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On Various Multifractal Spectra

Jacques Lévy Véhel and Claude Tricot

Abstract. We introduce two classes of multifractal spectra, called respectively dimension and continuous spectra. Dimension spectra offer an interesting alternative to the classical Hausdorff spectrum: They are much easier to estimate yet still give relevant information about the geometry of the Hölder function. Continuous spectra are a generalization of the large deviation spectrum that allow to obtain partition free results. Both classes of spectra allow to perform efficient multifractal analysis in an experimental framework.

1. Background and Notations

Multifractal analysis has developed in many directions since its introduction. Progress has been accomplished concerning the domain of validity of the multifractal formalism, both in the deterministic and random frameworks [1, 2, 3, 8]. The analysis has been extended to functions [7] or sequences of capacities [12] in addition to measures. The paper by P. Mörters in this volume gives an account on the multifractal analysis of certain measures related to Brownian paths. More refined spectra and estimation procedures have been defined [4, 9].

The numerical computation of a multifractal spectrum on sampled data remains however a challenging task. We introduce in this work two new classes of spectra, called *dimension spectra* and *continuous spectra*. Our aim is to facilitate the estimation problem, so as to obtain meaningful numerical results even when no assumption on the data structure is made (*i.e.*, in a non-parametric frame).

It is well-known that the Hausdorff multifractal spectrum f_h (see section 2 for definitions) is very hard to calculate in general. Apart from very restricted classes of mathematical models, the exact value of the Hausdorff dimension is difficult to obtain theoretically, and almost impossible to estimate on experimental data. The sets E_α which form a partition of the support are practically inaccessible in a discrete framework. To the contrary, the *box dimension* Δ can be estimated in experimental situations. But Δ is of no use here since in all interesting situations the E_α are either empty, or dense in an interval: In this case $\Delta(E_\alpha) = 1$, and the spectrum f_Δ is trivial. One can try to estimate f_h with the help of the large deviation

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spectrum f_g, which is most of the times easier to work with. The inequality $f_h \leq f_g$ is always true. However, f_g measures an information which is essentially different from f_h: The Hausdorff spectrum is defined as a dimension function which emphasizes the *geometric* structure of the singularities of a function or measure, whilst f_g yields *statistical* information. We introduce in this paper two spectra, denoted $f_d^{\lim}$ and $f_d^{\limsup}$ where d is any dimension. As is the case for f_h or f_Δ, they are defined by using set dimensions. On the other hand, they share many properties with f_g. For instance, $f_d^{\lim}$ and $f_d^{\limsup}$ are upper semi-continuous functions, and, conversely, every upper semi-continuous function is the $f_d^{\lim}$ or $f_d^{\limsup}$ spectrum for some signal. The major motivation for defining these dimension spectra is that $f_\Delta^{\limsup}$ is both easy to evaluate in practical situations and "more precise" than f_g, i.e. one always has $f_h \leq f_\Delta^{\limsup} \leq f_g$.

Another path consists in focusing on the large deviation spectrum f_g. Although originally introduced as a way to estimate f_h, it has soon been realized that f_g is of independent interest, specially in applications (see for instance [10, 11] for applications in image processing and Internet traffic modelling). A drawback of the large deviation spectrum is that its very definition relies on an arbitrary partitioning of the support of the signal. Different partitions will in general lead to different spectra. We introduce two variants of f_g. Both are *continuous* spectra, and thus allow to get rid of the discretization. As a consequence, these partition-free spectra contain a more intrinsic multifractal information. Moreover, one of them, denoted $\tilde{f}_g^c$, is defined using only one limit ($\eta \to 0$), instead of the two limiting operations ($n \to \infty$ and $\varepsilon \to 0$) used for f_g. We define the corresponding continuous Legendre spectra, and show that they are, under mild conditions, the concave envelopes of the continuous large deviation spectra.

To gain generality, we introduce our spectra for *abstract set functions*, rather than for Hölder exponents of measures or functions of a real variable, as is done classically. More precisely, let $X([0,1])$ be the metric set of all closed sub-intervals (including the singletons) of $[0,1]$ (the extension to $\mathbb{R}$ and $\mathbb{R}^n$ is straightforward). We shall base our multifractal analysis on the study of a function $A : X([0,1]) \longrightarrow \mathbb{R}^+ \cup \{+\infty\}$. The interpretation of A in the classical frame is as follows (the length of an interval u is denoted by $|u|$):

- For the analysis of a Borelian measure μ, take $A(u) = \log \mu(u) / \log |u|$;
- For the analysis of a function z, take $A(u) = \log v_z(u) / \log |u|$, where $v_z(u)$ measures the "variation" of z in u. Common choices are the increment $|z(u_{\max}) - z(u_{\min})|$ (where $u = [u_{\min}, u_{\max}]$), the oscillation $\sup_{t \in u} z(t) - \inf_{t \in u} z(t)$, or, when $u = u_n^k$ is the dyadic interval $[k2^{-n}, (k+1)2^{-n}]$, the wavelet coefficient of z at scale n and location k[1].

Section 2 defines the dimension spectra $f_d^{\lim}$ and $f_d^{\limsup}$. We study their main properties in Section 3: Domain of definition, relationships, maximum. The inverse problem (given an upper semi-continuous function F, find A whose spectrum $f_d^{\lim}$

[1]This choice requires care, since the resulting spectra will depend on the analyzing wavelet.

or $f_d^{\limsup}$ is equal to F) is solved in Section 4 in two cases: d is σ-stable (like the Hausdorff dimension), or $d = \Delta$. Section 5 shows explicit computations of dimension spectra. Finally, we define and study the continuous spectra f_g^c and $\tilde{f}_g^c$ and the corresponding Legendre spectra in Section 6.

2. New dimension spectra

For $x \in [0,1]$, $u_n(x)$ denotes the dyadic interval $u_n^k = [k2^{-n}, (k+1)2^{-n}]$ which contains x (take the right one if there are two such intervals). For any real number α, $N_\alpha(\varepsilon, n)$ denotes the number of 2^{-n}-dyadic intervals such that $|A(u) - \alpha| \le \varepsilon$, and $I_\alpha(\varepsilon, n)$ their union. Recall the definition of the *large deviation spectrum* f_g:

$$f_g(\alpha) = \lim_{\varepsilon \to 0} (\limsup_{n \to \infty} \frac{\log N_\alpha(\varepsilon, n)}{n \log 2}), \tag{1}$$

with the convention that $\log N_\alpha(\varepsilon, n)/n \log 2 = -\infty$ if $N_\alpha(\varepsilon, n) = 0$.
Prior to defining the dimension spectra, let us clarify our notion of a dimension.

Definition 1. *We call dimension a function $d : \mathcal{P}([0,1]) \to \mathbb{R}^+ \cup \{-\infty\}$, such that*
(i) $E \subset F \Rightarrow d(E) \le d(F)$
(ii) $d(\emptyset) = -\infty$, $d(\{x\}) = 0$ for any x, and $d(E) = 1$ for any E such that $|E| > 0$.

A dimension may have the following properties:

Definition 2. *The dimension d is* stable *if*

$$d(E \cup F) = \max\{d(E), d(F)\}$$

*for all sets E, F in $[0,1]$. It is σ-*stable *if*

$$d(\bigcup_n E_n) = \sup\{d(E_n)\}$$

for any countable set family (E_n).

Let us now define our new spectra. For any real number x in $[0,1]$, set $\alpha_n(x) = A(u_n(x))$. For any real α, let

$$E_\alpha(\varepsilon, N) = \{x, n \ge N \Rightarrow |\alpha_n(x) - \alpha| \le \varepsilon\}.$$

Note that $E_\alpha(\varepsilon, N)$ increases with N, so that $\bigcup_N E_\alpha(\varepsilon, N)$ may be written as $\sup_N E_\alpha(\varepsilon, N)$. Also, $E_\alpha(\varepsilon, N) = \bigcap_{n \ge N} I_\alpha(\varepsilon, n)$. Let

$$\begin{aligned} E_\alpha(\varepsilon) &= \sup_N E_\alpha(\varepsilon, N) = \{x, \exists N \text{such that } n \ge N \Rightarrow |\alpha_n(x) - \alpha| \le \varepsilon\} \\ &= \liminf_{N \to \infty} I_\alpha(\varepsilon, N). \end{aligned}$$

Since the sets $E_\alpha(\varepsilon)$ decrease with ε, one may define

$$E_\alpha = \lim_{\varepsilon \to 0} E_\alpha(\varepsilon) = \{x, \alpha_n(x) \to_{n \to \infty} \alpha\}.$$

Definition 3. *For any dimension d and any real α, define the following spectra:*

$$f_d(\alpha) = d(E_\alpha) = d(\lim_{\varepsilon\to 0}\sup_N E_\alpha(\varepsilon, N)) \tag{2}$$

$$f_d^{\lim}(\alpha) = \lim_{\varepsilon\to 0} d(E_\alpha(\varepsilon)) = \lim_{\varepsilon\to 0} d(\sup_N E_\alpha(\varepsilon, N)) \tag{3}$$

$$f_d^{\lim\sup}(\alpha) = \lim_{\varepsilon\to 0}\sup_N d(E_\alpha(\varepsilon, N)). \tag{4}$$

When d is the Hausdorff dimension h, $f_d = f_h$ is the usual Hausdorff spectrum.

