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Mireille Bossy, Jean-François Jabir

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Abstract

We construct a confined Langevin type process aimed to satisfy a mean no-permeability condition at the boundary. This Langevin process lies in the class of conditional McKean Lagrangian stochastic models studied in [5]. The confined process considered here is a first construction of solutions to the class of Lagrangian stochastic equations with boundary condition issued by the so-called PDF methods for Computational Fluid Dynamics. We prove the well-posedness of the confined system when the state space of the Langevin process is a half-space.

Keywords McKean Langevin equation, Lagrangian stochastic model, mean no-permeability condition, specular boundary condition.

1 Introduction

We consider the nonlinear Langevin equation (1.1), describing the time evolution of the position and velocity (X, U) of a particle with the position process X confined in the upper-half plane $\mathbb{R}^{d-1} \times [0, +\infty)$,

$$\begin{cases} X_t = X_0 + \int_0^t U_s ds, \\ U_t = U_0 + \int_0^t B\left[X_s, U_s; \rho_s\right] ds + W_t + K_t, \\ K_t = -\sum_{0<s\leq t} 2\left(U_{s^-} \cdot n_{\mathcal{D}}\right) n_{\mathcal{D}} \mathbb{1}_{\{X_s \in \partial\mathcal{D}\}}, \\ \rho_t \text{ is the probability density of } (X_t, U_t), \text{ for all } t \in (0, T]. \end{cases} \tag{1.1}$$

Here $\mathcal{D} = \mathbb{R}^{d-1} \times (0, +\infty)$, $\partial\mathcal{D} = \mathbb{R}^{d-1} \times \{0\}$ and $n_{\mathcal{D}} = (0, \ldots, 0, -1)$ is the outward normal unit vector.

In this paper, we prove the well-posedness of the equation (1.1) on a finite time interval $[0, T]$. Moreover, we prove that the confined solution (X, U) satisfies a mean version of the so-called no-permeability condition, that we write formally in this introduction as

$$\mathbb{E}\left[(U_t \cdot n_{\mathcal{D}})/X_t = x\right] = 0, \text{ for a.e. } (t, x) \in (0, T] \times \partial\mathcal{D}. \tag{1.2}$$

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More precisely, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider a d-dimensional standard Brownian motion W and a $\mathcal{D} \times \mathbb{R}^d$-valued random variable (X_0, U_0) independent of W. We prove that there exists a unique solution, (X, U) valued in $\mathcal{C}([0,T]; \overline{\mathcal{D}}) \times \mathbb{D}([0,T]; \mathbb{R}^d)$, to the nonlinear McKean equation (1.1). In particular, the related sequence of hitting times

$$\tau_n = \inf\{\tau_{n-1} < t \leq T \text{ s.t. } X_t \in \partial\mathcal{D}\}, \text{ for } n \geq 1, \; \tau_0 = 0, \tag{1.3}$$

is well-defined and the sum $\sum_{0<s\leq t} \mathbb{1}_{\{X_s \in \partial\mathcal{D}\}}$ acts over a countable set of times. Then the confining term K is a càdlàg process reflecting the velocity of the outgoing particles. The drift coefficient B is the mapping from $\mathcal{D} \times \mathbb{R}^d \times L^1(\mathcal{D} \times \mathbb{R}^d)$ to $\mathbb{R}^d$ defined by

$$B[x, u; \gamma] = \begin{cases} \dfrac{\displaystyle\int_{\mathbb{R}^d} b(v, u)\gamma(x, v)\, dv}{\displaystyle\int_{\mathbb{R}^d} \gamma(x, v)\, dv} & \text{if } \displaystyle\int_{\mathbb{R}^d} \gamma(x, v)\, dv \neq 0, \\ 0 \text{ elsewhere,} & \end{cases} \tag{1.4}$$

where $b : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a given bounded and continuous interaction kernel. As noticed in Bossy, Jabir and Talay [5], formally the drift coefficient $B[x, u; \rho_t]$ in (1.1) is the function

$$(t, x, u) \mapsto \mathbb{E}\left[b(U_t, u)/X_t = x\right],$$

and the system (1.1) involves a conditional expectation in its drift term.

This present work is a first step in the analysis of the Lagrangian stochastic models involving a prescribed behaviour of the velocity when the particle hits the boundary $\partial\mathcal{D}$. The boundary condition (1.2) provides a representative example in the class of boundary conditions related to the Probability Density Function (PDF) methods proposed by S.B. Pope for Computational Fluid Dynamics applications (see Pope [15] and references therein). The PDF methods have been developed for the simulation of turbulent flows. They are based on the Lagrangian stochastic modelling of the fluid motion through a system of Stochastic Differential Equations (SDEs) which describe the dynamics of a generic fluid-particle. These SDEs are nonlinear in the sense of McKean and involve singular interaction kernels. We refer to [5] for a short survey on mathematical issues related to the Lagrangian stochastic models and the well-posedness of (1.1) when $\mathcal{D} = \mathbb{R}^d$ (i.e. $K = 0$).

For the Lagrangian modelling of near-wall turbulent flows, Dreeben and Pope [7] suggested a reflection procedure of the velocities at the boundaries according to a prescribed boundary layer model. Here we formalize those ideas and construct a first example of confined Lagrangian models satisfying (1.2).

The paper is organized as follows. In Section 2, we state our main results on the well-posedness of (1.1). We further show that the trace at the boundary $\gamma(\rho)$ of the time-marginal densities $(\rho_t; t \in (0, T])$ of the solution of (1.1) satisfies the so-called specular boundary condition (see (2.5) below). This implies the mean no-permeability condition (1.2). The rest of the paper is devoted to the proofs. As a preliminary step, in Section 3, we construct a confined version of the Brownian motion and its primitive (i.e. $B = 0$). In Section 4, we combine arguments from Sznitman [17], on the well-posedness of McKean nonlinear SDEs with reflection of Skorokhod type, with the approach used in [5] to the proof of the well-posedness of (1.1). Section 5 is devoted to the study of the trace $\gamma(\rho)$.

Notation

- For any point $x \in \mathbb{R}^d$, we write $x = (x', x^{(d)})$ where x' denotes the $(d-1)^{\text{th}}$ first coordinates and $x^{(d)}$ denotes the d^{th} coordinate. The surface measure $d\sigma_{\mathcal{D}}$ related to $\partial\mathcal{D}$ is $d\sigma_{\mathcal{D}} = dx'$.

- For all $t > 0$, $Q_t = (0,t) \times \mathcal{D} \times \mathbb{R}^d$, $\Sigma_t = (0,t) \times \partial\mathcal{D} \times \mathbb{R}^d$.

- We denote by $\mathbb{D}([0,T];\mathbb{R}^q)$ the space of all $\mathbb{R}^q$-valued càdlàg functions equipped with the Skorokhod topology. For $z \in \mathbb{D}([0,T];\mathbb{R}^q)$, $\triangle z_t$ and z_{t^-} denote, respectively the jump and the left-hand limit of z at time t.

- For any metric space E, we denote by $\mathcal{M}(E)$ the set of probability measures equipped with the weak topology.

- For any point $(t,z,\nu) \in (0,+\infty) \times \mathbb{R}^d \times \mathbb{R}^d$, $g_d(t;z,\nu;\zeta,\mu)$ denotes the transition density of the d-dimensional Brownian motion's primitive and its Brownian motion $(z + \nu t + \int_0^t W_s ds, \nu + W_t)$, starting at (z,ν). A straightforward computation leads to the explicit expression:

$$g_d(t;z,\nu;\zeta,\mu) = \left(\frac{\sqrt{3}}{\pi t^2}\right)^d \exp\left(-\frac{6|\zeta - z - t\nu|^2}{t^3} + \frac{6((\zeta - z - t\nu)\cdot(\mu-\nu))}{t^2} - \frac{2|\mu-\nu|^2}{t}\right). \tag{1.5}$$

2 Main results

From now on, we assume that the distribution μ_0 of (X_0,U_0) and the kernel b in (1.1) satisfy the following hypotheses (H).

$(H\text{-}i)$ The initial measure μ_0 has its support in $\mathcal{D}\times\mathbb{R}^d$ and $\int_{\mathcal{D}\times\mathbb{R}^d} \left(|x| + |u|^2\right) \mu_0(dx,du) < +\infty$.

$(H\text{-}ii)$ $b : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is uniformly bounded and continuous.

2.1 On the well-posedness of (1.1)

Let $\mathcal{E}$ be the sample space

$$\mathcal{E} := \mathcal{C}([0,T];\overline{\mathcal{D}}) \times \mathbb{D}([0,T];\mathbb{R}^d) \times \mathbb{D}([0,T];\mathbb{R}^d).$$

We equip $\mathcal{E}$ with the Skorokhod topology, so that $\mathcal{E}$ is a closed subset of $\mathbb{D}([0,T];\mathbb{R}^{3d})$ and further it is a Polish space (see Jacod and Shiryaev [10]). We denote by $(x_t,u_t,k_t; t \in [0,T])$ the canonical process on $\mathcal{E}$, and by $(\mathcal{B}_t; t \in [0,T])$ the associated canonical filtration. The martingale problem related to (1.1) is then formulated as follows.

Definition 2.1. *A probability measure $\mathbb{P} \in \mathcal{M}(\mathcal{E})$ is a solution to the martingale problem (MP) if the following hold.*

(i) $\mathbb{P} \circ (x_0,u_0,k_0)^{-1} = \mu_0 \otimes \delta_0$, where δ_0 denotes the Dirac mass at the origin on $\mathbb{R}^d$.

(ii) For all $t \in (0,T]$, $\mathbb{P} \circ (x_t,u_t)^{-1}$ admits a positive Lebesgue density ρ_t.

(iii) For all $f \in \mathcal{C}_b^2(\mathbb{R}^{2d})$, the process

$$\begin{aligned} &f(x_t,u_t-k_t) - f(x_0,u_0) - \int_0^t (u_s \cdot \nabla_x f(x_s,u_s-k_s))\, ds \\ &- \int_0^t \left[(B\left[x_s,u_s;\rho_s\right] \cdot \nabla_u f(x_s,u_s-k_s)) + \frac{1}{2}\triangle_u f(x_s,u_s-k_s)\right] ds \end{aligned} \tag{2.1}$$

is a continuous $\mathbb{P}$-martingale w.r.t. $(\mathcal{B}_t; t \in [0,T])$.

(iv) $\mathbb{P}$*-a.s., the set* $\{t \in [0,T]$ *s.t.* $x_t^{(d)} = 0\}$ *is at most countable, and*

$$k_t = -2 \sum_{0<s\le t} (u_{s^-} \cdot n_{\mathcal{D}}) n_{\mathcal{D}} \mathbb{1}_{\{x_s \in \partial\mathcal{D}\}}, \ \forall\, t \in [0,T].$$

Theorem 2.2. *Assume* (H)*. Then there exists a unique solution to the martingale problem (MP).*

Under the solution $\mathbb{P}$ of (MP), one may observe that the canonical process $(x_t, u_t, k_t; t \in [0,T])$ satisfies (1.1).

Section 4 is devoted to the proof of Theorem 2.2. As in [5], the existence of a solution is based on a classical particle approximation method of a smoothed problem, here in the framework of the Skorokhod topology.

2.2 On the mean no-permeability condition

First, we prove the regularity of the time-marginal densities $(\rho_t; t \in (0,T])$ of the solution to (MP), and the existence of the trace $\gamma(\rho)$ at the boundary Σ_T. We give some related integrability properties.

Theorem 2.3. *Assume* (H)*. Let* $\mathbb{P}$ *be the solution to (MP).*

(a) The time-marginal densities (ρ_t) *of* $\mathbb{P}$ *are Hölder-continuous: for* $x, x_0 \in \overline{\mathcal{D}}$*, there exist some positive constants* β*,* α*,* α_1 *and* C *such that for a.e.* $0 < t_0 < t \le T$*,* $u \in \mathbb{R}^d$*,*

$$|\rho_t(x,u) - \rho_t(x_0,u)| \le C t_0^{-\alpha} (1 \vee (t-t_0)^{\alpha_1}) |x - x_0|^{\beta}.$$

(b) We have $\sum_{n\in\mathbb{N}} \mathbb{P}(\tau_n \le T) < +\infty$ *and the trace function* $\gamma(\rho)$ *defined by*

$$\gamma(\rho)(t,x,u) := \lim_{\delta\to 0^+} \rho_t((x',\delta),u), \text{ for all } x = (x',0) \in \partial\mathcal{D}, \text{ for a.e. } (t,u) \in (0,T)\times\mathbb{R}^d, \tag{2.2}$$

satisfies, for all bounded measurable functions f *on* Σ_T*,*

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}}\left[\sum_{n\in\mathbb{N}} \Big(f(\tau_n, x_{\tau_n}, u_{\tau_n}) - f(\tau_n, x_{\tau_n}, u_{\tau_n^-}) \Big) \mathbb{1}_{\{\tau_n \,\le\, T\}}\right] \\ &= -\int_{\Sigma_T} (u\cdot n_{\mathcal{D}})\, \gamma(\rho)(s,(x',0),u) f(s,(x',0),u) ds\, dx' du. \end{aligned} \tag{2.3}$$

(c) The following properties hold for the trace function $\gamma(\rho)$*:*

$$\int_{\mathbb{R}^d} |(u\cdot n_{\mathcal{D}})| \gamma(\rho)(t,x,u)\, du < +\infty, \ dt \otimes d\sigma_{\mathcal{D}}\text{-a.e. on } (0,T)\times\partial\mathcal{D}, \tag{2.4a}$$

$$\int_{\mathbb{R}^d} \gamma(\rho)(t,x,u)\, du > 0, \ dt \otimes d\sigma_{\mathcal{D}}\text{-a.e. on } (0,T)\times\partial\mathcal{D}. \tag{2.4b}$$

In view of Theorem 2.3, the conditional expectation of the normal velocity component at the boundary can be explicited as: for $dt \otimes d\sigma_{\mathcal{D}}$-a.e (t,x) in $(0,T)\times\partial\mathcal{D}$,

$$\mathbb{E}_{\mathbb{P}}\left[(U_t \cdot n_{\mathcal{D}})/X_t = x\right] = \frac{\displaystyle\int_{\mathbb{R}^d} (u\cdot n_{\mathcal{D}})\gamma(\rho)(t,x,u) du}{\displaystyle\int_{\mathbb{R}^d} \gamma(\rho)(t,x,u) du}.$$

We show that $\gamma(\rho)$ satisfies the so-called specular boundary condition arising in the kinetic theory of gases (see, *e.g.*, Cercignani [6]). The boundary condition (1.2) is then established with the following.

