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# Perturbations for Delaunay and weighted Delaunay 3D Triangulations

Olivier Devillers      Monique Teillaud \*

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## Abstract

The Delaunay triangulation and the weighted Delaunay triangulation are not uniquely defined when the input set is degenerate. We present a new symbolic perturbation that allows to always define these triangulations in a unique way, as soon as the points are not all coplanar. No flat tetrahedron exists in the defined triangulation. The perturbation scheme is easy to code; It is implemented in CGAL, and guarantees that both vertex insertion and vertex removal are fully robust.

## 1 Introduction

The Delaunay triangulation of a set  $\mathcal{S}$  of points *in general position* in  $\mathbb{R}^3$  is usually defined as the partition of the convex hull of the points consisting of the tetrahedra whose circumscribing balls do not contain any point of  $\mathcal{S}$  in their interior. Here, general position means that there are no four coplanar points and no five cospherical points. When  $\mathcal{S}$  is degenerate, but when its points are not all coplanar, the Delaunay complex is the partition of its convex hull into 3D polytopes whose vertices lie on a sphere that does not enclose any other point of  $\mathcal{S}$ . Each of these polytopes can be triangulated in several ways, yielding Delaunay triangulations of  $\mathcal{S}$  [4].

The regular triangulation, or weighted Delaunay triangulation, is a generalization of the Delaunay triangulation when the sites in  $\mathcal{S}$  are spheres [3]. It is also not unique when the sites are in a degenerate position.

The method proposed in this paper allows to uniquely choose a Delaunay or regular triangulation, even when degeneracies occur.

While algorithmic research papers in computational geometry usually leave the handling of degeneracies to the reader, the issue must be solved when it comes to actually implementing an algorithm. While *ad hoc* tricks can be used when implementing a simple incremental algorithm, more care must be taken when updating a 3D triangulation after a vertex is removed, as will be detailed later in this introduction. This practical difficulty was encountered

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when coding this functionality in the 3D triangulation package of the CGAL library [1, 15], which motivated this research.

## 1.1 Definitions

Let us first recall definitions now, and introduce terminology used in the rest of the paper.

A sphere of center  $p \in \mathbb{R}^3$  and radius  $r \geq 0$  is denoted as  $s = (p, r)$ . It can also be seen as the *weighted point*  $p$  with weight  $r^2$ . The *power product* of two spheres  $s_0 = (p_0, r_0)$  and  $s_1 = (p_1, r_1)$  is defined as

$$\Pi(s_0, s_1) = \|p_0 p_1\|^2 - r_0^2 - r_1^2,$$

where  $\|p_0 p_1\|$  denotes the Euclidean distance between  $p_0$  and  $p_1$ . The spheres  $s_0$  and  $s_1$  are orthogonal iff  $\Pi(s_0, s_1) = 0$ . If  $\Pi(s_0, s_1) > 0$  (i.e.  $s_0$  and  $s_1$  do not intersect, or the angle in which they intersect is strictly smaller than  $\frac{\pi}{2}$ ), we say that  $s_0$  and  $s_1$  are *suborthogonal*. If  $\Pi(s_0, s_1) < 0$ , then we say that  $s_0$  and  $s_1$  are *superorthogonal*. Four spheres whose centers

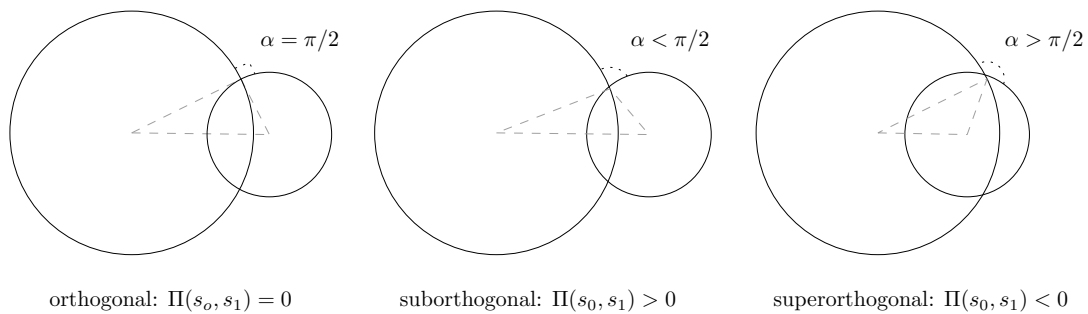


Figure 1: Orthogonal, suborthogonal and superorthogonal circles in the plane; When circles intersect,  $\Pi(s_0, s_1) = 2r_0 r_1 \cos \alpha$

are not coplanar have a unique common orthogonal sphere. Figures 1 and 2 illustrate these notions in 2D.

