On the Identification of the Pointwise Holder Exponent of the Generalized Multifractional Brownian Motion

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Abstract

The Generalized Multifractional Brownian Motion (GMBM) is a continuous Gaussian process that extends the classical Fractional Brownian Motion (FBM) and Multifractional Brownian Motion (MBM) [30, 31, 10, 4, 5]. As is the case for the MBM, the Hölder regularity of the GMBM varies from point to point. However, and this is the main interest of the GMBM, contrarily to the MBM, these variations may be very erratic: As shown in [1], the pointwise Hölder exponent $\{\alpha_X(t)\}_t$ of the GMBM may be any liminf of continuous functions with values in a compact of $(0,1)$. This feature makes the GMBM a good candidate to model complex data such as textured images or multifractal processes. For the GMBM to be useful in applications, it is necessary that its Hölder exponents may be estimated from discrete data. This work deals with the problem of identifying the pointwise Hölder function H of the GMBM: While it does not seem easy to do so when H is an arbitrary liminf of continuous functions, we obtain below the following *a priori* unexpected result: *as soon as the pointwise Hölder function of the GMBM belongs to the first class of Baire (i.e when $\{\alpha_X(t)\}_t$ is a limit of continuous functions), it may be estimated almost surely at any point t.* We also derive a Central Limit Theorem for our estimator. Thus, even very irregular variations of the Hölder regularity of the GMBM may be detected and estimated in practice. This has important consequences in applications of the GMBM to signal and image processing. It may also lead to new methods for the practical computation of multifractal spectra. We illustrate our results on both simulated and real data.
AMS Mathematics Subjets Classifications (1991): 60G15, 60G17, 60G18.
Key words: Gaussian process, fractional Brownian motion, generalized multifractional Brownian motion, pointwise Hölder exponent, Identification.

1 Introduction and background

The celebrated Fractional Brownian motion (FBM) was first introduced by Kolmogorov in 1940, in a Hilbertian framework [24]. The seminal paper of Mandelbrot and Van Ness popularized the FBM by showing its relevance for the modelling of natural phenomena such as hydrology or finance [30]. FBM is a continuous and centered Gaussian process, denoted $\{B_H(t)\}_{t\in\mathbb{R}^d}$. It depends of one parameter (the Hurst parameter) $H \in (0,1)$. In, e.g., the book of Samorodnitsky and Taqqu [33], it is shown that FBM can be represented, for every $t \in \mathbb{R}^d$ as

$$B_H(t) = \mathrm{Re}\Big(\int_{\mathbb{R}^d} \frac{(e^{it.\xi}-1)}{|\xi|^{H+d/2}}\,\widetilde{W}(\xi)\Big), \tag{1.1}$$

where Re(.) denotes the real part and where the complex isotropic random measure $d\widetilde{W}$ satisfies

$$d\widetilde{W} = dW_1 + i dW_2, \tag{1.2}$$

dW_1 and dW_2 being two independent real-valued Brownian measures (throughout the article, the symbol $|.|$ will either denotes the Euclidian norm on $\mathbb{R}^d$ or the absolute value on $\mathbb{R}$). When $H = 1/2$, FBM reduces to Brownian Motion. FBM is therefore an extension of the Wiener process and shares many of its properties.

A major difference, which is one of the main interests of FBM, is that, contrarily to Brownian Motion, its increments are correlated. They even display *long range dependence* when $H > 1/2$ (see [33] for a definition). FBM has been used in a number of areas, most recently in telecommunications (see for instance [35]). The monograph of Doukhan, Oppenheim and Taqqu [14] offers a systematic treatment of FBM, as well as an overview of different areas of applications. Another important property of FBM is that its pointwise Hölder exponent $\{\alpha_{B_H}(t)\}_{t\in\mathbb{R}^d}$ can be prescribed via its Hurst parameter. Indeed, one has (a.s.) for every $t \in \mathbb{R}^d$,

$$\alpha_{B_H}(t) = H.$$

Recall that the pointwise Hölder exponent of a stochastic process $\{X(t)\}_{t\in\mathbb{R}^d}$ whose trajectories are continuous and nowhere differentiable is the stochastic process $\{\alpha_X(t)\}_{t\in\mathbb{R}^d}$ defined for every t as

$$\alpha_X(t) = \sup\left\{\alpha, \limsup_{h\to 0} \frac{|X(t+h)-X(t)|}{|h|^\alpha} = 0\right\}.$$

It allows to measure the local variations of regularity of $\{X(t)\}_{t\in\mathbb{R}^d}$.

Remark

In general, $\alpha_X(t)$ is a random quantity. However, when $\{X(t)\}_{t\in\mathbb{R}^d}$ is a continuous Gaussian process, this quantity assumes, for each fixed t, an almost sure value. This fact is a simple consequence of the zero-one law (see for instance [6]). All the stochastic processes that will be considered in this article are Gaussian. Their Hölder exponent at any fixed but arbitrary point will therefore be "deterministic".

The fact that the pointwise Hölder exponent of FBM remains the same all along its trajectory restricts its applications in several situations. Let us give an example in the field of image synthesis: FBM has frequently been used for generating artificial mountains [34]. Such a modelling assumes that the irregularity of the mountain is everywhere the same. This is not realistic, since it does not take into account erosion or other meteorological phenomena which smooth some parts of the mountains more than others. Multifractional Brownian Motion (MBM) was introduced, independently in [31] and [10], to overcome these limitations. Roughly speaking, it is obtained by replacing the Hurst parameter H of FBM, by a smooth function $t \mapsto H(t)$. More precisely, MBM can be defined as follows.

Definition 1.1 *(Harmonizable representation of MBM) Let $H(.)\ : \mathbb{R}^d \to [a,b] \subset (0,1)$ be a β-Hölder function (i.e for all t_1, t_2, one has $|H(t_1)-H(t_2)| \le c|t_1-t_2|^\beta$) satisfying the technical assumption*

$$\sup_t H(t) < \beta.$$

The MBM with functional parameter $H(.)$ is the continuous Gaussian process $\{Z(t)\}_{t\in\mathbb{R}^d}$ defined for every $t \in \mathbb{R}^d$ as,

$$Z(t) = Re\Big(\int_{\mathbb{R}^d} \frac{(e^{it.\xi}-1)}{|\xi|^{H(t)+d/2}}\, d\widetilde{W}(\xi)\Big), \tag{1.3}$$

where $d\widetilde{W}$ is the complex-valued stochastic measure introduced in (1.2).

MBM is an extension of FBM at least for the following two reasons.

- When $H(t) = H$ for all t, then MBM reduces to an FBM with parameter H.
- At any point t, MBM is Locally Asymptotically Self-Similar with index $H(t)$ [10], more precisely,

$$\lim_{\rho\to 0+} \text{ law } \left\{\frac{Z(t+\rho u)-Z(t)}{\rho^{H(t)}}\right\}_{u\in\mathbb{R}^d} = \text{ law } \{B_{H(t)}(u)\}_{u\in\mathbb{R}^d},$$

  where $\{B_{H(t)}(u)\}_{u\in\mathbb{R}^d}$ is an FBM with parameter $H(t)$. In fact, this property means that at any point t, there is an FBM with parameter $H(t)$ tangent to the MBM. We refer to the recent works of Falconer [15, 16] for an extensive study of the notion of tangent process.

Similarly to FBM the pointwise Hölder regularity of MBM can be prescribed via its functional parameter. Namely, for every $t \in \mathbb{R}^d$, (a.s.)

$$\alpha_Z(t) = H(t).$$

A problem remains with MBM: because $H(.)$ must be a Hölder function, its Hölder function (i.e. the function $t \mapsto \alpha_Z(t)$) cannot evolve irregularly in time. This is a strong limitation in applications such as turbulence, finance, telecommunications and textured image modelling. Indeed, in such applications, numerical evidences have shown that the pointwise Hölder regularity changes widely from point to point. Note that it is not possible to force discontinuities in the pointwise Hölder exponent of MBM by simply taking a discontinuous $H(.)$: it has been proved by Ayache and Taqqu in [6] that when the function $H(.)$ is discontinuous, then the trajectories of MBM, are themselves, with probability 1, discontinuous. A more refined approach is necessary to obtain a Gaussian process with controlled but very erratic Hölder function.

Daoudi, Jaffard, Lévy Véhel and Meyer have completely described the class of pointwise Hölder functions of continuous functions over an arbitrary compact cube [13, 27]. They have shown that this class is that of all lim inf of sequences of nonnegative continuous functions. Recently, the authors ([4, 5]) have introduced a continuous Gaussian process whose pointwise Hölder exponent can be of the most general form, i.e. any lim inf of a sequence of continuous and nonnegative functions. This process is called the Generalized Multifractional Brownian Motion (GMBM), since it extends both FBM and MBM. Roughly speaking, it is obtained by substituting to the Hurst parameter of FBM a sequence of Lipschitz functions. The Definition of GMBM is more or less inspired from that of the Generalized Weierstrass function [13]. In order to be able to give it, we need first to introduce some notations. We note in passing that another approach for obtaining erratic Hölder functions through a generalization of the mBm is described in [19, 20]. Also, a rather different approach for constructing processes with both strongly varying local regularity and long range dependence, based on the use of pseudo-differential operators, is developped in, e.g., [23, 25, 26].

Let $f_{-1} \in L^1(\mathbb{R})$ be a function such that its Fourier transform $\hat{f}_{-1}$ is C^d and ranges in $[0,1]$. Assume in addition that for every $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$

$$\hat{f}_{-1}(\xi) = \begin{cases} 1 \text{ if for all } i,\ |\xi_i| \leq 1 \\ \\ 0 \text{ if for some } i,\ |\xi| \geq 5/4. \end{cases} \tag{1.4}$$

For all $n \in \mathbb{N}$, we denote by f_n the function of $L^1(\mathbb{R})$, defined by its Fourier transform as follows: For all $\xi \in \mathbb{R}$,

$$\hat{f}_n(\xi) = \hat{f}_{-1}(2^{-n-1}\xi) - \hat{f}_{-1}(2^{-n}\xi). \tag{1.5}$$

Observe that for each $n \in \mathbb{N}$ and all $\xi \in \mathbb{R}^d$

$$\hat{f}_n(\xi) = \hat{f}_0(2^{-n}\xi) \tag{1.6}$$

and

$$\sum_{n=0}^{\infty} \hat{f}_{n-1}(\xi) = 1. \tag{1.7}$$

The functions $\hat{f}_n$ are compactly supported. Moreover,

$$\text{supp } \hat{f}_{-1} \subset D_1 \tag{1.8}$$

and for all $n \in \mathbb{N}$,

$$\text{supp } \hat{f}_n \subset D_{n+2} \setminus D_n, \tag{1.9}$$

where for every $n \in \mathbb{N}$, D_n denotes the compact cube

$$D_n = [-2^n, 2^n]^d. \tag{1.10}$$

Definition 1.2 *Let $[a,b] \subset (0,1)$ be an arbitrary but fixed interval. An* admissible sequence *$(H_n(.))_{n \in \mathbb{N}}$ is a sequence of Lipschitz functions defined on $[0,1]$ and ranging in $[a,b]$ with Lipschitz constants δ_n verifying, for all $n \in \mathbb{N}$,*

$$\delta_n \leq c_1 2^{n\alpha}, \tag{1.11}$$

where $c_1 > 0$ and $\alpha \in (0,a)$ are constants.

Remarks

- We recall that any liminf of a sequence of continuous functions ranging in $[a, b]$ is also a liminf of an admissible sequence [13].
- The problem of constructing an admissible sequence of Lipschitz functions $(H_n(.))_n$ converging to the pointwise Hölder function $H(.)$ has been extensively discussed by the authors in [4]. A general method for obtaining such sequences has been given in the proof of Proposition 1 of [4]. For the sake of concreteness, let us consider here the special case where the pointwise Hölder exponent $H(.)$ takes a finite number of values. Set for instance:

$$H(t) = \sum_{i=1}^{p} c_i \chi_{[d_{i-1}, d_i)}(t) + a,$$

where a and the c_i are positive reals. Then for any n big enough one may simply take for every t,

$$H_n(t) = \begin{cases} c_i + a, \text{ for all } i = 1, \dots, n \text{ and } t \in [d_{i-1}, d_i - \frac{1}{n}] \\ a, \text{ for all } t \in (-\infty, d_0 - \frac{1}{n}] \cup [d_p, +\infty) \\ \text{an affine function otherwise.} \end{cases}$$

We are now in a position to recall the definition of GMBM. For the sake of simplicity, the processes we will consider in the remainder of this article will be defined on $[0, 1]^d$.

Definition 1.3 *Let $(H_n(.))_{n \in \mathbb{N}}$ be an admissible sequence. The* Generalized Multifractional Field *(GMF) with parameter the sequence $(H_n(.))_{n \in \mathbb{N}}$ is the continuous Gaussian field $\{Y(x, y)\}_{(x,y) \in [0,1]^d \times [0,1]^d}$ defined for all (x, y) as*

$$Y(x, y) = Re \left(\int_{\mathbb{R}^d} \left[\sum_{n=0}^{\infty} \frac{(e^{ix\xi} - 1)}{|\xi|^{H_n(y)+1/2}} \hat{f}_{n-1}(\xi) \right] d\widetilde{W}(\xi) \right), \tag{1.12}$$

where $d\widetilde{W}$ is the stochastic measure introduced in (1.2). The Generalized Multifractional Brownian Motion *(GMBM) with parameter the sequence $(H_n(.))_{n \in \mathbb{N}}$ is the continuous Gaussian process $\{X(t)\}_{t \in [0,1]^d}$ defined as the restriction of $\{Y(x, y)\}_{(x,y) \in [0,1]^d \times [0,1]^d}$ to the diagonal: for all $t \in [0, 1]^d$,*

$$X(t) = Y(t, t). \tag{1.13}$$

GMBM is an extension of FBM and MBM at least for the following two reasons.

- When all the Lipschitz functions $H_n(.)$ are equal to the same function $H(.)$ (resp. to the same real H), then Relation (1.7) implies that GMBM reduces to MBM with parameter $H(.)$ (resp. to FBM with parameter H).
- According to Proposition 3 in [4], under some technical conditions on $(H_n(.))_{n \in \mathbb{N}}$, at any point t, the GMBM is Locally Asymptotically Self-Similar with index $H(t) = \liminf_{n \to \infty} H_n(t)$.

One of the main interests of GMBM is that similarly to FBM and MBM its pointwise Hölder exponent can be prescribed via its parameter $(H_n(.))_{n \in \mathbb{N}}$. Namely, for every $t \in \mathbb{R}^d$, (a.s.)

$$\alpha_X(t) = H(t) = \liminf_{n \to \infty} H_n(t). \tag{1.14}$$

Let us now explain the main objective of our work. The rationale behind the definition of the GMBM is that the variations of the pointwise regularity of many natural processes display the two following features:

- They hold some important information, useful for the processing of the data. Typical examples include financial data analysis and medical image modelling. In the former case, points with smaller Hölder exponent correspond to time instants where the risk is larger (see section 4.3 for a regularity analysis of a financial log). In the latter case, smaller exponents are the signature of highly textured regions, or of edge points.
- They are very erratic in time/space. This happens for instance in the case of medical images, such as MR images of the brain or mammographies, where microcalcifications induce strong localized irregularities.