Corollary 2.4. *The trace function $\gamma(\rho)$ defined in (2.2) satisfies the specular boundary condition:*

$$\gamma(\rho)(t,x,u) = \gamma(\rho)(t,x,u-2(u\cdot n_{\mathcal{D}})n_{\mathcal{D}}),\ \ dt\otimes d\sigma_{\mathcal{D}}\otimes du\text{-a.e. on } \Sigma_T. \tag{2.5}$$

Moreover, for a.e. (t,x) in $(0,T)\times\partial\mathcal{D}$,

$$\int_{\mathbb{R}^d}(u\cdot n_{\mathcal{D}})\gamma(\rho)(t,x,u)du = 0, \tag{2.6}$$

and the mean no-permeability condition (1.2) is fulfilled.

Proof. In (2.3), choosing f of the form

$$f(s,x,u) = \phi(s,x,u) + \phi(s,x,u-2(u\cdot n_{\mathcal{D}})n_{\mathcal{D}}),$$

we observe that $f(\tau_n, x_{\tau_n}, u_{\tau_n}) = f(\tau_n, x_{\tau_n}, u_{\tau_n^-})$, $\mathbb{P}$-a.s and

$$\int_{\Sigma_T}(u\cdot n_{\mathcal{D}})\left(\phi(s,x,u)+\phi(s,x,u-2(u\cdot n_{\mathcal{D}}))\right)\gamma(\rho)(s,x,u)\,ds\,d\sigma_{\mathcal{D}}(x)du = 0.$$

Define $\psi(x;\cdot):\mathbb{R}^d\longrightarrow\mathbb{R}^d$ as

$$\psi(x;u) = u - 2(u\cdot n_{\mathcal{D}})n_{\mathcal{D}}.$$

Since, for all $x\in\partial\mathcal{D}$, $\psi(x;\cdot)$ is a continuously differentiable involution on $\mathbb{R}^d$ and since its Jacobian determinant is equal to -1, the change of variable $u\mapsto\psi(x;u)$ gives

$$\int_{\Sigma_T}(u\cdot n_{\mathcal{D}})\left(\gamma(\rho)(s,x,u)-\gamma(\rho)(s,x,u-2(u\cdot n_{\mathcal{D}}))\right)\phi(s,x,u)\,ds\,d\sigma_{\mathcal{D}}(x)du = 0,$$

for any bounded measurable function ϕ on Σ_T, from which we obtain (2.5). Moreover we have

$$\begin{aligned}\int_{\mathbb{R}^d}\Big(u\cdot n_{\mathcal{D}}\Big)\gamma(\rho)(t,x,u)\,du &= \int_{\mathbb{R}^d}\Big(u\cdot n_{\mathcal{D}}\Big)\gamma(\rho)(t,x,\psi(x;u))du\\ &= \int_{\mathbb{R}^d}\Big(\psi(x;u)\cdot n_{\mathcal{D}}\Big)\gamma(\rho)(t,x,u)du,\end{aligned}$$

and (2.6) follows by noticing that $(\psi(x;u)\cdot n_{\mathcal{D}}) = -(u\cdot n_{\mathcal{D}})$. ☐

Theorem 2.3 is proved in Section 5.

3 Preliminaries: the confined Langevin process

Throughout this section, we assume (*H-i*). We consider the confined equation (1.1) in the case where $b=0$, namely

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t U_s ds,\\ U_t = U_0 + W_t + \displaystyle\sum_{0<s\le t} 2U_{s^-}^{(d)} n_{\mathcal{D}}\mathbb{1}_{\{X_s^{(d)}=0\}},\ \forall\, t\ge 0,\end{cases} \tag{3.1}$$

with (X',U') denoting the $(d-1)^{\text{th}}$ first components of (X,U) and $(X^{(d)},U^{(d)})$ denoting the d^{th} component.

In this section, we show the existence in law and the pathwise uniqueness of the reflected process satisfying (3.1) (Proposition 3.1). Further, we provide some estimates on the semi-group related to the solution to (3.1) (Proposition 3.3).

3.1 Well-posedness of Equation (3.1)

In equation (3.1), (U', X') is a $(d-1)$-dimensional Brownian motion and its associated primitive. Thus, we only need to consider the d^{th} equation of a one-dimensional Brownian motion primitive reflected in $\mathbb{R}^+$ as

$$\begin{cases} X_t^c = X_0^{(d)} + \int_0^t U_s^c ds, \\ U_t^c = U_0^{(d)} + W_t^c - 2 \sum_{0<s\leq t} U_{s-}^c \mathbb{1}_{\{X_s^c = 0\}}, \ \forall\, t \geq 0. \end{cases} \tag{3.2}$$

On a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t; t \geq 0), \mathbb{Q})$, endowed with a one dimensional standard $\mathcal{F}_t$-Brownian motion W^f and such that (X_0, U_0) is a r.v. with law $\mathbb{Q}((X_0, U_0) \in dy\, dv) = \mu_0(dy, dv)$, we consider the free Langevin system (X^f, U^f) satisfying

$$\begin{cases} X_t^f = X_0^{(d)} + \int_0^t U_s^f ds, \\ U_t^f = U_0^{(d)} + W_t^f, \ \forall\, t \geq 0. \end{cases} \tag{3.3}$$

We also consider the family of probability measures $\{\mathbb{Q}_{y,v}\}_{(y,v)\in\mathbb{R}^2}$ on $(\Omega, \mathcal{F})$, defined by

$$\mathbb{Q}_{y,v}(A) = \mathbb{Q}\left(A/(X_0^f, U_0^f) = (y, v)\right), \ \forall A \in \mathcal{F}.$$

As mentioned by Bertoin [3], it is straightforward to check that $|X^f|$ describes the reflected Langevin process in the half line, in the sense that a particle arriving at 0 with velocity $v < 0$ bounces back with velocity $v > 0$. Here, we also need to explicit the construction of U^c in terms of the free Langevin process. McKean [14] has shown that if $(y, v) \neq (0, 0)$ then, $\mathbb{Q}_{y,v}$-almost surely, the paths $t \mapsto (X_t^f, U_t^f)$ never cross $(0, 0)$. Thus, under the assumption $(H\text{-}i)$, the sequence of passage times

$$\tau_n^f = \inf\{t > \tau_{n-1}^f \text{ s.t. } X_t^f = 0\}, \text{ for } n \geq 1, \ \tau_0^f = 0, \tag{3.4}$$

is well-defined. The law of the passage times is explicited by Lachal [12, Corollary 3]: for all $n \geq 1$, and $(y, v) \in \mathbb{R} \times \mathbb{R}$ with $y \neq 0$,

$$\begin{aligned} \mathbb{Q}_{y,v}\left(\tau_n^f \in dt\right) = \Bigg(\int_0^t \int_0^{+\infty} & \frac{1}{(2\pi)^{3/2}\sqrt{s}} \exp(-\frac{z^2}{s}) h(t-s; y, v, z) \\ & \times \left(\int_0^{+\infty} \gamma \sinh\left(\frac{\pi\gamma}{2}\right) \frac{\left[2\cosh(\frac{\pi\gamma}{3}) - 1\right]}{\left(2\cosh(\frac{\pi\gamma}{3})\right)^n} K_{i\gamma/2}(\frac{z^2}{s}) d\gamma \right) dz\, ds \Bigg) dt, \end{aligned} \tag{3.5}$$

where $h(t; y, v, z) = \frac{2\sqrt{3}}{\pi t^2} \exp\{-\frac{6y^2}{t^3} - \frac{6yv}{t^2} - \frac{2(v^2+z^2)}{t}\} \cosh(\frac{2z}{t^2}(3y + tv))$ and $K_{i\gamma}$ denotes the modified Bessel function. Since $K_{i\gamma/2}(z)$ is nonnegative for $z \geq 0$, (3.5) gives that for all $t_0 > 0$,

$$\mathbb{Q}_{y,v}\left(\tau_n^f \leq t_0\right) \leq \frac{1}{2^{n-1}} \mathbb{Q}_{y,v}\left(\tau_1^f \leq t_0\right) \leq \frac{1}{2^{n-1}}, \ \forall\, n \geq 1. \tag{3.6}$$

Therefore, $\mathbb{Q}_{y,v}\left(\tau_n^f \leq t_0\right)$ tends to 0 as n grows to infinity, and the sequence $\{\tau_n^f\}_{n\in\mathbb{N}}$ grows to infinity. Defining the sign function as

$$sgn(x) = \begin{cases} 1 \text{ if } x > 0, \\ -1 \text{ elsewhere}, \end{cases}$$

the right-hand limit process $t \mapsto \mathcal{S}_t^f = sgn(X_t^f)_+$ is well defined.

Proposition 3.1. *Assume (H-i). On the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t; t \geq 0), \mathbb{Q})$, there exists a one dimensional standard $\mathcal{F}_t$-Brownian motion W^c such that*

$$(X_t^c = |X_t^f|, U_t^c = U_t^f \mathcal{S}_t^f; 0 \leq t \leq T) \tag{3.7}$$

is a solution to (3.2)*. In particular, $|U_t^c| = |U_0 + W_t^f|$, and the paths $t \mapsto -2\sum_{0<s\leq t} U_{s^-}^c \mathbb{1}_{\{X_s^c=0\}}$ are positive and non-decreasing. The sequence of hitting times*

$$\tau_n = \inf\{t > \tau_{n-1} \text{ s.t. } X_t^c = 0\}, \text{ for } n \geq 1,\ \tau_0 = 0, \tag{3.8}$$

is well-defined and grows to infinity. The pathwise uniqueness holds for the solution of (3.2).

Proof. We consider the càdlàg $\mathcal{F}_t$-adapted process U^c and its primitive X^c:

$$\begin{cases} X_t^c = X_0^f + \int_0^t U_s^c ds, \\ U_t^c = U_t^f \mathcal{S}_t^f,\ \forall\ t \geq 0. \end{cases}$$

Since $\mathcal{S}^f$ is a pure jump process, by the integration by parts formula,

$$U_t^c = U_t^f \mathcal{S}_t^f = U_0^c + \int_0^t \mathcal{S}_{s^-}^f\, dU_s^f + \sum_{0<s\leq t} U_s^f \triangle \mathcal{S}_s^f,\ \forall\ t \geq 0.$$

By Lévy's characterization, $(W_t^c := \int_0^t \mathcal{S}_{s^-}^f dU_s^f, t \geq 0)$ is a standard $\mathcal{F}_t$-Brownian motion. Moreover, on the set $\{t \geq 0 \text{ s.t. } X_t^f = 0\}$, if $U_t^f > 0$ then $\mathcal{S}_{t^-}^f = -1$ and $\triangle\mathcal{S}_t^f = 2$, while if $U_t^f < 0$, $\mathcal{S}_{t^-}^f = 1$ and $\triangle\mathcal{S}_t^f = -2$. Then,

$$\sum_{0<s\leq t} U_s^f \triangle \mathcal{S}_s^f = -2 \sum_{0<s\leq t} U_{s-}^c \mathbb{1}_{\{X_s^f=0\}}.$$

The set $\{t \geq 0 \text{ s.t. } X_t^f = 0\}$ being countable, we may replace $\mathcal{S}_t^f$ by $sgn(X_t^f)$ in the dynamics of $|X^f|$. Then,

$$|X_t^f| = |X_0^f| + \int_0^t sgn(X_s^f)\ U_s^f ds = |X_0^f| + \int_0^t U_s^c ds.$$

Then $\mathbb{Q}$-a.s., $X^c = |X^f|$, and $\{\tau_n\}_{n\in\mathbb{N}} = \{\tau_n^f\}_{n\in\mathbb{N}}$. Consequently, (X^c, U^c) satisfies (3.2).