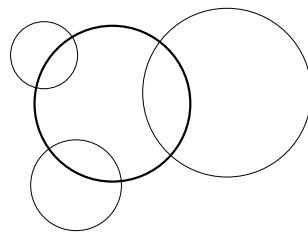


Figure 2: In 2D, three circles whose centers are not collinear have a unique common orthogonal circle

Let now  $\mathcal{S}$  be a set of  $n$  spheres and  $S$  be a sphere in  $\mathbb{R}^3$ .  $S$  is said to be “*empty*” with respect to  $\mathcal{S}$  if for any site  $s$  in the set  $\mathcal{S}$ , the spheres  $S$  and  $s$  are suborthogonal. Given

four sites  $s_i = (p_i, r_i), i \in I$  ( $I \subset \mathbb{N}$  and  $|I| = 4$ ) in  $\mathcal{S}$  whose centers are not coplanar, let  $T_I$  be the tetrahedron whose vertices are the four centers  $p_i, i \in I$ . We define the *sphere*  $S_I$  of  $T_I$  as the sphere orthogonal to the four sites  $s_i, i \in I$ . A tetrahedron whose sphere is empty with respect to  $\mathcal{S} \setminus \{s_i, i \in I\}$  is said to be *regular*. The *regular triangulation*  $\mathcal{RT}(\mathcal{S})$  is the partition of the convex hull  $CH(\mathcal{P})$  of  $\mathcal{P} = \{p \in \mathbb{R}^3, s = (p, r) \in \mathcal{S}\}$  formed by all regular tetrahedra constructed from sites of  $\mathcal{S}$ . An example in 2D is shown in Figure 3.

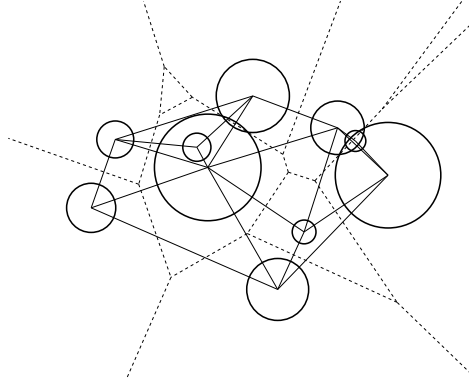


Figure 3: Regular triangulation of a set of circles in the plane (their power diagram is shown dashed)

If all radii are equal to zero, then the regular triangulation is the Delaunay triangulation. Also, more generally if all radii are equal, the regular triangulation of the spheres is the Delaunay triangulation of their centers. The regular triangulation is also called *weighted Delaunay triangulation*. The dual of the regular triangulation is known as the *power diagram* or *weighted Voronoi diagram* or *Laguerre diagram*.

The above definition of  $\mathcal{RT}(\mathcal{S})$  in fact assumes that the sites of  $\mathcal{S}$  are in *general position*. The general position assumption states that no four sites have coplanar centers and no five sites admit a common orthogonal sphere.

When the set  $\mathcal{S}$  is degenerate, we distinguish two types of regular tetrahedra. The sphere  $S_I$  can be orthogonal to other sites than the  $s_i, i \in I$ , that define  $T_I$  (which is a degeneracy). If  $S_I$  is suborthogonal or orthogonal to all sites in  $\mathcal{S}$ , then  $T_I$  is called *weakly regular*. A tetrahedron  $T_I$  whose sphere is suborthogonal to all sites of  $\mathcal{S} \setminus \{s_i, i \in I\}$  is called *strongly regular*. Note that the centers  $p_i, i \in I$  are not coplanar, otherwise  $S_I$  is not defined.

## 1.2 Contribution

We assume that the centers of the sites in  $\mathcal{S}$  are not all coplanar. There are always weakly regular tetrahedra, and their union is the convex hull of  $\mathcal{P}$ . When degeneracies occur, some weakly regular tetrahedra may overlap, and the set of strongly regular tetrahedra does not necessarily fill the convex hull; It can be completed in several ways by a subset of the weakly regular tetrahedra, which leads to several valid regular triangulations of  $\mathcal{S}$ .

Our goal here is to propose a method to define *in a unique way* the regular triangulation of a set of spheres of  $\mathbb{R}^3$  whose centers are not all coplanar, even in the presence of degeneracies.