3. Properties and relationships

Let us describe a few basic properties verified by dimension spectra.

3.1. Domain of definition

It is clear that the spectra defined above all range in $[0,1]\cup\{-\infty\}$. Let $D = \overline{\mathrm{Im}(A)}$ be the closure of the image of the function A. For every $\alpha \notin D$, there exists ε_0 such that $\varepsilon \le \varepsilon_0 \implies E_\alpha(\varepsilon, N) = \emptyset$. Therefore $f_d(\alpha) = f_d^{\lim}(\alpha) = f_d^{\lim\sup}(\alpha) = -\infty$. Also, $N_\alpha(\varepsilon, n) = 0$, so that $f_g(\alpha) = -\infty$. Thus, while all the spectra are defined on $\mathbb{R}$, their "support" (i.e. the set of α for which the spectrum belongs to $[0,1]$) is included in D.

3.2. Inequalities

Since $E_\alpha \subset E_\alpha(\varepsilon)$, we have

$$f_d(\alpha) \le f_d^{\lim}(\alpha) \tag{5}$$

for all α. Also, $d(E_\alpha(\varepsilon, N)) \le d(\sup_N E_\alpha(\varepsilon, N))$ implies that

$$f_d^{\lim\sup}(\alpha) \le f_d^{\lim}(\alpha). \tag{6}$$

We will see that there is no relationship in general between f_d and $f_d^{\lim\sup}$. If d is σ-stable, then $d(\sup_N E_\alpha(\varepsilon, N)) = \sup_N d(E_\alpha(\varepsilon, N))$, so that $f_d^{\lim}$ is identical to $f_d^{\lim\sup}$. In this particular case, inequalities (5) and (6) reduce to

$$f_d(\alpha) \le f_d^{\lim}(\alpha) = f_d^{\lim\sup}(\alpha). \tag{7}$$

The spectra f_d and f_g cannot be compared without specifying the dimension d. Let us take for d the *box dimension*, which is defined for any bounded set E as

$$\Delta(E) = \limsup_{n\to\infty} \frac{\log \omega(2^n, E)}{n \log 2}$$

where $\omega(2^n, E)$ denotes the number of 2^n-dyadic intervals covering E. This dimension is stable, but not σ-stable. For all $n \ge N$, $\omega(2^n, E_\alpha(\varepsilon, N)) \le N_\alpha(\varepsilon, n)$, so that for all α, ε, N, $\Delta(E_\alpha(\varepsilon, N)) \le f_g(\alpha)$. Therefore

$$f_\Delta^{\lim\sup}(\alpha) \le f_g(\alpha). \tag{8}$$

There is no relationship in general between f_g and $f_\Delta^{\lim}$.

Lemma 1. *Let d_1, d_2, be such that for all $E \subset [0,1]$, $d_1(E) \le d_2(E)$. Then for every A and α:*

$$f_{d_1}(\alpha) \le f_{d_2}(\alpha)\ ,\ f_{d_1}^{\lim}(\alpha) \le f_{d_2}^{\lim}(\alpha)\ ,\ f_{d_1}^{\limsup}(\alpha) \le f_{d_2}^{\limsup}(\alpha).$$

We leave the proof to the reader.

If h denotes the Hausdorff dimension, it is well known that $h(E) \le \Delta(E)$ for all E. Gathering previous results, we get the following sequence of inequalities:

Proposition 1. *For any set function A,*

$$f_h(\alpha) \le f_h^{\limsup}(\alpha) = f_h^{\lim}(\alpha) \le f_\Delta^{\limsup}(\alpha) \le \min(f_\Delta^{\lim}(\alpha), f_g(\alpha)). \tag{9}$$

When the (strong) multifractal formalism holds, $f_h(\alpha) = f_g(\alpha)$ for all α, so that all the above spectra coincide, with the possible exception of $f_\Delta^{\lim}$. Incidentally, this result explains the *a priori* unexpected fact that a naive numerical estimation of f_h on a multinomial measure yields acceptable results: Indeed, estimating simply the box dimension of the sets $E_\alpha(\varepsilon, N)$ gives a correct approximation in this case.

More generally, i.e. without assuming the multifractal formalism, (9) shows that $f_\Delta^{\limsup}(\alpha)$ is always a better approximation to f_h than f_g. In addition, it is not more difficult to estimate.

3.3. Maximum of a spectrum

Since $d([0,1]) = 1$, every spectrum has a maximum not larger than 1. In general, the upper bound depends on A and d. To make this precise, let us introduce new sets. Let

$$S(\varepsilon, N) = \{x, m \ge N, n \ge N \Rightarrow |\alpha_m(x) - \alpha_n(x)| \le \varepsilon\}.$$

Note that $S(\varepsilon, N)$ increases with N, so that $\bigcup_N S(\varepsilon, N) = \sup_N S(\varepsilon, N)$. Let

$$S(\varepsilon) = \sup_N S(\varepsilon, N) = \{x, \exists N \text{such that } m \ge N, n \ge N \Rightarrow |\alpha_m(x) - \alpha_n(x)| \le \varepsilon\}.$$

Since the sets $S(\varepsilon)$ decrease as $\varepsilon \to 0$, one may define

$$S = \lim_{\varepsilon \to 0} S(\varepsilon).$$

These constructions are similar to those of $E_\alpha(\varepsilon, N)$, $E_\alpha(\varepsilon)$, E_α, except that they are independent of α. Note that for all $\varepsilon' < \varepsilon$,

$$S(\varepsilon) \subset \{x, \limsup \alpha_n(x) - \liminf \alpha_n(x) \le \varepsilon\} \subset S(\varepsilon')$$

and

$$S = \{x, \alpha_n(x) \text{ converges}\}.$$

Now define the dimensional indices:

$$d_0 = d(S) = d(\lim_{\varepsilon \to 0} \sup_N S(\varepsilon, N)).$$

$$d_0^{\lim} = \lim_{\varepsilon \to 0} d(S(\varepsilon)) = \lim_{\varepsilon \to 0} d(\sup_N S(\varepsilon, N))$$

$$d_0^{\limsup} = \lim_{\varepsilon\to 0} \sup_N d(S(\varepsilon, N)).$$

Proposition 2. *For every α,*

$$f_d(\alpha) \le d_0\,,\ f_d^{\lim}(\alpha) \le d_0^{\lim}\,,\ f_d^{\limsup}(\alpha) \le d_0^{\limsup}. \tag{10}$$

Proof Use the inclusions $E_\alpha \subset S$, $E_\alpha(\varepsilon) \subset S(2\varepsilon)$, $E_\alpha(\varepsilon, N) \subset S(2\varepsilon, N)$. ∎

Proposition 3. *If the set D is bounded, and d is a stable dimension, then $f_d^{\limsup}$ and $f_d^{\lim}$ reach the above upper bounds. In other words,*

$$d_0^{\limsup} \in f_d^{\limsup}(D) \quad \text{and} \quad d_0^{\lim} \in f_d^{\lim}(D).$$

Proof 1. We first show that there exists $\alpha_1 \in D$ such that $d_0^{\limsup} \le f_d^{\limsup}(\alpha_1)$.

Let $N \in \mathbb{N}$, $\varepsilon > 0$. For any $x \in S(\varepsilon, N)$ and any $n \ge N$, $|\alpha_n(x) - \alpha(x)| \le \varepsilon$, where $\alpha(x) = \frac{1}{2}(\liminf \alpha_n(x) + \limsup_n \alpha_n(x))$. Therefore $x \in E_{\alpha(x)}(\varepsilon, N)$. This implies that $S(\varepsilon, N) \subset \bigcup_{\alpha\in\mathbb{R}} E_\alpha(\varepsilon, N)$.