The uniqueness result is a consequence of (*H-i*). Consider $(\widetilde{X}, \widetilde{U})$, another solution to (3.2) defined on the same probability space, endowed with the same Brownian motion. Since $\widetilde{X}_0 > 0$, we can define the first passage time at zero $\widetilde{\tau}_1$ of $\widetilde{X}$, and we observe that $\widetilde{\tau}_1 = \tau_1$ due to the continuity of X^c and $\widetilde{X}$. It follows that $U_{\tau_1\wedge\widetilde{\tau}_1}^c = \widetilde{U}_{\tau_1\wedge\widetilde{\tau}_1}$, so that (X^c, U^c) and $(\widetilde{X}, \widetilde{U})$ are equal up to τ_1. By induction, one checks that this assertion holds true up to τ_n for all $n \in \mathbb{N}$. As τ_n tends to $+\infty$ $\mathbb{Q}$-a.s., (X^c, U^c) and $(\widetilde{X}, \widetilde{U})$ are equal on $\mathbb{R}^+$. □

Remark 3.2. *The explicit construction in Proposition 3.1 has a straightforward extension for the Langevin process with bounded drift: for any $\mathbb{R}^d$-valued bounded measurable function on Q_T, $\beta(t, x, u) = (\beta', \beta^{(d)})(t, x, u)$, from the unique weak $\mathbb{R}^{2d}$-valued solution of*

$$\begin{cases} Y_t = X_0 + \int_0^t V_s ds, \\ V_t = U_0 + \int_0^t \widetilde{\beta}(s, Y_s, V_s) ds + \widetilde{W}_t,\ \forall\ t \in [0, T], \end{cases} \tag{3.9}$$

with $\widetilde{\beta}(t,x,u) := (\beta', sgn(x^{(d)})\beta^{(d)})(t,(x',|x^{(d)}|),(u', sgn(x^{(d)})u^{(d)}))$*, one deduces that*

$$(\mathcal{X}_t,\mathcal{U}_t; t\in[0,T]) = ((Y'_t,|Y_t^{(d)}|),(V'_t, sgn(Y_{t^+}^{(d)})V_t^{(d)}); t\in[0,T])$$

is the weak solution in $\mathcal{D}\times\mathbb{R}^d$ *to*

$$\begin{cases} \mathcal{X}_t = X_0 + \displaystyle\int_0^t \mathcal{U}_s ds, \\ \mathcal{U}_t = U_0 + \displaystyle\int_0^t \beta(s,\mathcal{X}_s,\mathcal{U}_s)ds + W_t - \sum_{0<s\le t} 2\left(\mathcal{U}_{s^-}\cdot n_{\mathcal{D}}(\mathcal{X}_s)\right) n_{\mathcal{D}}(\mathcal{X}_s)\mathbb{1}_{\{\mathcal{X}_s\in\partial\mathcal{D}\}}, \ \forall\, t\in[0,T]. \end{cases}$$

3.2 On the semi-group of the confined Langevin process

Following the approach of Yamada and Watanabe (see, *e.g.*, Karatzas and Schreve [11]), the pathwise uniqueness of the solution to (3.1) implies its uniqueness in law. According to Ethier and Kurtz [8], the uniqueness in law yields to the strong Markov property for the process (X,U), valued in $\overline{\mathcal{D}}\times\mathbb{R}^d$, solution of (3.1), and it is straightforward to compute the density $\Gamma(t;y,v;x,u)$ of (X_t,U_t) under the probability measure $\mathbb{Q}_{y,v}$, for $(y,v)\in\mathcal{D}\times\mathbb{R}^d$. For any $x=(x',x^{(d)})\in\mathbb{R}^{d-1}\times[0,+\infty)$, $u=(u',u^{(d)})\in\mathbb{R}^d$, $y=(y',y^{(d)})\in\mathbb{R}^{d-1}\times(0,+\infty)$ and $v=(v',v^{(d)})\in\mathbb{R}^d$, we have

$$\Gamma(t;y,v;x,u) = g_{d-1}(t;y',v';x',u')g^c(t;y^{(d)},v^{(d)};x^{(d)},u^{(d)}), \tag{3.10}$$

where $g_{d-1}(t;y',v';x',u')$ is the density law of the $(d-1)$-dimensional Brownian motion and its primitive starting from (y',v'), explicited in (1.5), and $g^c(t;y^{(d)},v^{(d)};x^{(d)},u^{(d)})$ is the transition density of the confined d^{th} component (X^c,U^c) satisfying (3.2).

In view of (3.7), we get that for all $f\in\mathcal{C}_b(\mathbb{R}^+\times\mathbb{R})$,

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}_{y^{(d)},v^{(d)}}}\left[f\left(X_t^c,\ U_t^c\right)\right] \\ &= \mathbb{E}_{\mathbb{Q}_{y^{(d)},v^{(d)}}}\left[f\left(X_t^f,U_t^f\right)\mathbb{1}_{\{X_t^f>0\}}\right] + \mathbb{E}_{\mathbb{Q}_{y^{(d)},v^{(d)}}}\left[f\left(-X_t^f,-U_t^f\right)\mathbb{1}_{\{X_t^f<0\}}\right], \end{aligned}$$

as $\{X_t^f=0\}$ is $\mathbb{Q}_{y^{(d)},v^{(d)}}$-negligible. Therefore,

$$g^c(t;y^{(d)},v^{(d)};\zeta,\nu) = g_1(t;y^{(d)},v^{(d)};\zeta,\nu), + g_1(t;y^{(d)},v^{(d)};-\zeta,-\nu), \tag{3.11}$$

for all $t>0$, a.e. (ζ,ν) in $[0,+\infty)\times\mathbb{R}$ and $(y^{(d)},v^{(d)})\in(0,+\infty)\times\mathbb{R}$.

Let us define the semi-group $(S_t;t>0)$ associated to the transition Γ by

$$S_t(f)(y,v) = \mathbb{E}_{\mathbb{Q}_{y,v}}\left[f(X_t,U_t)\right] = \int_{\mathcal{D}\times\mathbb{R}^d}\Gamma(t;y,v;x,u)f(x,u)dx\,du. \tag{3.12}$$

For $q\in\mathbb{N}^*$, let $\mathcal{A}_q$ be the second order differential operator defined on $C^{1,2}(\mathbb{R}^q\times\mathbb{R}^q)$ by

$$\mathcal{A}_q(f)(y,v) = (v\cdot\nabla_y f(y,v)) + \frac{1}{2}\triangle_v f(y,v).$$

Proposition 3.3. *(i) For all* $t>0$ *and* $f\in\mathcal{C}_c(\mathcal{D}\times\mathbb{R}^d)$*, the function* $G_{t,f}$ *defined by*

$$G_{t,f}(s,y,v) = S_{t-s}(f)(y,v), \ \textit{for}\ (s,y,v)\in[0,t)\times\mathcal{D}\times\mathbb{R}^d, \tag{3.13}$$

is a classical solution to the following Cauchy problem:

$$\begin{cases} \partial_s G_{t,f} + \mathcal{A}_d(G_{t,f}) = 0, \ \textit{in}\ [0,t)\times\mathcal{D}\times\mathbb{R}^d, \\ G_{t,f}(s,y,v) = G_{t,f}(s,y,v-2(v\cdot n_{\mathcal{D}}(y))n_{\mathcal{D}}(y)), \ \textit{in}\ [0,t)\times\partial\mathcal{D}\times\mathbb{R}^d, \\ \lim\limits_{s\to t^-} G_{t,f}(s,y,v) = f(y,v), \ \textit{in}\ \mathcal{D}\times\mathbb{R}^d. \end{cases} \tag{3.14}$$

(ii) There exists a constant $C > 0$ such that, for all $t > 0$,

$$\sup_{(y,v)\in\mathcal{D}\times\mathbb{R}^d} \int_{\mathcal{D}\times\mathbb{R}^d} |\nabla_v \Gamma(t;y,v;x,u)|\, dx\, du \leq \frac{C}{\sqrt{t}}. \tag{3.15}$$

Proof. Since the transition probability density function Γ in (3.10) is the fundamental solution of the differential operator $\partial_t - \mathcal{A}_d$, we only need to check that the specular boundary condition in (3.14) holds true. Owing to the smoothness of $y^{(d)} \mapsto g^c(t; y^{(d)}, v^{(d)}; x^{(d)}, u^{(d)})$, one also has

$$\lim_{y^{(d)}\to 0^+} G_{t,f}(s,y,v) = \int_{\mathcal{D}\times\mathbb{R}^d} f(x,u) g_{d-1}(t-s; y', v'; x', u') g^c(t-s; 0, v^{(d)}; x^{(d)}, u^{(d)}) dx\, du.$$

Hence, for all $t > 0$ and $(s,y,v) \in [0,t) \times \partial\mathcal{D} \times \mathbb{R}^d$,

$$G_{t,f}(s,y,v) = \int_{\mathcal{D}\times\mathbb{R}^d} f(x,u) g_{d-1}(t-s; y', v'; x', u') g^c(t-s; 0, v^{(d)}; x^{(d)}, u^{(d)}) dx\, du. \tag{3.16}$$

In view of (1.5) and (3.11), one observes that $g_1(t; 0, v^{(d)}; -x^{(d)}, -u^{(d)}) = g_1(t; 0, -v^{(d)}; x^{(d)}, u^{(d)})$ and

$$\begin{aligned} g^c(t; 0, v^{(d)}; x^{(d)}, u^{(d)}) &= g_1(t; 0, v^{(d)}; x^{(d)}, u^{(d)}) + g_1(t; 0, -v^{(d)}; x^{(d)}, u^{(d)}) \\ &= g^c(t; 0, -v^{(d)}; x^{(d)}, u^{(d)}). \end{aligned}$$

Coming back to (3.16), we deduce that $G_{t,f}$ satisfies the specular boundary condition in (3.14).

The estimate (3.15) is obtained by a straightforward computation on the explicit expression of $\nabla_v \Gamma(t; y, v; x, u)$ in terms of $\partial_v g_1$ in (1.5). $\square$

4 Proof of Theorem 2.2

In this section, the hypotheses (H) are implicitly assumed in all the stated propositions and lemmas.

We construct a solution to the martingale problem (MP) by means of the convergence of a smoothed and confined interacting particle system $\{(X^{i,\epsilon,N}, U^{i,\epsilon,N}, K^{i,\epsilon,N})\}_{i=1,\dots,N}$ defined by

$$\begin{cases} X_t^{i,\epsilon,N} = X_0^i + \displaystyle\int_0^t U_s^{i,\epsilon,N} ds, \\ U_t^{i,\epsilon,N} = U_0^i + \displaystyle\int_0^t \frac{\sum_{j=1}^N b(U_s^{j,\epsilon,N}, U_s^{i,\epsilon,N}) \phi_\epsilon(X_s^{i,\epsilon,N} - X_s^{j,\epsilon,N})}{\sum_{j=1}^N \left(\phi_\epsilon(X_s^{i,\epsilon,N} - X_s^{j,\epsilon,N}) + \epsilon\right)} ds + W_t^i + K_t^{i,\epsilon,N}, \\ K_t^{i,\epsilon,N} = -2 \displaystyle\sum_{0<s\leq t} \left(U_{s^-}^{i,\epsilon,N} \cdot n_{\mathcal{D}}\right) n_{\mathcal{D}} \mathbb{1}_{\left\{X_s^{i,\epsilon,N} \in \partial\mathcal{D}\right\}}, \; i = 1, \dots, N, \end{cases} \tag{4.1}$$

where $\{(X_0^i, U_0^i, W^i)\}_{i\geq 1}$ are independent copies of (X_0, U_0, W) on a given probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t; t \geq 0), \mathbb{Q})$. The sequence $\{\phi_\epsilon; \epsilon > 0\}$ is a family of mollifiers defined by $\phi_\epsilon(x) := \epsilon^{-d}\phi(\frac{x}{\epsilon})$, for some $\phi \in \mathcal{C}_c^1(\mathcal{D})$ such that $\phi \geq 0$ and $\int_{\mathbb{R}^d} \phi(x) dx = 1$. For $i = 1, \dots, N$, the processes $(X^{i,\epsilon,N}, U^{i,\epsilon,N}, K^{i,\epsilon,N})$ are valued in $\mathcal{E}$. The existence and uniqueness in law for the linear equation (4.1) derives from the Girsanov's Theorem and Proposition 3.1.

For a fixed $\epsilon > 0$, when $N \to +\infty$, the particle system (4.1) is aimed to propagate the chaos with a limit law given by the unique solution $\mathbb{P}^\epsilon$ of the following martingale problem.

Definition 4.1. *A probability measure $\mathbb{P}^\epsilon \in \mathcal{M}(\mathcal{E})$ is a solution to the martingale problem (MP_ϵ) if the following hold.*

(i) $\mathbb{P}^\epsilon \circ (x_0, u_0, k_0)^{-1} = \mu_0 \otimes \delta_0$.

(ii) For all $t \in (0,T]$, $\mathbb{P}^\epsilon \circ (x_t, u_t)^{-1}$ *admits a Lebesgue density* ρ_t^ϵ.

(iii) For all $f \in \mathcal{C}_b^2(\mathbb{R}^2)$, *the process*

$$
\begin{aligned}
f(x_t, u_t - k_t) - f(x_0, u_0) - \int_0^t (u_s \cdot \nabla_x f(x_s, u_s - k_s))\, ds \\
- \int_0^t \left[(B_\epsilon[x_s, u_s; \rho_s^\epsilon] \cdot \nabla_u f(x_s, u_s - k_s)) + \frac{1}{2} \triangle_u f(x_s, u_s - k_s) \right] ds
\end{aligned}
\tag{4.2}
$$

is a continuous $\mathbb{P}^\epsilon$*-martingale w.r.t.* $(\mathcal{B}_t; t \in [0,T])$.

