(f'/2) = x - 2(1-b).$$

This shows that when a tends to 1, and b tends to zero, there is (only) a C^0 continuity between the adjacent pieces.

In a joint work (in preparation) with D. Bembe, we consider another curve associated to f which is regular but does not have the same shape: the real algebraic curve in the plane (x, a) defined by the bivariate polynomial of degree n that we denoted above by $P_a(f)$. We study the relation between this curve and the so-called virtual roots of f introduced in [12] and [4].

4.3 From discrete to continuous

I made several experiments, and noticed that the patterns exhibited by the stems of most of our random polynomials presented common features:

- Long quasi lines joining the external real roots of f to the axis $(x = 0)$,
- Curves (of smaller size) joining the inner real roots to the axis $(x = 0)$,

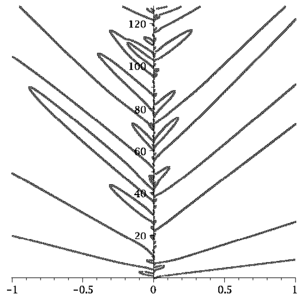

Figure 9: Stem of a SO(2) polynomial of degree 128

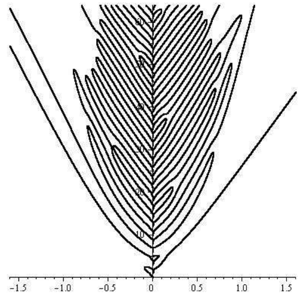

Figure 10: Stem of a NC Chebyshev of degree 64

- Closed curves, often shaped like an ear, starting and ending at $(x = 0)$.
- Stems of a same family of random polynomials share more similarities.

Remark: In our pictures, the line $x = 0$ is a singularity and an axis of almost symmetry of the patterns. This is coherent with the considered random polynomials with centered distribution of coefficients. Our choice of fractional derivatives respects this symmetry.

4.4 Similarity of graphs

The graph of a (random) polynomial bears interesting features. Algebraically and visually its shape is corelated to the roots of f and its derivatives through extrema, inflection points, and generalized inflections. We compared the

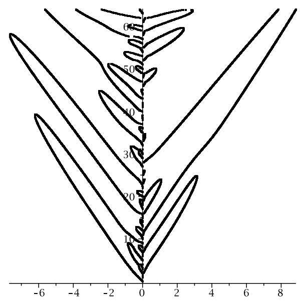

Figure 11: Stem attached to a random matrix

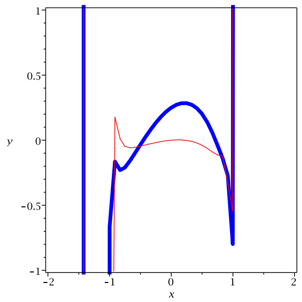

Figure 12: Graphs of f , xf' for a Kac polynomial

graphs of f and f' or xf', for our random polynomials, expecting that randomness acts as a filter: details are blurred and similarities are magnified.

We made several experiments with Maple for polynomials f of medium degrees. For a Kac polynomials of degree 64, we rescaled f, and xf', restricted the graphs to $x \in [-1.2, 1.2]$ and $y \in [-1, 1]$. As illustrated in Figure 12 we found that, very often, the graph of xf' is similar to the graph of f but shrunk towards the origin. This is coherent with the pattern exhibited by the variation diagram of f. The problem is how to quantify the observed transformation.

4.5 Rolle theorem

Generically, there are an odd number of roots of f' between two successive positive roots x_1 and x_2 of f. We aim to analyze the pairing between the roots established by the stem of f, the previous figures suggest that for our random polynomials, almost surely the stem connects the root x_2 of f to one of the roots of f'. Stems of random polynomials may have points with horizontal tangents, but we observed that at these points the graph is convex. With other words, we propose this property as a conjecture.

CONJECTURE 2. *For the chosen families of random polynomials, almost surely the stem between the roots of f and xf' does not connect two roots of f. But it may connect two roots of f'.*

Another interesting task would be to quantify the ratio defined by the consecutive roots of f and f'. Such estimates were given obtained by P. Andrews [1] for hyperbolic polynomials.