For every α there exists $k \in \mathbb{Z}$ such that $[\alpha-\varepsilon, \alpha+\varepsilon] \subset [2(k-1)\varepsilon, 2(k+1)\varepsilon]$, so that $\bigcup_{\alpha\in\mathbb{R}} E_\alpha(\varepsilon, N) \subset \bigcup_{k\in\mathbb{Z}} E_{2k\varepsilon}(2\varepsilon, N)$. If D is bounded, then all sets $E_{2k\varepsilon}(2\varepsilon, N)$ are empty but a finite number of them. Using the stability of d, we deduce that

$$d(S(\varepsilon, N)) \le \max_{k\in\mathbb{Z}} d(E_{2k\varepsilon}(2\varepsilon, N)).$$

From this inequality it follows that for all ε, N there exists a real number $\beta(\varepsilon, N)$ such that

$$d(S(\varepsilon, N)) \le d(E_{\beta(\varepsilon,N)}(2\varepsilon, N)).$$

Since the distance from $\beta(\varepsilon, N)$ to D is not more than 2ε, the sequence $(\beta(\varepsilon, N))_N$ has a limiting value $\beta(\varepsilon)$ (for instance $\beta(\varepsilon) = \limsup_N \beta(\varepsilon, N)$). Let N_k be a subsequence such that $\beta(\varepsilon, N_k) \to \beta(\varepsilon)$. If k is large enough, $[\beta(\varepsilon, N_k) - 2\varepsilon, \beta(\varepsilon, N_k) + 2\varepsilon] \subset [\beta(\varepsilon) - 3\varepsilon, \beta(\varepsilon) + 3\varepsilon]$, so that

$$E_{\beta(\varepsilon,N_k)}(2\varepsilon, N_k) \subset E_{\beta(\varepsilon)}(3\varepsilon, N_k).$$

Therefore $d(S(\varepsilon, N_k)) \le d(E_{\beta(\varepsilon)}(3\varepsilon, N_k))$. Since the two sides of this inequality increase as $k \to +\infty$, we obtain

$$\sup_N d(S(\varepsilon, N)) \le \sup_N d(E_{\beta(\varepsilon)}(3\varepsilon, N)).$$

The function $\beta(\varepsilon)$ has a limiting value α_1 in D (for instance $\alpha_1 = \limsup_{\varepsilon\to 0} \beta(\varepsilon)$). Let ε_i be a sequence such that $\beta(\varepsilon_i)$ converges to α_1. Let $\eta > 0$. For i large enough, $\varepsilon_i \le \eta$ and $|\beta(\varepsilon_i) - \alpha_1| \le \eta$. Therefore $[\beta(\varepsilon_i) - 3\varepsilon_i, \beta(\varepsilon_i) + 3\varepsilon_i] \subset [\alpha_1 - 4\eta, \alpha_1 + 4\eta]$. This implies that for all $N \in \mathbb{N}$,

$$E_{\beta(\varepsilon_i)}(3\varepsilon_i, N)) \subset E_{\alpha_1}(4\eta, N),$$

so that

$$\sup_N d(S(\varepsilon_i, N)) \le \sup_N d(E_{\alpha_1}(4\eta, N)).$$

When i tends to ∞, the left hand side tends to $d_0^{\limsup}$. When η tends to 0, the right hand side tends to $f_d^{\limsup}(\alpha_1)$. This proves the required inequality.

2. Let us now show that there exists $\alpha_2 \in D$ such that $d_0^{\lim} \leq f_d^{\lim}(\alpha_2)$. The proof goes along the same lines, but it is somewhat simpler.

First check that $S(\varepsilon) \subset \bigcup_{\alpha\in\mathbb{R}} E_\alpha(\varepsilon) \subset \bigcup_{k\in\mathbb{Z}} E_{2k\varepsilon}(2\varepsilon)$. Deduce that

$$d(S(\varepsilon)) \leq \max_k d(E_{2k\varepsilon}(2\varepsilon)).$$

Now choose $\gamma(\varepsilon)$ such that $d(S(\varepsilon)) \leq d(E_{\gamma(\varepsilon)}(2\varepsilon))$, and a limiting value $\alpha_2 \in D$ of $\gamma(\varepsilon)$. Let $\varepsilon_i \to 0$ be such that $\gamma(\varepsilon_i) \to \alpha_2$. Let $\eta > 0$. If i is large enough, show that $E_{\gamma(\varepsilon_i)}(2\varepsilon_i) \subset E_{\alpha_2}(3\eta)$. This implies that $d(S(\varepsilon_i)) \leq d(E_{\alpha_2}(3\eta))$. Deduce that $d_0^{\lim} \leq f_d^{\lim}(\alpha_2)$. ■

Particular cases:
1. If d is σ-stable, then the spectra $f_d^{\lim}$ and $f_d^{\limsup}$ are the same and $d_0^{\lim} = d_0^{\limsup}$.
2. If $|S| > 0$, then $|S(\varepsilon)| > 0$ for all ε and $S(\varepsilon, N)| > 0$ for N large enough. In this case, $d_0 = d_0^{\lim} = d_0^{\limsup} = 1$. Both $f_d^{\lim}$ and $f_d^{\limsup}$ reach the value 1. This is also the maximum of f_g.

Remark: Regarding the spectrum f_d, one can show the following: If $\hat{f}_d$ denotes the spectrum

$$\hat{f}_d(\alpha) = \lim_{\varepsilon\to 0} d(\{x/\alpha_n(x) \text{ converges and } |\lim \alpha_n(x) - \alpha| \leq \varepsilon\}),$$

then

$$\sup_\alpha f_d(\alpha) \leq d_0 \leq \sup_\alpha \hat{f}_d(\alpha).$$

This result is less precise than those of Proposition 3. The following example shows that the difference between f_d and the other spectra may be as large as possible.

Example: Consider the generalized Weierstrass function

$$W(x) = \sum_{k=1}^{\infty} \lambda^{-kx} \sin(\lambda^k x),$$

with $\lambda > 1$ and $x \in [0,1]$. It is proved in [13] that $\alpha(x) = x$ for all x. Setting $A(u) = \log v_W(u)/\log|u|$, where $v_W(u)$ is the oscillation of W in u, it is easy to check that the support of all spectra is $[0,1]$. Since E_α contains only one point for every $\alpha \in [0,1]$, $f_d(\alpha) = 0$ identically in $[0,1]$. On the other hand, $\alpha_n(x) \to x$ for all $x \in [0,1]$, thus $d_0 = 1$ and $f_d^{\limsup}(\alpha) = f_d^{\lim}(\alpha) = f_g(\alpha) = 1$ for all α in $[0,1]$.

3.4. Semi-continuity

We show in this section that $f_d^{\limsup}$, $f_d^{\lim}$ and f_g share a semi-continuity property. Once again f_d does not have this property in general (for a study of the structural properties of f_d with $d = h$, see [12]).

Recall that a function $f : D \subset \mathbb{R} \to \mathbb{R}$ is *upper semi-continuous* if for all $x \in D$, for all sequences (x_n) of D converging to x,

$$\limsup_{n\to\infty} f(x_n) \leq f(x). \tag{11}$$

Another way to express this property is as follows:

$$\forall x \in D, \forall \varepsilon > 0, \exists \eta(x,\varepsilon) : |x-y| \leq \eta \Rightarrow f(y) \leq f(x) + \varepsilon.$$

Proposition 4. *The functions f_g, $f_d^{\lim}$, $f_d^{\limsup}$ are upper semi-continuous.*

Proof 1. Let (α_k) be a sequence in D converging to α. Let $\varepsilon > 0$. For any η, $0 < \eta < \varepsilon$, there exists $K(\eta)$ such that $k \geq K(\eta) \Rightarrow [\alpha_k - \eta, \alpha_k + \eta] \subset [\alpha - \varepsilon, \alpha + \varepsilon]$, so that for any n, $N_{\alpha_k}(\eta, n) \leq N_\alpha(\varepsilon, n)$. Therefore,

$$k \geq K(\eta) \Rightarrow \limsup_{n\to\infty} \frac{\log N_{\alpha_k}(\eta, n)}{n \log 2} \leq \limsup_{n\to\infty} \frac{\log N_\alpha(\varepsilon, n)}{n \log 2}.$$

This gives

$$\limsup_{k\to\infty} (\limsup_{n\to\infty} \frac{\log N_{\alpha_k}(n,\eta)}{n\log 2}) \leq \limsup_{n\to\infty} \frac{\log N_\alpha(n,\varepsilon)}{n \log 2}.$$

Since $N_{\alpha_k}(n,\eta)$ decreases as η tends to 0, we deduce that

$$\limsup_{k\to\infty} (\lim_{\eta\to 0} (\limsup_{n\to\infty} \frac{\log N_{\alpha_k}(n,\eta)}{n\log 2})) \leq \limsup_{n\to\infty} \frac{\log N_\alpha(n,\varepsilon)}{n \log 2}.$$