More precisely, we introduce a predicate that, given four sites  $s_i, i \in I$ , whose centers are not coplanar, and a fifth site  $s$ , never answers that  $s$  is orthogonal to  $S_I$ . If the spheres are orthogonal, the predicate decides whether they should be considered as suborthogonal or superorthogonal. The regular triangulation is uniquely defined as the set of tetrahedra whose spheres are empty, as answered by this predicate. This definition is actually used in the 3D Delaunay and regular triangulations of CGAL [17].

### 1.3 Application

This work was initially motivated by handling vertex removal in the Delaunay triangulation of a set of 3D points [10]. The fact that a Delaunay triangulation is not defined uniquely for degenerate sets of points allow the algorithm to choose between different weakly Delaunay triangulations. This becomes problematic when a given choice is made without taking into account previous choices, and might be inconsistent with them. In particular, inconsistencies between choices made during the construction of the triangulation and choices made during the removal of a vertex can cause the failure of the vertex removal. Let us quickly review this problem here: When a vertex  $v$  is removed from the Delaunay triangulation  $\mathcal{DT}(\mathcal{S})$  of  $\mathcal{S}$ , the tetrahedra incident to  $v$  are removed, which creates a polyhedral hole, and the interior of this polyhedron  $H$  must be triangulated with Delaunay tetrahedra. Triangulating  $H$  is exactly the inverse operation of inserting  $v$  in  $\mathcal{DT}(\mathcal{S} \setminus \{v\})$ . After the removal, the Delaunay triangulation is the triangulation that would have been obtained if  $v$  had never been inserted.

The difficulty arises when there are at least four cocircular points  $p_1, p_2, p_3, p_4$  (i.e. points that are both cospherical and coplanar) on the boundary of  $H$ . Indeed, in this case, for any point  $p_5$  on  $H$ , the five points  $p_1, p_2, p_3, p_4, p_5$  are cospherical, and there are two possible triangulations of this set of points, corresponding to the two different choices for the diagonal of the convex polygon  $(p_1, p_2, p_3, p_4)$  which is a facet of  $H$  (or a sub-facet in the case when there are more than four cocircular points), see Figure 4. Depending on this choice, we will get a different triangulation of the facet. But the outside of  $H$  is already triangulated, which induces triangulations of all the facets of  $H$ . If a different triangulation of the facet  $(p_1, p_2, p_3, p_4)$  is chosen when triangulating the interior of  $H$ , then an inconsistency occurs in the triangulation of  $\mathcal{S} \setminus \{v\}$ .

The problem can be seen as a special instance of the following question: *Is it always possible to compute a Delaunay triangulation of a given polyhedron  $H$  in such a way that the triangulation of its facets is respected?* Let us consider a straight prism  $H$  with triangular basis such that its six vertices are cospherical. Assume that its rectangular facets are triangulated as shown in Figure 5(left). Let us now try to triangulate the interior of  $H$ . The six vertices of  $H$  are exactly in the same configuration regarding their incidences on  $H$ . Take one of them, say  $p$  without loss of generality, then it can easily be seen that any possible tetrahedron having this vertex and any other three vertices of  $H$  will have an edge that crosses an existing edge on  $H$ . This is a variant of the famous Schönhardt polyhedron:

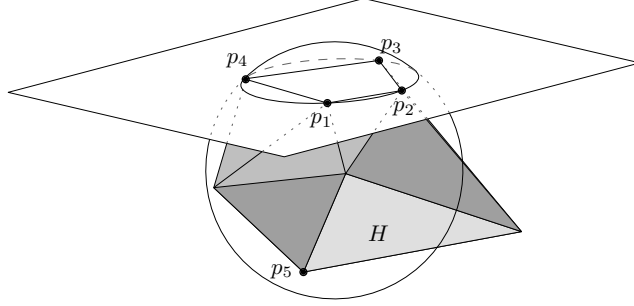


Figure 4: Cocircular points on the boundary of the hole

a polyhedron that cannot be triangulated [18].

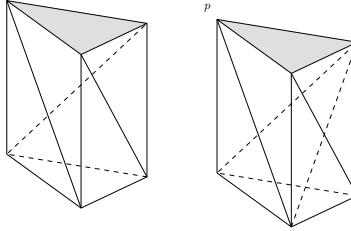


Figure 5: A polyhedron that cannot be triangulated

The symbolic perturbation technique introduced in this paper allows us to define the 3D Delaunay triangulation of a set of points uniquely, which avoids the above inconsistency problem.

## 1.4 Overview

After a quick review of related work on symbolic perturbation techniques (Section 2), we present in Section 3 a symbolic perturbation allowing to define the regular triangulation of a set of spheres in a unique way, and we discuss its properties. Section 4 quickly shows that the perturbation can be used by several standard algorithms to compute the uniquely defined triangulation.