(iv) $\mathbb{P}^\epsilon$*-a.s., the set* $\{t \in [0,T] \text{ s.t. } x_t^{(d)} = 0\}$ *is at most countable, and*

$$
k_t = -2 \sum_{0<s\leq t} (u_{s^-} \cdot n_{\mathcal{D}}) n_{\mathcal{D}} \mathbb{1}_{\{x_s \in \partial\mathcal{D}\}}, \ \forall\, t \in [0,T].
$$

The kernel B_ϵ is defined for $(x, u, \gamma) \in \mathcal{D} \times \mathbb{R}^d \times L^1(\mathcal{D} \times \mathbb{R}^d)$, by

$$
B_\epsilon[x, u; \gamma] = \frac{\displaystyle\int_{\mathcal{D}\times\mathbb{R}^d} b(v,u) \phi_\epsilon(x-y) \gamma(y,v)\, dy\, dv}{\displaystyle\int_{\mathcal{D}\times\mathbb{R}^d} \phi_\epsilon(x-y)\gamma(y,v) dy dv + \epsilon}.
\tag{4.3}
$$

In what follows, we also refer to B_ϵ for the kernel $(x, u, m) \mapsto B_\epsilon[x, u; m]$, for $(x, u) \in \mathcal{D} \times \mathbb{R}^d$ and $m \in \mathcal{M}(\mathcal{D} \times \mathbb{R}^d)$, by substituting $\gamma(y,v)dydv$ for $m(dy, dv)$ in definition (4.3). Note that, on $\mathcal{D} \times \mathbb{R}^d \times L^1(\mathcal{D} \times \mathbb{R}^d)$, B_ϵ is a mollified version of the kernel B defined in (1.4).

Theorem 2.2 is a consequence of the following proposition.

Proposition 4.2. *Let* $\bar{\mu}^{\epsilon,N} := \frac{1}{N} \sum_{i=1}^N \delta_{\left\{X^{i,\epsilon,N}, U^{i,\epsilon,N}, K^{i,\epsilon,N}\right\}}$ *be the empirical measure associated to (4.1). Then,*

(i) for all $\epsilon > 0$, *the sequence* $(\bar{\mu}^{\epsilon,N}, N \geq 1)$ *converges weakly to the unique solution* $\mathbb{P}^\epsilon$ *of the martingale problem* (MP_ϵ);

(ii) when ϵ *tends to* 0, $\mathbb{P}^\epsilon$ *converges to the unique solution of the martingale problem (MP).*

As shown in Sznitman [18], *(i)* ensures that the laws $\mathbb{P}^{\epsilon,N}$ of the particles $\{(X^{i,\epsilon,N}, U^{i,\epsilon,N}, K^{i,\epsilon,N}), 1 \leq i \leq N\}$ are $\mathbb{P}^\epsilon$-chaotic: for all integers $k \geq 2$ and all finite families $(\psi_l, 1 \leq l \leq k)$ of functions in $\mathcal{C}_b(\mathcal{C}([0,T]; \overline{\mathcal{D}}) \times \mathbb{D}([0,T]; \mathbb{R}^d) \times \mathbb{D}([0,T]; \mathbb{R}^d))$,

$$
\langle \mathbb{P}^{\epsilon,N}, \psi_1 \otimes \cdots \psi_k \otimes 1 \cdots \rangle \to \prod_{l=1}^k \langle \mathbb{P}^\epsilon, \psi_l \rangle, \text{ when } N \to +\infty.
$$

The proof of Proposition 4.2 is organized as follows. First, the uniqueness results for the martingale problems (MP) and (MP_ϵ) are stated in Proposition 4.3. Second we prove *(i)* in Section 4.2. Finally, we show the convergence result *(ii)* with Proposition 4.10. The proofs adapt the convergence techniques for martingale problems, from the *free* McKean Lagrangian case in [5] to the present case of *a confined* McKean Lagrangian diffusion with jumps.

4.1 Uniqueness results for the martingale problems $(\boldsymbol{MP})$ and$(\boldsymbol{MP}_\epsilon)$

It is enough to prove uniqueness results for the time-marginals $(\rho_t; t \in (0,T])$ and $(\rho^\epsilon_t; t \in (0,T])$ related to (MP) and (MP_ϵ) respectively. Indeed, this will ensure that any two weak solutions to (MP) or (MP_ϵ) are equipped with the same drift B or B_ϵ respectively. Owing to the boundedness of b, the uniqueness results for the martingale problems will follow by a change of probability measure. We identify the mild equations solved by the time-marginals, and prove uniqueness results by adapting the step-proofs of [5].

Consider $(S^*_t; t \in (0,T])$, the adjoint of $(S_t; t \in (0,T])$ in (3.12), defined by

$$S^*_t(\mu)(x,u) = \int_{\mathcal{D}\times\mathbb{R}^d} \Gamma(t; y, v; x, u)\mu(dy, dv). \tag{4.4}$$

In view of (3.10), for all $t \in (0,T]$, S^*_t is a linear operator from $\mathcal{M}(\mathcal{D}\times\mathbb{R}^d)$ to $L^1(\mathcal{D}\times\mathbb{R}^d)$. In addition, we define the operator $(S'_t; t \in (0,T])$ by

$$S'_t(f)(x,u) = \int_{\mathcal{D}\times\mathbb{R}^d} (\nabla_v\Gamma(t; y, v; x, u) \cdot f(y,v))\, dy\, dv. \tag{4.5}$$

According to (3.15), for any $t > 0$, $S'_{t-\cdot}$ defines a linear mapping from $L^\infty((0,t); L^1(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d))$ to $L^1(Q_t)$ such that for all $\gamma \in L^\infty((0,t); L^1(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d))$,

$$\int_{Q_t} \left|S'_{t-s}(\gamma(s))(y,v)\right| ds\, dy\, dv \le \int_0^t \frac{C}{\sqrt{t-s}} \|\gamma(s)\|_{L^1(\mathcal{D}\times\mathbb{R}^d)} ds. \tag{4.6}$$

Proposition 4.3. *(i) The time-marginals $(\rho_t; t \in (0,T])$ and $(\rho^\epsilon_t; t \in (0,T])$ of $\mathbb{P}$ and $\mathbb{P}^\epsilon$, solutions to the martingale problem (MP) and (MP$_\epsilon$) respectively, satisfy the following mild equations in $L^1(\mathcal{D}\times\mathbb{R}^d)$:*

$$\forall\ t \in (0,T]\,,\ \rho_t = S^*_t(\mu_0) + \int_0^t S'_{t-s}(\rho_s(\cdot)B[\cdot;\rho_s])ds, \tag{4.7}$$

$$\forall\ t \in (0,T]\,,\ \rho^\epsilon_t = S^*_t(\mu_0) + \int_0^t S'_{t-s}(\rho^\epsilon_s(\cdot)B_\epsilon[\cdot;\rho^\epsilon_s])ds. \tag{4.8}$$

(ii) The mild equations (4.7) and (4.8) have at most one solution.

Proof. We prove only *(i)*, the proof of *(ii)* being a straightforward replication of the proof of Lemma 4.5 in [5].

For a fixed $t \in (0,T]$ and $f \in \mathcal{C}_c\left(\mathcal{D}\times\mathbb{R}^d\right)$, let $G_{t,f}$ be the classical solution of (3.14) defined in (3.13). As mentioned above, for $\mathbb{P}$ solving (MP), (x,u,k) satisfies (1.1). Therefore,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[G_{t,f}(t, x_t, u_t)\right] =& \mathbb{E}_{\mathbb{P}}\left[G_{t,f}(0, x_0, u_0)\right] + \int_0^t \mathbb{E}_{\mathbb{P}}\left[(B\left[x_s, u_s; \rho_s\right] \cdot \nabla_u G_{t,f}(s, x_s, u_s))\right] ds\\
&+ \mathbb{E}_{\mathbb{P}}\left[\int_0^t \left(\partial_s + \mathcal{A}_d\right)(G_{t,f})(s, x_s, u_s) ds\right]\\
&+ \mathbb{E}_{\mathbb{P}}\left[\sum_{0<s\le t} \left(G_{t,f}(s, x_s, u_s) - G_{t,f}(s, x_s, u_{s-})\right) \mathbb{1}_{\{x_s\in\partial\mathcal{D}\}}\right].
\end{aligned}$$

According to Proposition 3.3-*(i)*, the equality above writes

$$\begin{aligned}
\int_{\mathcal{D}\times\mathbb{R}^d} f(x,u)\rho_t(x,u) dx\, du =& \int_{\mathcal{D}\times\mathbb{R}^d} G_{t,f}(0, x, u)\mu_0(dx, du)\\
&+ \int_{Q_t} \left(B\left[x, u; \rho_s\right] \cdot \nabla_u G_{t,f}(s, x, u)\right) \rho_s(x,u) ds\, dx\, du.
\end{aligned}$$

Noticing that $\int_{\mathcal{D}\times\mathbb{R}^d}\Gamma(t;y,v;x,u)dy\,dv=1$, we have

$$\int_{\mathcal{D}\times\mathbb{R}^d} G_{t,f}(0,x,u)\mu_0(dx,du)=\int_{\mathcal{D}\times\mathbb{R}^d} S_t^*(\mu_0)(x,u)f(x,u)\,dx\,du,$$

with $S_t^*(\mu_0)\in L^1(\mathcal{D}\times\mathbb{R}^d)$. Similarly, owing to the estimate (4.6), it holds that

$$\int_{Q_t}\left(B\left[x,u;\rho_s\right]\cdot\nabla_u G_{t,f}(s,x,u)\right)\rho_s(x,u)ds\,dx\,du=\int_{Q_t} f(x,u)S'_{t-s}(\rho_s(\cdot)B[\cdot;\rho_s])(x,u)ds\,dx\,du.$$

Thus we deduce that, for all $t\in(0,T]$ and $f\in\mathcal{C}_c(\mathcal{D}\times\mathbb{R}^d)$,

$$\int_{\mathcal{D}\times\mathbb{R}^d} f(x,u)\left(\rho_t(x,u)-S_t^*(\mu_0)(x,u)-\int_0^t S'_{t-s}(\rho_s(\cdot)B\left[\cdot;\rho_s\right])(x,u)ds\right)dx\,du=0.$$

With Proposition 3.3-*(ii)*, we conclude that $(\rho_t;t\in(0,T])$ satisfy the mild equation (4.7).

In the case where the drift coefficient is B_ϵ, duplicating the same arguments, it is straightforward to establish (4.8). □

4.2 Existence result for (MP_ϵ)

In this section, we restrict ourselves to the case $d=1$ and $\mathcal{D}=(0,+\infty)$, the proof can be readily extended to the case $\mathcal{D}=\mathbb{R}^{d-1}\times(0,+\infty)$. When $d=1$, the particle system (4.1) writes

$$\begin{cases} X_t^{i,\epsilon,N}=X_0^i+\displaystyle\int_0^t U_s^{i,\epsilon,N}ds,\\ U_t^{i,\epsilon,N}=U_0^i+\displaystyle\int_0^t\frac{\displaystyle\sum_{j=1}^N b(U_s^{j,\epsilon,N},U_s^{i,\epsilon,N})\phi_\epsilon(X_s^{i,\epsilon,N}-X_s^{j,\epsilon,N})}{\displaystyle\sum_{j=1}^N\left(\phi_\epsilon(X_s^{i,\epsilon,N}-X_s^{j,\epsilon,N})+\epsilon\right)}ds+W_t^i+K_t^{i,\epsilon,N},\\ K_t^{i,\epsilon,N}=-2\displaystyle\sum_{0<s\le t}U_{s^-}^{i,\epsilon,N}\mathbb{1}_{\{X_s^{i,\epsilon,N}=0\}},\ i=1,\ldots,N.\end{cases}\tag{4.9}$$

The proof of Proposition 4.2-*(i)* proceeds in two steps.

Step 1. We prove a tightness result for the sequence of probability measures $\{\pi^{\epsilon,N}\}_{N\ge1}$ induced by $\bar{\mu}^{\epsilon,N}$ on $\mathcal{M}(\mathcal{E})$, by using Aldous's Tightness criterion.

Step 2. We check that all limit points of $\{\pi^{\epsilon,N}\}_{N\ge1}$ have full measure on the set of probability measures satisfying the properties *(i)-(iv)* of (MP_ϵ) in Definition 4.1.

We then deduce the existence of a probability measure $\mathbb{P}^\epsilon$ solution to (MP_ϵ). The uniqueness result in Proposition 4.3 implies that all converging subsequences of $\{\pi^{\epsilon,N}\}_{N\ge1}$ tend to $\delta_{\{\mathbb{P}^\epsilon\}}$. It follows that the entire sequence $\{\pi^{\epsilon,N}\}_{N\ge1}$ converges to $\delta_{\{\mathbb{P}^\epsilon\}}$, and enables us to conclude on Proposition 4.2-*(i)*.

Step 1. Tightness result As shown in Sznitman [18], the exchangeability of the particle system (4.9) induces the equivalence between the tightness property of $\{\pi^{\epsilon,N}\}_{N\ge1}$ and the tightness of the sequence $\{\mathbb{Q}\circ(X^{1,\epsilon,N},U^{1,\epsilon,N},K^{1,\epsilon,N})^{-1}\}_{N\ge1}$.