Figures 13 and 14 show the detail of a “good” (with respect to the conjecture) example of the stem of a polynomial of degree 6, and the corresponding deformation between the graphs of f and xf', formed by the fractional derivatives. In contrast, Figure 15 and 16 show the corresponding pictures for an example which does not correspond to the situation encountred with our random polynomials. More precisely, in the “bad” example the continuation between the inner roots of f and xf' does not remain on the real line but passes through the complex plane, this is pictured by the dotted curve. So, an explanation of the connection between real roots of f and f' might be found exploring what happens in the complex plane.

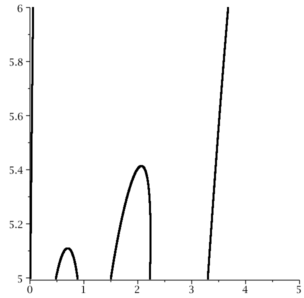

Figure 13: detail of a Stem

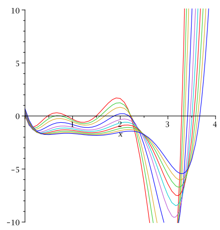

Figure 14: a “good” homotopy between f and xf'

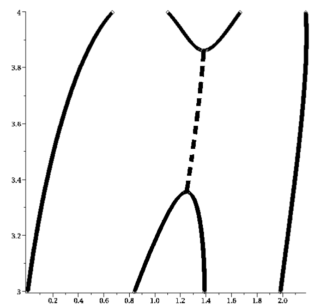

Figure 15: detail of a “bad” Stem

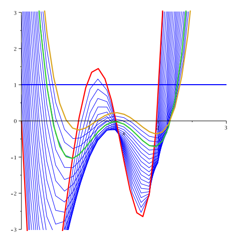

Figure 16: a “bad” homotopy between f and xf'

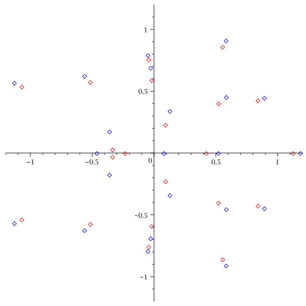

Figure 17: Roots of f and f'

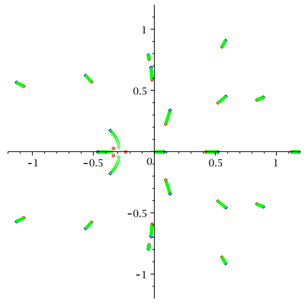

Figure 18: Pairing via fractional derivatives

5. COMPLEX CRITICAL POINTS

There is an important bibliography on the location of the critical points of a polynomial with respect to the location of its roots, going back to Gauss with Gauss-Lucas theorem. Several recent works concentrate on the following conjecture of Sendov, which has been proved for small degrees and in several special cases. Their main tools rely either on the implicit function theorem or on extremal polynomials, or on refinements of Gauss-Lucas theorem. See the book [16].

Sendov Conjecture: Let f be a polynomial having all its roots in the disk D. If z is a root of f, then the disk $z + D$ contains a root of f'.

I did not found mention in these researches, developed in analysis and approximation theory, of the case of polynomials with random coefficients.

5.1 Observations

I made experiments with the first classes (see section 2) of random polynomials, they exhibit interesting behaviors:

- for almost each root of f smaller disks $z+\epsilon D$, with $\epsilon << 1$, contain a critical point; in such a way that they describe a bijection between the roots of f and the roots of xf',

- one can restrict these disks to small sectors, which indicates a direction towards the real axis or towards the origin.

Figures 17 to 19 illustrate the relation between roots and critical points for an $SO(2)$ random polynomial of degree 32. Fractional derivatives are used to construct (as for real roots) an homotopy between the zero sets of f and xf'. The color chart is: the roots of f are blue, the roots of f' are red and the roots of the fractional derivatives are green. Figure 19 shows the "top" part of a complex analog of the variation diagram, notice the regular alignments towards the origin (it is slightly noised near the real axis).

5.2 Electrostatic attraction

The interpretation of the position of each critical point of f as an equilibrium of a logarithmic potential, where the roots of f are viewed as positively charged particles (or rods), goes back to F. Gauss. As reported in [16], the following equality to zero provides a quick proof of Gauss-Lucas theorem.