The GMBM is capable to finely model such processes, because one can prescribe its pointwise Hölder function, and this function may be arbitrarily erratic. In order for the GMBM to be useful in the above contexts, however, one needs to be able to estimate $H(t)$. Another application of the GMBM is in multifractal analysis. The so-called multifractal formalism has been introduced because physicists are convinced that one cannot estimate a very erratic pointwise Hölder exponent. Being able to identify $H(t)$ might lead to alternative methods for computing multifractal spectra. These and other applications show that estimating the pointwise Hölder exponent of the GMBM is important both from the theoretical and an applied points of view. Using the method of Generalized Quadratic Variations, we obtain below the following *a priori* unexpected result: *as soon as the pointwise Hölder exponent of GMBM belongs to the first class of Baire (i.e when $H(.)$ is a limit of continuous functions) one may estimate it at any point t almost surely.* Furthermore, under some conditions, a Central Limit Theorem holds for the estimator.

Remarks

- As the pointwise Hölder function of a typical natural signal is erratic, its structure is generally unknown. One therefore needs to employ *a nonparametric procedure* for estimating it.
- Generally speaking, the long range dependence structure of a stochastic process is governed by the "low frequencies" part of its Fourier spectrum, while its Hölder regularity is governed by the "high frequencies" part of this spectrum. In this respect, one of the advantages of GMBM is that, contrarily to FBM, different (functional) parameters, namely the first terms and the tail of the sequence $(H_n(.))_{n\in\mathbb{N}}$, rule the two ends of its Fourier spectrum. Thus, with GMBM, it is possible to have at the same time a very irregular local behavior (i.e. a small value for H) and long range dependence. This is not possible with FBM, which displays long range dependence only for $H > 1/2$. GMBM seems therefore adapted to model processes which display both those features, such as Internet traffic or certain highly textured images with strong global organization, as are e.g. MR images of the brain. Since different parameters rule the low and high frequencies of GMBM, its pointwise Hölder function cannot be identified by the methods of Heyde and Gay [21] or that of Robinson [32]. Indeed, all these methods rely on some properties of the "low frequencies" part of the Fourier spectrum. In view of the remark above, these methods could rather be adapted to compute the long range dependence exponent of GMBM.

A method commonly used in the literature for estimating a Hölder exponent is that of Quadratic Variations [18, 22, 9, 8, 7, 12]. Recall that, if for some integer $N \geq 1$, $\{X(\frac{p}{N});\, p \in \{0,\dots,N-1\}^d\}$ is a discretized trajectory of a process $\{X(t)\}_{t\in[0,1]^d}$, then the corresponding quadratic variations are defined as

$$V_N^{(1)} = \sum_{p\in\{0,\dots,N-1\}^d} \Big(\sum_{\epsilon\in\{0,1\}^d} (-1)^{\epsilon_1+\dots+\epsilon_d} X\Big(\frac{p+\epsilon}{N}\Big)\Big)^2, \tag{1.15}$$

where $p = (p_1,\dots,p_d)$, $\epsilon = (\epsilon_1,\dots,\epsilon_d)$ and $\frac{p+\epsilon}{N} = (\frac{p_1+\epsilon_1}{N},\dots,\frac{p_d+\epsilon_d}{N})$. Observe that the random variables $\sum_{\epsilon\in\{0,1\}^d}(-1)^{\epsilon_1+\dots+\epsilon_d}X(\frac{p+\epsilon}{N})$ are rectangular increments of order 1 of the process $\{X(t)\}_{t\in[0,1]^d}$. Guyon and Léon have noticed that the Quadratic Variations of an FBM with parameter H, satisfy a standard Central Limit Theorem when $H \in (0, 3/4)$ while they fail to satisfy such a Theorem when $H \in (3/4, 1)$ [18]. This is why Istas and Lang have proposed to replace them by the Generalized Quadratic Variations (GQVs) [22]. For the sake of simplicity, we will always suppose that they are of the form

$$V_N^{(2)} = \sum_{p\in\{0,\dots,N-2\}^d} \Big(\sum_{k\in F} d_k X\Big(\frac{p+k}{N}\Big)\Big)^2, \tag{1.16}$$

where $F = \{0,1,2\}^d$ and for all $k = (k_1,\dots,k_d) \in F$,

$$d_k = \prod_{l=1}^{d} e_{k_l}, \tag{1.17}$$

with $e_0 = 1$, $e_1 = -2$ and $e_2 = 1$. Observe that the random variables $\sum_{k\in F} d_k X(\frac{p+k}{N})$ are rectangular increments of order 2 of the process $\{X(t)\}_{t\in[0,1]^d}$. Next, let us fix $t = (t_1,\dots,t_d) \in [0,1]^d$, *the GQVs of* $\{X(t)\}_{t\in[0,1]^d}$ *localized around* t are defined as

$$V_N^{(2)}(t) = \sum_{p\in\nu_N(t)} \Big(\sum_{k\in F} d_k X\Big(\frac{p+k}{N}\Big)\Big)^2, \tag{1.18}$$

where

$$\nu_N(t) = \nu_N^1(t_1) \times \nu_N^2(t_2) \times \ldots \times \nu_N^d(t_d) \tag{1.19}$$

and for all $i = 1, \ldots, d$

$$\nu_N^i(t_i) = \Big\{p_i \in I\!N;\ 0 \le p_i \le N-2 \text{ and } \Big|t_i - \frac{p_i}{N}\Big| \le N^{-\gamma}\Big\}, \tag{1.20}$$

$\gamma \in (0,1)$ being fixed. Heuristically speaking $\nu_N(t)$ can be seen as a neighborhood of the point t. Under the assumption that $H(.)$ is a C^1 function, using the localized GQVs, Benassi Cohen and Istas have identified, when $d = 1$, the Hölder exponent of MBM at any point t [8]. We will also use the localized GQVs for identifying the Hölder exponent of the GMBM. However, there is some difference between our method and that of [8]: we show that, up to a negligible part, the GQVs of GMBM are equal to that of the process with stationary increments $\{Y(s,t)\}_{s\in[0,1]^d}$, where t is fixed (recall that Y is the GMF, see (1.12)). The stationarity of the increments makes these last GQVs easier to study.

Remark

In the special case of FBM, the estimation of the Hurst parameter H, only requires a parametric procedure. The Whittle estimator is therefore the most efficient one. However, an equally efficient estimator may be obtained by the method of the Generalized Quadratic Variations, even if the number of observations is small (this happens when one localizes the Generalized Quadratic Variations), as shown by Coeurjolly in Chapter 2 of his Phd Thesis [12].

At last, let us mention that some results on the identification of a multifractional process with a discontinuous pointwise Hölder exponent have been obtained in [7] and [2]. Both these papers use the method of the Generalized Quadratic Variations. The estimation of the piecewise constant Hölder exponent of the Step Fractional Brownian Motion has been performed in [7]. A model called Generalized Multifractional Gaussian Process, which is similar to GMBM and can be studied with the same methods, has been introduced in [2]. Under some restrictive assuptions, a kind of average of the values of the pointwise Hölder exponent of this model has been identified in [2].

The remainder of our article is organized as follows. In section 2, we will construct two strongly consistent estimators of the pointwise Hölder exponent of GMBM. In section 3, we will show that the Generalized Quadratic Variations of some classes of GMBMs satisfy a Central Limit Theorem. Such a result is important from a statistical point of view since it allows to construct tests. At last, in section 4, we give a method for simulating a GMBM, and we apply our estimation procedure to sampled, synthetic and real, data.

2 Two estimators of pointwise Hölder exponents of GMBMs

First a word about notations. From now on $t = (t_1, \ldots, t_d) \in [0,1]^d$ will be fixed and for every integer $N \ge 2$ $V_N(t)$ will the GQVs of the GMBM localized around t. Observe that

$$V_N(t) = \sum_{p\in\nu_N(t)} \Big(\sum_{k\in F} d_k Y\Big(\frac{p+k}{N}, \frac{p+k}{N}\Big)\Big)^2, \tag{2.1}$$

where $\{Y(x,y)\}_{(x,y)\in[0,1]^d\times[0,1]^d}$ is the GMF that we have introduced in Relation (1.12). The quantity $V_N(t)$ seems to be difficult to handle since the GMBM is with non stationary increments. However, thanks to Lemma 2.3 below, we will show that up to a negligible part, it is equal to $T_N(t)$, where $T_N(t)$ denotes the GQV localized around t of the process with stationary increments $\{Y(s,t)\}_{s\in[0,1]^d}$. Observe that

$$T_N(t) = \sum_{p\in\nu_N(t)} \left(\sum_{k\in F} d_k Y\Big(\frac{p+k}{N}, t\Big)\right)^2. \tag{2.2}$$

At last it is convenient to introduce

$$W_N(t) = \sum_{p\in\nu_N(t)} \left(\sum_{k\in F} d_k \Big(Y\Big(\frac{p+k}{N}, \frac{p+k}{N}\Big) - Y\Big(\frac{p+k}{N}, t\Big)\Big)\right)^2. \tag{2.3}$$

Let us now state the main results of this section.

Theorem 2.1 *Let $\{X(t)\}_{t\in[0,1]^d}$ be a GMBM with parameter an admissible sequence $(H_n(.))_{n\in\mathbb{N}}$ ranging in $[a,b]\subset(0,1-\frac{1}{2d})$. Fix $\gamma\in(b,1-\frac{1}{2d})$ and assume that the sequence of real numbers $(H_n(t))_{n\in\mathbb{N}}$ converges to the real number $H(t)$. Then, almost surely,*

$$\lim_{N\to\infty}\frac{1}{2}\left(d(1-\gamma)-\frac{\log V_N(t)}{\log N}\right)=H(t). \tag{2.4}$$

Theorem 2.2 *Let $\{X(t)\}_{t\in[0,1]^d}$ be a GMBM with parameter an admissible sequence $(H_n(.))_{n\in\mathbb{N}}$ ranging in $[a,b]\subset(0,1)$. Choose δ, γ such that $\delta-\gamma>1/2d$ and $\gamma>\delta b$. Set,*

$$\widetilde{V}_N(t)=\sum_{p\in\widetilde{\nu}_N(t)}\Big(\sum_{k\in F}d_kX\Big(\frac{p+k}{N^\delta}\Big)\Big)^2, \tag{2.5}$$

where

$$\widetilde{\nu}_N(t)=\widetilde{\nu}_N^1(t_1)\times\ldots\times\widetilde{\nu}_N^d(t_d) \tag{2.6}$$

and where for all $i=1,\ldots,d$,

$$\widetilde{\nu}_N^i(t_i)=\Big\{p_i\in\mathbb{N};\ 0\le p_i\le N-2 \text{ and } \Big|t_i-\frac{p_i}{N^\delta}\Big|\le N^{-\gamma}\Big\}. \tag{2.7}$$

Assume that the sequence of real numbers $(H_n(t))_{t\in\mathbb{N}}$ converges to the real number $H(t)$. Then, almost surely,

$$\lim_{N\to\infty}\frac{1}{2\delta}\left(d(1-\gamma)-\frac{\log\widetilde{V}_N(t)}{\log N}\right)=H(t). \tag{2.8}$$

We will only give the proof of Theorem 2.1 since that of Theorem 2.2 is similar. This proof mainly relies on the following four Lemmas. From now on, we set, for all integers $n\in\mathbb{N}$, $h_n=H_n(t)$ and $h=H(t)=\lim_{n\to\infty}H_n(t)$.

Lemma 2.3 *There exists a random variable $\tilde{C}_1>0$ with the following properties:*

- *all the moments of $\tilde{C}_1$ are finite,*
- *almost surely, for all $t\in[0,1]^d$ and for all integer $N\ge2$,*

$$W_N(t)\le\tilde{C}_1N^{d(1-\gamma)-2\gamma}. \tag{2.9}$$

Lemma 2.4 *For all $\epsilon_1>0$, there exist two constants $0<c_2\le c_3$ (depending only on t and ϵ_1) such that, for all integer N large enough,*

$$c_2N^{d(1-\gamma)-2h-2\epsilon_1}\le E(T_N(t))\le c_3N^{d(1-\gamma)-2h+2\epsilon_1}. \tag{2.10}$$

Lemma 2.5 *For all $\epsilon_2>0$, there exists a constant $c_4>0$ (depending only on t and ϵ_2) such that, for all integer $N\ge2$,*

$$Var(T_N(t))\le c_4N^{d(1-\gamma)-4h+4\epsilon_2}. \tag{2.11}$$

Lemma 2.6 *For all $t\in[0,1]^d$, almost surely,*

$$\lim_{N\to\infty}\frac{T_N(t)}{E(T_N(t))}=1. \tag{2.12}$$

To simplify the notations, we set $T_N=T_N(t)$, $V_N=V_N(t)$, $W_N=W_N(t)$ and $\nu_N=\nu_N(t)$.

Lemmas 2.3, 2.4, 2.5 and 2.6 will be proved below.

Proof of Theorem 2.1

¿From (2.1),

$$V_N = \sum_{p\in\nu_N}\left(\sum_{k\in F} d_k Y\left(\frac{p+k}{N},\frac{p+k}{N}\right)\right)^2$$
$$= \sum_{p\in\nu_N}\left(\sum_{k\in F} d_k\left[Y\left(\frac{p+k}{N},\frac{p+k}{N}\right) - Y\left(\frac{p+k}{N},t\right)\right] + \sum_{k\in F} d_k Y\left(\frac{p+k}{N},t\right)\right)^2.$$

(2.2), (2.3) and the triangular inequality in $\mathbb{R}^{\nu_N}$ (equipped with the Euclidean norm) then entail that

$$|T_N^{1/2} - W_N^{1/2}| \leq V_N^{1/2} \leq T_N^{1/2} + W_N^{1/2}. \tag{2.13}$$

Let us now show that, almost surely,

$$\lim_{N\to\infty}\frac{W_N}{T_N} = 0. \tag{2.14}$$

¿From (2.3), (2.9) and (2.10), we know that, almost surely,

$$0 \leq \frac{W_N}{E(T_N)} \leq \frac{\tilde{C}_1 N^{d(1-\gamma)-2\gamma}}{c_2 N^{d(1-\gamma)-2h-2\epsilon_1}} \leq \tilde{C}_5 N^{-2(\gamma-h+\epsilon_1)}. \tag{2.15}$$

Now, since $\gamma > b \geq h$, we find that, almost surely, when $\epsilon_1 > 0$ is small enough,

$$\lim_{N\to\infty}\frac{W_N}{E(T_N)} = 0. \tag{2.16}$$

Writing $\frac{W_N}{T_N} = \frac{E(T_N)}{T_N}\times\frac{W_N}{E(T_N)}$ and using (2.16) and Lemma 2.6, we get (2.14). Besides, it results from (2.13) that

$$\log T_N^{1/2} + \log\left|1 - \frac{W_N^{1/2}}{T_N^{1/2}}\right| \leq \log V_N^{1/2} \leq \log T_N^{1/2} + \log\left|1 + \frac{W_N^{1/2}}{T_N^{1/2}}\right|. \tag{2.17}$$

Note that, for all integer $N \geq 2$, one has almost surely $T_N > 0$: indeed, the random variable $\sum_{k\in F} d_k Y(\frac{k}{N}, t)$ is almost surely non-zero, since it is Gaussian and non-degenerated. Remark also that, from (2.14), the random variable $\log\left|1 - \frac{W_N^{1/2}}{T_N^{1/2}}\right|$ is, for all sufficiently large N, almost surely well-defined.