## 2 Symbolic Perturbation Techniques

Using symbolic perturbations is a general approach to work around degenerate cases [21, 13, 19]. The rough idea is to make the problem dependent on a parameter  $\varepsilon > 0$  such that:

- there exists  $\varepsilon_0 > 0$  such that the parameterized problem is in general position for  $\varepsilon \in (0, \varepsilon_0]$ ,

- if the original problem is in general position, the solution of the parameterized problem tends to the solution of the original problem when  $\varepsilon$  goes to zero,
- if the original problem is not in general position, the solution of the parameterized problem tends to a solution satisfying some wished properties when  $\varepsilon$  goes to zero. These properties depend on the problem and on needs for further use of the solution.

When using a perturbation, some properties can be lost in the result: to check whether they are satisfied, we must

- check that the parameterized solution satisfies the properties,
- check that the properties are still true at the limit.

In the case we are interested in, if  $\mathcal{S}$  is not in general position, the regular triangulation is not uniquely defined and the aim of a perturbation technique is to select one of the possible regular triangulations, by using the unique triangulation of a perturbed input.

Perturbing the input can have very serious drawbacks: if the points move with  $\varepsilon$ , then a non-flat tetrahedron can become flat at the limit [2]. By *flat* tetrahedron, we mean a tetrahedron whose four vertices are coplanar. Allowing flat tetrahedra is not acceptable. Indeed, these flat tetrahedra do not correspond with the definition of a regular triangulation, since the sphere of a flat tetrahedron is not defined: its four sites have either zero or an infinity of common orthogonal spheres. Flat tetrahedra do not correspond to the usual intuition either. Moreover, in the context of a CGAL package, this would lead to a heavier user code: before applying geometric operations to a tetrahedron, such as for instance computing its circumcenter, the user would have to check that the tetrahedron is not flat.

Edelsbrunner and Mücke write that directly using their general *Simulation of Simplicity* perturbation technique for Delaunay triangulations is a “real pain” [13, page 96, line 20] and suggest to use the transformation of the Delaunay triangulation into a convex hull in one dimension higher [6, 14, 9] and perturb the computation of the convex hull. However, such perturbation of the 4D convex hull, without taking into account the special structure of the points in  $\mathbb{R}^4$ , does not either give any guarantee on the fact that tetrahedra are non-flat.

In this paper, we propose to perturb *only the radii* of the 3D sites, which perturbs the points in 4D in the fourth direction only. One important advantage is that the 3D points do not move, thus if a tetrahedron belongs to the limit solution, the same tetrahedron is regular for a non-degenerate set of sites with the same centers but different radii, thus this tetrahedron is not flat.

The following section presents the perturbation in more detail and proves its correctness.

## 3 Perturbing the *power\_test* Predicate

### 3.1 The Predicates

The predicates *orient* and *power\_test* (resp. *orient* and *in\_sphere* ) are the only predicates necessary to determine the regular (resp. Delaunay) triangulation.

Let  $(x_a, y_a, z_a)$  denote the Cartesian coordinates of a point  $p_a$  of  $\mathbb{R}^3$ .  
 For any four points  $p_a, p_b, p_c, p_d$  in  $\mathbb{R}^3$ ,

$$\text{orient}(a, b, c, d) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_a & x_b & x_c & x_d \\ y_a & y_b & y_c & y_d \\ z_a & z_b & z_c & z_d \end{vmatrix}$$

is a degree three polynomial in the coordinates of the points, whose sign determines the orientation of the four points.

Let  $p_a, p_b, p_c, p_d$  be any four non-coplanar points in  $\mathbb{R}^3$ , and  $p_e$  a fifth point.

$$\left. \begin{array}{l} \text{in\_sphere}(p_a, p_b, p_c, p_d, p_e) \\ > 0 \text{ if } p_e \text{ is outside} \\ = 0 \text{ if } p_e \text{ is on the boundary of} \\ < 0 \text{ if } p_e \text{ is inside} \end{array} \right\} \text{ the ball circumscribing } p_a, p_b, p_c, p_d.$$

For any four spheres  $s_a, s_b, s_c, s_d$  of  $\mathbb{R}^3$  with non-coplanar centers  $p_a, p_b, p_c, p_d$ , and a fifth sphere  $s_e$  centered at  $p_e$ , the *power\_test* predicate generalizes *in\_sphere*:

$$\left. \begin{array}{l} \text{power\_test}(s_a, s_b, s_c, s_d, s_e) \\ > 0 \text{ if } s_e \text{ is suborthogonal} \\ = 0 \text{ if } s_e \text{ is orthogonal} \\ < 0 \text{ if } s_e \text{ is superorthogonal} \end{array} \right\} \text{ to the sphere orthogonal to } s_a, s_b, s_c, s_d.$$