Lemma 4.4. *$\{\mathbb{Q}\circ\left(X^{1,\epsilon,N},U^{1,\epsilon,N},K^{1,\epsilon,N}\right)^{-1}\}_{N\ge1}$ is tight on $\mathcal{E}$.*

Proof. We apply the Aldous criterion to the system $\{X^{1,\epsilon,N}, U^{1,\epsilon,N}, K^{1,\epsilon,N}\}_{N\geq 1}$. For the sake of completeness, let us recall this criterion.

Theorem 4.5. *(see, e.g., Billingsley [4].) Let $\{Y^n\}_{n\in\mathbb{N}}$ be a sequence of r.v. defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in $\mathbb{D}([0,T];\mathbb{R}^q)$. The sequence $\{P\circ(Y^n)^{-1}\}_{n\in\mathbb{N}}$ is tight on $\mathbb{D}([0,T];\mathbb{R}^q)$ if the following hold.*

(C1) For all $t\geq 0$, $P\circ(Y^n_t)^{-1}$ is tight on $\mathbb{R}^q$.

(C2) For all $\varepsilon>0$, $\eta>0$, there exist $\delta_0>0$ and $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ and for all discrete-valued $\sigma(Y^n_s; s\in[0,T])$-stopping times β such that $0\leq\beta+\delta_0\leq T$,

$$\sup_{\delta\in[0,\delta_0]} P\left(\left|Y^n_{\delta+\beta}-Y^n_\beta\right|\geq\eta\right)\leq\varepsilon.$$

We check *(C1)*. With $K^{i,\epsilon,N}_t$ obtained from the evolution equation of $U^{i,\epsilon,N}$, we easily get that

$$\begin{aligned}
&\sup_{N\geq 1}\mathbb{E}_{\mathbb{Q}}[|X^{1,\epsilon,N}_t|]+\sup_{N\geq 1}\mathbb{E}_{\mathbb{Q}}[|K^{1,\epsilon,N}_t|]\\
&\leq\mathbb{E}_{\mathbb{Q}}\left(|X^1_0|+|U^1_0|+|W^1_t|\right)+(1+T)\sup_{N\geq 1}\mathbb{E}_{\mathbb{Q}}\left[\sup_{\theta\in[0,T]}|U^{1,\epsilon,N}_\theta|\right]+T\|b\|_\infty.
\end{aligned}\tag{4.10}$$

Applying the Itô formula to $|U^{1,\epsilon,N}_t|^2$, the jump term vanishes and we get

$$|U^{1,\epsilon,N}_t|^2=|U^{1,\epsilon,N}_0|^2+2\int_0^T\left(D^{1,\epsilon,N}_s\cdot U^{1,\epsilon,N}_s\right)ds+2\int_0^T U^{1,\epsilon,N}_s dW^1_s+t,$$

with $(D^{1,\epsilon,N}_t; t\in[0,T])$ defined by

$$D^{1,\epsilon,N}_t=\frac{\sum_{j=1}^N b(U^{j,\epsilon,N}_t,U^{1,\epsilon,N}_t)\phi_\epsilon(X^{1,\epsilon,N}_t-X^{j,\epsilon,N}_t)}{\sum_{j=1}^N\left(\phi_\epsilon(X^{1,\epsilon,N}_t-X^{j,\epsilon,N}_t)+\epsilon\right)}.$$

By classical arguments and thanks to (H), this allows to show that there exists a positive constant C depending on T, $\mathbb{E}[|U_0|^2]$ and $\|b\|_\infty$ such that

$$\sup_{N\geq 1}\mathbb{E}_{\mathbb{Q}}[\sup_{\theta\in[0,T]}|U^{1,\epsilon,N}_\theta|^2]\leq C.\tag{4.11}$$

Then *(C1)* is fulfilled.

Now we check *(C2)*. For a given $N\geq 2$, let β be a $\sigma\left(X^{1,\epsilon,N},U^{1,\epsilon,N},K^{1,\epsilon,N}\right)$-stopping time with discrete values such that $\beta+\delta_0\leq T$. Then we observe that

$$\lim_{\delta_0\to 0^+}\sup_{\delta\in[0,\delta_0]}\mathbb{Q}\left(\left|X^{1,\epsilon,N}_{\delta+\beta}-X^{1,\epsilon,N}_\beta\right|\geq\eta\right)\leq\lim_{\delta_0\to 0^+}\frac{\delta_0}{\eta}\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T]}\left|U^{1,\epsilon,N}_t\right|\right]\leq\lim_{\delta_0\to 0^+}\frac{\delta_0 C}{\eta},$$

for a constant $C>0$, which does not depend on N and ϵ. Since b is bounded, we introduce the probability measure $\widetilde{\mathbb{Q}}$ defined by $d\widetilde{\mathbb{Q}}=Z^{1,\epsilon,N}_T d\mathbb{Q}$, where the $\mathcal{F}_t$-martingale $(Z^{1,\epsilon,N}_t; t\in[0,T])$ is given by

$$Z^{1,\epsilon,N}_t=\exp\left(-\int_0^t D^{1,\epsilon,N}_s\,dW^1_s-\int_0^t\left|D^{1,\epsilon,N}_s\right|^2 ds\right),$$

Girsanov's Theorem implies that $(W_t^{1,\epsilon,N} := W_t^1 + \int_0^t D_s^{1,\epsilon,N} ds; t \in [0,T])$ is a Brownian motion on $(\Omega, \mathcal{F}_T, \widetilde{\mathbb{Q}})$. Proposition 3.1 ensures that

$$\widetilde{\mathbb{Q}} \circ (X^{1,\epsilon,N}, U^{1,\epsilon,N}, K^{1,\epsilon,N})^{-1} = \mathbb{Q} \circ (X^c, U^c, -2 \sum_{0<s\leq \cdot} U^c_{s^-} \mathbb{1}_{\{X^c_s=0\}})^{-1},$$

where (X^c, U^c) is the solution of (3.2). For the jump component $K^{1,\epsilon,N}$, using the change of probability $d\widetilde{\mathbb{Q}} = Z_T^{1,\epsilon,N}\, d\mathbb{Q}$, as $\beta + \delta_0 \leq T$, we have

$$\mathbb{Q}\left(\left|K^{1,\epsilon,N}_{\delta+\beta} - K^{1,\epsilon,N}_{\beta}\right| \geq \eta\right) \leq \sqrt{\frac{1}{\eta}} \exp\left(T\|b\|_\infty^2/2\right) \sqrt{\mathbb{E}_{\mathbb{Q}}\left[\left|\sum_{\beta<s\leq\beta+\delta} -2U^c_{s^-}\mathbb{1}\{X^c_s = 0\}\right|\right]}.$$

According to Proposition 3.1, $s \mapsto -2U^c_{s^-}\mathbb{1}\{X^c_s = 0\}$ is non-negative. Moreover,

$$\sup_{\delta\in[0,\delta_0]} \mathbb{E}_{\mathbb{Q}}\left[\sum_{\beta<s\leq\beta+\delta} -2U^c_{s^-}\mathbb{1}\{X^c_s = 0\}\right] \leq \sqrt{T} + 2\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T]} |U^c_t|\right] < +\infty.$$

Owing to the right continuity of $t \mapsto \sum_{0<s\leq t} -2U^c_{s^-}\mathbb{1}_{\{X^c_s=0\}}$, and by the Dominated Convergence Theorem, one immediately gets that

$$\lim_{\delta_0\to 0^+} \mathbb{E}_{\mathbb{Q}}\left[\sum_{\beta<s\leq\beta+\delta_0} -2U^c_{s^-}\mathbb{1}\{X^c_s = 0\}\right] = 0.$$

Consequently, $\sup_{\delta\in[0,\delta_0]} \mathbb{Q}\left(\left|K^{1,\epsilon,N}_{\delta+\beta} - K^{1,\epsilon,N}_{\beta}\right| \geq \eta\right)$ tends to 0 when δ_0 goes to 0. For the velocity component, notice that $\mathbb{Q}(|U^{1,\epsilon,N}_{\delta+\beta} - U^{1,\epsilon,N}_{\beta}| \geq 3\eta)$ is bounded by

$$\mathbb{Q}\left(\left|K^{1,\epsilon,N}_{\delta+\beta} - K^{1,\epsilon,N}_{\beta}\right| \geq \eta\right) + \frac{1}{\eta}\mathbb{E}_{\mathbb{Q}}\left[|W^1_{\delta+\beta} - W^1_\beta|\right] + \frac{1}{\eta}\mathbb{E}_{\mathbb{Q}}\left[\left|\int_\beta^{\delta+\beta} D^{1,\epsilon,N}_s\, ds\right|\right].$$

We already studied the first term. The boundedness of b provides that

$$\mathbb{E}_{\mathbb{Q}}\left[\left|\int_\beta^{\delta+\beta} D^{1,\epsilon,N}_s\, ds\right|\right] \leq \delta\|b\|_\infty,$$

and, since β is discrete valued,

$$\mathbb{E}_{\mathbb{Q}}\left[|W^1_{\delta+\beta} - W^1_\beta|\right] \leq \mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T-\delta]} \left|W^1_{\delta+t} - W^1_t\right|\right] \leq \sqrt{\frac{2}{\pi}\delta}.$$

These estimates imply that

$$\lim_{\delta_0\to 0^+} \sup_{\delta\in[0,\delta_0]} \mathbb{Q}\left(\left|U^{1,\epsilon,N}_{\delta+\beta} - U^{1,\epsilon,N}_{\beta}\right| \geq 3\eta\right) = 0,$$

which ends the proof. □

Step 2. Identification of limit points. Let $\pi^{\epsilon,\infty}$ denote the limit of a converging subsequence of $\left\{\pi^{\epsilon,N}\right\}_{N\geq 1}$ that we still index by N for simplicity. Below, we show that all elements of the support of $\pi^{\epsilon,\infty}$ satisfy (MP_ϵ)-*(i)* to (MP_ϵ)-*(iv)*. First, since (X_0^i, U_0^i) are i.i.d. with law μ_0, (MP_ϵ)-*(i)* is clearly satisfied, for $\pi^{\epsilon,\infty}$-a.e. $m \in \mathcal{M}(\mathcal{E})$. Next we prove

Lemma 4.6. *For $\pi^{\epsilon,\infty}$-almost all $m \in \mathcal{M}(\mathcal{E})$, $m((x_0, u_0) \in dx\,du) = \mu_0(dx, du)$ and*

$$\begin{cases} x_t = x_0 + \displaystyle\int_0^t u_s ds, \\ u_t = u_0 + \displaystyle\int_0^t B_\epsilon[x_s, u_s; m_s]\, ds + w_t + k_t, \\ m_t \textit{ is the law of } (x_t, u_t) \textit{ under } m, \textit{ for all } t \in [0, T], \end{cases} \tag{4.12}$$

where $(w_t; t \in [0, T])$ is a Wiener process.

Proof. We preliminary check that $\pi^{\epsilon,\infty}$ has full measure on the set of all $m \in \mathcal{M}(\mathcal{E})$ such that the process $t \mapsto (x_t, u_t - k_t)$ is m-a.s. continuous. Since $\mathcal{C}([0, T]; \mathbb{R}^2)$ is closed for the Skorokhod topology, the set

$$D = \{(x, u, k) \in \mathcal{E} \text{ s.t. } t \mapsto (x_t, u_t - k_t) \text{ is continuous}\}$$

is closed in $\mathcal{E}$. Applying twice the Portemanteau Theorem (see, *e.g.*, Billingsley [4]), we get first that

$$m\,(D) \geq \limsup_{n \to +\infty} m_n\,(D)\,,$$

for all sequences m_n of $\mathcal{M}(\mathcal{E})$ converging weakly to m. Thus $\{m \in \mathcal{M}(\mathcal{E}) \text{ s.t. } m(D) = 1\}$ is a closed subset of $\mathcal{M}(\mathcal{E})$. And next

$$\pi^{\epsilon,\infty}\left(\{m \in \mathcal{M}(\mathcal{E}) \text{ s.t. } m(D) = 1\}\right) \geq \lim_{N \to +\infty} \mathbb{Q}\left(\bar{\mu}^{\epsilon,N}(D) = 1\right).$$

Since $\mathbb{Q}$-a.s., $\bar{\mu}^{\epsilon,N}$ has full measure on D, we get our claim.

For $f \in \mathcal{C}_b^2(\mathbb{R}^2)$ and for $\pi^{\epsilon,\infty}$-a.e. $m \in \mathcal{M}(\mathcal{E})$, the process

$$\begin{aligned} t \mapsto & f(x_t, u_t - k_t) - f(x_0, u_0) - \int_0^t \left(u_s \partial_x f(x_s, u_s - k_s) + \frac{1}{2}\partial_u^2 f(x_s, u_s - k_s)\right) ds \\ & - \int_0^t \left(B_\epsilon[x_s, u_s; m_s] \partial_u f(x_s, u_s - k_s)\right) ds \end{aligned} \tag{4.13}$$

is m-a.s. continuous, and we prove now that it is a martingale. To this aim, for all $0 \leq t_1 \leq \cdots \leq t_q \leq s < t \leq T$ and all finite families of functions $\{\psi_i\}_{1 \leq i \leq q}$ in $\mathcal{C}_b(\mathbb{R}^3)$, we want to show that

$$\mathbb{E}_{\pi^{\epsilon,\infty}}\left[\left|F_{t_1,t_2,\ldots,t_q,s,t}(m)\right|\right] = 0, \tag{4.14}$$

where $F_{t_1,t_2,\ldots,t_q,s,t} : \mathcal{M}(\mathcal{E}) \longrightarrow \mathbb{R}$ is defined by

$$\begin{aligned} & F_{t_1,t_2,\ldots,t_q,s,t}(m) \\ & = \mathbb{E}_m\Bigg[\prod_{i=1}^q \psi_i(x_{t_i}, u_{t_i}, k_{t_i})\Bigg(f(x_t, u_t - k_t) - f(x_s, u_s - k_s) \\ & \qquad - \int_s^t \left(u_\theta \partial_x f(x_\theta, u_\theta - k_\theta) + \frac{1}{2}\partial_u^2 f(x_\theta, u_\theta - k_\theta)\right) d\theta\Bigg)\Bigg] \\ & \quad - \mathbb{E}_m\left[\prod_{i=1}^q \psi_i(x_{t_i}, u_{t_i}, k_{t_i}) \int_s^t B_\epsilon\left[x_\theta, u_\theta; m_\theta\right] \partial_u f(x_\theta, u_\theta - k_\theta)\, d\theta\right] \\ & =: \mathbb{E}_m\left[\Phi_{t_1,t_2,\ldots,t_q,s,t}\right] + \mathbb{E}_m\left[\Psi_{t_1,t_2,\ldots,t_q,s,t}(m)\right]. \end{aligned} \tag{4.15}$$

Let us quote a result from Graham and Méléard [9].