Using (2.14) and (2.17), we get that, almost surely,

$$\liminf_{N\to\infty}\frac{\log T_N}{\log N} \leq \liminf_{N\to\infty}\frac{\log V_N}{\log N} \leq \limsup_{N\to\infty}\frac{\log V_N}{\log N} \leq \limsup_{N\to\infty}\frac{\log T_N}{\log N}. \tag{2.18}$$

Furthermore, from (2.10), one has, for all $\epsilon_1 > 0$,

$$d(1-\gamma) - 2h - 2\epsilon_1 \leq \liminf_{N\to\infty}\frac{\log E(T_N)}{\log N} \leq \limsup_{N\to\infty}\frac{\log E(T_N)}{\log N} \leq d(1-\gamma) - 2h + 2\epsilon_1.$$

As a consequence,

$$\lim_{N\to\infty}\frac{\log E(T_N)}{\log N} = d(1-\gamma) - 2h. \tag{2.19}$$

Finally, for all integer $N \geq 2$,

$$\frac{\log T_N}{\log N} = \frac{\log(T_N/E(T_N))}{\log N} + \frac{\log E(T_N)}{\log N},$$

(2.12) and (2.19) then entail that, almost surely,

$$\lim_{N\to\infty}\frac{\log T_N}{\log N} = d(1-\gamma) - 2h$$

and (2.18) ensures that almost surely,

$$\lim_{N\to\infty}\frac{\log V_N}{\log N} = d(1-\gamma) - 2h.$$

■

The proof of Lemma 2.3 relies on Lemma 2.7 and on Remark 2.8. Observe that Lemma 2.7 is the natural multidimensional extension of Proposition 1 in [1], this is why we have omited its proof. We have also omited the proof of Remark 2.8, since it is obvious.

Lemma 2.7 *[1] There exists a random variable $\tilde{C}_5 > 0$ with the following property: almost surely, for all $y, y' \in [0,1]^d$,*

$$\sup_{x\in[0,1]^d} |Y(x,y) - Y(x,y')| \le \tilde{C}_5 |y - y'|. \tag{2.20}$$

Furthermore, all the moments of $\tilde{C}_5$ are finite.

Remark 2.8 *There exists two constants $0 < c_6 \le c_7$ such that, for all $N \ge 2$,*

$$c_6 N^{d(1-\gamma)} \le card(\nu_N) \le c_7 N^{d(1-\gamma)}. \tag{2.21}$$

Proof of Lemma 2.3

¿From (2.3) and (2.20)

$$\begin{aligned} W_N &\le \sum_{p\in\nu_N} \left(\sum_{k\in F} |d_k| \left| Y\left(\frac{p+k}{N}, \frac{p+k}{N}\right) - Y\left(\frac{p+k}{N}, t\right) \right| \right)^2 \\ &\le \sum_{p\in\nu_N} \left(\sum_{k\in F} |d_k| \sup_{x\in[0,1]} \left| Y\left(x, \frac{p+k}{N}\right) - Y\Big(x,t\Big) \right| \right)^2, \\ &\le \tilde{C}_5^2 \sum_{p\in\nu_N} \left(\sum_{k\in F} |d_k| \left| \frac{p+k}{N} - t \right| \right)^2 \\ &\le \tilde{C}_5^2 \sum_{p\in\nu_N} \left(\sum_{k\in F} |d_k| \left| \frac{p}{N} - t \right| + \sum_{k\in F} |d_k| \frac{|k|}{N} \right)^2. \end{aligned}$$

Thus, using (1.19), (1.20) and (2.21),

$$\begin{aligned} W_N &\le \tilde{C}_5^2 N^{-2\gamma} \left(3\sqrt{d} \sum_{k\in F} |d_k| \right)^2 c_7 N^{d(1-\gamma)} \\ &\le \tilde{C}_1 N^{d(1-\gamma)-2\gamma}. \end{aligned}$$

■

Lemma 2.6 will result from Lemmas 2.4 and 2.5. The following Remark will be useful in the sequel.

Remark 2.9 *Set, for all $N \ge 2$ and all $p, p' \in \{0, \dots, N-2\}^d$,*

$$I_N(p,p') = E\left(\sum_{k,k'\in F} d_k d_{k'} Y\left(\frac{p+k}{N}, t\right) \overline{Y\left(\frac{p'+k'}{N}, t\right)} \right). \tag{2.22}$$

Then

$$I_N(p,p') = 16^d \int_{\mathbb{R}^d} e^{i(\frac{p-p'}{N}).\xi} g_N(\xi)\, d\xi, \tag{2.23}$$

where

$$g_N(\xi) = \begin{cases} 0 \ \ if\ \xi = 0, \\ \prod_{l=1}^{d} \sin^4(\xi_l/2N) \left(\sum_{n=0}^{\infty} \frac{\hat{f}_{n-1}(\xi)}{|\xi|^{h_n + d/2}} \right)^2 \ \ otherwise. \end{cases} \tag{2.24}$$

Proof of Remark 2.9

For all $p, p' \in \{0, \ldots, N-2\}^d$

$$\theta(\xi) = \sum_{k,k' \in F} d_k d_{k'} \left(e^{i(\frac{p+k}{N}).\xi} - 1\right)\left(e^{-i(\frac{p'+k'}{N}).\xi} - 1\right).$$

Observe that since $\sum_{k\in F} d_k = 0$ one has that

$$\theta(\xi) = \Big(\sum_{k\in F} d_k e^{i(\frac{p+k}{N}).\xi}\Big)\Big(\sum_{k'\in F} d_{k'} e^{-i(\frac{p'+k'}{N}).\xi}\Big).$$

Then using (1.17) one gets

$$\theta(\xi) = e^{i(\frac{p-p'}{N}).\xi}\Big|\sum_{k\in F} d_k e^{ik.\xi/N}\Big|^2 = e^{i(\frac{p-p'}{N}).\xi}\prod_{l=1}^{d}|e^{i\xi_l/N}-1|^4 = 16^d e^{i(\frac{p-p'}{N}).\xi}\prod_{l=1}^{d}\sin^4(\xi_l/2N). \quad (2.25)$$

At last (2.22) and (2.25) entail that

$$\begin{aligned} I_N(p,p') &= \int_{\mathbb{R}^d}\theta(\xi)\left(\sum_{n=0}^{\infty}\frac{\hat f_{n-1}(\xi)}{|\xi|^{h_n+1/2}}\right)^2 d\xi \\ &= 16^d\int_{\mathbb{R}^d} e^{i(\frac{p-p'}{N}).\xi}\prod_{l=1}^{d}\sin^4(\xi_l/2N)\left(\sum_{n=0}^{\infty}\frac{\hat f_{n-1}(\xi)}{|\xi|^{h_n+d/2}}\right)^2 d\xi. \end{aligned}$$

■

Proof of Lemma 2.4

¿From (2.2), (2.21) and (2.22), one gets that, for all integer $N \geq 2$,

$$c_6 N^{d(1-\gamma)} I_N(0,0) \leq E(T_N) \leq c_7 N^{d(1-\gamma)} I_N(0,0). \quad (2.26)$$

Since $h = \lim_{n\to\infty} h_n$, for all $\epsilon > 0$, there exists n_2 such that, for all integer $n \geq n_2+1$,

$$h - \epsilon \leq h_n \leq h - \epsilon. \quad (2.27)$$

The triangular inequality in $L^2(\mathbb{R}^d)$ yields

$$\begin{aligned} &\left(\int_{\mathbb{R}^d}\prod_{l=1}^{d}\sin^4(\xi_l/2N)\left(\sum_{n=n_2+1}^{\infty}\frac{\hat f_{n-1}(\xi)}{|\xi|^{h_n+d/2}}\right)^2 d\xi\right)^{1/2} - \left(\int_{\mathbb{R}^d}\prod_{l=1}^{d}\sin^4(\xi_l/2N)\left(\sum_{n=0}^{n_2}\frac{\hat f_{n-1}(\xi)}{|\xi|^{h_n+d/2}}\right)^2 d\xi\right)^{1/2} \\ &\leq \frac{I_N^{1/2}(0,0)}{4^d} \\ &\leq \left(\int_{\mathbb{R}^d}\prod_{l=1}^{d}\sin^4(\xi_l/2N)\left(\sum_{n=n_2+1}^{\infty}\frac{\hat f_{n-1}(\xi)}{|\xi|^{h_n+d/2}}\right)^2 d\xi\right)^{1/2} + \left(\int_{\mathbb{R}^d}\prod_{l=1}^{d}\sin^4(\xi_l/2N)\left(\sum_{n=0}^{n_2}\frac{\hat f_{n-1}(\xi)}{|\xi|^{h_n+d/2}}\right)^2 d\xi\right)^{1/2}. \end{aligned}$$

Using (2.27) and the inclusions (1.8) and (1.9), we get

$$\begin{aligned} &\left(\int_{\mathbb{R}^d}\frac{\prod_{l=1}^{d}\sin^4(\xi_l/2N)}{|\xi|^{2h+d+2\epsilon}}\left(\sum_{n=n_2+1}^{\infty}\hat f_{n-1}(\xi)\right)^2 d\xi\right)^{1/2} - \frac{N^{-2d}}{4^d}\left(\int_{\mathbb{R}^d}\left(\sum_{n=0}^{n_2}\frac{\hat f_{n-1}(\xi)}{|\xi|^{h_n}}\right)^2|\xi|^{3d}\,d\xi\right)^{1/2} \\ &\leq \frac{I_N^{1/2}(0,0)}{4^d} \qquad (2.28) \\ &\leq \left(\int_{\mathbb{R}^d}\frac{\prod_{l=1}^{d}\sin^4(\xi_l/2N)}{|\xi|^{2h+d-2\epsilon}}\left(\sum_{n=n_2+1}^{\infty}\hat f_{n-1}(\xi)\right)^2 d\xi\right)^{1/2} + \frac{N^{-2d}}{4^d}\left(\int_{\mathbb{R}^d}\left(\sum_{n=0}^{n_2}\frac{\hat f_{n-1}(\xi)}{|\xi|^{h_n}}\right)^2|\xi|^{3d}\,d\xi\right)^{1/2}. \end{aligned}$$

Thus it results from (1.7) and (2.28) that

$$\frac{I_N^{1/2}(0,0)}{4^d} \leq \left(\int_{\mathbb{R}^d}\frac{\prod_{l=1}^{d}\sin^4(\xi_l/2N)}{|\xi|^{2h+d-2\epsilon}}\,d\xi\right)^{1/2} + c_8 N^{-2d}.$$

Then setting for $l = 1, \ldots, d$, $u_l = \xi_l / N$, in the last integral, we obtain that

$$\frac{I_N^{1/2}(0,0)}{4^d} \le c_9 N^{-(h-\epsilon)} + c_8 N^{-2d} \le c_{10} N^{-(h-\epsilon)}. \tag{2.29}$$

Using (2.26), we thus obtain the last inequality in (2.10). Let us now prove the first inequality in (2.10). From (1.7) and the triangular inequality in $L^2(\mathbb{R}^d)$, we have, for all sufficiently large integer N

$$\begin{aligned}
&\left(\int_{\mathbb{R}^d} \frac{\prod_{l=1}^d \sin^4(\xi_l/2N)}{|\xi|^{2h+d+2\epsilon}} \left(\sum_{n=n_2+1}^{\infty} \hat{f}_{n-1}(\xi)\right)^2 d\xi\right)^{1/2} \\
&\ge \left(\int_{\mathbb{R}^d} \frac{\prod_{l=1}^d \sin^4(\xi_l/2N)}{|\xi|^{2h+d+2\epsilon}} d\xi\right)^{1/2} - \left(\int_{\mathbb{R}^d} \frac{\prod_{l=1}^d \sin^4(\xi_l/2N)}{|\xi|^{2h+d+2\epsilon}} \left(\sum_{n=0}^{n_2} \hat{f}_{n-1}(\xi)\right)^2 d\xi\right)^{1/2} \\
&\ge N^{-(h+\epsilon)} \left(\int_{\mathbb{R}^d} \frac{\sin^4(u/2)}{|u|^{2h+d+2\epsilon}} d\xi\right)^{1/2} - \frac{N^{-2d}}{4^d} \left(\int_{\mathbb{R}^d} |\xi|^{3d-2h-2\epsilon} \left(\sum_{n=0}^{n_2} \hat{f}_{n-1}(\xi)\right)^2 d\xi\right)^{1/2} \\
&\ge c_{11} N^{-(h+\epsilon)}.
\end{aligned}$$

This last inequality and (2.28) entail that, for all sufficiently large integer N

$$\frac{I_N^{1/2}(0,0)}{4^d} \ge c_{12} N^{-(h+\epsilon)}. \tag{2.30}$$

Finally, the first inequality of (2.10) results from (2.30) and (2.26). ■

To prove Lemma 2.5 we need some preliminary results.