It is well known that *power\_test* can be computed in the following way:

$$\text{power\_test}(s_a, s_b, s_c, s_d, s_e) = \text{sign} \frac{\text{pow\_det}(s_a, s_b, s_c, s_d, s_e)}{\text{orient}(p_a, p_b, p_c, p_d)}$$

where

$$\text{pow\_det}(s_a, s_b, s_c, s_d, s_e) = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ x_a & x_b & x_c & x_d & x_e \\ y_a & y_b & y_c & y_d & y_e \\ z_a & z_b & z_c & z_d & z_e \\ x_a^2 + y_a^2 + z_a^2 - r_a^2 & x_b^2 + y_b^2 + z_b^2 - r_b^2 & x_c^2 + y_c^2 + z_c^2 - r_c^2 & x_d^2 + y_d^2 + z_d^2 - r_d^2 & x_e^2 + y_e^2 + z_e^2 - r_e^2 \end{vmatrix}.$$

The predicate  $\text{sign pow\_det}(s_a, s_b, s_c, s_d, s_e)$  in  $\mathbb{R}^3$  can be seen as an orientation predicate in  $\mathbb{R}^4$ , if each site  $s = ((x, y, z), r)$  of  $\mathbb{R}^3$  is mapped onto a point  $\pi(s)$  of  $\mathbb{R}^4$  [14, 9], where

$$\pi(s) = (x, y, z, x^2 + y^2 + z^2 - r^2).$$



### 3.2 The Perturbation

We assume that the sites of  $\mathcal{S} = \{s_1, \dots, s_n\}$  are indexed in some way.

Given four sites  $s_i, s_j, s_k, s_l$  whose centers are not coplanar, the decision whether the tetrahedron appears in the regular triangulation  $\mathcal{RT}(\mathcal{S})$  is clear when the *power\_test* test against any other site  $s_m \in \mathcal{S} \setminus \{s_i, s_j, s_k, s_l\}$  answers ‘suborthogonal’ or ‘superorthogonal’.

When a case of orthogonality appears, the five sites have a common orthogonal sphere, i.e. a degeneracy occurs, and the decision must be made using other criteria. Five sites in  $\mathbb{R}^3$  have a common orthogonal sphere if and only if their images by  $\pi$  lie in the same hyperplane of  $\mathbb{R}^4$ . We define a symbolic perturbation of the *power\_test* test that consists in adding respectively some values to the fourth coordinate of  $\pi(s_i), \pi(s_j), \pi(s_k), \pi(s_l), \pi(s_m)$  so that these points are not in the same hyperplane any more in  $\mathbb{R}^4$ . Then the predicate answers either ‘suborthogonal’ or ‘superorthogonal’, instead of ‘orthogonal’.

More precisely, we add  $\varepsilon^{\sigma(i)}$  to the fourth coordinate of each point  $\pi(s_i), i = 1, \dots, n$  in  $\mathbb{R}^4$ , where  $\sigma$  is some permutation of  $(1, \dots, n)$ . The quantity each point is perturbed with depends on its index. The choice of the permutation  $\sigma$  will be discussed in Section 3.3. The determinant  $pow\_det(s_i, s_j, s_k, s_l, s_m)$  is perturbed into

$$pow\_det_\varepsilon(s_i, s_j, s_k, s_l, s_m) = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ x_i & x_j & x_k & x_l & x_m \\ y_i & y_j & y_k & y_l & y_m \\ z_i & z_j & z_k & z_l & z_m \\ t_i + \varepsilon^{\sigma(i)} & t_j + \varepsilon^{\sigma(j)} & t_k + \varepsilon^{\sigma(k)} & t_l + \varepsilon^{\sigma(l)} & t_m + \varepsilon^{\sigma(m)} \end{vmatrix}$$

where

$$t_\star = x_\star^2 + y_\star^2 + z_\star^2 - r_\star^2 \text{ for } \star = i, j, k, l, m.$$

Developing with respect to the last row yields a polynomial in  $\varepsilon$

$$\begin{aligned} pow\_det_\varepsilon(s_i, s_j, s_k, s_l, s_m) = & pow\_det(s_i, s_j, s_k, s_l, s_m) \\ & + orient(p_j, p_k, p_l, p_m)\varepsilon^{\sigma(i)} - orient(p_i, p_k, p_l, p_m)\varepsilon^{\sigma(j)} \\ & + orient(p_i, p_j, p_l, p_m)\varepsilon^{\sigma(k)} - orient(p_i, p_j, p_k, p_m)\varepsilon^{\sigma(l)} \\ & + orient(p_i, p_j, p_k, p_l)\varepsilon^{\sigma(m)}. \end{aligned}$$