Lemma 4.7. *Let w denote the canonical process in $\mathbb{D}([0,T];\mathbb{R}^p)$. Let $\mathcal{P} \in \mathcal{M}(\mathcal{M}(\mathbb{D}([0,T];\mathbb{R}^p)))$. Then the set*

$$\mathbf{D}_{\mathcal{P}} = \{\theta \in [0,T] \ s.t. \ \mathcal{P}\left(\{m \in \mathcal{M}\left(\mathbb{D}([0,T];\mathbb{R}^p)\right) \ s.t. \ m\left(|\triangle w_\theta| > 0\right) > 0\}\right) > 0\}$$

is at most countable.

For $t_1 \le \cdots \le t_q \le s \le t \le T$, let us consider a bounded approximation $\Phi^{\kappa}_{t_1,t_2,\ldots,t_q,s,t}$ of $\Phi_{t_1,t_2,\ldots,t_q,s,t}$, substituting u_θ with $-\kappa \vee (u_\theta \wedge \kappa)$ in (4.15); namely

$$\begin{aligned}\Phi^{\kappa}_{t_1,t_2,\cdots,t_n,s,t} := \prod_{i=1}^{q} \psi_i(x_{t_i}, u_{t_i}, k_{t_i}) \Bigg(& f(x_t, u_t - k_t) - f(x_s, u_s - k_s) \\ & - \int_s^t \left(-\kappa \vee (u_\theta \wedge \kappa) \partial_x f(x_\theta, u_\theta - k_\theta) + \frac{1}{2}\partial_u^2 f(x_\theta, u_\theta - k_\theta) \right) d\theta \Bigg).\end{aligned}$$

According to Lemma 4.7, there exists a countable set of times $\mathbf{D}_{\pi^{\epsilon,\infty}}$ such that, for all $0 \le t_1 \le \cdots \le t_q \le s < t \le T$ outside $\mathbf{D}_{\pi^{\epsilon,\infty}}$, the mappings $\omega \mapsto \Phi_{t_1,t_2,\cdots,t_q,s,t}(\omega)$ and $\omega \mapsto \Psi_{t_1,t_2,\ldots,t_q,s,t}(\widetilde{m}, \omega)$ are m-a.s. continuous on the sample space $\mathcal{E}$, for $\pi^{\epsilon,\infty}$-a.e. m and $\widetilde{m}$. Outside $\mathbf{D}_{\pi^{\epsilon,\infty}}$, it is clear that $\omega \mapsto \Phi^{\kappa}_{t_1,t_2,\cdots,t_q,s,t}$ is continuous, and so $m \mapsto \mathbb{E}_m\left[\Phi^{\kappa}_{t_1,t_2,\cdots,t_q,s,t}\right]$ is continuous on $\mathcal{M}(\mathcal{E})$. The continuity of $m \mapsto \mathbb{E}_m\left[\Psi_{t_1,t_2,\cdots,t_q,s,t}(m)\right]$ is obtained as a simple extension of the unconfined situation treated in the proof of [5, Proposition 5.6]. Then, by Fatou's Lemma and the weak convergence of $\pi^{\epsilon,N}$, we get

$$\begin{aligned}&\mathbb{E}_{\pi^{\epsilon,\infty}} \left|\mathbb{E}_m\left(\Phi_{t_1,t_2,\cdots,t_q,s,t} + \Psi_{t_1,t_2,\cdots,t_q,s,t}(m)\right)\right| \\ &\le \liminf_{\kappa \to +\infty} \lim_{N \to +\infty} \mathbb{E}_{\pi^{\epsilon,N}} \left|\mathbb{E}_m\left(\Phi^{\kappa}_{t_1,t_2,\cdots,t_q,s,t} + \Psi_{t_1,t_2,\cdots,t_q,s,t}(m)\right)\right|.\end{aligned} \tag{4.16}$$

But,

$$\begin{aligned}&\lim_{N\to+\infty} \mathbb{E}_{\mathbb{Q}} \left|\mathbb{E}_{\bar{\mu}^{\epsilon,N}}\left(\Phi^{\kappa}_{t_1,t_2,\cdots,t_q,s,t} + \Psi_{t_1,t_2,\cdots,t_q,s,t}(\bar{\mu}^{\epsilon,N})\right)\right| \\ &\le \lim_{N\to+\infty} \mathbb{E}_{\mathbb{Q}}\left|F_{t_1,t_2,\cdots,t_q,s,t}(\bar{\mu}^{\epsilon,N})\right| + \prod_{i=1}^{q} \|\psi_i\|^2_\infty \|\partial_x f\|_\infty \lim_{N\to+\infty} \frac{2}{\kappa N} \sum_{i=1}^{N} \mathbb{E}_{\mathbb{Q}}[\sup_{\theta\in[0,T]} |U^{i,\epsilon,N}_\theta|^2].\end{aligned}$$

Under $\mathbb{Q}$, $F_{t_1,t_2,\ldots,t_q,s,t}(\bar{\mu}^{\epsilon,N})$ writes

$$\frac{1}{N}\sum_{j=1}^{N}\prod_{i=1}^{q} \psi_i(X^{j,\epsilon,N}_{t_i}, U^{j,\epsilon,N}_{t_i}, K^{j,\epsilon,N}_{t_i}) \int_s^t \partial_u f(X^{j,\epsilon,N}_\theta, U^{j,\epsilon,N}_\theta - K^{j,\epsilon,N}_\theta) dW^j_\theta.$$

Owing to the independence of the Brownian family $\{W^j\}_{j\in\mathbb{N}^*}$ and the exchangeability of the particle system, it holds that

$$\mathbb{E}_{\mathbb{Q}}\left[\left(F_{t_1,t_2,\ldots,t_q,s,t}(\bar{\mu}^{\epsilon,N})\right)^2\right] \le (t-s)\|\partial_u f\|^2_\infty \frac{1}{N}\prod_{i=1}^{q}\|\psi_i\|^2_\infty.$$

Consequently, $\mathbb{E}_{\mathbb{Q}}[|F_{t_1,t_2,\ldots,t_q,s,t}(\bar{\mu}^{\epsilon,N})|]$ tends to 0 when N goes to $+\infty$. Furthermore, using the momentum estimate (4.11), one obtains that

$$\lim_{\kappa\to+\infty}\lim_{N\to+\infty}\frac{2}{\kappa N}\sum_{i=1}^{N}\mathbb{E}_{\mathbb{Q}}\left[\sup_{\theta\in[0,T]}\left|U^{i,\epsilon,N}_\theta\right|^2\right] \le \lim_{\kappa\to+\infty}\frac{2}{\kappa}\left(\sup_{N\ge1}\mathbb{E}_{\mathbb{Q}}\left[\sup_{\theta\in[0,T]}\left|U^{1,\epsilon,N}_\theta\right|^2\right]\right) = 0.$$

Coming back to (4.16), we get that for all $0 \leq t_1 \leq \cdots \leq t_q \leq s \leq t \leq T$ outside $\mathbf{D}_{\pi^{\epsilon,\infty}}$,

$$\mathbb{E}_{\pi^{\epsilon,\infty}} \left| \mathbb{E}_m \left[\Phi_{t_1,t_2,\cdots,t_q,s,t} + \Psi_{t_1,t_2,\cdots,t_q,s,t}(m) \right] \right| = 0.$$

Now, any t_i in $[0,T]$ can be approximated with a decreasing sequence $\{t_i^l\}_{l\in\mathbb{N}}$ outside $\mathbf{D}_{\pi^{\epsilon,\infty}}$. Owing to the right-continuity of $(t_1, t_2, \cdots, t_q, s, t) \mapsto F_{t_1,t_2,\cdots,t_q,s,t}(m)$ for all $m \in \mathcal{E}$, and Fatou's Lemma, (4.14) holds true, and we conclude on our claim. □

We complete *Step* 2 by identifying $(k_t; t \in [0,T])$.

Lemma 4.8. *For $\pi^{\epsilon,\infty}$-a.e. $m \in \mathcal{M}(\mathcal{E})$, $(x_t, u_t, k_t; t \in [0,T])$ satisfies m-a.s.:*

(a) For all jump times $t \in [0,T]$ of u, $\triangle u_t = -2u_{t^-}$.

(b) k is a non-decreasing function, and $k_t = \int_0^t \mathbb{1}_{\{x_s = 0\}} \, dk_s, \ \forall \ t \in [0,T]$.

(c) The set $\{x_t \ \in [0,T]$ s.t. $x_t = 0\}$ is at most countable.

As a consequence of Lemma 4.8, $(k_t; t \in [0,T])$ in (4.12) is a pure jump process, which stands for the cumulative jump part of the velocity $(u_t; t \in [0,T])$ on the null set of $t \mapsto x_t$; namely $k_t = -2\sum_{0<s\leq t} u_{s^-} \mathbb{1}_{\{x_s=0\}}$ for all t. The existence of the time-marginal densities of m follows by applying Girsanov's Theorem. This ends *Step 2*.

Proof of Lemma 4.8. Let us introduce the set $\mathcal{H}$ which consists in all elements (x, u, k) of $\mathcal{E}$ such that u and k satisfies the properties *(a)* and *(b)*. We observe the following.

Lemma 4.9. *Under the Skorokhod topology, $\mathcal{H}$ is a closed subset of $\mathcal{E}$.*

We postpone the proof of this statement at the end of this section. With the observation that $\pi^{\epsilon,N}(\mathcal{M}(\mathcal{H})) = 1$ for all $N \geq 1$, the Portemanteau Theorem gives

$$\pi^{\epsilon,\infty}\left(\mathcal{M}(\mathcal{H})\right) \geq \limsup_{N\to+\infty} \pi^{\epsilon,N}\left(\mathcal{M}(\mathcal{H})\right) = 1.$$

Consequently, for $\pi^{\epsilon,\infty}$-a.e. $m \in \mathcal{M}(\mathcal{E})$, $(x_t, u_t, k_t; t \in [0,T])$ satisfies *(a)* and *(b)*. To prove *(c)*, we fix $m \in \mathcal{M}(\mathcal{H})$. Since $(w_t; t \in [0,T])$ in (4.12) is a $\mathcal{B}_t$-Brownian motion and b is bounded, we define $\widetilde{m}$ by $d\widetilde{m} = z_T^\epsilon dm$ with $(z_t^\epsilon; t \in [0,T])$ given by

$$z_t^\epsilon = \exp\left(-\int_0^t B_\epsilon[x_s, u_s; m_s] dw_s - \frac{1}{2}\int_0^t |B_\epsilon[x_s, u_s; m_s]|^2 \, ds \right).$$

By Girsanov Theorem, $(\widetilde{w}_t^\epsilon := \int_0^t B_\epsilon[x_s, u_s; m_s] \, ds + w_t; t \in [0,T])$ is a Wiener process under $\widetilde{m}$, and

$$\begin{cases} x_t = x_0 + \displaystyle\int_0^t u_s ds, \\ u_t = u_0 + \widetilde{w}_t^\epsilon + k_t, \ \forall \ t \in [0,T]. \end{cases} \tag{4.17}$$

By Assumption *(H-i)*, $x_0 > 0$ $\widetilde{m}$-a.s., and thus the first hitting time τ_1 at 0 is positive. In particular, $u_{\tau_1^-} < 0$. Thus we can define $\tau_2 = \inf\{t > \tau_1$ s.t. $x_t = 0\}$, and according to *(a)* and *(b)*, we observe that $k_{\tau_1} = -2u_{\tau_1^-}$. Repeating this argument, we prove that the sequence $\{\tau_n; n \in \mathbb{N}\}$ exists, and further that $(x, u, k, \widetilde{w}^\epsilon)$ under $\widetilde{m}$ is a solution in law of (3.2). By Proposition 3.1, we conclude that τ_n grows to infinity, and thus that *(c)* holds true. □