Let ε tend to 0 to get

$$\limsup_{k\to\infty} f_g(\alpha_k) \leq f_g(\alpha). \tag{12}$$

2. Similarly, for any N:

$$k \geq K(\eta) \Rightarrow E_{\alpha_k}(\eta, N) \subset E_\alpha(\varepsilon, N).$$

Letting N tend to ∞, and using the increasing property of d:

$$k \geq K(\eta) \Rightarrow d(E_{\alpha_k}(\eta)) \leq d(E_\alpha(\varepsilon)).$$

Therefore,

$$\limsup_{k\to\infty} d(E_{\alpha_k}(\eta)) \leq d(E_\alpha(\varepsilon)).$$

Since $d(E_{\alpha_k}(\eta))$ decreases as $\eta \to 0$,

$$\limsup_{k\to\infty} (\lim_{\eta\to 0} d(E_{\alpha_k}(\eta))) \leq d(E_\alpha(\varepsilon)).$$

Let ε tend to 0 to get $\limsup_{k\to\infty} f_d^{\lim}(\alpha_k) \leq f_d^{\lim}(\alpha)$. The same type of arguments hold for $f_d^{\limsup}$. ■

Notation For any $f : \mathbb{R} \to \mathbb{R}$, we denote by $\tilde{f}$ the *upper semi-continuous envelope* of f, that is

$$\tilde{f}(\alpha) = \lim_{\varepsilon\to 0} (\sup\{f(\beta), |\beta - \alpha| \leq \varepsilon\}).$$

Inequality (5) and Proposition 4 imply that for all α:

$$\tilde{f}_d(\alpha) \leq f_d^{\lim}(\alpha). \tag{13}$$

In particular, one always has $\tilde{f}_h(\alpha) \leq f_h^{\lim}(\alpha) \leq f_g$.

3.5. Spectrum of a maximum

Proposition 5. *Let A, B be two functions* $: X([0,1]) \to \mathbb{R}$, *and* $C = \max\{A, B\}$. *Let $f_g(\alpha, A)$, $f_g(\alpha, B)$, $f_g(\alpha, C)$ be the corresponding spectra. Then, for all α,*

$$f_g(\alpha, C) \leq \max\{f_g(\alpha, A), f_g(\alpha, B)\} \tag{14}$$

Proof Let $\alpha_n(x) = A(u_n(x))$, $\beta_n(x) = B(u_n(x))$, $\gamma_n(x) = C(u_n(x))$. The relation

$$|\gamma_n(x) - \alpha| \leq \varepsilon \Rightarrow \text{ either } |\alpha_n(x) - \alpha| \leq \varepsilon \text{ or } |\beta_n(x) - \alpha| \leq \varepsilon,$$

valid for all α, n, x, implies the following (with obvious notations):

$$N_\alpha^C(n, \varepsilon) \leq N_\alpha^A(n, \varepsilon) + N_\alpha^B(n, \varepsilon),$$

hence relation (14). ∎

Proposition 6. *If d is stable, the same result as (14) holds for f_d, $f_d^{\lim}$, $f_d^{\lim\sup}$.*

Proof Using similar notations, we observe that

$$E_\alpha^C(\varepsilon, N) \subset E_\alpha^A(\varepsilon, N) \cup E_\alpha^B(\varepsilon, N).$$

The stability of d implies

$$d(E_\alpha^C(\varepsilon, N)) \leq \max\{d(E_\alpha^A(\varepsilon, N)), d(E_\alpha^B(\varepsilon, N))\},$$

so that

$$f_d^{\lim\sup}(\alpha, C) \leq \max\{f_d^{\lim\sup}(\alpha, A), f_d^{\lim\sup}(\alpha, B)\}.$$

The proof for the other spectra uses similar arguments. ∎

4. The inverse problem for spectra

We have seen that the three spectra $f_d^{\lim\sup}$, $f_d^{\lim}$ and f_g are upper semi-continuous. It is natural to enquire whether the converse holds, i.e. whether any u.s.c. function F is the spectrum of some function A. The main result of this section answers by the affirmative for $f_d^{\lim\sup}$ and $f_d^{\lim}$. The case of f_g is treated in [5].

4.1. Generalities on u.s.c. functions

Let us recall some well-known facts. For $E \subset \mathbb{R}$, one has: E is closed $\Leftrightarrow 1_E$ is u.s.c, where 1_E is the characteristic function of E. If f is an upper semi-continuous function defined on a compact set D and ranging in $\mathbb{R}$, then:

- f is bounded from above and reaches its maximum.
- f can be written as $f = \inf\{g \geq f, \text{g continuous on } D\}$.

We shall need the following result:

Lemma 2. *The function $f : D \to \mathbb{R}$ is u.s.c. iff there exists a countable set E dense in D such that*

1. *The restriction $f_{|E}$ is u.s.c.*
2. *In $D - E$, f can be obtained by semi-continuity:*

$$\forall x \in D - E,\ f(x) = \lim_{\varepsilon \to 0} \sup\{f(y) : |x - y| \leq \varepsilon, y \in E\}.$$

Proof $\Rightarrow$ Assume that f is u.s.c.. Construct E as follows: In every dyadic interval $[k\,2^{-n},(k+1)2^{-n}]$, choose a point $x_{k,n}$ where f reaches its maximum. Take $E=\{x_{k,n}: k\in\mathbb{Z}, n\in\mathbb{N}\}$. The restriction $f_{|E}$ is u.s.c.. Let $x\in D-E$. For every n, there exists $x_{k,n}$ in the interval $u_n(x)$ such that $f(x)\le f(x_{k,n})$. Therefore $f(x)\le \sup\{f(y)/y\in E, |x-y|\le 2^{-n}\}$. Letting $n\to+\infty$, we obtain

$$f(x)\le \limsup_{\varepsilon\to 0}\{f(y)/y\in E, |x-y|\le\varepsilon\}.$$

Since f is u.s.c., we obtain an equality.

$\Leftarrow$ It suffices to show that, if (x_n) is a sequence of $D-E$ converging to x (which is necessarily in D), then relation (11) is verified. For all $x'\in D$ and $\varepsilon>0$, let:

$$s(x',\varepsilon)=\sup\{f(y)/|x'-y|\le\varepsilon, y\in E\}.$$

The function $\varepsilon\to s(x',\varepsilon)$ is non decreasing by construction. Using Assumption 2, $s(x_n,\varepsilon)$ converges to $f(x_n)$ as $\varepsilon\to 0$. Thus, for all n, there exists ε small enough to ensure that $f(x_n)\le s(x_n,\varepsilon)\le f(x_n)+1/2n$. There exists $y_n\in E$ such that $|x_n-y_n|\le\min\{\varepsilon,1/n\}$ and $f(y_n)\le s(x_n,\varepsilon)\le f(y_n)+1/2n$. Then $|f(y_n)-f(x_n)|\le 1/n$. The sequence (y_n) tends to x. If $x\in E$, then $f(x)\ge\limsup_n f(y_n)$ from Assumption 1. If $x\in D-E$, the same result stems from Assumption 2. Then $f(x)\ge\limsup_n f(y_n)=\limsup_n f(x_n)$. ■

Lemma 2 shows that, as is the case for continuous functions, a u.s.c. function f is fully determined by its values on a countable set E. However, contrarily to continuous functions, the set E here depends on f.

4.2. Construction of A when d is σ-stable

Let a compact set $D\subset\mathbb{R}$, and a semi-continuous function $F: D\longrightarrow[0,1]\cup\{-\infty\}$ such that $\sup_D F=1$, be given. We want to construct a function A, the $f_d^{\lim}$ or $f_d^{\limsup}$ spectrum of which is equal to F. By semi-continuity, the set $F^{-1}([0,1])$ is closed. Then we may assume that $F(D)\subset[0,1]$ without loss of generality. The function F can be extended to $\mathbb{R}$ by defining $F(x)=-\infty$ if $x\notin D$.

Lemma 2 shows that it suffices to match the values of F on a countable set $E=\{\alpha_k\}$ dense in D. On $D-E$, F is obtained by semi-continuity. We can assume that $F(\alpha_0)=1$, and that $\alpha_i\ne\alpha_j$ if $i\ne j$. We will construct a function $A: X([0,1])\longrightarrow\mathbb{R}$ such that

$$f_d(\alpha_k)=f_d^{\lim}(\alpha_k)=f_d^{\limsup}(\alpha_k)=F(\alpha_k) \tag{15}$$

for all $k\ge 0$. This implies that

$$f_d^{\lim}(\alpha)=f_d^{\limsup}(\alpha)=F(\alpha) \tag{16}$$

for all $\alpha\in D$.