Lemma 2.10 *Let $h_1, \ldots h_n$ be C^d functions defined over an open set $U \subset \mathbb{R}^d$. For every $\xi = (\xi_1, \ldots, \xi_d) \in U$, we set*

$$g(\xi) = \prod_{l=1}^n h_l(\xi). \tag{2.31}$$

Then for all $1 \le u \le d$ and $\xi \in U$, we have

$$\partial_1 \ldots \partial_u g(\xi) = \sum_{\epsilon_{1,1}+\ldots+\epsilon_{1,n}=1} \sum_{\epsilon_{2,1}+\ldots+\epsilon_{2,n}=1} \cdots \sum_{\epsilon_{u,1}+\ldots+\epsilon_{u,n}=1} \prod_{l=1}^n \partial_1^{\epsilon_{1,l}} \ldots \partial_{u-1}^{\epsilon_{u-1,l}} \partial_u^{\epsilon_{u,l}} h_l(\xi), \tag{2.32}$$

where for all i and l, $\epsilon_{i,l} \in \{0,1\}$ and with the convention that:

- *for any $1 \le m \le d$, $\partial_m^1 = \partial_m$ is the partial derivative with respect of ξ_m and ∂_m^0 is the identity map.*
- *for any $1 \le i \le u$ and any sequence $\{a(\epsilon_{i,1}, \ldots \epsilon_{i,n});\ (\epsilon_{i,1}, \ldots \epsilon_{i,n}) \in \{0,1\}^n\}$, $\sum_{\epsilon_{i,1}+\ldots+\epsilon_{i,n}=1} a(\epsilon_{i,1}, \ldots \epsilon_{i,n})$ denotes the sum of all terms $a(\epsilon_{i,1}, \ldots \epsilon_{i,n})$ such that $\epsilon_{i,1} + \ldots + \epsilon_{i,n} = 1$ (observe that only one $\epsilon_{i,l}$ is equal to 1 and the others are equal to zero). For example, if $n = 3$,*

$$\sum_{\epsilon_{i,1}+\epsilon_{i,2}+\epsilon_{i,3}=1} a(\epsilon_{i,1}, \epsilon_{i,2}, \epsilon_{i,3}) = a(1,0,0) + a(0,1,0) + a(0,0,1).$$

Proof of Lemma 2.10

We will prove this Lemma by induction on u. Let us first suppose that $u = 1$. It is clear that

$$\partial_1 g(\xi) = \sum_{\epsilon_{1,1}+\ldots+\epsilon_{1,n}=1} \prod_{l=1}^n \partial_1^{\epsilon_{1,l}} h_l(\xi).$$

Next let us suppose that for some integer $u \ge 2$, one has

$$\partial_1 \ldots \partial_{u-1} g(\xi) = \sum_{\epsilon_{1,1}+\ldots+\epsilon_{1,n}=1} \cdots \sum_{\epsilon_{u-1,1}+\ldots+\epsilon_{u-1,n}=1} \prod_{l=1}^n \partial_1^{\epsilon_{1,l}} \ldots \partial_u^{\epsilon_{u-1,l}} h_l(\xi).$$

Since

$$\partial_u \Big(\prod_{l=1}^n \partial_1^{\epsilon_{1,l}} \ldots \partial_{u-1}^{\epsilon_{u-1,l}} h_l \Big)(\xi) = \sum_{\epsilon_{u,1}+\ldots+\epsilon_{u,n}=1} \prod_{l=1}^n \partial_1^{\epsilon_{1,l}} \ldots \partial_{u-1}^{\epsilon_{u-1,l}} \partial_u^{\epsilon_{u,l}} h_l(\xi)$$

we obtain (2.32). ■

Remark 2.11 *For every $\xi \in \mathbb{R}^d$ and for every integers $N \geq 2$ and $n \geq 0$, let us set*

$$k_N(\xi) = \prod_{l=1}^{d} \sin^4(\xi_l/2N). \tag{2.33}$$

and

$$\lambda_n(\xi) = \begin{cases} 0 \text{ if } \xi = 0 \\ |\xi|^{-h_n - d/2} = (\xi_1^2 + \ldots + \xi_d^2)^{-h_n/2 - d/4} \text{ otherwise.} \end{cases} \tag{2.34}$$

Then, the function g_N, which has been introduced in (2.24), can be written for every $\xi \in \mathbb{R}^d$ as,

$$g_N(\xi) = k_N(\xi)\Big(\sum_{n=0}^{\infty} \hat{f}_{n-1}(\xi)\lambda_n(\xi)\Big)^2. \tag{2.35}$$

Moreover, for every non vanishing $\xi \in \mathbb{R}^d$ and for all $1 \leq u \leq d$, $|\partial_1 \ldots \partial_u g_N(\xi)|$ is bounded by a sum of terms of the form

$$\begin{aligned} |\partial_1^{\epsilon_{1,1}} \ldots \partial_u^{\epsilon_{u,1}} k_N(\xi)| \quad &\times \quad \Big(\sum_{n=0}^{\infty} |\partial_1^{\epsilon_{1,2}} \ldots \partial_u^{\epsilon_{u,2}} \hat{f}_{n-1}(\xi)||\partial_1^{\epsilon_{1,3}} \ldots \partial_u^{\epsilon_{u,3}} \lambda_n(\xi)|\Big) \\ &\times \quad \Big(\sum_{n'=0}^{\infty} |\partial_1^{\epsilon_{1,4}} \ldots \partial_u^{\epsilon_{u,4}} \hat{f}_{n'-1}(\xi)||\partial_1^{\epsilon_{1,5}} \ldots \partial_u^{\epsilon_{u,5}} \lambda_{n'}(\xi)|\Big), \end{aligned} \tag{2.36}$$

where for all $1 \leq i \leq u$ and $1 \leq l \leq 5$, $\epsilon_{i,l} \in \{0,1\}$ and

$$\sum_{i=1}^{u} \epsilon_{i,l} = 1. \tag{2.37}$$

Proof of Remark 2.11
It follows from (2.35) that for every $\xi \in \mathbb{R}^d$,

$$g_N(\xi) = \sum_{n,n' \in \mathbb{N}} k_N(\xi)\hat{f}_{n-1}(\xi)\lambda_n(\xi)\hat{f}_{n'-1}(\xi)\lambda_{n'}(\xi).$$

Then Lemma 2.10 entails Remark 2.11. ∎

Let us now compute the partial derivatives of the functions k_N, λ_n and $\hat{f}_n$.

Remark 2.12 *For all $\xi \in \mathbb{R}^d$, for $j = 2$ or $j = 4$ and for all $n \in \mathbb{N}$*

$$\partial_1^{\epsilon_{1,1}} \ldots \partial_u^{\epsilon_{u,1}} k_N(\xi) = \Big(\frac{2}{N}\Big)^{\sum_{i=1}^u \epsilon_{i,1}} \prod_{i=1}^{u} sin^{4-\epsilon_{i,1}}(\xi_i/2N) \times \prod_{i=1}^{u} \cos^{\epsilon_{i,1}}(\xi_i/2N) \times \prod_{l=u+1}^{d} \sin^4(\xi_l/2N), \tag{2.38}$$

for $j = 2$ or $j = 4$

$$\partial_1^{\epsilon_{1,j}} \ldots \partial_u^{\epsilon_{u,j}} \hat{f}_n(\xi) = 2^{-n\sum_{i=1}^u \epsilon_{i,j}} \partial_1^{\epsilon_{1,j}} \ldots \partial_u^{\epsilon_{u,j}} \hat{f}_0(2^{-n}\xi) \tag{2.39}$$

and for all non vanishing $\xi \in \mathbb{R}^d$,

$$\partial_1^{\epsilon_{1,j+1}} \ldots \partial_u^{\epsilon_{u,j+1}} \lambda_n(\xi) = \frac{\prod_{l=1}^u (-h_n - d/2 - 2\sum_{i=1}^{l-1} \epsilon_{i,j+1})^{\epsilon_{l,j+1}} \xi_l^{\epsilon_{l,j+1}}}{|\xi|^{h_n + d/2 + 2\sum_{i=1}^u \epsilon_{i,j+1}}}, \tag{2.40}$$

with the convention $\sum_{i=1}^{-1} \epsilon_{i,j+1} = 0$.

Proof of Remark 2.12
Relations (2.38) and (2.39) are obvious. We just have to use Relation (1.6) for obtaining Relation (2.39). Let us prove Relation (2.40) by induction on u. It is clear that for all non vanishing $\xi \in \mathbb{R}^d$ and all $n \in \mathbb{N}$,

$$\partial_1^{\epsilon_{1,j+1}} \lambda_n(\xi) = \frac{(-h_n - d/2)^{\epsilon_{1,j+1}} \xi_1^{\epsilon_{1,j+1}}}{|\xi|^{h_n + d/2 + 2\epsilon_{1,j+1}}}.$$

Let us now suppose that for an arbitrary $u \geq 2$,

$$\partial_1^{\epsilon_{1,j+1}} \ldots \partial_{u-1}^{\epsilon_{u-1,j+1}} \lambda_n(\xi) = \frac{\prod_{l=1}^{u-1} (-h_n - d/2 - 2\sum_{i=1}^{l-1} \epsilon_{i,j+1})^{\epsilon_{l,j+1}} \xi_l^{\epsilon_{l,j+1}}}{|\xi|^{h_n + d/2 + 2\sum_{i=1}^{u-1} \epsilon_{i,j+1}}}.$$

Then since the partial derivative with respect of ξ_u of the function $\xi \mapsto |\xi|^{-h_n - d/2 - 2\sum_{i=1}^{u-1} \epsilon_{i,j+1}}$ is equal to $\xi \mapsto \frac{(-h_n - d/2 - 2\sum_{i=1}^{u-1} \epsilon_{i,j+1})\xi_u}{|\xi|^{h_n + d/2 + 2\sum_{i=1}^{u-1} \epsilon_{i,j+1} + 2}}$, we obtain Relation (2.40). ■

¿From now on, if A is an arbitrary subset of $\mathbb{R}^d$, then χ_A will denote its indicator, namely the function such $\chi_A(\xi) = 1$ if $\xi \in A$ and $\chi_A(\xi) = 0$ else. Recall that for every $n \in \mathbb{N}$, $D_n = [-2^n, 2^n]^d$.

Remark 2.13 *Using the same notations as in Remark 2.11, there is a constant $c > 0$, such that for all $1 \leq u \leq d$, all non vanishing $\xi \in \mathbb{R}^d$ and for $j = 2$ or $j = 4$*

$$\sum_{n=0}^{\infty} |\partial_1^{\epsilon_{1,j}} \ldots \partial_u^{\epsilon_{u,j}} \hat{f}_{n-1}(\xi)||\partial_1^{\epsilon_{1,j+1}} \ldots \partial_u^{\epsilon_{u,j+1}} \lambda_n(\xi)|$$
$$\leq c \frac{\chi_{D_0}(\xi)}{|\xi|^{h_0 + d/2 + \sum_{i=1}^{u} \epsilon_{i,j}}} + c \sum_{n=0}^{\infty} \frac{2^{-n\sum_{i=1}^{u} \epsilon_{i,j}} \chi_{D_{n+1} \setminus D_n}(\xi)}{|\xi|^{\min(h_n, h_{n+1}) + d/2 + \sum_{i=1}^{u} \epsilon_{i,j+1}}}. \quad (2.41)$$

For any integer $p \geq 1$,

$$\sum_{n=p}^{\infty} |\partial_1^{\epsilon_{1,j}} \ldots \partial_u^{\epsilon_{u,j}} \hat{f}_{n-1}(\xi)||\partial_1^{\epsilon_{1,j+1}} \ldots \partial_u^{\epsilon_{u,j+1}} \lambda_n(\xi)| \leq c \sum_{n=p}^{\infty} \frac{2^{-n\sum_{i=1}^{u} \epsilon_{i,j}} \chi_{D_{n+1} \setminus D_n}(\xi)}{|\xi|^{\min(h_n, h_{n+1}) + d/2 + \sum_{i=1}^{u} \epsilon_{i,j+1}}}, \quad (2.42)$$

$$\Big(\sum_{n=0}^{\infty} |\partial_1^{\epsilon_{1,2}} \ldots \partial_u^{\epsilon_{u,2}} \hat{f}_{n-1}(\xi)||\partial_1^{\epsilon_{1,3}} \ldots \partial_u^{\epsilon_{u,3}} \lambda_n(\xi)|\Big)\Big(\sum_{n'=0}^{\infty} |\partial_1^{\epsilon_{1,4}} \ldots \partial_u^{\epsilon_{u,4}} \hat{f}_{n'-1}(\xi)||\partial_1^{\epsilon_{1,5}} \ldots \partial_u^{\epsilon_{u,5}} \lambda_{n'}(\xi)|\Big)$$
$$\leq c \frac{\chi_{D_0}(\xi)}{|\xi|^{2h_0 + d + \sum_{i=1}^{u} (\epsilon_{i,3} + \epsilon_{i,5})}} + c \sum_{n=0}^{\infty} \frac{2^{-n\sum_{i=1}^{u} (\epsilon_{i,2} + \epsilon_{i,4})} \chi_{D_{n+1} \setminus D_n}(\xi)}{|\xi|^{2\min(h_n, h_{n+1}) + d + \sum_{i=1}^{u} (\epsilon_{i,3} + \epsilon_{i,5})}}. \quad (2.43)$$

For any integer $p \geq 1$,

$$\Big(\sum_{n=p}^{\infty} |\partial_1^{\epsilon_{1,2}} \ldots \partial_u^{\epsilon_{u,2}} \hat{f}_{n-1}(\xi)||\partial_1^{\epsilon_{1,3}} \ldots \partial_u^{\epsilon_{u,3}} \lambda_n(\xi)|\Big)\Big(\sum_{n'=p}^{\infty} |\partial_1^{\epsilon_{1,4}} \ldots \partial_u^{\epsilon_{u,4}} \hat{f}_{n'-1}(\xi)||\partial_1^{\epsilon_{1,5}} \ldots \partial_u^{\epsilon_{u,5}} \lambda_{n'}(\xi)|\Big)$$
$$\leq c \sum_{n=p}^{\infty} \frac{2^{-n\sum_{i=1}^{u} (\epsilon_{i,2} + \epsilon_{i,4})} \chi_{D_{n+1} \setminus D_n}(\xi)}{|\xi|^{2\min(h_n, h_{n+1}) + d + \sum_{i=1}^{u} (\epsilon_{i,3} + \epsilon_{i,5})}}. \quad (2.44)$$

Proof of Remark 2.13

Let us first prove Relation (2.41). Observe that as the sequence $(h_n)_{n \in \mathbb{N}}$ is bounded, there is a constant $c_1 > 0$, such that for all $n \in \mathbb{N}$, $1 \leq u \leq d$ and $\epsilon_{l,k} \in \{0, 1\}$

$$\Big|\prod_{l=1}^{u} \Big(-h_n - d/2 - 2\sum_{i=1}^{l-1} \epsilon_{i,j+1}\Big)^{\epsilon_{l,j+1}}\Big| \leq c_1 \quad (2.45)$$

In addition one has that for all non vanishing $\xi \in \mathbb{R}^d$,

$$\frac{\prod_{l=1}^{u} |\xi_l|^{\epsilon_{l,j+1}}}{|\xi|^{h_n + d/2 + 2\sum_{i=1}^{u} \epsilon_{i,j+1}}} \leq \frac{\prod_{l=1}^{u} |\xi|^{\epsilon_{l,j+1}}}{|\xi|^{h_n + d/2 + 2\sum_{i=1}^{u} \epsilon_{i,j+1}}} \leq |\xi|^{-h_n - d/2 - \sum_{i=1}^{u} \epsilon_{i,j+1}}. \quad (2.46)$$

Using Relations (2.40), (2.45) and (2.46), one obtains that for every $n \in \mathbb{N}$, $1 \leq u \leq d$ and non vanishing $\xi \in \mathbb{R}^d$,

$$|\partial_1^{\epsilon_{1,j+1}} \ldots \partial_u^{\epsilon_{u,j+1}} \lambda_n(\xi)| \leq c_1 |\xi|^{-h_n - d/2 - \sum_{i=1}^{u} \epsilon_{i,j+1}}. \quad (2.47)$$

Next, since for all $1 \leq u \leq d$, $\partial_1^{\epsilon_{1,j}} \ldots \partial_u^{\epsilon_{u,j}} \hat{f}_{-1}$ and $\partial_1^{\epsilon_{1,j}} \ldots \partial_u^{\epsilon_{u,j}} \hat{f}_0$ are continuous functions, with support, respectively, in the domains D_1 and $D_2 \setminus D_0$, there is a constant $c_2 > 0$ such that for all $\xi \in \mathbb{R}^d$,

$$|\partial_1^{\epsilon_{1,j}} \ldots \partial_u^{\epsilon_{u,j}} \hat{f}_{-1}(\xi)| \leq c_2 \chi_{D_1}(\xi). \tag{2.48}$$

and for every $n \in \mathbb{N}$,

$$\begin{aligned} |\partial_1^{\epsilon_{1,j}} \ldots \partial_u^{\epsilon_{u,j}} \hat{f}_n(\xi)| &\leq 2^{-n \sum_{i=1}^u \epsilon_{i,j}} |\partial_1^{\epsilon_{1,j}} \ldots \partial_u^{\epsilon_{u,j}} \hat{f}_0(2^{-n}\xi) \\ &\leq c_2 2^{-n \sum_{i=1}^u \epsilon_{i,j}} \chi_{D_{n+2} \setminus D_n}(\xi). \end{aligned} \tag{2.49}$$