Denote by x_j the complex roots of the polynomial f assumed

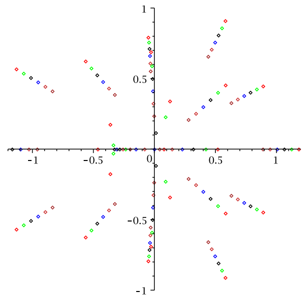

Figure 19: Truncated $SO(2)$ complex VD

two by two distinct and distinct from z, another complex number. Then by logarithmic derivation and conjugation we deduce:

$$f'(z) = 0 \;\Rightarrow\; \sum \frac{z - x_j}{|z - x_j|^2} \;=\; 0.$$

The vector in the complex plane $\frac{-z+x_j}{|z-x_j|^2}$ is viewed as a force applied to z directed towards x_j proportional to the reciprocal of the distance. Summing these forces, the point z (viewed as an electron) is attracted by the roots system of f (viewed as positively charged particles). When the limit distribution of the roots of f is uniform in angles (also called rotational symmetric, i.e. only depends on the radius), the resulting electrostatic force on a point z inherits a limit symmetry, and tends to be directed towards the origin.

Let us denote by L_1 the real line joining the origin to a root x_k and by L_2 the real line orthogonal to L_1 through the origin. The number and distribution of roots below and above L_1 (respectively L_2) are asymptotically "almost" balanced. So, with a good probability, an equilibrium z_k can be found "near" L_1 with the vector $x_k z_k$ oriented towards the origin. One can expect as well that the more x_k is far from the origin, the smallest the vector $x_k z_k$ should be. This is what we observed in our experiments as illustrated with Figures 17 and 18.

The previous balanced count of forces is noised when we approach the real axis, because there is another axial symmetry due to complex conjugation, and a positive probability of real roots. This breaks the rotational symmetry, conse-

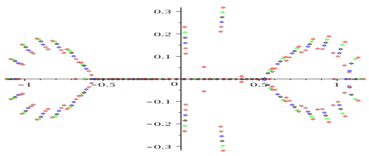

Figure 20: Higher derivatives, CN Chebyshev

quently the resulting electrostatic force is now also directed towards the real axis. The attraction toward the real axis is magnified when we consider the set of roots of a NC Chebyshev polynomial (see Figure 4), since it contains many real points and "only" a central limit symmetry. In this case, we observe that through successive pairing and after a rather small number of derivations, most complex roots of f give rise to a real root of a higher derivative of f. This process is illustrated in Figure 20 with a CN Chebyshev polynomial of degree 50, The colors (red, green, black, blue, orange, brown) correspond to the derivation orders (0,1,2,3,4,5).

CONJECTURE 3. *For f a Kac, $SO(2)$, Weyl random polynomial or the characteristic polynomial of a random matrix as in section 2, the presented continuation process realizes a bijection such that each critical point z_k of f is attached to a root x_k. Moreover in the limit distribution of (x_k, z_k) when n tends to infinity, almost surely the vectors $x_k z_k$ point towards the origin.*

For the other random cases with only a limit central symmetry, we also conjecture a pairing but the limit orientation of the $x_k z_k$ will be dependant on the point x_k. Our intuition is as follows. Consider all the roots of f except x_k, the resulting electrostatic force will be regular in a small disc around x_k, the average of these forces in the disc is a good candidate for the researched direction, then z_k will be positioned on the corresponding line to realize the equilibrium.

6. TENTATIVE EXPLANATIONS

Let us summarize. In section 3 we described intriguing alignments of points, observed on the collection of all roots of a real polynomial f and its derivatives, organized in a 2D diagram (VD).

Trying to explain the curved parts of this phenomena, we presented in section 4 an original use of fractional derivatives. It allows an interpolation of the discrete set into a 2D curve which we called the stem. This construction gave rise to experiments, to other intriguing observations and to two questions: Why does the C^0 interpolation look so regular ? Is there any correlation between the distribution of the complex roots of f and its stem ? These question lead us to new observations on the location of critical points of our random polynomials and again to new questions and conjectures.