When the spheres  $s_i, s_j, s_k, s_l, s_m$  have a common orthogonal sphere, the constant term  $pow\_det(s_i, s_j, s_k, s_l, s_m)$  of  $pow\_det_\varepsilon(s_i, s_j, s_k, s_l, s_m)$  vanishes to zero. At least the last coefficient  $orient(p_i, p_j, p_k, p_l)$  of the polynomial is non-null, since the corresponding sites define a common orthogonal sphere, so, the polynomial  $pow\_det_\varepsilon$  is not identically zero. In the special case of a Delaunay triangulation, in fact at most one of the coefficients  $orient()$  can be zero, otherwise two subsets with four points would consist of four coplanar points; either the five points would be coplanar, which contradicts our hypothesis, or the three points shared by the two subsets would be collinear, which is impossible since these three points are cospherical.

The coefficients of  $pow\_det_\varepsilon$  are examined in order of increasing exponents of  $\varepsilon$ , until the first non-null coefficient is found, which determines the sign of  $pow\_det_\varepsilon(s_i, s_j, s_k, s_l, s_m)$ .

A tetrahedron formed by four sites whose centers are non-coplanar, and whose common orthogonal sphere is suborthogonal, as answered by the perturbed *power\_test*, to all other

sites of  $\mathcal{S}$ , is called *PP-regular* (for “perturbed predicate” regular).  $\mathcal{RT}(\mathcal{S})$  can now be defined as the set of all PP-regular tetrahedra:

**Theorem 1.** *The set of PP-regular tetrahedra  $\mathcal{RT}(\mathcal{S})$  defines a triangulation.*

*Proof.* By definition, a strongly regular tetrahedron is PP-regular and a PP-regular tetrahedron is weakly regular. The result comes readily from usual reasoning on symbolic perturbations. There exists a small enough  $\epsilon > 0$  such that a tetrahedron is PP-regular in  $\mathcal{S}$  if and only if the same tetrahedron with perturbed weights is strongly regular in the set of sites with perturbed weights. Since the strongly regular tetrahedra define a 3D triangulation of the sites with perturbed weights, we can conclude.  $\square$

**Remarks.**

For the special case of the Delaunay triangulation of point sites, the perturbation reduces to computing the regular triangulation for a set of spheres with radii going to zero when  $\epsilon$  goes to zero. This is an analogy with the “sliver exudation” method [8, 7] that consists in associating radii (or weights) to points, chosen so that the *almost* flat tetrahedra that are unavoidable in a Delaunay triangulation disappear in the regular triangulation. In our case, the tetrahedra that are avoided are not almost flat, but really flat, and the weights are symbolic.

The technique proposed in this paper is a perturbation technique on the regular triangulation problem. When used to compute a Delaunay triangulation, a possible drawback is that, since the perturbed problem is no longer a Delaunay but a regular triangulation problem, a property proved in the context of non-degenerate Delaunay triangulations may be not verified by a triangulation produced by our method. As an example, in 2D, the combinatorial triangulation of Figure 6a is known to be impossible to realize as a Delaunay triangulation of a non-degenerate set of points [12] although it is realizable as a weakly Delaunay triangulation (Figure 6b). Since this triangulation can be viewed as a (strongly) regular triangulation with a relevant choice of weights (Figure 6c), our perturbation scheme may actually produce it by a suitable ordering of the points. A perturbation technique transforming the problem into a non-degenerate Delaunay triangulation problem would never produce such “forbidden” triangulation, but might be unable to avoid flat triangles. An attempt to obtain a *true* 3D Delaunay triangulation without flat tetrahedra has been done by Sugihara [20]. A relevant ordering to perturb the point coordinates is searched to avoid the creation of flat tetrahedra, such an ordering is found easily in many situations but it is not guarantee to exist and may need  $O(n^4)$  computation time.

### 3.3 Choosing the Permutation $\sigma$

A feature of the technique is that to assign to each site a function of  $\epsilon$  as radius, we use a permutation on the indices of the sites. We show in this section that in fact we can define the permutation by any total comparison order on the sites. Relying on an ordering of sites is both a drawback and an advantage: it is a drawback because the result depends on that

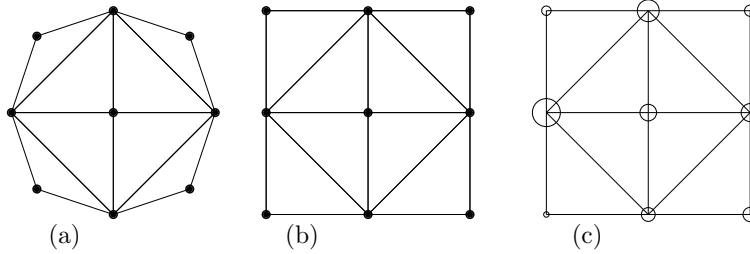


Figure 6: Realizability of the Delaunay triangulation

ordering; it is an advantage because we can force the result to satisfy specific properties, by just choosing the ordering appropriately.