Next it follows from Relations (2.47), (2.48) and (2.49) that there is a constant $c_4 > 0$, such that for all $\xi \in \mathbb{R}^d$ and for $j = 2$ or $j = 4$,

$$\begin{aligned} &\sum_{n=0}^{\infty} |\partial_1^{\epsilon_{1,j}} \ldots \partial_u^{\epsilon_{u,j}} \hat{f}_{n-1}(\xi)||\partial_1^{\epsilon_{1,j+1}} \ldots \partial_u^{\epsilon_{u,j+1}} \lambda_n(\xi)| \\ \leq\; & c_4 \frac{\chi_{D_1}(\xi)}{|\xi|^{h_0 + d/2 + \sum_{i=1}^u \epsilon_{i,j+1}}} + c_4 \sum_{n=0}^{\infty} \frac{2^{-n \sum_{i=1}^u \epsilon_{i,j}} \chi_{D_{n+2} \setminus D_n}(\xi)}{|\xi|^{h_{n+1} + d/2 + \sum_{i=1}^u \epsilon_{i,j+1}}} \\ \leq\; & c_4 \frac{\chi_{D_0}(\xi)}{|\xi|^{h_0 + d/2 + \sum_{i=1}^u \epsilon_{i,j+1}}} + c_4 \frac{\chi_{D_1 \setminus D_0}(\xi)}{|\xi|^{h_0 + d/2 + \sum_{i=1}^u \epsilon_{i,j+1}}} \\ & + c_4 \sum_{n=0}^{\infty} \frac{2^{-n \sum_{i=1}^u \epsilon_{i,j}} \chi_{D_{n+1} \setminus D_n}(\xi)}{|\xi|^{h_{n+1} + d/2 + \sum_{i=1}^u \epsilon_{i,j+1}}} + c_4 \sum_{n=0}^{\infty} \frac{2^{-n \sum_{i=1}^u \epsilon_{i,j}} \chi_{D_{n+2} \setminus D_{n+1}}(\xi)}{|\xi|^{h_{n+1} + d/2 + \sum_{i=1}^u \epsilon_{i,j+1}}} \\ =\; & c_4 \frac{\chi_{D_0}(\xi)}{|\xi|^{h_0 + d/2 + \sum_{i=1}^u \epsilon_{i,j+1}}} + c_4 \frac{\chi_{D_1 \setminus D_0}(\xi)}{|\xi|^{h_0 + d/2 + \sum_{i=1}^u \epsilon_{i,j+1}}} \\ & + c_4 \sum_{n=0}^{\infty} \frac{2^{-n \sum_{i=1}^u \epsilon_{i,j}} \chi_{D_{n+1} \setminus D_n}(\xi)}{|\xi|^{h_{n+1} + d/2 + \sum_{i=1}^u \epsilon_{i,j+1}}} + c_4 \sum_{n=1}^{\infty} \frac{2^{-(n-1) \sum_{i=1}^u \epsilon_{i,j}} \chi_{D_{n+1} \setminus D_n}(\xi)}{|\xi|^{h_n + d/2 + \sum_{i=1}^u \epsilon_{i,j+1}}} \\ \leq\; & c \frac{\chi_{D_0}(\xi)}{|\xi|^{h_0 + d/2 + \sum_{i=1}^u \epsilon_{i,j+1}}} + c \sum_{n=0}^{\infty} \frac{2^{-n \sum_{i=1}^u \epsilon_{i,j}} \chi_{D_{n+1} \setminus D_n}(\xi)}{|\xi|^{\min(h_n, h_{n+1}) + d/2 + \sum_{i=1}^u \epsilon_{i,j+1}}}. \end{aligned}$$

Similarly, one can show that Relation (2.42) holds. At last, Relations (2.43) and (2.44) are straightforward consequences of Relations (2.41) and (2.42) since all the sets D_0 and $D_{n+1} \setminus D_n$, $n \in \mathbb{N}$ are disjoint. ∎

Lemma 2.14 *For all integer $N \geq 2$ and for any $p = (p_1, \ldots, p_d) \in \{0, \ldots, N-2\}^d$ and $p' = (p'_1, \ldots, p'_d) \in \{0, \ldots, N-2\}^d$, with $p \neq p'$, by reordering the p_i and the p'_i, one may suppose that there exists $1 \leq m \leq d$, such that for all $i = 1, \ldots, m$, $p_i \neq p'_i$ and for all $i = m+1, \ldots, d$, $p_i = p'_i$. Then, one has that for all $1 \leq u \leq m$,*

$$|I_N(p,p')| = \frac{16^d N^u}{\prod_{l=1}^u |p_l - p'_l|} \Big| \int_{\mathbb{R}^d} e^{i(p-p').\xi/N} \partial_1 \ldots \partial_u g_N(\xi)\, d\xi \Big|. \tag{2.50}$$

Proof of Lemma 2.14

We will prove this Lemma by induction on u. It follows (2.23) and Fubini Theorem that

$$\begin{aligned} \frac{I_N(p,p')}{16^d} &= \int_{\mathbb{R}^d} e^{i \sum_{k=1}^d (p_k - p'_k)\xi_k/N} g_N(\xi_1, \ldots, \xi_d)\, d\xi_1 \ldots d\xi_d \\ &= \int_{\mathbb{R}^{d-1}} e^{i \sum_{k=2}^d (p_k - p'_k)\xi_k/N} \Big(\int_{\mathbb{R}} e^{i(p_1 - p'_1)\xi_1/N} g_N(\xi_1, \xi_2, \ldots, \xi_d)\, d\xi_1 \Big) d\xi_2 \ldots d\xi_d. \end{aligned} \tag{2.51}$$

Then by integrating by parts and using Remarks 2.11 and 2.13 one obtains that for almost all $(\xi_2, \ldots \xi_d) \in \mathbb{R}^{d-1}$,

$$\int_{\mathbb{R}} e^{i(p_1 - p'_1)\xi_1/N} g_N(\xi_1, \xi_2, \ldots, \xi_d)\, d\xi_1 = -\frac{N}{i(p_1 - p'_1)} \int_{\mathbb{R}} e^{i(p_1 - p'_1)\xi_1/N} \partial_1 g_N(\xi_1, \xi_2, \ldots, \xi_d)\, d\xi_1. \tag{2.52}$$

Relations (2.51) and (2.52) imply that Lemma 2.14 holds when $u = 1$. Next suppose that this Lemma holds for some $u \leq m-1$. Then one may show that it also holds for $u+1$, by replacing in Relations (2.51) and (2.52) g_N by $\partial_1 \ldots \partial_u g_N$ and ∂_1 by ∂_{u+1}. ∎

Lemma 2.15 *For any arbitrarily small $\epsilon > 0$, there is a constant $c > 0$ such that for all integers $N \geq 2$ and $1 \leq m \leq d$,*

$$\int_{\mathbb{R}^d} |\partial_1 \dots \partial_m g_N(\xi)|\, d\xi \leq c N^{-m-2(h-\epsilon)}. \tag{2.53}$$

Proof of Lemma 2.15

First, let us notice that as $\lim_{n\to\infty} h_n = h$, there is $n_2 \in \mathbb{N}$, such that for all $n \geq n_2 + 1$,

$$h - \epsilon \leq h_n \leq h + \epsilon. \tag{2.54}$$

Now let us introduce some notations. For $j = 2$ or $j = 4$ and for $K = n_2$ or $K = \infty$ and for all non vanishing $\xi \in \mathbb{R}^d$, let us set

$$I_{j,K}(\xi) = \sum_{n=0}^{K} |\partial_1^{\epsilon_{1,j}} \dots \partial_m^{\epsilon_{m,j}} \hat{f}_{n-1}(\xi)||\partial_1^{\epsilon_{1,j+1}} \dots \partial_m^{\epsilon_{m,j+1}} \lambda_n(\xi)|, \tag{2.55}$$

and

$$L_j(\xi) = \sum_{n=n_2+1}^{\infty} |\partial_1^{\epsilon_{1,j}} \dots \partial_m^{\epsilon_{m,j}} \hat{f}_{n-1}(\xi)||\partial_1^{\epsilon_{1,j+1}} \dots \partial_m^{\epsilon_{m,j+1}} \lambda_n(\xi)|. \tag{2.56}$$

Observe that one has

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\partial_1 \dots \partial_m g_N(\xi)|\, d\xi \\
\leq\ & \int_{\mathbb{R}^d} |\partial_1^{\epsilon_{1,1}} \dots \partial_m^{\epsilon_{m,1}} k_N(\xi)| I_{2,n_2}(\xi) I_{4,n_2}(\xi)\, d\xi + \int_{\mathbb{R}^d} |\partial_1^{\epsilon_{1,1}} \dots \partial_m^{\epsilon_{m,1}} k_N(\xi)| I_{2,n_2}(\xi) I_{4,\infty}(\xi)\, d\xi \\
&+ \int_{\mathbb{R}^d} |\partial_1^{\epsilon_{1,1}} \dots \partial_m^{\epsilon_{m,1}} k_N(\xi)| I_{2,\infty}(\xi) I_{4,n_2}(\xi)\, d\xi + \int_{\mathbb{R}^d} |\partial_1^{\epsilon_{1,1}} \dots \partial_m^{\epsilon_{m,1}} k_N(\xi)| L_2(\xi) L_4(\xi)\, d\xi.
\end{aligned} \tag{2.57}$$

¿From now on our aim will be to bound each of these integrals. To simplify our notations, let us set for all $l = 1, \dots, 5$

$$\beta_l = \sum_{i=1}^{m} \epsilon_{i,l}. \tag{2.58}$$

Observe that since for $1 \leq i \leq m$,

$$\sum_{l=1}^{5} \epsilon_{i,l} = 1, \tag{2.59}$$

Clearly one has that

$$\beta_1 + \beta_3 + \beta_5 \leq \sum_{l=1}^{5} \beta_l = m \leq d. \tag{2.60}$$

Next Relations (2.48), (2.49), (2.40) and (2.41) imply that there is a constant $c > 0$, such that for all $1 \leq u \leq d$ and all non vanishing $\xi \in \mathbb{R}^d$, one has

$$I_{j,n_2}(\xi) \leq c|\xi|^{-d/2-\beta_{j+1}} \max(|\xi|^{-a}, |\xi|^{-b}) \chi_{D_{n_2+1}}(\xi) \text{ and } I_{j,\infty} \leq c|\xi|^{-d/2-\beta_{j+1}} \max(|\xi|^{-a}, |\xi|^{-b}). \tag{2.61}$$

Thus, when $K_2 = n_2$ or $K_4 = n_2$, using Relations (2.38), (2.60) and (2.61) one obtains that

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\partial_1^{\epsilon_{1,1}} \dots \partial_m^{\epsilon_{m,1}} k_N(\xi)| I_{2,K_2}(\xi) I_{4,K_4}(\xi)\, d\xi \\
&\leq c_1 \Big(\frac{2}{N}\Big)^{\beta_1} \int_{D_{n_2+1}} \prod_{i=1}^{m} |\sin^{4-\epsilon_{i,1}}(\xi_i/2N)| \times \prod_{i=m+1}^{d} \sin^4(\xi_i/2N) \times |\xi|^{-d-\beta_3-\beta_5} \times \max(|\xi|^{-2a}, |\xi|^{-2b})\, d\xi \\
&\leq c_2 N^{-4d} \int_{D_{n_2+1}} |\xi|^{3d-\beta_1-\beta_3-\beta_5} \max(|\xi|^{-2a}, |\xi|^{-2b})\, d\xi \\
&\leq c_3 N^{-4d} \int_{D_{n_2+1}} |\xi|^{2d} \max(|\xi|^{-2a}, |\xi|^{-2b})\, d\xi \\
&\leq c_4 N^{-4d},
\end{aligned} \tag{2.62}$$

where the constant $c_4 > 0$ is finite. Next, let us give an upper bound of $\rho = \int_{\mathbb{R}^d} |\partial_1^{\epsilon_{1,1}} \dots \partial_m^{\epsilon_{m,1}} k_n(\xi)| L_2(\xi) L_4(\xi)\, d\xi$. Using Relation (2.44) with $p = n_2 + 1$ and $u = m$, one obtains that for all non vanishing $\xi \in \mathbb{R}^d$,

$$L_2(\xi) L_4(\xi) \le c_5 \sum_{n=n_2+1}^{\infty} \frac{2^{-n(\beta_2+\beta_4)} \chi_{D_{n+1}\setminus D_n}(\xi)}{|\xi|^{2\min(h_n, h_{n+1})+d+\beta_3+\beta_5}}.$$

Then Relations (2.38) and (2.54) imply that

$$\rho \le c_{10} N^{-\beta_1} \sum_{n=1}^{\infty} 2^{-n(\beta_2+\beta_4)} \int_{D_n \setminus D_{n-1}} \frac{\prod_{i=1}^m |\sin^{4-\epsilon_{i,1}}(\xi_i/2N)| \prod_{i=m+1}^d \sin^4(\xi_i/2N)}{|\xi|^{2h-2\epsilon+d+\beta_3+\beta_5}}\, d\xi. \tag{2.63}$$

Next let $n_0 \ge 1$, be the integer such that

$$2^{-n_0-1} < 1/N \le 2^{-n_0}, \tag{2.64}$$

let

$$\rho_1 = \sum_{n=1}^{n_0} 2^{-n(\beta_2+\beta_4)} \int_{D_n \setminus D_{n-1}} \frac{\prod_{i=1}^m |\sin^{4-\epsilon_{i,1}}(\xi_i/2N)| \prod_{i=m+1}^d \sin^4(\xi_i/2N)}{|\xi|^{2h-2\epsilon+d+\beta_3+\beta_5}}\, d\xi \tag{2.65}$$

and let

$$\rho_2 = \sum_{n=n_0+1}^{\infty} 2^{-n(\beta_2+\beta_4)} \int_{D_n \setminus D_{n-1}} \frac{\prod_{i=1}^m |\sin^{4-\epsilon_{i,1}}(\xi_i/2N)| \prod_{i=m+1}^d \sin^4(\xi_i/2N)}{|\xi|^{2h-2\epsilon+d+\beta_3+\beta_5}}\, d\xi. \tag{2.66}$$

First we will give an upper bound of ρ_2. Relation (2.64) implies that

$$\rho_2 \le N^{-\beta_2-\beta_4} \int_{\mathbb{R}^d} \frac{\prod_{i=1}^m |\sin^{4-\epsilon_{i,1}}(\xi_i/2N)| \prod_{i=m+1}^d \sin^4(\xi_i/2N)}{|\xi|^{2h-2\epsilon+d+\beta_3+\beta_5}}\, d\xi. \tag{2.67}$$

Next setting in this last integral, for all $i = 1, \dots, d$, $v_i = \xi_i/N$, one obtains that

$$\rho_2 \le N^{-2(h-\epsilon)-\sum_{l=2}^5 \beta_l} \int_{\mathbb{R}^d} \frac{\prod_{i=1}^m |\sin^{4-\epsilon_{i,1}}(v_i/2)| \prod_{i=m+1}^d \sin^4(v_i/2)}{|v|^{2h-2\epsilon+d+\beta_3+\beta_5}}\, dv.$$