Let (I_k) be a family of closed, non degenerate, disjoint intervals in $[0,1]$. For all $k\ge 1$, let $C_k\subset I_k$ be a compact set such that

$$d(C_k)=F(\alpha_k).$$

Let $C_0 = [0,1] - \bigcup_{k=1}^{\infty} C_k$. Since C_0 contains an open set, $d(C_0) = 1$. For any dyadic interval u, define

$$A(u) = \alpha_{l(u)} \ , \ \text{where} \ \ l(u) = \begin{cases} 0 \text{ if } u \subset C_0 \\ \min\{k \geq 1/u \cap C_k \neq \emptyset\} \text{ otherwise} \end{cases}$$

Since the intervals I_k are disjoint, there exists for every $k \geq 1$ an integer N_k such that for all $n \geq N_k$, for all $x \in C_k$, $u_n(x) \subset C_0 \cup C_k$. Then $A(u_n(x)) = \alpha_k$. Therefore $E_{\alpha_k} = C_k$ and

$$f_d(\alpha_k) = F(\alpha_k).$$

For all $\varepsilon > 0$,

$$E_{\alpha_k}(\varepsilon) = \bigcup \{C_i : |\alpha_i - \alpha_k| \leq \varepsilon\}, \tag{17}$$

so that

$$d(E_{\alpha_k}(\varepsilon)) \geq \sup\{d(C_i) : |\alpha_i - \alpha_k| \leq \varepsilon\}.$$

Since d is σ-stable, this is an equality, so that

$$f_d^{\lim}(\alpha_k) = \lim_{\varepsilon \to 0} \sup\{F(\alpha_i) : |\alpha_i - \alpha_k| \leq \varepsilon\}.$$

The right hand side member is equal to $F(\alpha_k)$. Finally, the set C_k is included in $E_{\alpha_k}(\varepsilon, N)$ for all $N \geq N_k$, so that $F(\alpha_k) \leq f_d^{\limsup}(\alpha_k)$. Using (6), we obtain (15) for $k \geq 1$. For $k = 0$, $f_d^{\lim}(\alpha_0) = f_d^{\limsup}(\alpha_0) = F(\alpha_0) = 1$.

4.3. Construction of A when $d = \Delta$

We will use the same A as before, with extra conditions on the sequences (I_k) and (C_k).

Lemma 3. *Given a dyadic interval* $[0, 2^{-N}]$*, and a real number* $\delta \in (0,1)$*, there exists a compact set* C *such that* $\Delta(C) = \delta$*, and for all* $n \geq N$*:*

$$\omega(2^n, C) \leq 2^{(n-N)\delta}. \tag{18}$$

Proof For any sequence (ω_n) of integers such that

$$\omega_N = 1 \, , \, \omega_n \leq \omega_{n+1} \leq 2\omega_n, \tag{19}$$

there are infinitely many ways to construct a compact set C as the limit of embedded coverings by ω_n dyadic intervals; such a set verifies $\omega(2^n, C) = \omega_n$. Let us define

$$\omega_n = 2^{E[(n-N)\delta]},$$

where $E[.]$ denotes the integer part. It is clear that $\omega_n \leq \omega_{n+1}$. On the other hand,

$$\delta \leq 1 \Longrightarrow (n - N + 1)\delta \leq (n - N)\delta + 1,$$

so that

$$E[(n - N + 1)\delta] \leq E[(n - N)\delta] + 1,$$

which gives $\omega_{n+1} \leq 2\omega_n$. Then the conditions given in (19) are fulfilled. This proves the existence of the set C. Finally, (18) is also verified, and

$$\Delta(C) = \limsup \frac{\log \omega_n}{n \log 2} = \delta.$$

■

Lemma 4. *Let $I_k = [2^{-2k}, 2^{-2k+1}]$, $\delta_k \in [0,1]$, and C_k be a compact set included in I_k such that $\Delta(C_k) = \delta_k$ and for all $n \geq 2^{2k}$,*

$$\omega(2^n, C_k) \leq 2^{(n-2k)\delta_k}. \tag{20}$$

Then for any strictly increasing sequence (k_i) of integers:

$$\Delta(\bigcup_i C_{k_i}) = \sup_i \Delta(C_{k_i}). \tag{21}$$

The existence of C_k is proved in Lemma 3.

Proof Let $\delta = \sup_i \Delta(C_{k_i})$. The inequality $\Delta(\bigcup_i C_{k_i}) \geq \delta$ is trivial. For the reverse inequality, choose for any n the integer i_n such that

$$2k_{i_n} \leq n \leq 2k_{i_n+1}.$$

Then $\cup_{i \geq i_n+1} C_{k_i} \subset [0, 2^{-n}]$, and

$$\omega(2^n, \bigcup_i C_{k_i}) \leq 1 + \sum_{i=1}^{i_n} \omega(2^n, C_{k_i}).$$

Using (20),

$$i \leq i_n \Rightarrow \omega(2^n, C_{k_i}) \leq 2^{(n-2k_i)\delta} \leq 2^{n\delta}.$$

Since $i_n \leq n$,

$$\omega(2^n, \bigcup_i C_{k_i}) \leq 1 + n2^{n\delta}.$$

Thus

$$\Delta(\bigcup_i C_{k_i}) \leq \limsup \frac{\log(1+n2^{n\delta})}{n \log 2} = \delta.$$

■

Let us now take (I_k) and (C_k) as in Lemma 4, with $\delta_k = F(\alpha_k)$, and come back to Equation (17). Using (21) we obtain:

$$\Delta(E_{\alpha_k}(\varepsilon)) = \sup\{\Delta(C_i)/|\alpha_i - \alpha_k| \leq \varepsilon\},$$

so that

$$f_\Delta^{\lim}(\alpha_k) = \lim_{\varepsilon \to 0} \sup\{F(\alpha_i)/|\alpha_i - \alpha_k| \leq \varepsilon\} = F(\alpha_k).$$

We conclude as before.

5. Examples of spectra

In this section, we provide various examples of computation of the new spectra. They are meant to show that the inequalities between the spectra may be strict. We choose for d the box dimension, as it is the one most often used in applications.

Example 1: $f_\Delta(\alpha) < f_\Delta^{\lim}(\alpha)$ for all α. Let $A([a,b]) = a$ for all $[a,b] \subset [0,1]$. Then $D = [0,1]$, and for all $x \in [0,1]$, $\alpha(x) = x$. For all N,

$$E_\alpha(\varepsilon, N) = E_\alpha(\varepsilon) = [\alpha - \varepsilon, \alpha + \varepsilon] \quad , \quad E_\alpha = \{\alpha\}.$$

Therefore $\Delta(E_\alpha) = 0$, $\Delta(E_\alpha(\varepsilon)) = \Delta(E_\alpha(\varepsilon, N)) = 1$. Finally, $N_\alpha(\varepsilon, n) \simeq 2\varepsilon/2^{-n}$. For all $\alpha \in D$, we obtain

$$f_\Delta(\alpha) = 0 \ , \ f_\Delta^{\lim}(\alpha) = f_\Delta^{\lim\sup}(\alpha) = 1 \ , \ f_g(\alpha) = 1. \tag{22}$$

It is not possible to find a function or a measure whose Hölder regularity is exactly A above. However, it is easily checked that the set function defined on intervals by $C([a,b]) = |b-a|^a$ extends to a Choquet capacity on the Borel subsets of $[0,1]$, to which a multifractal analysis may be applied (see [12]). Alternatively, one may relax the condition $A([a,b]) = a$. Indeed, the computations above still apply when, for all $u \subset [0,1]$,

$$A(u) + O\left(\frac{1}{\log|u|}\right) \in u \tag{23}$$

uniformly with respect to $|u|$. This situation is illustrated by the generalized Weierstrass function W (see Section 3.3), with $A(u) = \log v_W(u)/\log|u|$ and $v_W(u)$ the oscillation of W in u. There exists constants $C_1 \neq 0$ and C_2 such that:

$$\forall\, u = [a,b], \quad C_1|u|^b \leq v_W(u) \leq C_2|u|^a.$$

Then (23) is verified, and the spectra take the values shown in (22).

Example 2: $f_\Delta^{\lim\sup}(\alpha) < f_\Delta^{\lim}(\alpha)$ for some α. This example shows that $f_d^{\lim\sup}$ may be different from $f_d^{\lim}$ when d is not σ-stable.