Notice that when coding the standard incremental algorithm (see Section 4.2), a simple way to handle degeneracies is to always consider that the *last inserted site* is suborthogonal to all spheres of tetrahedra where the result of the *power\_test* test is 0. This minimizes the number of updates done on the triangulation during the insertion. This can in fact be seen as *implicitly* implementing the symbolic perturbation proposed in Section 3.2, where the permutation is defined as  $\sigma(i) = n - i, i = 1, \dots, n$  when inserting the points in the order of their indices: at each step, the last point  $s_m$  is more perturbed than all the previous ones, so, when it has a common orthogonal sphere with four sites  $s_i, s_j, s_k, s_l, i, j, k, l \in \{1, \dots, m-1\}$  defining an already existing tetrahedron  $(p_i, p_j, p_k, p_l)$ , it is considered as being suborthogonal to the sphere of the tetrahedron. This corresponds to looking at the sign of the coefficient of the monomial of smallest degree  $\sigma(m) = n - m$  in  $pow\_det_\epsilon$ , which is positive. However, note that this requires to store the order of insertion of each point in the corresponding vertex to be able to update the triangulation in a consistent way when performing vertex removal. This was the choice made in CGAL for the Delaunay triangulation in earlier releases. Some users reported this choice as being annoying for their application [11]. In more recent releases, we chose the lexicographical ordering on the coordinates of the points. The same choice was made later, when the vertex removal in the regular triangulation was implemented.

Any other choice could have been made. Choosing the lexicographical ordering of points has the advantage of giving an intrinsic definition of the regular triangulation, even in degenerate cases. On the other hand, it may lead to a slower construction of the triangulation for some very degenerate input. Let us note that the triangulation is not preserved through transformations such as symmetries with respect to coordinate planes for instance. We may think of leaving the choice of the order to the CGAL user in the future.

## 4 Algorithms

The definition of regular triangulation given above is actually independent from the choice of the algorithm used to construct the triangulation. While a few algorithms require additional predicates of higher degrees [16], the predicates *orient* and *power\_test* are the most basic

predicates used when computing a regular triangulation. As mentioned earlier, the *power\_test* predicate can only be used on sites having non-coplanar centers, which is not a strong constraint since the common orthogonal sphere is well-defined only for four sites with non-coplanar centers. Note that a potential divide-and-conquer approach should make sure that the divide steps always produce sets of sites whose centers are not all coplanar.

We review below a few standard algorithms that can easily benefit from the perturbation technique presented above.

## 4.1 Naive Algorithm

The naive algorithm considers all sets of four sites with non-coplanar centers and checks them with the power test predicate against every other site. When more than four sites have a common orthogonal sphere, the non-perturbed predicate does not allow to decide whether a tetrahedron should be kept in the triangulation. The use of the symbolic perturbation makes the decision simple and guarantees that the result is a triangulation.

## 4.2 Incremental Algorithm

The CGAL 3D triangulation package implements a standard incremental algorithm [5]. For  $s_e \in \mathcal{S}$ , let us assume that the triangulation of  $\mathcal{S}' = \mathcal{S} \setminus \{s_e\}$  was constructed.

Let us first consider the case when the center  $p_e$  of  $s_e$  is contained in the convex hull of the centers of the sites in  $\mathcal{S}'$ . The tetrahedra whose sphere is superorthogonal to  $s_e$  are removed from the triangulation. This creates a polyhedral hole  $H_e$  that is star-shaped with respect to  $p_e$ . The new triangulation  $\mathcal{RT}(\mathcal{S})$  is obtained by adding all tetrahedra formed by  $p_e$  and a triangular facet of  $H_e$ . The decision whether to delete a tetrahedron  $(p_a, p_b, p_c, p_d)$  from the regular triangulation  $\mathcal{RT}(\mathcal{S}')$  is made using the *power\_test* predicate. Since the triangulation  $\mathcal{RT}(\mathcal{S}')$  admits no flat tetrahedron by definition, the predicate *power\_test*  $(s_a, s_b, s_c, s_d, s_e)$  is used only when its first four arguments have actually non-coplanar centers.