Next, using the inequality $|\sin(v_i/2)| \le |v|$ and Relation (2.60) one gets that

$$\rho_2 \le N^{-2(h-\epsilon)-\sum_{l=2}^5 \beta_l} \Big(\int_{|v|\le 1} |v|^{2d-2h+2\epsilon}\, dv + \int_{|v|>1} |v|^{-2h+2\epsilon-d-\beta_3-\beta_5}\, dv \Big). \tag{2.68}$$

Next let us give an upper bound of ρ_1. Using the inequality $|\sin(\xi_i/2N)| \le N^{-1}|\xi|$ and Relations (2.58), (2.60) and (2.64), one obtains the following inequalities, where $\lambda_d(D_n \setminus D_{n-1})$ denotes the Lebesgue measure of the set $D_n \setminus D_{n-1}$.

$$\begin{aligned} \rho_1 &\le N^{-4d+\beta_1} \sum_{n=1}^{n_0} 2^{-n(\beta_2+\beta_4)} \int_{D_n \setminus D_{n-1}} |\xi|^{3d-2(h-\epsilon)-\beta_1-\beta_3-\beta_5}\, d\xi \\ &\le cN^{-4d+\beta_1} \sum_{n=1}^{n_0} 2^{n(3d-2h+2\epsilon-\sum_{l=1}^5 \beta_l)} \lambda(D_n \setminus D_{n-1}) \\ &\le c' N^{-4d+\beta_1} \sum_{n=1}^{n_0} 2^{n(4d-2h+2\epsilon-m)} \le c'' N^{-4d+\beta_1} 2^{n_0(4d-2h+2\epsilon-m)} \le c'' N^{-m+\beta_1-2(h-\epsilon)}. \end{aligned} \tag{2.69}$$

Next Relations (2.63), (2.65), (2.66), (2.68) and (2.69) entail that

$$\rho \le cN^{-m-2(h-\epsilon)} \tag{2.70}$$

At last, it follows from Relations (2.57), (2.62) and (2.70) that there is a constant $c > 0$ such that for all $1 \le m \le d$ and $N \ge 2$,

$$\int_{\mathbb{R}^d} |\partial_1 \dots \partial_m g_N(\xi)|\, d\xi \le cN^{-m-2(h-\epsilon)}.$$

■

Lemma 2.16 *For any arbitrarily small $\epsilon > 0$, there is a constant $c > 0$ such that for all integer $N \geq 2$, all $p = (p_1, \dots, p_d) \in \{0, \dots, N-2\}^d$ and $p' = (p'_1, \dots, p'_d) \in \{0, \dots, N-2\}^d$, one has*

$$|I_N(p,p')| \leq cN^{-2h+2\epsilon}[\prod_{l=1}^{d}(1+|p_l - p'_l|)]^{-1}. \tag{2.71}$$

Proof of Lemma 2.16

Using Relation (2.29) one can easily show that Lemma 2.16 holds when $p = p'$. Next, observe that there is a constant $c' > 0$ such that for all $p_l \in \mathbb{Z}$ and $p'_l \in \mathbb{Z}$ satisfying $p_l \neq p'_l$, one has

$$|p_l - p'_l| \leq c'(1 + |p_l - p'_l|)^{-1}. \tag{2.72}$$

At last, Relation (2.72) and Lemmas 2.14 and 2.15 imply that Lemma 2.16 holds when $p \neq p'$. ■

We are now in a position to prove Lemma 2.5.

Proof of Lemma 2.5

Since the process $\{Y(s,t)\}_{s\in[0,1]^d}$ is Gaussian and centered, one has for all integer $N \geq 2$,

$$Var(T_N) = 2 \sum_{p,p' \in \nu_N} (I_N(p,p'))^2.$$

Then using Lemma 2.16, it follows that for any arbitrarily small $\epsilon > 0$, there is a constant $c' > 0$ such that

$$Var(T_N) \leq c' N^{-4h+4\epsilon} \sum_{p\in\nu_N} \sum_{p'\in\mathbb{Z}^d} \prod_{l=1}^{d}(1+|p_l - p'_l|)^{-2} \leq c'' card(\nu_N) N^{-4h+4\epsilon}, \tag{2.73}$$

where $c'' = c'\left(\sum_{q\in\mathbb{Z}}(1+|q|)^{-2}\right)^d$. At last Relation (2.73) and Remark 2.8 entail Lemma 2.5. ■

To prove Lemma 2.6, we shall use the following remark.

Remark 2.17 *There exists a constant $c > 0$ such that for all integer $N \geq 2$,*

$$E[(T_N - E(T_N))^4] \leq cVar^2(T_N). \tag{2.74}$$

This property results from the Gaussianity of the process $\{Y(s,t)\}_{s\in[0,1]}$. For a proof, see for instance [9] pages 42 and 43.

Proof of Lemma 2.6

We shall apply the Borel-Cantelli Lemma. One has, for all $\eta > 0$ and all integer $N \geq 2$

$$\begin{aligned} P\left(\left|\frac{T_N}{E(T_N)} - 1\right| \geq \eta\right) &= P\left(\left|\frac{T_N}{E(T_N)} - 1\right|^4 \geq \eta^4\right) \\ &= P(|T_N - E(T_N)|^4 \geq \eta^4 E^4(T_N)). \end{aligned}$$

Markov inequality and Remark 2.17 entail that, for all integer $N \geq 2$

$$\begin{aligned} P\left(\left|\frac{T_N}{E(T_N)} - 1\right| \geq \eta\right) &\leq \frac{E(|T_N - E(T_N)|^4)}{\eta^4 E^4(T_N)} \\ &\leq c\frac{Var^2(T_N)}{\eta^4 E^4(T_N)}. \end{aligned}$$

Applying Lemmas 2.4 and 2.5 with $\epsilon_1 = \epsilon_2 = \epsilon$, where $\epsilon > 0$ is arbitrarily small, one gets

$$P\left(\left|\frac{T_N}{E(T_N)} - 1\right| \geq \eta\right) \leq c'\frac{N^{2d(1-\gamma)-8h+8\epsilon}}{N^{4d(1-\gamma)-8h-8\epsilon}} = c'N^{-(2d(1-\gamma)-16\epsilon)}. \tag{2.75}$$

Since $\gamma \in (0, 1 - \frac{1}{2d})$, one has, for $\epsilon > 0$ sufficiently small, $2d(1-\gamma) - 16\epsilon > 1$. Thus

$$\sum_{N=2}^{\infty} P\left(\left|\frac{T_N}{E(T_N)} - 1\right| \geq \eta\right) < \infty.$$

■

3 A Central Limit Theorem for the Generalized Quadratic Variations of some class of GMBMs

Central Limit Theorems are quite useful since they allow to construct tests. In this section, our goal will be to prove the following Central Limit Theorem for the GQVs of GMBMs. For the sake of simplicity, we have restricted to GMBMs defined over $[0,1]$, but our results can be extended to GMBMs defined over an arbitrary compact cube of $\mathbb{R}^d$.

Theorem 3.1 *Let $\{X(s)\}_{s\in[0,1]}$ be a GMBM with parameter the admissible sequence $(H_n(.))_{n\in\mathbb{N}}$. Let $t\in[0,1]$ be a point satisfying the following property:*

$(\mathcal{P})$ there is $n_1 = n_1(t)\in\mathbb{N}$ such that for all $n\geq n_1$, $H_n(t)=H(t)$, with $H(t)\in(0,1/8)$.

As usual, we denote $V_N(t)$ with $N\geq 2$ the GQVs of $\{X(s)\}_{s\in[0,1]}$ localized around t. We assume that $\gamma\in(\frac{1+4H(t)}{3},\frac{1}{2})$ and we set

$$\mu_N(t) = E(V_N(t)) \text{ and } S_N(t)=\sqrt{Var(V_N(t))}. \tag{3.1}$$

Then, one has that

$$V_N(t)=\mu_N(t)+S_N(t)\epsilon_N(t),$$

where the random variable $\epsilon_N(t)$ converges in distribution to a $\mathcal{N}(0,1)$ Gaussian variable as $N\to\infty$.

Let us first introduce some notations:

- As previously, we set $h=H(t)$ and for all $n\in\mathbb{N}$, $h_n=H_n(t)$.
- $\{B_h(s)\}_{s\in[0,1]}$ denotes an FBM with Hurst parameter h.
- For every integer $N\geq 2$ and for every $p\in\{0,\ldots,N-2\}$, $\Delta_N X(p)$, $\Delta_{t,N}Y(p)$ and $\Delta_N B_h(p)$ are respectively the increments of order 2 of the processes $\{X(s)\}_{s\in[0,1]}$, $\{Y(s,t)\}_{s\in[0,1]}$ and $\{B_h(s)\}_{s\in[0,1]}$ defined as

$$\Delta_N X(p)=\sum_{k=0}^{2} d_k X\Big(\frac{p+k}{N}\Big)=\sum_{k=0}^{2} Y\Big(\frac{p+k}{N},\frac{p+k}{N}\Big), \tag{3.2}$$

$$\Delta_{t,N} Y(p)=\sum_{k=0}^{2} d_k Y\Big(\frac{p+k}{N},t\Big), \tag{3.3}$$

and

$$\Delta_N B_h(p)=\sum_{k=0}^{2} d_k B_h\Big(\frac{p+k}{N}\Big). \tag{3.4}$$

Recall that $d_0=1$, $d_1=-2$ and $d_2=1$.

- $(J_N(p,p'))_{p,p'\in\nu_N(t)}$, $(I_N(p,p'))_{p,p'\in\nu_N(t)}$, $(K_N(p,p'))_{p,p'\in\nu_N(t)}$ are respectively the covariance matrices of the centered Gaussian vectors $(\Delta_N X(p))_{p\in\nu_N(t)}$, $(\Delta_{t,N}Y(p))_{p\in\nu_N(t)}$ and $(\Delta_N B_h(p))_{p\in\nu_N(t)}$. Thus one has for every $p,p'\in\nu_N(t)$,

$$J_N(p,p')=E(\Delta_N X(p)\Delta_N X(p')), \tag{3.5}$$

$$I_N(p,p')=E(\Delta_{t,N} Y(p)\Delta_{t,N} Y(p')), \tag{3.6}$$

and

$$K_N(p,p')=E(\Delta_N B_h(p)\Delta_N B_h(p')). \tag{3.7}$$

To prove Theorem 3.1 we need some preliminary results. The following Remark is a direct consequence of some results in [22]. This is why we omit its proof.

Remark 3.2 *Theorem 3.1 is implied by the following property:*

$$\lim_{N\to\infty}\frac{\lambda_N(t)}{S_N(t)}=0, \tag{3.8}$$

where $\lambda_N(t)$ is the spectrum of the covariance matrix $(J_N(p,p'))_{p,p'\in\nu_N(t)}$ i.e $\lambda_N(t)$ is the maximum of the eigenvalues of this matrix. One generally bound $\lambda_N(t)$ by the quantity

$$\beta_N(t)=\max_{p\in\nu_N(t)}\sum_{p'\in\nu_N(t)}|J_N(p,p')|, \tag{3.9}$$

which is less difficult to handle. Relation (3.8) therefore results from,

$$\lim_{N\to\infty}\frac{\beta_N(t)}{S_N(t)}=0. \tag{3.10}$$

The following two Lemmas will allow us to bound $\beta_N(t)$.

Lemma 3.3 *There is a contant $c_2>0$ such that the inequality*

$$|J_N(p,p')-I_N(p,p')|\le c_2N^{-\gamma}, \tag{3.11}$$

holds for all integer $N\ge 2$ and for every $p,p'\in\nu_N(t)$.

Proof of Lemma 3.3
One has

$$\begin{aligned}\theta &= |J_N(p,p')-I_N(p,p')| = |E(\Delta_N X(p)\Delta_N X(p'))-E(\Delta_{t,N}Y(p)\Delta_{t,N}Y(p'))|\\ &\le E(|\Delta_N X(p)||\Delta_N X(p')-\Delta_{t,N}Y(p')|)+E(|\Delta_{t,N}Y(p')||\Delta_N X(p)-\Delta_{t,N}Y(p)|).\end{aligned}$$

Then using Cauchy-Schwarz inequality and Relations (3.2) and (3.3), one obtains that,

$$\begin{aligned}\theta \le\ & \Big(\sum_{k=0}^{2}|d_k|\Big\|Y\Big(\frac{p'+k}{N},\frac{p'+k}{N}\Big)-Y\Big(\frac{p'+k}{N},t\Big)\Big\|_2\Big)\Big(\sum_{k=0}^{2}|d_k|\Big\|X\Big(\frac{p+k}{N}\Big)\Big\|_2\Big)\\ &+\Big(\sum_{k=0}^{2}|d_k|\Big\|Y\Big(\frac{p+k}{N},\frac{p+k}{N}\Big)-Y\Big(\frac{p+k}{N},t\Big)\Big\|_2\Big)\Big(\sum_{k=0}^{2}|d_k|\Big\|Y\Big(\frac{p'+k}{N},t\Big)\Big\|_2\Big)\end{aligned} \tag{3.12}$$

Next, observe that the functions $x\mapsto\|X(x)\|_2$ and $x\mapsto\|Y(x,t)\|_2$ being continuous on $[0,1]$, they are bounded on this interval. At last, Lemma 3.3 follows from Lemma 2.7 and Relation (1.20). ■

Lemma 3.4 *There is a constant $c_3>0$ (depending on t) such that the inequality*

$$|K_N(p,p')-I_N(p,p')|\le c_3N^{-4}, \tag{3.13}$$

holds for all integer $N\ge 2$ and for every $p,p'\in\nu_N(t)$.