To stop this flow of questions, we rely on the interpretation of a critical point as an equilibrium position under electrostatic forces. Indeed this viewpoint allows the use of the density functions of complex and real roots of a random polynomial f. Such density functions have been studied by several authors for classical families of random polynomials, e.g. for Kac polynomials we already cited [17].

The strategy is to perform a mean field approximation to establish as follows a pairing between a root of f and a root of f'. One can estimate an equilibrium z_k near a fixed root x_k, relying on the density functions to average the attraction corresponding to the other roots of f. If the root x_k is real, complex conjugation obliges its image z_k to be real. Notice that the continuation curve between x_k and z_k, defined by the homotopy, needs not be real, we only conjectured this property with a good probability for some random polynomials.

This process will allow for random polynomials at each derivation step to "get down" towards the real axis and at least for rotational symmetric limit distribution of complex roots, towards the origin. Note that in order to prove that the continuation defined via fractional derivatives follows the same dynamic, we need to develop a generalized attraction interpretation. When a couple of conjugate complex roots of a fractional derivative reach the real axis they form a double root: in the stem of f, this event corresponds to a summit of an ear shaped curve.

We are still far from a rigorous presentation of our interpretation and all its consequences. Improvements of Gauss-Lucas, use of majorization techniques, images of uniform distributions laws, might eventually explain the features exhibited by the variation diagrams, but it is a long way.

As a conclusion, I experimented, introduced new tools, presented pictures, observed phenomena. I sketched a possible strategy, which opens several problems and an exploratory research project.

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7. REFERENCES

[1] Andrews,P: Where not to find the critical points of a polynomial -variation on a Putnam theme. Amer. Math. Monthly, 102, pp 155-158 (1995).

[2] Bharucha-Reid, A. T. and Sambandham, M.: Random Polynomials. Academic Press, N.Y. (1986).

[3] Bochnack, J. and Coste, M. and Roy, M-F.: Real Algebraic Geometry. Springer (1998).

[4] Coste, M and Lajous, T and Lombardi, H and Roy, M-F : Generalized Budan-Fourier theorem and virtual roots. Journal of Complexity, 21, 478-486 (2005).

[5] Diaconis, P: Patterns in eigenvalues: the 70th Josiah Willard Gibbs lecture. Bull. Amer. Math. Soc. 40, 155-178. (2003)

[6] Edelman,A and Kostlan,E: How many zeros of a random polynomial are real? Bulletin AMS, 32(1):137, (1995).

[7] Emiris, I and Galligo, A and Tsigaridas, E: Random polynomials and expected complexity of bissection methods for real solving. *Proceedings of the ISSAC'2010 conference,* pp 235-242, ACM NY, (2010).

[8] Farahmand,K: Topics in random polynomials. Pitman research notes in mathematics series 393, Addison Wesley, (1998).

[9] Galligo,A and Miclo, L: On the cut-off phenomenon for the transitivity of subgroups. Submitted for publication (2009).

[10] Galligo,A and Poteaux, A: Computing Monodromy via Continuation methods on Random Reimann Surfaces. Theoretical Computer Science, to be published in (2011).

[11] Galligo,A and vanHoeij, M: Approximate bivariate factorization, a geometric viewpoint. In *Proceedings of the 2007 SNC conference*, pp 1-10, ACM NY, (2007).

[12] Gonzales-Vega, L and Lombardi, H and Mahé, L : Virtual roots of real polynomials. J. Pure Appl. Algebra,124, pp 147-166,(1998).

[13] Kac, M: On the Average Number of Real Roots of a Random Algebraic Equation. Bull. Amer. Math. Soc. 49, pp 314-320, (1943).

[14] Mehta, M-L: Random matrices. Pure and Applied Mathematics, 142. Elsevier/Academic Press, Amsterdam, (2004)

[15] Oldham, K and Spanier, J: The fractional calculus. Academic Press Inc. NY, (1974).

[16] Rahman, Q.I and Schmeisser, G.: Analytic theory of polynomials. Oxford Univ. Press (2005).

[17] Shepp, L.A. and Vanderbei, R.J. The complex zeros of random polynomials. Transactions of the AMS vol 347, 11, (1995).