The convex hull is managed as follows. In CGAL the unbounded cell outside the convex hull is subdivided into “tetrahedra”, by considering that each convex hull facet is incident to an infinite cell having as fourth vertex an auxiliary vertex called the *infinite vertex*. In that way, each facet is incident to exactly two tetrahedra and special cases at the boundary of the convex hull are simple to deal with. The triangulations that are manipulated are triangulations of the combinatorial sphere  $\mathbb{S}^3$ .

The definition of *power\_test* used for the regular triangulation is then extended to infinite cells. For four sites  $s_a, s_b, s_c, s_d$  whose centers are oriented positively, and a fifth site  $s_e$ , *power\_test*  $(s_a, s_b, s_c, s_d, s_e) = \text{sign } \text{pow\_det}(s_a, s_b, s_c, s_d, s_e)$ . We define *power\_test*  $(s_a, s_b, s_c, \infty, s_e)$  as the limit of the sign of *pow\_det* $(s_a, s_b, s_c, s_d, s_e)$  when the center  $p_d$  of  $s_d$  goes to infinity (staying in the same half-space defined by  $p_a, p_b, p_c$ ) and its radius is zero. Let  $P_{abc}$  denotes the plane through  $p_a, p_b$ , and  $p_c$ . Geometrically, the common orthogonal “ball” of  $s_a, s_b, s_c$  and  $\infty$  is the union of the open half-space limited by  $P_{abc}$  with the open disk in  $P_{abc}$  that is orthogonal to the circles  $s_a \cap P_{abc}$ ,  $s_b \cap P_{abc}$ , and  $s_c \cap P_{abc}$ . The common orthogonal “sphere” of  $s_a, s_b, s_c$ , and  $\infty$  is reduced to the common orthogonal circle of  $s_a, s_b$ , and  $s_c$  in

$P_{abc}$ . The actual implementation of *power\_test* uses this geometric interpretation, looking at the side of  $p_e$  with respect to the plane  $P_{abc}$ , and the angle of  $s_e$  with the circle in the case of coplanarity. The perturbation scheme explained in Section 3 is then applied on the predicate of comparison of a site with a circle defined by three sites of the convex hull. With this predicate adapted to infinite cells, the insertion of a site outside the convex hull works exactly as when the inserted point lies inside the convex hull.

### 4.3 Gift-wrapping-like Algorithm

Assume for a while that a PP-regular tetrahedron  $(p_a, p_b, p_c, p_d)$  has been found, then its neighbor  $(p_a, p_b, p_c, p_e)$  in  $\mathcal{RT}(\mathcal{S})$  through facet  $(p_a, p_b, p_c)$  can be determined in linear time. We first initialize  $s_e$  with any site having its center on the other side of the plane  $P_{abc}$  than  $p_d$ . If no such site exists, then  $(p_a, p_b, p_c)$  is a convex hull facet, otherwise for each site  $s \in \mathcal{S}$ , if *power\_test* evaluates  $s$  as superorthogonal to  $S_{\{a,b,c,e\}}$  then the new value of  $s_e$  is  $s$ . At the end of this loop, there is no site in  $\mathcal{S}$  superorthogonal to  $S_{\{a,b,c,e\}}$  and the right value for  $s_e$  has been found. This simple procedure allows to construct the regular triangulation, as we defined it, from neighbor to neighbor.

The construction of the first PP-regular tetrahedron can be achieved by selecting two points  $p_a$  and  $p_b$  defining an edge of the convex hull, which is easy even in case of degeneracies. Then, referring to the convex hull management presented in Section 4.2, one can say that  $(s_a, s_b, \infty)$  is a facet of the regular triangulation and we can complete it into a Delaunay tetrahedron as in the previous paragraph, using the specialized definition of *power\_test* for  $\infty$ .

## 5 Conclusion

We propose a symbolic perturbation technique that allows to define the regular triangulation of a set of 3D sites in a unique way even in degenerate configurations, as long as the centers of the sites are not all coplanar.

The definition can be used for various algorithms. The perturbation is implemented in CGAL, which allows CGAL to provide a vertex removal that works even in degenerate situations. The code for perturbing the *power\_test* predicate is extremely simple. As far as we know, CGAL is the only publicly available software proposing fully dynamic 3D Delaunay and regular triangulations [17].

Note that the same technique can obviously be used in 2D, even if the degenerate cases can also be solved otherwise. The advantage is for instance that in the case of a square grid, choosing the lexicographical ordering of points ensures that all diagonals of squares are slanted in the same way, which is expected by most users.

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