Let $\gamma > 0$ and $F = \{k^{-\gamma} : k \geq 1\}$. Then $\Delta(F) = 1/(\gamma+1)$. The left extremity of u is denoted by $u_{\min}$. Let $p : [0,1] \longrightarrow \mathbb{R}$ be a strictly increasing, continuous function such that $p(0) = 0$. For all $u \in X([0,1])$, let

$$A(u) = \begin{cases} 0 & \text{if } u \cap F \neq \emptyset \text{ and } p(|u|) < u_{\min} \\ 1 & \text{otherwise} \end{cases}$$

If $x \in F$, $u_n(x) \cap F \neq \emptyset$. There exists a smallest integer $N(x)$ such that $n \geq N(x) \Rightarrow p(2^{-n}) \leq u_n(x)_{\min}$, so that $\alpha_n(x) = 0$. For all $\varepsilon = 2^{-n}$, the set $E_0(\varepsilon, N) = \{k^{-\gamma}/N(k^{-\gamma}) \leq N\}$ is finite, and $E_0(\varepsilon) = F$. If $x \notin F$, and n is large enough, $\alpha_n(x) = 1$. Let us take two real numbers $c < d$ and an integer K such that $2^{-\gamma} < c - 2^{-K} \leq d + 2^{-K} < 1$. For all $x \in [c,d]$ and $n \geq K$, then $u_n(x) \cap F = \emptyset$, so that $\alpha_n(x) = 1$. Therefore $[c,d] \subset E_1(2^{-K}, N)$ for all $N \geq K$. We deduce that for all $\varepsilon < 2^{-K}$, $\sup_N \Delta(E_1(\varepsilon, N)) = 1$. Therefore,

$$f_\Delta^{\lim\sup}(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0 \\ 1 & \text{if } \alpha = 1 \end{cases}$$

and

$$f_\Delta^{\lim}(\alpha) = f_\Delta(\alpha) = \begin{cases} \Delta(F) & \text{if } \alpha = 0 \\ 1 & \text{if } \alpha = 1\ . \end{cases}$$

These results do not depend on the function p.

To illustrate this case in the frame of classical multifractal analysis, we deal with a slightly more complex but similar situation, and consider the measure defined as

$$\mu = \mathcal{L} + \sum_{k=1}^{\infty} a_k \delta_{k^{-\gamma}}$$

where $\mathcal{L}$ is the Lebesgue measure on $[0,1]$, δ_x is the Dirac mass at x and (a_k) is a sequence of real numbers decreasing to 0 and such that $\sum_{k=1}^{\infty} a_k$ converges. Let $A(u) = \log \mu(u)/\log|u|$. If $x = k^{-\gamma}$, then $\mu(u_n(x)) = 2^{-n} + a_k$ if n is large enough. Since $a_k > 0$, $\alpha(u_n(k^{-\gamma})) \to 0$. But this convergence is not uniform with respect to k, so that for every N the set $E_0(\varepsilon, N)$ is finite as before. If $x \notin F$, and $x \neq 0$, then $\mu(u_n(x)) = 2^{-n}$ when n large, so that $\alpha(u_n(x)) \to 1$. The results on the spectra are the same.

Example 3: $f_\Delta^{\lim}(\alpha) \neq f_g(\alpha)$ for some value of α. We need to show that there is no general relationship between these two spectra. Let β, γ, δ, ω be such that $0 < \delta < \min\{\beta, 1\}$ and $0 < \omega < \gamma/(\gamma+1)$. Let $F = \{k^{-\gamma}\}$ as in Example 2. For all $u \in X([0,1])$, let

$$A(u) = \begin{cases} 0 & \text{if } |u|^\beta \leq u_{\min} \leq |u|^\delta, \\ & \text{or } |u|^\omega < u_{\min} \text{ and } u \cap F \neq \emptyset \\ 1 & \text{otherwise.} \end{cases}$$

The function $p(x)$ of Example 2 is replaced by x^ω. Results on the spectra f_Δ, $f_\Delta^{\lim}$, $f_\Delta^{\lim\sup}$ are unchanged. Let us compute f_g.

The number of dyadic intervals which do not meet F is of the order of 2^n. Therefore $f_g(1) = 1$.

Let us evaluate the number K_n of dyadic intervals u_n such that $u_n \cap F \neq \emptyset$ and $(u_n)_{\min} > 2^{-n\omega}$. Note that the minimum distance between two points of F in $[x, 1]$ is $\rho(x) \simeq k^{-\gamma} - (k+1)^{-\gamma} \simeq k^{-\gamma-1}$, where $k^{-\gamma} \simeq x$. Therefore $\rho(x) \simeq x^{(\gamma+1)/\gamma}$. Letting $x = 2^{-n\omega}$, $\rho(2^{-n\omega})$ is equivalent to $2^{-n\omega(\gamma+1)/\gamma}$ which is larger than 2^{-n} since $\omega < \gamma/(\gamma+1)$. We deduce that $K_n \simeq \text{Card}(F \cap [2^{-n\omega}, 1])$. Since $K_n^{-\gamma} \simeq 2^{-n\omega}$, then $K_n \simeq 2^{n\omega/\gamma}$. Since $\delta < 1$, the number of dyadic intervals u_n such that $2^{-n\beta} \leq (u_n)_{\min} \leq 2^{-n\delta}$ is $K_n' \simeq 2^n(2^{-n\delta} - 2^{-n\beta}) \simeq 2^{n(1-\delta)}$.

The number of u_n verifying $A(u_n) = 0$ is equivalent to $K_n + K_n'$. Therefore

$$f_g(\alpha) = \begin{cases} \max\{\frac{\omega}{\gamma}, 1-\delta\} & \text{if } \alpha = 0 \\ 1 & \text{if } \alpha = 1\ . \end{cases}$$

This result does not depend on β.

For a numerical application, take $\gamma = 1$, $\omega = 1/3$, $\delta = 2/3$: Then $f_g(0) = 1/3$, $f_\Delta^{\lim}(0) = 1/2$. Let now $\gamma = 1$, $\omega = 1/3$, $\delta = 1/3$: Then $f_g(0) = 2/3$, $f_\Delta^{\lim}(0) = 1/2$. The spectrum $f_\Delta^{\lim}$ can thus be larger or smaller than f_g for some values of α.

To exhibit a function whose Hölder regularity is similar to the above function A, we shall use a technique based on wavelets. Fix a wavelet ψ in the Schwartz class such that the functions $t \to \psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$, $j, k \in \mathbb{Z}$ form an orthonormal basis of L^2, and define a function z by its wavelet coefficients in this basis, $c_{j,k} = 2^j \int \psi_{j,k}(t)z(t)dt$ (note that we use here an L^∞ normalization rather than an L^2 one). We set:

$$c_{j,k} = \begin{cases} 1 & \text{if } 2^{-n\beta} \leq k2^{-n} \leq 2^{-n\delta}, \\ & \text{or } 2^{-n\omega} < k2^{-n} \text{ and } u_n^k \cap F \neq \emptyset \\ 2^{-n} & \text{otherwise} \end{cases}$$

where $u_n^k = [k2^{-n}, (k+1)2^{-n}]$. One immediately checks that z is in L^2 if $\beta < \gamma$. Choose now as is usual $A(u_n^k) = |\log(c_{n,k})|/n$. This yields the desired behaviour. Note that z depends on the wavelet ψ.

6. Continuous spectra

The definition of the large deviation spectrum f_g has two drawbacks: First it depends on the choice of the interval partitions (usually the dyadic intervals), and second it uses two limiting operations (n tends to $+\infty$, then ε tends to 0) which makes it difficult to evaluate from a given set of data (see [9] for more on this topic). This section introduce variants of f_g, denoted by f_g^c and $\tilde{f}_g^c$, and called the continuous large deviation spectra. They are independent of any interval partition. Moreover $\tilde{f}_g^c$ uses only one limiting operation. These spectra are helpful in numerical applications of multifractal analysis.

As before, A is a function $X([0,1]) \longrightarrow \mathbb{R}^+ \cup \{+\infty\}$. Recall that for any set family $\mathcal{F}$, $\cup\mathcal{F}$ denotes the union of all sets in $\mathcal{F}$. For any measurable set E of the line, $|E|$ is its Lebesgue measure. Let $\eta \in (0,1)$. We introduce families of intervals:

$$\mathcal{R}_\eta = \{u \in X([0,1]) \text{ such that } |u| = \eta\}$$

$$\mathcal{R}_\eta(\alpha) = \{u \in X([0,1]) \text{ such that } |u| = \eta \text{ and } A(u) = \alpha\}$$

$$\mathcal{R}_\eta^\varepsilon(\alpha) = \{u \in X([0,1]) \text{ such that } |u| = \eta \text{ and } |A(u) - \alpha| \leq \varepsilon\}.$$

By convention $\log 0/\log\eta = +\infty$.