Proof of Lemma 4.9. Let $\{\zeta^n = (x^n, u^n, k^n); n \in \mathbb{N}\}$ be a sequence in $\mathcal{H}$, converging to $\zeta = (x, u, k)$ in $\mathcal{E}$. According to the definition of the Skorokhod topology (see, *e.g.*, Jacod and Shiryaev [10]), there exists a sequence of continuous increasing functions $\{\lambda_n\}_{n\in\mathbb{N}}$ defined on $[0, T]$ such that for all n, $\lambda_n(0) = 0$, $\lambda_n(T) = T$,

$$\begin{aligned}&\lim_{n\to+\infty} \sup_{t\in[0,T]} |\lambda_n(t) - t| = 0, \quad \lim_{n\to+\infty} \sup_{t\in[0,T]} |\zeta^n(\lambda_n(t)) - \zeta(t)| = 0,\\ &\text{and for all } t \in [0, T], \quad \lim_{n\to+\infty} |\Delta\zeta^n(\lambda_n(t)) - \Delta\zeta(t)| = 0.\end{aligned} \tag{4.18}$$

According to (4.18), $\triangle u$ and k are given by

$$\begin{aligned}\triangle u(t) &= \lim_{n\to+\infty} \triangle u^n(\lambda_n(t)) = -\lim_{n\to+\infty} 2u^n(\lambda_n(t)^-) = -2u(t^-), \ \forall\, t \in [0, T],\\ k(t) - k(s) &= \lim_{n\to+\infty} (k^n(\lambda_n(t)) - k^n(\lambda_n(s))) \geq 0, \ \forall\, 0 \leq s < t \leq T.\end{aligned}$$

To conclude that (x, u, k) belongs to $\mathcal{H}$, it remains to show that $k(t) = \int_0^t \mathbb{1}_{\{x(s)\,=\,0\}} dk(s)$, for all $t \in [0, T]$. Since $x \geq 0$ on $[0, T]$, we prove that $\int_0^T x(s)\, dk(s) = 0$. Let us check that

$$\lim_{n\to+\infty} \int_0^T x^n(s) dk^n(s) = \int_0^T x(s)\, dk(s). \tag{4.19}$$

As $\int_0^T x^n(s)dk^n(s) = 0$ for all $n \in \mathbb{N}$, this will give our claim. Using the change of variable $s \mapsto \lambda_n^{-1}(s)$, and since x and x^n are continuous,

$$\begin{aligned}&\left|\int_0^T x^n(s)dk^n(s) - \int_0^T x(s)\, dk(s)\right|\\ &\leq \left|\int_0^T (x^n(\lambda^n(s)) - x(s))\, d(k^n \circ \lambda_n)(s)\right| + \left|\int_0^T x(s)\, d(k^n \circ \lambda_n)(s) - \int_0^T x(s)dk(s)\right|\\ &\leq \max_{t\in[0,T]} |x^n(\lambda_n(t)) - x(t)|\, |k^n(T)| + \left|\int_0^T x(s)d(k^n \circ \lambda_n)(s) - \int_0^T x(s)dk(s)\right|.\end{aligned}$$

According to (4.18), $\max_{t\in[0,T]} |x^n(\lambda_n(t)) - x(t)|$ tends to 0 as n tends to $+\infty$, the measure $d(k^n \circ \lambda_n)$ converges weakly to dk and (4.19) follows. ☐

4.3 Existence result for $(\boldsymbol{MP})$

Now we construct a solution to the martingale problem (MP). We only sketch the main steps, which combine arguments in [5, Section 5] with those in Section 4.2.

Proposition 4.10. *The solution $\mathbb{P}^\epsilon$ to the martingale problem (MP$_\epsilon$) converges to the solution of the martingale problem (MP).*

Proof. We mimic the proof of Lemma 4.4 to check that $\mathbb{P}^\epsilon$ is tight on $\mathcal{E}$. Let $\mathbb{P}^\epsilon$ denote a converging subsequence, and $\mathbb{P}$ its limit. Following the proof-steps of [5, Proposition 5.6], we further verify that $\mathbb{P}$ satisfies Conditions *(i)* and *(ii)* of (MP). From *(iii)*, *(iv)* follows simply by replicating the proof of Lemma 4.8.

For the martingale property *(iii)*, replicating the main steps of the proof of Lemma 4.6, we observe that *(iii)* follows by proving that

$$\begin{aligned}&\lim_{\epsilon\to0^+} \mathbb{E}_{\mathbb{P}^\epsilon}\left[\prod_{i=1}^n \psi_i(x_{t_i}, u_{t_i}, k_{t_i}) \int_s^t \left(B_\epsilon\left[x_\theta, u_\theta; \rho_\theta^\epsilon\right] \cdot \nabla_u f(x_\theta, u_\theta - k_\theta)\right) d\theta\right]\\ &= \mathbb{E}_{\mathbb{P}}\left[\prod_{i=1}^n \psi_i(x_{t_i}, u_{t_i}, k_{t_i}) \int_s^t \left(B\left[x_\theta, u_\theta; \rho_\theta\right] \cdot \nabla_u f(x_\theta, u_\theta - k_\theta)\right) d\theta\right],\end{aligned}$$

for all $0 \le t_1 \le \cdots \le t_q \le s < t \le T$ outside $\mathbf{D}_{\mathbb{P}} := \{\theta \in [0,T] \text{ s.t. } \mathbb{P}(|\triangle u_\theta| + |\triangle k_\theta| > 0)\}$. As outlined in [5], the above convergence can be proved by adapting some arguments from Stroock and Varadhan [16], as long as we show that

$$\lim_{|h|,|\delta|\to 0} \limsup_{\epsilon\to 0^+} \int_{\mathcal{D}\times\mathbb{R}^d} |\rho_t^\epsilon(x+h,u+\delta) - \rho_t^\epsilon(x,u)|\, dx\, du = 0,\ \forall\, t \in (0,T]. \tag{4.20}$$

Following the proof of [5, Lemma 5.9], (4.20) is obtained with the mild equation (4.8) and the property that

$$(y,v) \mapsto \int_{\mathcal{D}\times\mathbb{R}} |\nabla_v \Gamma(\theta; y, v; x+h, u+\delta) - \nabla_v \Gamma(\theta; y, v; x, u)|\, dx\, du$$

is bounded and continuous for all $\theta > 0$, $h \in \mathcal{D}$ and $\delta \in \mathbb{R}^d$ (this assertion is deduced from Proposition 3.3-*(ii)*). □

5 Proof of Theorem 2.3

Let $\mathbb{P}$ be a solution to the martingale problem (MP) and let $(\rho_t;\ t \in (0,T])$ be the related time-marginal densities.

Proof of *(a)*. According to Remark 3.2 and since we have the uniqueness in law for the solution of (3.9), for all $0 < t_0 < t \le T$, a.e. $(x,u) \in \overline{\mathcal{D}} \times \mathbb{R}^d$,

$$\rho_t(x,u) = \Big(\widetilde{\rho}_t(x,u) + \widetilde{\rho}_t((x', -x^{(d)}), (u', -u^{(d)}))\Big)\, \mathbb{1}_{\{x^{(d)} \ge 0\}},$$

where $(\widetilde{\rho}_t;\ t \in (0,T])$ are the time-marginal densities of the free Langevin process (3.9) with drift $\widetilde{\beta}$ constructed from $\beta(t,x,u) := B[x,u;\rho_t]$. Moreover, $(\widetilde{\rho}_t;\ t \in (0,T])$ is solution to the linear mild equation

$$\begin{aligned}\widetilde{\rho}_t(x,u) &= \int_{\mathbb{R}^{2d}} g_d(t-t_0; y, v; x, u)\widetilde{\rho}_{t_0}(y,v)dydv \\ &\quad + \int_{t_0}^t \int_{\mathbb{R}^{2d}} (\widetilde{\beta}(s,y,v) \cdot \nabla_v g_d(t-s; y, v; x, u))\widetilde{\rho}_s(y,v)dydvds.\end{aligned}$$

Then, for all $0 < t_0 < t \le T$, a.e. $u \in \mathbb{R}^d$, $x, x_0 \in \mathbb{R}^{2d}$,

$$\begin{aligned}&|\widetilde{\rho}_t(x,u) - \widetilde{\rho}_t(x_0,u)| \\ &\le \int_{\mathbb{R}^{2d}} |g_d(t-t_0; y, v; x, u) - g_d(t-t_0; y, v; x_0, u)|\, \widetilde{\rho}_{t_0}(y,v)dy\, dv \\ &\quad + \|b\|_\infty \int_{t_0}^t \int_{\mathbb{R}^{2d}} |\nabla_v g_d(t-s; y, v; x, u) - \nabla_v g_d(t-s; y, v; x_0, u)|\, \widetilde{\rho}_s(y,v)dy\, dv\, ds.\end{aligned} \tag{5.1}$$

The proof of the Hölder-continuity is then based on the following regularity result on the kernel g_d.

Lemma 5.1. *For $p > 4d+2$, let $h \in L^p(\mathbb{R}^{2d})$ and $H \in L^p((0,T)\times\mathbb{R}^{2d})$. Then, for all $(x,x_0) \in \mathbb{R}^{2d}$, and for all $0 < t_0 < t \le T$, it holds that*

$$\int_{\mathbb{R}^{2d}} |g_d(t-t_0; y, v; x, u) - g_d(t-t_0; y, v; x_0, u)|\, |h(y,v)|\, dy\, dv \le c|x-x_0|^{\frac{1}{3}-\frac{4d+1}{3p}} \|h\|_{L^p(\mathbb{R}^{2d})},$$

$$\begin{aligned}&\int_{(t_0,t)\times\mathbb{R}^{2d}} |\nabla_v g_d(t-s; y, v; x, u) - \nabla_v g_d(t-s; y, v; x_0, u)|\, |H(s,y,v)|\, dy\, dv\, ds \\ &\le c|x-x_0|^{\frac{1}{3}-\frac{4d+2}{3p}} \|H\|_{L^p((t_0,t)\times\mathbb{R}^{2d})}.\end{aligned}$$

The proof of Lemma 5.1 is a straightforward consequence of Proposition 5.2 (using the step-proof of Theorem 1.2) in Manfredini and Polidoro [13].

To apply Lemma 5.1, we exhibit an L^p estimate on $\widetilde{\rho}_t$, for $t > 0$: for the solution $(\mathbb{P}, Y, V, \widetilde{W})$ to (3.9), we consider $\widetilde{\mathbb{P}}$ defined by $d\widetilde{\mathbb{P}} = Z_T d\mathbb{P}$ with

$$Z_t = \exp\left(-\int_0^t (\widetilde{\beta}(s, Y_s, V_s) \cdot d\widetilde{W}_s) - \frac{1}{2}\int_0^t \left|\widetilde{\beta}(s, Y_s, V_s)\right|^2 ds\right).$$

Then for all bounded measurable functions F, and for all $l > 1$,

$$\left|\int_{\mathbb{R}^{2d}} F(x,u)\widetilde{\rho}_t(x,u)dx\,du\right| = \left|\mathbb{E}_{\widetilde{\mathbb{P}}}\left[Z_T^{-1}F(Y_t,V_t)\right]\right| \leq C\left[\mathbb{E}_{\widetilde{\mathbb{P}}}\left|F(Y_t,V_t)\right|^l\right]^{\frac{1}{l}},$$

for a constant $C > 0$ depending only on $\|b\|_\infty$ and l. Moreover,

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left[|F(Y_t,V_t)|^l\right] = \int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^{2d}} g_d(t;y,v;x,u)\mu_0(dy,dv)\right)|F(x,u)|^l dx\,du,$$

and for $F \in L^{lq}(\mathbb{R}^{2d})$, for (r,q) such that $\frac{1}{r} + \frac{1}{q} = 1$, we obtain that

$$\left|\int_{\mathbb{R}^{2d}} F(x,u)\widetilde{\rho}_t(x,u)dx\,du\right| \leq C\left\|\int_{\mathbb{R}^{2d}} g_d(t;y,v;\cdot,\cdot)\mu_0(dy,dv)\right\|_{L^r(\mathbb{R}^{2d})}^{\frac{1}{l}}\|F\|_{L^{lq}(\mathbb{R}^{2d})}.$$

Note that for all $r > 1$, for all $t > 0$ and $(y,v) \in \mathbb{R}^{2d}$, $\|g_d(t;y,v;\cdot,\cdot)\|_{L^r(\mathbb{R}^{2d})} \leq C(r)t^{-2d(1-1/r)}$. This gives

$$\left|\int_{\mathbb{R}^{2d}} F(x,u)\widetilde{\rho}_t(x,u)dx\,du\right| \leq Ct^{-\frac{2d}{lq}}\|F\|_{L^{lq}(\mathbb{R}^{2d})}.$$

Choosing $lq = p/(p-1)$ and $l = 2$, we get $\|\widetilde{\rho}_t\|_{L^p(\mathbb{R}^{2d})} \leq Ct^{-\frac{2d(p-1)}{p}}$.

By Lemma 5.1, for $x, x_0 \in \mathbb{R}^{2d}$, and $p > 4d+2$, there exists a constant $C > 0$ such that, for a.e. $0 < t_0 < t \leq T$, $u \in \mathbb{R}^d$,

$$\begin{aligned}
&|\widetilde{\rho}_t(x,u) - \widetilde{\rho}_t(x_0,u)| \\
&\leq C\left(|x-x_0|^{\frac{1}{3}-\frac{4d+1}{3p}}\|\widetilde{\rho}_{t_0}\|_{L^p(\mathbb{R}^{2d})} + |x-x_0|^{\frac{1}{3}-\frac{4d+2}{3p}}\|\widetilde{\rho}\|_{L^p((t_0,t)\times\mathbb{R}^{2d})}\right) \\
&\leq Ct_0^{-\frac{2d(p-1)}{p}}\left(|x-x_0|^{\frac{1}{3}-\frac{4d+1}{3p}} + |x-x_0|^{\frac{1}{3}-\frac{4d+2}{3p}}(t-t_0)^{\frac{1}{p}}\right)
\end{aligned}$$

and *(a)* follows.