Proof of Lemma 3.4
First, observe that it follows from Relations (1.7) and (1.9) that for all non vanishing $\xi\in I\!R^d$,

$$\begin{aligned}\Big(\sum_{n=0}^{\infty}\frac{\hat f_{n-1}(\xi)}{|\xi|^{h_n}}\Big)^2 &= \Big(\sum_{n=0}^{n_1+1}\frac{\hat f_{n-1}(\xi)}{|\xi|^{h_n}}+\sum_{n=n_1+2}^{\infty}\frac{\hat f_{n-1}(\xi)}{|\xi|^{h}}\Big)^2\\ &= \Big(\sum_{n=0}^{n_1+1}\frac{\hat f_{n-1}(\xi)}{|\xi|^{h_n}}\Big)^2+\Big(\sum_{n=n_1+2}^{\infty}\frac{\hat f_{n-1}(\xi)}{|\xi|^{h}}\Big)^2+\frac{2\hat f_{n_1}(\xi)}{|\xi|^{h_{n_1+1}}}\Big(\sum_{n=n_1+2}^{\infty}\frac{\hat f_{n-1}(\xi)}{|\xi|^{h}}\Big)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{|\xi|^{2h}} &= \Big(\sum_{n=0}^{\infty}\frac{\hat f_{n-1}(\xi)}{|\xi|^{h}}\Big)^2\\ &= \Big(\sum_{n=0}^{n_1+1}\frac{\hat f_{n-1}(\xi)}{|\xi|^{h}}\Big)^2+\Big(\sum_{n=n_1+2}^{\infty}\frac{\hat f_{n-1}(\xi)}{|\xi|^{h}}\Big)^2+\frac{2\hat f_{n_1}(\xi)}{|\xi|^{h}}\Big(\sum_{n=n_1+2}^{\infty}\frac{\hat f_{n-1}(\xi)}{|\xi|^{h}}\Big)\end{aligned}$$

Thus, since $h_{n_1+1} = h$, one obtains that for every non vanishing $\xi \in I\!R$,

$$\Big|\Big(\sum_{n=0}^{n_1+1} \frac{\hat{f}_{n-1}(\xi)}{|\xi|^{h_n}}\Big)^2 - \frac{1}{|\xi|^{2h}}\Big| \le \Big(\sum_{n=0}^{n_1+1} \frac{\hat{f}_{n-1}(\xi)}{|\xi|^{h_n}}\Big)^2 + \Big(\sum_{n=0}^{n_1+1} \frac{\hat{f}_{n-1}(\xi)}{|\xi|^{h}}\Big)^2.$$

Next, using Remark 2.9 and the inequality $|\sin(\xi/2N)| \le N^{-1}|\xi|$, one gets that for all integer $N \ge 2$ and every $p, p' \in \{0, \dots, N-2\}$,

$$|K_N(p,p') - I_N(p,p')| \le N^{-4} \int_{I\!R} |\xi|^3 \Big[\Big(\sum_{n=0}^{n_1+1} \frac{\hat{f}_{n-1}(\xi)}{|\xi|^{h_n}}\Big)^2 + \Big(\sum_{n=0}^{n_1+1} \frac{\hat{f}_{n-1}(\xi)}{|\xi|^{h}}\Big)^2\Big]\, d\xi$$

and this last integral is clearly finite. ■

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1

Thanks to Remark 3.2 it is sufficient to prove that

$$\lim_{N\to\infty} \frac{\beta_N(t)}{S_N(t)} = 0. \tag{3.14}$$

Using Remarks 2.8, Lemmas 3.3 and 3.4, one obtains that there is a constant $c_1 > 0$, such that for all integer $N \ge 2$,

$$\beta_N(t) \le c_1\Big(N^{1-2\gamma} + \max_{p\in\nu_N(t)} |K_N(p,p')|\Big). \tag{3.15}$$

Next, let us give a lower bound of S_N. It follows from the Gaussianity of the process $\{X(s)\}_{s\in[0,1]}$, from Relation (3.10) and from the stationarity of the increments of the process $\{Y(s,t)\}_{s\in[0,1]}$, that

$$\begin{aligned} S_N(t) &\ge \Big(\sum_{p\in\nu_N(t)} \Big(J_N(p,p)\Big)^2\Big)^{1/2} \\ &\ge \Big(\sum_{p\in\nu_N(t)} \Big(I_N(p,p)\Big)^2\Big)^{1/2} - \Big(\sum_{p\in\nu_N(t)} \Big(I_N(p,p) - J_N(p,p)\Big)^2\Big)^{1/2} \qquad (3.16)\\ &\ge N^{1/2-\gamma/2} I_N(0,0) - c_2 N^{1/2-3\gamma/2}. \qquad (3.17) \end{aligned}$$

Next Relations (2.30), (3.16) and the assumption $\gamma > \frac{1+4h}{3} > 2h$ imply that there is a constant $c_3 > 0$, such that for all integer $N \ge 2$,

$$S_N(t) \ge c_3 N^{1/2-\gamma/2-2h-2\epsilon}. \tag{3.18}$$

Next, similarly to Relation (2.23) one can show that for any integer $N \ge 2$ and all $p, p' \in \nu_N(t)$, one has

$$K_N(p,p') = 16 \int_{I\!R} e^{i(p-p')\xi/N} \frac{\sin^4(\xi/2N)}{|\xi|^{2h+1}}\, d\xi. \tag{3.19}$$

Then by setting $u = \xi/N$ in this last integral and by integrating twice by part, it follows that there is constant $c_6 > 0$, which is independent on N, p and p', such that,

$$|K_N(p,p')| \le c_6 N^{-2h}(1 + |p-p'|)^{-2}. \tag{3.20}$$

Next Relations (3.18), (3.20) and the assumption $\gamma < 1/2$ entail that

$$\lim_{N\to\infty} \frac{\max_{p\in\nu_N(t)} \sum_{p'\in\nu_N(t)} |K_N(p,p')|}{S_N(t)} = 0. \tag{3.21}$$

Next Relations (3.15) and (3.18) imply that

$$\begin{aligned} 0 \le \frac{\beta_N(t)}{S_N(t)} &\le \frac{c_7(N^{1-2\gamma} + \max_{p\in\nu_N(t)} \sum_{p'\in\nu_N(t)} |K_N(p,p')|}{S_N(t)} \\ &\le c_7\Big(N^{1/2+2h+2\epsilon-3\gamma/2} + \frac{\max_{p\in\nu_N(t)} \sum_{p'\in\nu_N(t)} |K_N(p,p')|}{S_N(t)}\Big). \qquad (3.22) \end{aligned}$$

At last Relation (3.14) follows from (3.21), (3.22) and the assumption $\gamma > \frac{1+4h}{3}$. ■

4 Numerical Experiments

We elaborate briefly in this section on the applied aspects of our work. In order to test our estimator, we first need to generate sample paths of GMBMs. While synthesis methods exist for the mBm ([31]), the question of simulating GMBMs has been left unaddressed so far. We propose in subsection 4.1 a procedure based on discretizing the integrals defining the GMBM. Next, in subsection 4.2, we display results on the identification of $H(t)$ using the estimator proposed above, in simple cases where the Hölder function is discontinuous. Finally, subsection 4.3 deals with an application to financial data analysis.

4.1 Numerical Simulation of GMBM

For the sake of simplicity, throughout this section we restrict to GMBMs defined on the interval $[0,1]$ and we suppose that $\hat{f}_{-1}$ is the C^1 function defined for every $\xi \in \mathbb{R}$ as

$$\hat{f}_{-1}(\xi) = \begin{cases} 1 \text{ if } |\xi| \leq 1, \\ 0 \text{ if } |\xi| \geq 5/4, \\ \cos^2(2\pi(\xi-1)) \text{ if } \xi \in [1, 5/4], \\ \cos^2(2\pi(\xi+1)) \text{ if } \xi \in [-5/4, -1]. \end{cases} \tag{4.1}$$

Since $\hat{f}_{-1}$ is an even function the corresponding GMBMs can be represented as

$$X(t) = -4X_1(t) - 2X_2(t), \tag{4.2}$$

where

$$X_1(t) = \int_0^{+\infty} \Big[\sum_{n=0}^{\infty} \frac{\sin^2(t\xi/2)}{|\xi|^{H_n(t)+1/2}} \hat{f}_{n-1}(\xi)\Big]\, dW_1(\xi) \tag{4.3}$$

and

$$X_1(t) = \int_0^{+\infty} \Big[\sum_{n=0}^{\infty} \frac{\sin(t\xi)}{|\xi|^{H_n(t)+1/2}} \hat{f}_{n-1}(\xi)\Big]\, dW_2(\xi), \tag{4.4}$$

dW_1 and dW_2 being two independent real valued Brownian measures. For simulating the GMBM, we shall use a discretization of the stochastic integrals (4.3) and (4.4). Set

$$\delta = 2^{-p}, \tag{4.5}$$

where $p \geq 0$ is a fixed integer. One has

$$\begin{aligned} X_1(t) &= \sum_{m=0}^{\infty} \sum_{q=0}^{2^m-1} \int_{(2^m+q)\delta}^{(2^m+q+1)\delta} \Big[\sum_{n=0}^{\infty} \frac{\sin^2(t\xi/2)}{|\xi|^{H_n(t)+1/2}} \hat{f}_{n-1}(\xi)\Big]\, dW_1(\xi) \\ &\simeq \sum_{m=0}^{\infty} \sum_{q=0}^{2^m-1} \sum_{n=0}^{\infty} \frac{\sin^2(2^{-1}(2^m+q)\delta t)}{((2^m+q)\delta)^{H_n(t)+1/2}} \hat{f}_{n-1}((2^m+q)\delta)(W_1((2^m+q+1)\delta) - W_1((2^m+q)\delta)). \end{aligned}$$

Using (1.4) and (4.1), it follows that the stochastic integral (4.3) can be approximated by the random series

$$\begin{aligned} & 2^{-p/2} \sum_{l=0}^{5.2^{p-2}} \frac{\sin^2(2^{-p-1}lt)}{(2^{-p}l)^{H_0(t)+1/2}} \hat{f}_{-1}(2^{-p}l)\epsilon_l \\ & + 2^{-p/2} \sum_{n=0}^{\infty} \sum_{q=0}^{2^{n+p}-1} \frac{\sin^2((2^{n-1}+q2^{-p-1})t)}{(2^{n-1}+q2^{-p-1})^{H_n(t)+1/2}} \hat{f}_0(1+q2^{-n-p})\epsilon_{n,q} \\ & + 2^{-p/2} \sum_{n=0}^{\infty} \sum_{q=0}^{2^{n+p}-1} \frac{\sin^2((2^{n}+q2^{-p-1})t)}{(2^{n}+q2^{-p-1})^{H_n(t)+1/2}} \hat{f}_0(1+q2^{-n-1-p})\epsilon_{n+1,q}, \end{aligned}$$

where the ϵ_l and the $\epsilon_{n,q}$ are independent $\mathcal{N}(0,1)$ Gaussian variables. Similarly one can show that the stochastic integral (4.4) can be approximated by the random series

$$
\begin{aligned}
&2^{-p/2}\sum_{l=0}^{5.2^{p-2}}\frac{\sin(2^{-p}lt)}{(2^{-p}l)^{H_0(t)+1/2}}\hat{f}_{-1}(2^{-p}l)\eta_l\\
&+2^{-p/2}\sum_{n=0}^{\infty}\sum_{q=0}^{2^{n+p}-1}\frac{\sin((2^n+q2^{-p})t)}{(2^{n-1}+q2^{-p-1})^{H_n(t)+1/2}}\hat{f}_0(1+q2^{-n-p})\eta_{n,q}\\
&+2^{-p/2}\sum_{n=0}^{\infty}\sum_{q=0}^{2^{n+p}-1}\frac{\sin((2^{n+1}+q2^{-p})t)}{(2^{n}+q2^{-p-1})^{H_n(t)+1/2}}\hat{f}_0(1+q2^{-n-1-p})\eta_{n+1,q},
\end{aligned}
$$

where the η_l and the $\eta_{n,q}$ are independent $\mathcal{N}(0,1)$ Gaussian variables.

4.2 Hölder Exponent Estimation of Simulated GMBM

We show three examples of simulated GMBMs along with the estimation of their Hölder function. These examples correspond to situations of practical interest, where one needs to detect a) a sudden jump in Hölder regularity, b) an irregular point on a regular background, and c) a regular point on an irregular background.

The sample path displayed on figure 1 is obtained with a sequence of functions H_n converging to a step function having a discontinuity at 0.6: $H(t) = 0.3$ for $t \leq 0.6$, $H(t) = 0.7$ for $t > 0.6$. The sequence H_n is shown on figure 1 along with the GMBM. Figure 2 displays the estimated $H(t)$. As can be seen, the discontinuity is clearly detected.

The second example deals with an irregular point on a regular background, i.e. a sequence of H_n converging to the function $H(t) = 0.7$ for $t \neq 0.6$, $H(0.6) = 0.25$. Again, figure 3 shows the sample path of the GMBM along with the sequence H_n. The estimated Hölder function is displayed on figure 4.

Finally, we consider the more difficult case of a regular point on an irregular background. The sequence H_n converges to $H(t) = 0.2$ for $t \neq 0.6$, $H(0.6) = 0.8$. The sample path of the GMBM and the sequence H_n are on figure 5. Figure 6 displays the estimated Hölder function.

In both cases, the estimator is able to detect the point of interest with good accuracy.

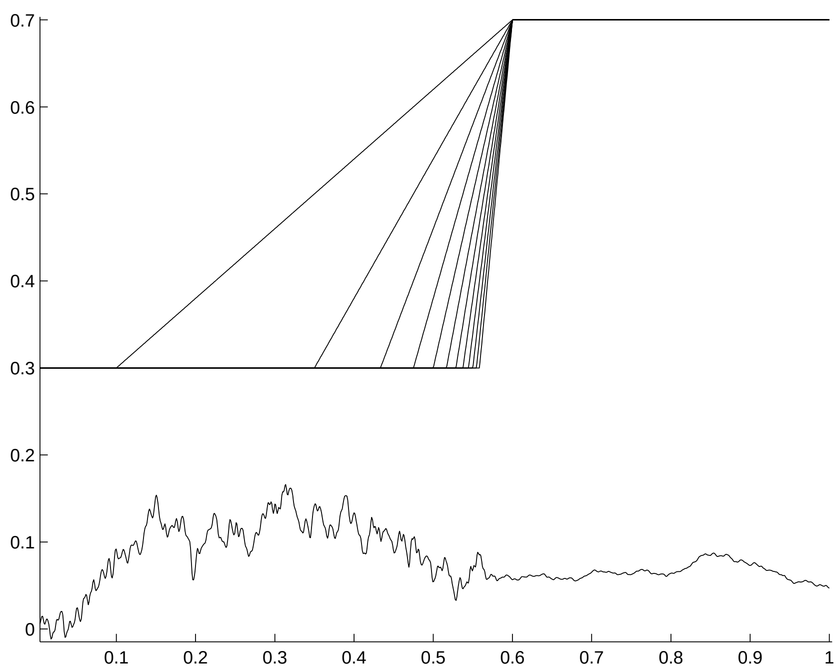

Figure 1: Simulated GMBM and associated sequence H_n converging to a step function.

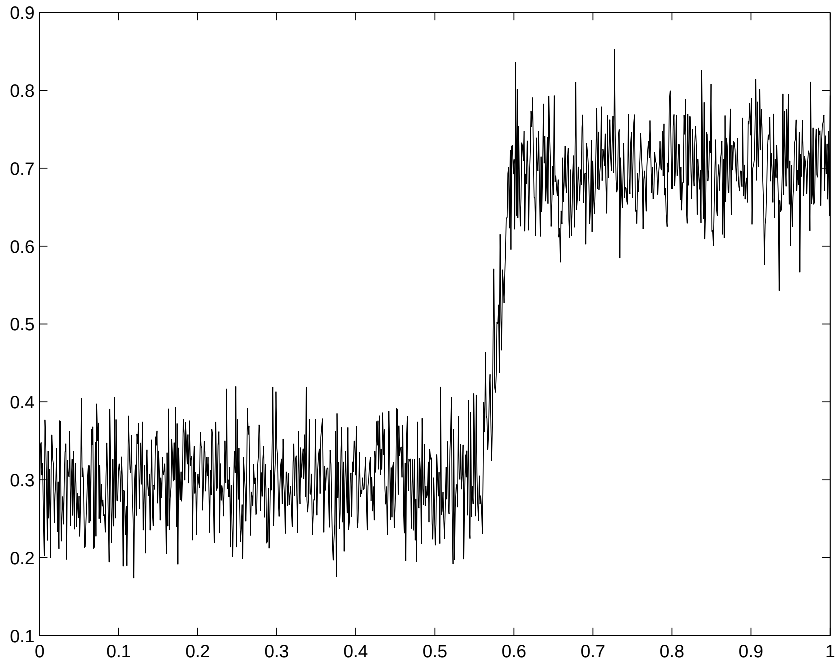

Figure 2: Estimated Hölder function of the GMBM in figure 1.