Definition 4. *The continuous large deviation spectra are*

$$f_g^c(\alpha) = \lim_{\varepsilon\to 0} \limsup_{\eta\to 0} \frac{\log\left(|\bigcup\mathcal{R}_\eta^\varepsilon(\alpha)|/\eta\right)}{|\log\eta|} = \lim_{\varepsilon\to 0} \limsup_{\eta\to 0} \left(1 - \frac{\log|\bigcup\mathcal{R}_\eta^\varepsilon(\alpha)|}{\log\eta}\right)$$

$$\tilde{f}_g^c(\alpha) = \limsup_{\eta\to 0} \frac{\log\left(|\bigcup\mathcal{R}_\eta(\alpha)|/\eta\right)}{|\log\eta|} = \limsup_{\eta\to 0} \left(1 - \frac{\log|\bigcup\mathcal{R}_\eta(\alpha)|}{\log\eta}\right)$$

The integer $N_\alpha(\varepsilon, n)$ used in the definition of f_g has been replaced by a mean number $|\bigcup\mathcal{R}_\eta^\varepsilon(\alpha)|/\eta$ in the definition of f_g^c. In many applications, A is continuous and $\mathcal{R}_\eta(\alpha)$ is non empty for a whole range of values of α: This allows to define the spectrum $\tilde{f}_g^c$, which avoids the use of the "ε-tolerance".

Recall that, when the so-called weak multifractal formalism holds, f_g can be obtained as the Legendre transform of the following function:

$$\tau(q) = \liminf_{n\to\infty} \frac{\log S_n(q)}{-n}$$

where, for all $q \in \mathbb{R}$, $S_n(q)$ is defined with dyadic intervals as follows:

$$S_n(q) = \sum_{k=0}^{2^n-1} 2^{-nqA(u_n^k)} \tag{24}$$

with the convention $0^q = 0$. One denotes by f_l the Legendre transform of τ:

$$f_l(\alpha) := \tau^*(\alpha) = \inf_{q\in\mathbb{R}} (q\alpha - \tau(q)).$$

The function f_l is called the *Legendre multifractal spectrum*, and the equality $f_l = f_g$, when it holds, is essentially a consequence of Ellis theorem (see [6]).

Let us introduce similar notions in a continuous framework. An interval family is a *packing* if all of its intervals are disjoint. For any $q \in \mathbb{R}$ and for any interval family $\mathcal{R}$, let

$$H^q(\mathcal{R}) = \sup\{\sum_{u\in\mathcal{R}'} |u|^{qA(u)}, \mathcal{R}' \subset \mathcal{R},\ \mathcal{R}' \text{ is a packing}\}.$$

By convention $H^q(\emptyset) = 0$. Here are some basic properties of $H^q(\mathcal{R})$:

Lemma 5. *Let $\mathcal{R}_1, \mathcal{R}_2$ be two families of interval.*
1. $\mathcal{R}_1 \subset \mathcal{R}_2 \Longrightarrow H^q(\mathcal{R}_1) \le H^q(\mathcal{R}_2)$;
2. $H^q(\mathcal{R}_1 \cup \mathcal{R}_2) \le H^q(\mathcal{R}_1) + H^q(\mathcal{R}_2)$;
3. If $|u| = \eta$ and $\alpha \le A(u) \le \beta$ for all $u \in \mathcal{R}$, then

$$\forall q \ge 0, \qquad \frac{1}{2}|\bigcup \mathcal{R}|\, \eta^{\beta q - 1} \le H^q(\mathcal{R}) \le |\bigcup \mathcal{R}|\, \eta^{\alpha q - 1}. \tag{25}$$

Proof Property 1 comes from the fact that for any packing $\mathcal{R}_3 \subset \mathcal{R}_1$ we have $\mathcal{R}_3 \subset \mathcal{R}_2$. For Property 2, take a packing $\mathcal{R}'$ in $\mathcal{R}_1 \cup \mathcal{R}_2$. Let $\mathcal{R}'_1 = \mathcal{R}' \cap \mathcal{R}_1$ and $\mathcal{R}'_2 = \mathcal{R}' \cap \mathcal{R}_2$. Then

$$\sum_{u\in\mathcal{R}'} |u|^{qA(u)} \le \sum_{u\in\mathcal{R}'_1} |u|^{qA(u)} + \sum_{u\in\mathcal{R}'_2} |u|^{qA(u)} \le H^q(\mathcal{R}_1) + H^q(\mathcal{R}_2).$$

For Property 3, it suffices to consider the case where $\bigcup \mathcal{R}$ is an interval. Let M_η be the maximum number of intervals of $\mathcal{R}$ covering $\bigcup \mathcal{R}$. Show that

$$\eta M_\eta \le |\bigcup \mathcal{R}| \le 2\eta M_\eta.$$

Then use the inequalities $\eta^{\beta q} \le |u|^{qA(u)} \le \eta^{\alpha q}$ for all $u \in \mathcal{R}$ and $q > 0$. ■

To define a continuous counterpart to $S_n(q)$, we simply take $H^q_\eta := H^q(\mathcal{R}_\eta)$. It can be written as

$$H^q_\eta = \sup\{\sum_{u\in\mathcal{R}'} \eta^{qA(u)}, \mathcal{R}' \text{ is a packing of } [0,1] \text{ by intervals of length } \eta\}.$$

The relevant quantity that corresponds to $\tilde{f}_g^c$ is:

$$\begin{aligned} J_\eta^q &= \sup_\alpha H^q(\mathcal{R}_\eta(\alpha)) \\ &= \sup_\alpha \sup\{\sum_{u\in\mathcal{R}'} \eta^{\alpha q},\ \mathcal{R}' \text{ is a packing such that } |u| = \eta \text{ and } A(u) = \alpha\}. \end{aligned}$$

Definition 5. *Let*

$$\tau^c(q) = \liminf_{\eta\to 0} \frac{\log H_\eta^q}{\log \eta}$$

and

$$\tilde{\tau}^c(q) = \liminf_{\eta\to 0} \frac{\log J_\eta^q}{\log \eta}.$$

The continuous Legendre spectra *are defined as* $f_l^c = (\tau^c)^*$ *and* $\tilde{f}_l^c = (\tilde{\tau}^c)^*$.

Here are some obvious properties of $f_g^c, \tilde{f}_g^c, f_l^c, \tilde{f}_l^c$.

Proposition 7.
1. *f_l^c and $\tilde{f}_l^c$ are concave.*
2. *$\forall\ \alpha, \tilde{f}_g^c(\alpha) \le f_g^c(\alpha)$ and $f_g(\alpha) \le f_g^c(\alpha)$.*
3. *If μ is a multinomial measure, $f_g^c(\alpha) = \tilde{f}_g^c(\alpha) = f_l^c(\alpha) = \tilde{f}_l^c(\alpha) = f_g(\alpha)$.*

A fundamental property of the continuous spectrum is that, under a mild restriction, f_l^c is the concave envelope of f_g^c:

Proposition 8.
1. *For all $\alpha, f_l^c(\alpha) \ge (f_g^c(\alpha))^{**}$ and $\tilde{f}_l^c(\alpha) \ge (\tilde{f}_g^c(\alpha))^{**}$.*
2. *Assume that $f_g^c = -\infty$ outside a compact interval. Then:*

$$\forall\alpha, \qquad f_l^c(\alpha) = (f_g^c(\alpha))^{**}.$$

For part 2, we will need a corollary of the next Lemma, which is of independent interest:

Lemma 6. *Let $\mathcal{X}(\mathbb{R})$ be the family of all closed intervals in $\mathbb{R}$. Let $F : \mathcal{X}(\mathbb{R}) \to \mathbb{R} \cup \{-\infty\}$ be such that*

$$I \subset J \Rightarrow F(I) \le F(J).$$

For any $\alpha \in \mathbb{R}$, define $f(\alpha) = \lim_{\varepsilon\to 0} F([\alpha-\varepsilon, \alpha+\varepsilon])$. Then, for any compact interval J and for any real q,

$$\lim_{\varepsilon\to 0} \inf_{\alpha\in J} \{\alpha\, q - F([\alpha-\varepsilon, \alpha+\varepsilon])\} = \inf_{\alpha\in J} \{\alpha\, q - f(\alpha)\}. \tag{26}$$

Proof Fix $q \in \mathbb{R}$. Let $l_\varepsilon = \inf_{\alpha\in J}\{q\alpha - F([\alpha-\varepsilon, \alpha+\varepsilon])\}$, $l = \inf_{\alpha\in J}\{q\alpha - f(\alpha)\}$. Since $F([\alpha-\varepsilon, \alpha+\varepsilon])$ decreases to $f(\alpha)$ as $\varepsilon \to 0$, l_ε increases and is bounded by l. Therefore we must show that $l \le \lim_\varepsilon l_\varepsilon$. For all $\varepsilon > 0$, there exists $\alpha(\varepsilon) \in J$ such that

$$q\,\alpha(\varepsilon) - F([\alpha(\varepsilon)-\varepsilon, \alpha(\varepsilon)+\varepsilon]) \le l_\varepsilon + \varepsilon. \tag{27}$$