Proof of *(b)*. We start by proving that the measure $\sum_{n\in\mathbb{N}} \mathbb{P} \circ \left(\tau_n, x_{\tau_n}, u_{\tau_n^-}\right)^{-1}$ is finite on the set $\{(t,x,u) \in (0,T) \times \partial\mathcal{D} \times \mathbb{R}^d \text{ s.t. } (u \cdot n_{\mathcal{D}}) > 0\}$, namely:

$$\sum_{n\in\mathbb{N}} \mathbb{P}(\tau_n \leq T) < +\infty. \tag{5.2}$$

We proceed as in the proof of Lemma 4.8 by removing the drift term B thanks to the change of probability measure $d\widetilde{\mathbb{P}} = z_T d\mathbb{P}$, with $(z_t; t \in [0,T])$ defined as

$$z_t = \exp\left(-\int_0^t (B[x_s, u_s; \rho_s] \cdot dw_s) - \frac{1}{2}\int_0^t |B[x_s, u_s; \rho_s]|^2 ds\right).$$

With $\widetilde{w}$ defined similarly to $\widetilde{w}^\epsilon$ in (4.17), $(x, u, \widetilde{w})$ solves (3.1) under $\widetilde{\mathbb{P}}$ and the law of τ_n is given by (3.5). Then

$$\mathbb{E}_{\mathbb{P}}\left[\sum_{n\in\mathbb{N}} \mathbb{1}_{\{\tau_n \leq T\}}\right] = \mathbb{E}_{\widetilde{\mathbb{P}}}\left[z_T^{-1}\sum_{n\in\mathbb{N}} \mathbb{1}_{\{\tau_n \leq T\}}\right] \leq C\sqrt{\mathbb{E}_{\widetilde{\mathbb{P}}}\left[\left(\sum_{n\in\mathbb{N}} \mathbb{1}_{\{\tau_n\leq T\}}\right)^2\right]},$$

for some finite constant $C > 0$. Since the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ is non-decreasing, we also get

$$\left(\sum_{n\in\mathbb{N}} \mathbb{1}_{\{\tau_n \leq T\}}\right)^2 \leq 2\sum_{n\in\mathbb{N}}\left(\sum_{m=0}^{n} \mathbb{1}_{\{\tau_m \leq T\}}\right)\mathbb{1}_{\{\tau_n \leq T\}} \leq 2\sum_{n\in\mathbb{N}}(n+1)\mathbb{1}_{\{\tau_n \leq T\}}.$$

Using (3.6), one observes that

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left[\left(\sum_{n\in\mathbb{N}} \mathbb{1}_{\{\tau_n \leq T\}}\right)^2\right] \leq 2\sum_{n\in\mathbb{N}}(n+1)\widetilde{\mathbb{P}}(\tau_n \leq T) \leq 2\sum_{n\in\mathbb{N}}\frac{(n+1)}{2^{n-1}}.$$

We deduce (5.2).

For (2.2), one may observe that, for any $f \in \mathcal{C}_c^\infty(Q_T)$,

$$\begin{aligned}&\int_{(0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}^d} u^{(d)}\gamma(\rho)(t,(x',0),u)f(t,(x',0),u)dt\,dx'\,du\\ &= \lim_{\delta\to0^+}\frac{1}{\delta}\int_0^\delta\int_{(0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}^d} u^{(d)}\rho_t((x',x^{(d)}),u)f(t,(x',x^{(d)}),u)dt\,dx'\,du\,dx^{(d)}.\end{aligned} \tag{5.3}$$

For a fixed $\delta > 0$, we set $\beta_\delta(y) := 1 - (\frac{y}{\delta}\wedge 1)$, a.e. differentiable on $(0, +\infty)$. The Itô's formula induces

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}}\left[\sum_{n\in\mathbb{N}}\left(f(\tau_n, x_{\tau_n}, u_{\tau_n}) - f(\tau_n, x_{\tau_n}, u_{\tau_n^-})\right)\mathbb{1}_{\{\tau_n \leq T\}}\right] - \frac{1}{\delta}\int_0^T \mathbb{E}_{\mathbb{P}}\left[u_{s^-}^{(d)}\mathbb{1}_{\{x_s^{(d)}\leq\delta\}}f(s,x_s,u_{s^-})\right]ds\\ &= \mathbb{E}_{\mathbb{P}}\left[\beta_\delta(x_T^{(d)})f(t,x_T,u_T) - \beta_\delta(x_0^{(d)})f(0,x_0,u_0) - \int_0^T \beta_\delta(x_s^{(d)})\left(\partial_s f + \mathcal{A}_{\rho_s}(f)\right)(s,x_s,u_{s^-})ds\right].\end{aligned}$$

Recalling that $\{s \in [0,T]\,;\, x_s^{(d)} = 0\}$ is at most countable $\mathbb{P}$-a.s., we obtain that

$$\begin{aligned}&\lim_{\delta\to0^+}\frac{1}{\delta}\int_0^\delta\left(\int_{(0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}^d} u^{(d)}\rho(t,(x',x^{(d)}),u)f(t,x,u)dt\,dx'\,du\right)dx^{(d)}\\ &= \mathbb{E}_{\mathbb{P}}\left[\sum_{n\in\mathbb{N}}\left(f(\tau_n, x_{\tau_n}, u_{\tau_n}) - f(\tau_n, x_{\tau_n}, u_{\tau_n^-})\right)\mathbb{1}_{\{\tau_n \leq T\}}\right],\end{aligned}$$

and coming back to (5.3), for any $f \in \mathcal{C}_c(\Sigma_T)$,

$$\begin{aligned}&\int_{(0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}^d} u^{(d)}\gamma(\rho)(t,(x',0),u)f(t,(x',0),u)dt\,dx'\,du\\ &= \mathbb{E}_{\mathbb{P}}\left[\sum_{n\in\mathbb{N}}\left(f(\tau_n, x_{\tau_n}, u_{\tau_n}) - f(\tau_n, x_{\tau_n}, u_{\tau_n^-})\right)\mathbb{1}_{\{\tau_n \leq T\}}\right].\end{aligned} \tag{5.4}$$

Thanks to (5.2), the relation above can be extended to any bounded measurable function f on Σ_T. This ends the proof of *(b)*.

Proof of *(c)*. According to (5.4) and (5.2), we immediately deduce that

$$\|u^{(d)}\gamma(\rho)\|_{L^1(\Sigma_T)} = \sup_{\substack{f\in C_c(\Sigma_T);\\ \sup_{\zeta\in\Sigma_T}|f(\zeta)|=1}} \int_{\Sigma_T} u^{(d)}\gamma(\rho)(t,(x',0),u)f(t,(x',0),u)dt\,dx'\,du < +\infty,$$

and (2.4a) follows.

To prove (2.4b), let us first observe that when $b=0$, the result is obvious since

$$\gamma(\rho)(t,(x',0),u) = \int_{\mathcal{D}\times\mathbb{R}^d} \Gamma(t;y,v;(x',0),u))\mu_0(dy,dv) > 0,$$

with $\Gamma(t;y,v;(x',0),u))$ given in (3.10). Next, for the case with drift, we observe that it suffices to show that for all set $A\in\mathcal{B}((0,T)\times\mathbb{R}^{d-1})$ such that A has positive $dt\otimes dx'$-finite measure,

$$\int_{A\times\{0\le(u\cdot n_{\mathcal{D}})\}} \gamma(\rho)(t,(x',0),u)dt\;dx'du > 0. \tag{5.5}$$

Fix such a set A. Since $\gamma(\rho)(t,x,u)$ is nonnegative,

$$\int_{A\times\{0\le(u\cdot n_{\mathcal{D}})\}} \gamma(\rho)(t,(x',0),u)dt\;dx'du \ge \int_{A\times\{0\le(u\cdot n_{\mathcal{D}})<1\}} (u\cdot n_{\mathcal{D}})\gamma(\rho)(t,(x',0),u)dt\;dx'du.$$

According to (2.3), it follows that

$$\int_{A\times\{0\le(u\cdot n_{\mathcal{D}})\}} \gamma(\rho)(t,(x',0),u)dtdx'du \ge \sum_{n\in\mathbb{N}} \mathbb{P}\left(\left(\tau_n,x'_{\tau_n}\right)\in A, 0<(u_{\tau_n^-}\cdot n_{\mathcal{D}})<1\right).$$

As $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent measures, we only need to check that

$$\sum_{n\in\mathbb{N}} \widetilde{\mathbb{P}}\left((\tau_n,x'_{\tau_n},u^{(d)}_{\tau_n})\in A\times(0,1)\right) > 0.$$

Equivalently, using (2.3)

$$\begin{aligned}&\sum_{n\in\mathbb{N}} \widetilde{\mathbb{P}}\left((\tau_n,x'_{\tau_n})\in A, 0<u^{(d)}_{\tau_n}<1\right)\\ &= \int_{\mathcal{D}\times\mathbb{R}^d}\left(\int_A\int_{\mathbb{R}^{d-1}}\int_0^1 u^{(d)}\Gamma(t;y,v;(x',0),(u',u^{(d)}))du^{(d)}du'dx'\right)\mu_0(dy,dv).\end{aligned}$$

The result follows, observing that $\Gamma(t;y,v;(x',0),(u',u^{(d)}))>0$.

6 Conclusion and perspectives

In this work, we have constructed a confined conditional McKean Lagrangian process $((X_t,U_t);t\in[0,T])$ satisfying the mean no-permeability condition

$$\mathbb{E}\left[(U_t\cdot n_{\mathcal{D}})/X_t=x\right]=0,\ \text{for } (t,x)\in(0,T]\times\partial\mathcal{D}.$$

This study is motivated by the application of Lagrangian stochastic models to the downscaling problem in Computational Fluid Dynamic (CFD).

In [1],[2], the authors construct a PDF method applied to the downscaling problem in meteorology. The goal is to compute a finer scale wind prediction from a coarse one given in a bounded domain $\mathcal{D}$. To this aim, the authors propose a Lagrangian stochastic model for the atmospheric flow description and construct a particle algorithm to solve this fine resolution. This Lagrangian model is confined in $\mathcal{D}$ and must satisfy a Dirichlet condition of the type

$$\mathbb{E}\left[U_t/X_t = x\right] = V_{\text{coarse}}(t,x), \text{ for } (t,x) \in (0,T) \times \partial\mathcal{D}, \tag{6.1}$$

where V_{coarse} is a given velocity field. This application in CFD motivates at least two future extensions of the present work: the case of a more general domain $\mathcal{D}$, and the case of the non homogeneous boundary condition (6.1).

References

[1] Frédéric Bernardin, Mireille Bossy, Claire Chauvin, Philippe Drobinski, Antoine Rousseau, and Tamara Salameh. Stochastic downscaling method: application to wind refinement. *Stoch. Environ. Res Risk. Assess*, 23(6):851–859, 2009.

[2] Frédéric Bernardin, Mireille Bossy, Claire Chauvin, Jean-François Jabir, and Antoine Rousseau. Stochastic Lagrangian method for downscaling problems in Computational Fluid Dynamics. *ESAIM: M2AN*, 44(5):885–920, 2010.

[3] Jean Bertoin. Reflecting a Langevin process at an absorbing boundary. *Ann. Probab.*, 35(6):2021–2037, 2007.

[4] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[5] Mireille Bossy, Jean-François Jabir, and Denis Talay. On conditional McKean Lagrangian stochastic models. *Probab. Theory Relat. Fields*, Published online May, 26 2010.

[6] Carlo Cercignani. *The Boltzmann equation and its applications*, volume 67 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1988.

[7] Thomas D. Dreeben and Stephen B. Pope. Wall-function treatment in pdf methods for turbulent flows. *Phys. Fluids*, 9(9):2692–2703, 1997.

[8] Stewart N. Ethier and Thomas G. Kurtz. *Markov processes: Characterization and convergence*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986.

[9] Carl Graham and Sylvie Méléard. Stochastic particle approximations for generalized Boltzmann models and convergence estimates. *Ann. Probab.*, 25(1):115–132, 1997.

[10] Jean Jacod and Albert N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1987.

[11] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1988.

[12] Aimé Lachal. Les temps de passage successifs de l'intégrale du mouvement brownien. *Ann. Inst. H. Poincaré Probab. Statist.*, 33(1):1–36, 1997.

[13] Maria Manfredini and Sergio Polidoro. Interior regularity for weak solutions of ultraparabolic equations in divergence form with discontinuous coefficients. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)*, 1(3):651–675, 1998.

[14] H. P. McKean, Jr. A winding problem for a resonator driven by a white noise. *J. Math. Kyoto Univ.*, 2:227–235, 1963.

[15] Stephen B. Pope. *Turbulent flows.* Cambridge University Press, Cambridge, 2000.

[16] Daniel W. Stroock and S. R. Srinivasa Varadhan. *Multidimensional diffusion processes*, volume 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1979.

[17] Alain-Sol Sznitman. Nonlinear reflecting diffusion process, and the propagation of chaos and fluctuations associated. *J. Funct. Anal.*, 56(3):311–336, 1984.

[18] Alain-Sol Sznitman. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 165–251. Springer, Berlin, 1991.