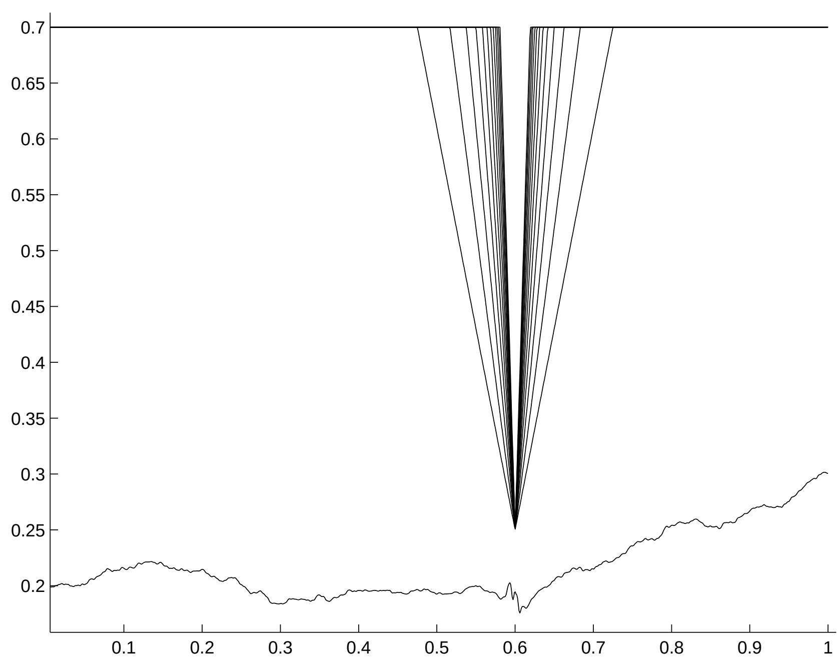

Figure 3: Simulated GMBM and associated sequence H_n converging to $H(t) = 0.7$ for $t \neq 0.6$, $H(0.6) = 0.25$.

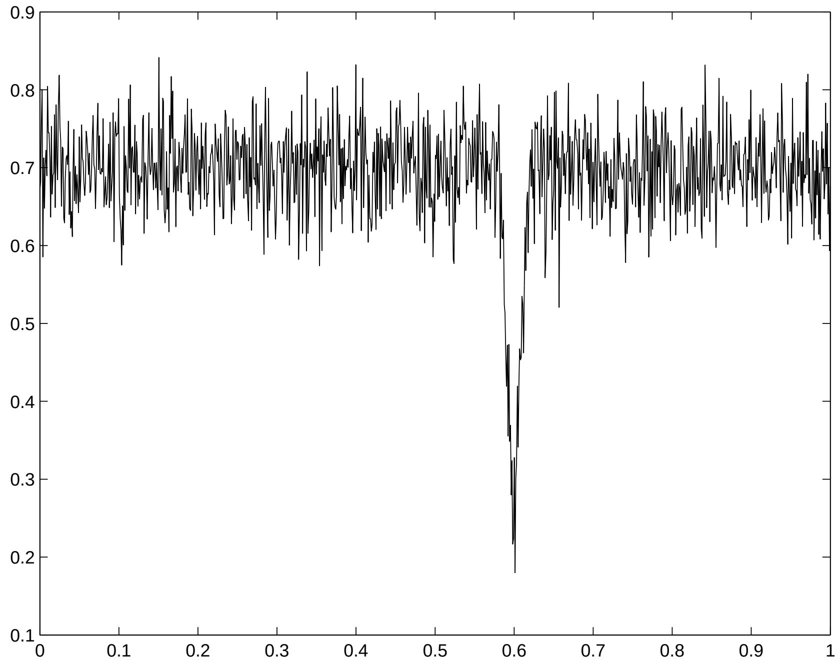

Figure 4: Estimated Hölder function of the GMBM in figure 3.

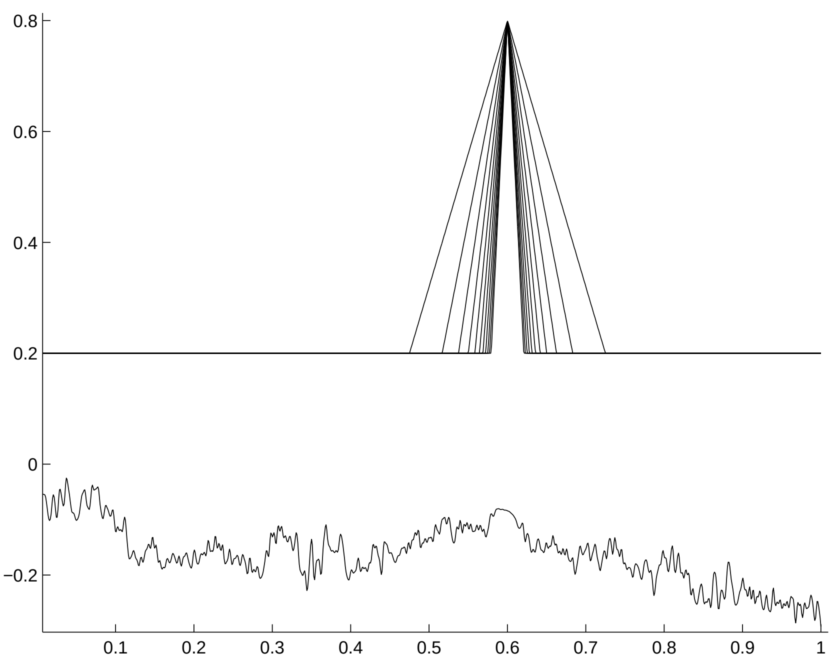

Figure 5: Simulated GMBM and associated sequence H_n converging to $H(t) = 0.2$ for $t \neq 0.6$, $H(0.6) = 0.8$.

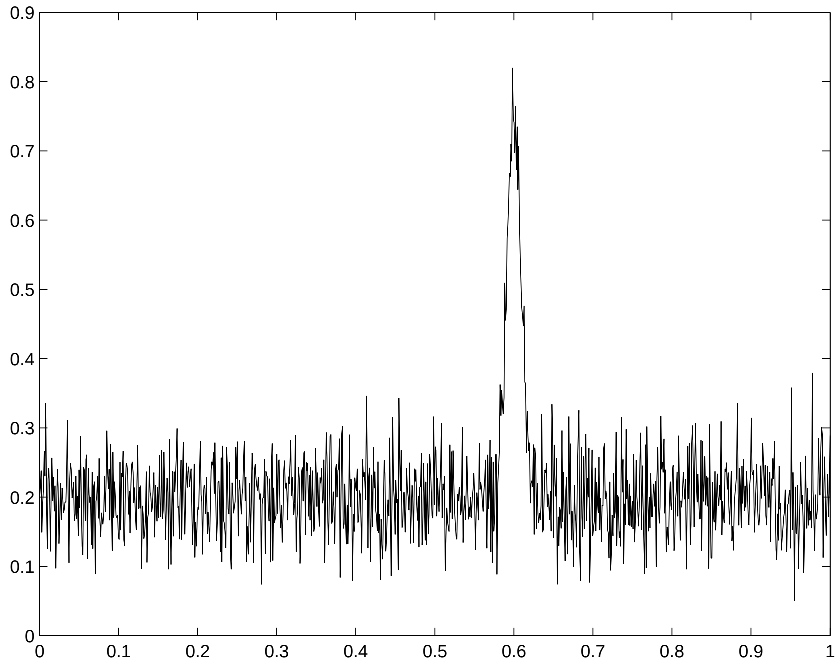

Figure 6: Estimated Hölder function of the GMBM in figure 5.

4.3 Analysis of Financial Data

We end this section with the analysis of a financial record. It is well-known that stock market logs are very irregular. Moreover, this irregularity is a function of time, and we expect that, for instance, at "quiet" periods, the market should evolve smoothly, resulting in a large value of $H(t)$, while krachs translate into sudden changes corresponding to small exponents.

We analyze in this section a log of the Nikkei225 index during the period 01/01/1980 to 05/11/2000. The log consists in 5313 daily values corresponding to that period. As financial analysts do not work directly on the prices, but on their logarithms, we shall deal with the logarithm of the Nikkei225 index record, which is displayed on figure 7. The signal is clearly quite erratic. Note in particular the large variations around the points 1780, 2040, 2650, 2760 or 3200. Although we are not able to verify whether these data may actually be well modelled with a GMBM, simple tests show that they are approximatively Gaussian. As we now show, a local regularity analysis based on the estimator proposed above allows to highlight significant events in the log. The estimated Hölder function is displayed on figure 8. As can be seen on the figure, most values of the Hölder exponents are between 0.2 and 0.8, with a few peaks up to 1. Recall that lower exponents correspond to more irregular parts of the signal. Looking at the original data, it appears obvious that the log is nowhere smooth, which is consistent with the values of the exponents. What is more interesting is that important events in the log have a specific signature in the Hölder function : Periods where "things happen" are characterized by a sudden increase in regularity, which reaches 1, followed by very small values, e.g. below 0.2, which correspond to low regularity. Let us take some examples. The most prominent feature of the Hölder function is the peak at abscissa 2018 with amplitude 1. Note also that the points with the lowest values in regularity of the whole log are located just after this peak: The Hölder exponent is around 0.2 at abscissa roughly between 2020 and 2050, and around 0.05 at abscissa between 2075 and 2100. Both values are well below the mean of the Hölder function, which is 0.4 (its variance of is 0.036). As a matter of fact, only 10 percent of the points of the signal have an exponent smaller than 0.2. Now the famous October 19 1987 krach corresponds to abscissa 2036, right in the middle on the first low regularity period after the peak. The days with smallest regularity in the whole log are thus, as expected, located in the weeks following the krach, and one can assess precisely which days were more erratic. However, if one looks at figure 7, these features do no show as clearly: Although the krach is easily seen as a strong downward variation at abscissa 2036, the area around this point does not appear to be more "special" than, for instance, the last part of the log.

Consider now another region which contains many points with small Hölder exponents along with a few isolated regular points (i.e. with exponent close to 1). Look at the area between abscissa 4450 and 4800: This roughly corresponds to the "Asian crisis" period, which approximately took place between January 1997 and June 1998 (there are no precisely defined dates for the beginning and end of the crisis. Some authors place the beginning of the crisis mid-1997, and the end by late 1999, or even later). On the graph of the original log of the Nikkei225, one can see that this period is quite erratic, with some rapid variations and pseudo-cycles (this behaviour arguably seems to extend between points 3500 and maybe the end of the trace). Looking now at the Hölder function, one notices that there are two peaks with exponents around one in the considered period (there is an additional such point around abscissa 4300, which, however, is not followed by points with low values of regularity -e.g. smaller than 0.15-, but is preceded by such points, between abscissa 4255 and 4285). The first peak is around 4455, and is followed by irregular points between 4465 and 4475. The second is around 4730. This region, between abscissa 4450 and 4800, has a large proportion of irregular points: 12 percent of its points have an exponent smaller than 0.15. This is three times the proportion observed in the whole log. In addition, this area is the one with highest density of points with exponent smaller than 0.15 (we exclude in these calculations the first and last points of the log, because of border effects).

Although the analysis above is very crude, it shows that estimating the Hölder regularity based on a modelling with a GMBM yields interesting insights on the data.

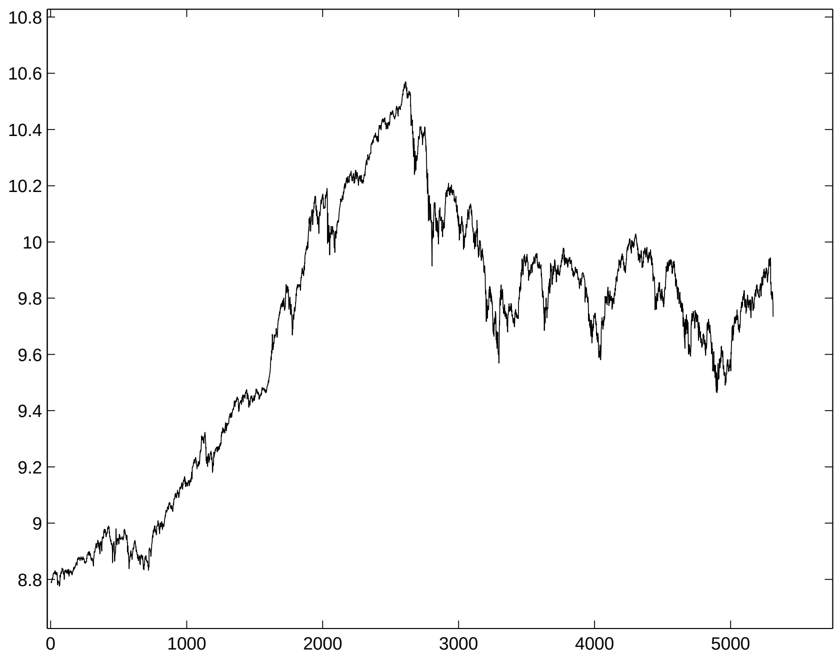

Figure 7: Logarithm of the Nikkei225 index during the period 01/01/1980 to 05/11/2000

Acknowledgements

The authors are thankful to P. Bertrand for many fruitful discussions. They also would like to thank the editors and the anonymous referees for their valuable comments, remarks and suggestions which have greatly helped them to improve the earlier version of their paper.

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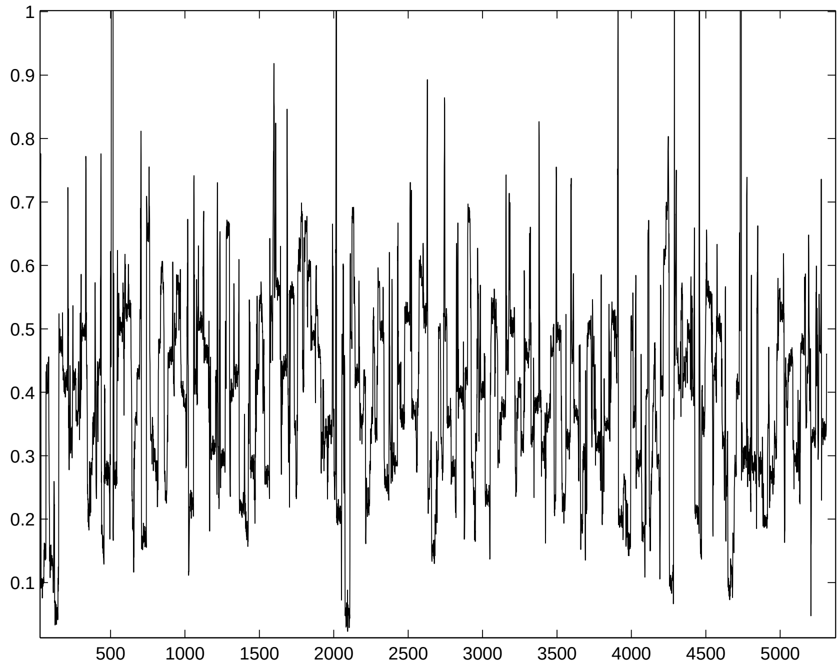

Figure 8: Estimated Hölder function of the logarithm of the Nikkei225

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