Let β be a limit value of the sequence $(\alpha(1/n))$ in the compact J, and $r > 0$. Let $N \geq 2/r$. There exists $n \geq N$ such that $|\alpha(1/n) - \beta| \leq r/2$. Since $1/n \leq r/2$,

$$[\alpha(\frac{1}{n}) - \frac{1}{n}, \alpha(\frac{1}{n}) + \frac{1}{n}] \subset [\beta - \varepsilon, \beta + \varepsilon]$$

so that $F([\alpha(1/n) - 1/n, \alpha(1/n) + 1/n]) \leq F([\beta - r, \beta + r])$. We obtain

$$q\,\beta - F([\beta - r, \beta + r]) \leq q(\alpha(\frac{1}{n}) + \frac{r}{2}) - F([\alpha(\frac{1}{n}) - \frac{1}{n}, \alpha(\frac{1}{n}) + \frac{1}{n}]) \tag{28}$$

$$\leq l_{1/n} + \frac{1}{n} + q\frac{r}{2}. \tag{29}$$

Since n can be taken to be arbitrarily large, inequality (29) implies

$$q\,\beta - F([\beta - r, \beta + r]) \leq \lim_{\varepsilon \to 0} l_\varepsilon + q\frac{r}{2}.$$

When $r \to 0$, $F([\beta - r, \beta + r])$ tend to $f(\beta)$, so that $q\,\beta - f(\beta) \leq \lim_{\varepsilon \to 0} l_\varepsilon$. ∎

This result may be applied directly to $f_g(\alpha, \varepsilon)$ or to $f_g^c(\alpha, \varepsilon)$, when these functions have compact support. Therefore:

Corollary 1. *If there exists a compact interval J such that $f_g(\alpha, \varepsilon) = -\infty$ for all $\alpha \notin J$ and $\varepsilon \leq 1$, then*

$$\lim_{\varepsilon \to 0} \inf_\alpha \{q\alpha - f_g(\alpha, \varepsilon)\} = \inf_\alpha \{q\alpha - f_g(\alpha)\}.$$

If there exists a compact interval J such that $f_g^c(\alpha, \varepsilon) = -\infty$ for $\alpha \notin J$ and $\varepsilon \leq 1$:

$$\lim_{\varepsilon \to 0} \inf_\alpha \{q\alpha - f_g^c(\alpha, \varepsilon)\} = \inf_\alpha \{q\alpha - f_g^c(\alpha)\}.$$

Remark: The corollary remains true even when $f_g^c(\alpha)$ (resp. $f_g(\alpha)$) does not have a compact support (see [5]).

Proof of Proposition 8

We treat the case $q \geq 0$.

1. We shall prove the equivalent statements:

$$\tau^c(q) \leq \inf_\alpha (q\alpha - f_g^c(\alpha)) \text{ and } \tilde{\tau}^c(q) \leq \inf_\alpha (q\alpha - \tilde{f}_g^c(\alpha)).$$

For all α and $\varepsilon > 0$, $\mathcal{R}_\eta^\varepsilon(\alpha) \subset \mathcal{R}_\eta$, so that

$$H^q(\mathcal{R}_\eta^\varepsilon(\alpha)) \leq H_\eta^q. \tag{30}$$

Relation (25) implies that

$$H^q(\mathcal{R}_\eta^\varepsilon(\alpha)) \geq \frac{1}{2}|\bigcup \mathcal{R}_\eta^\varepsilon(\alpha)|\,\eta^{q(\alpha+\varepsilon)-1}. \tag{31}$$

For all $\delta > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$,

$$f_g^c(\alpha) \leq \limsup_{\eta \to 0} \frac{\log(|\bigcup \mathcal{R}_\eta^\varepsilon(\alpha)|/\eta)}{|\log \eta|} + \frac{\delta}{2}.$$

For all $\eta_0 > 0$, there exists $\eta \leq \eta_0$ such that

$$f_g^c(\alpha) \leq \frac{\log(|\bigcup \mathcal{R}_\eta^\varepsilon(\alpha)|/\eta)}{|\log \eta|} + \delta,$$

so that

$$|\bigcup \mathcal{R}_\eta^\varepsilon(\alpha)| \geq \eta^{-f_g^c(\alpha)+\delta+1}. \tag{32}$$

Gathering (30), (31), (32), we get: $H_\eta^q \geq \frac{1}{2}\eta^{q(\alpha+\varepsilon)-f_g^c(\alpha)+\delta}$. Taking the logarithm on both sides and using the definition of τ^c as a $\liminf$:

$$\tau^c(q) \leq q\alpha - f_g^c(\alpha) + q\varepsilon + \delta.$$

Let $\varepsilon \to 0$, then $\delta \to 0$ to obtain $\tau^c(q) \leq \inf_\alpha(q\alpha - f_g^c(\alpha))$.

Let us now consider $\tilde{f}_g^c(\alpha)$. For all α, $J_\eta^q \geq \frac{1}{2}H^q(\mathcal{R}_\eta(\alpha)) \geq \frac{1}{2}|\bigcup \mathcal{R}_\eta(\alpha)|\,\eta^{q\alpha-1}$. We deduce that for all $\delta > 0$, $\eta_0 > 0$, there exists $\eta \leq \eta_0$ such that

$$J_\eta^q \geq \frac{1}{2}\eta^{q\alpha-\tilde{f}_g^c(\alpha)+\delta}.$$

Therefore $\tilde{\tau}^c(q) \leq q\alpha - \tilde{f}_g^c(\alpha) + \delta$.

2. Let $\varepsilon > 0$. Let $L_\varepsilon(q) = \inf_\alpha\{q\alpha - f_g^c(\alpha,\varepsilon)\}$. For every α there exists $\eta(\alpha,\varepsilon) < 1$ such that

$$\eta \leq \eta(\alpha,\varepsilon) \Rightarrow \frac{\log(|\bigcup \mathcal{R}_\eta^\varepsilon(\alpha)|/\eta)}{|\log \eta|} \leq f_g^c(\alpha,\varepsilon) + \varepsilon.$$

Using (25):

$$\begin{aligned} H^q(\mathcal{R}_\eta^\varepsilon(\alpha)) &\leq |\bigcup \mathcal{R}_\eta^\varepsilon(\alpha)|\eta^{q\alpha-|q|\varepsilon-1} \\ &\leq \eta^{q\alpha-f_g^c(\alpha,\varepsilon)-\varepsilon(1+|q|)} \\ &\leq \eta^{L_\varepsilon(q)-\varepsilon(1+|q|)}. \end{aligned}$$

Let α_1, ..., α_K be a finite sequence such that $J \subset \bigcup_i[\alpha_i - \varepsilon, \alpha_i + \varepsilon]$. For $\eta \leq \min_i\{\eta(\alpha_i,\varepsilon)\}$:

$$H_\eta^q \leq K\,\eta^{L_\varepsilon(q)-\varepsilon(1+|q|)}.$$

Taking logarithms:

$$\frac{\log H_\eta^q}{\log \eta} \geq L_\varepsilon(q) - \varepsilon(1+|q|) + \frac{\log K}{\log \eta}.$$

Now let η and ε tend to 0 to get $\tau^c(q) \geq \lim_{\varepsilon\to 0} L_\varepsilon(q)$. Use corollary 1 to conclude.

The case $q < 0$ goes along the same lines. ■

Remark: The same technique shows that $f_l = (f_g)^{**}$ when f_g has compact support.

We end this section with some easily proved properties of τ^c and $\tilde{\tau}^c$.

Proposition 9.
1. τ^c *and* $\tilde{\tau}^c$ *are increasing and concave.*
2. $\tau^c(0) = \tilde{\tau}^c(0) = -\Delta(\mathrm{Supp}(A))$, *where* $\mathrm{Supp}(A) = \mathbb{R} - \cup\{u, |u| > 0, A(u) = +\infty\}$.
3. If $A(u) = \log \mu(u)/\log |u|$ *with* μ *a probability measure, then* $\tau^c(1) = \tilde{\tau}^c(1) = 0$.
4. For any sequence (η_n) *tending to zero such that* $\lim_{n\to\infty} \log \eta_n / \log \eta_{n+1} = 1$, $\tau^c(q) = \liminf_{n\to\infty} \log H^q_{\eta_n} / \log \eta_n$ *and* $\tilde{\tau}^c(q) = \liminf_{n\to\infty} \log J^q_{\eta_n} / \log \eta_n$.

The last property is important for numerical applications, since it allows to evaluate τ^c and $\tilde{\tau}^c$ using discrete sequences such as $\eta_n = 2^{-n}$.

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