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Carine Lucas, James C. Mcwilliams, Antoine Rousseau

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Abstract

This paper introduces a rigorous derivation of the quasi-hydrostatic quasi-geostrophic (QHQG) equations of large scale ocean as the Rossby number goes to zero. We follow classical techniques for the derivation of the quasi-geostrophic (QG) equations (as in [BB94]), but the primitive equations that we consider account for the nontraditional rotating terms, as in [LPR10]. We end up with a slightly different QG model with a tilted vertical direction, which has been illustrated in previous works using the primitive equations (see [PMHA97, She04, WB06, GZMvH08, SKR02]), and for which we prove local and global existence results.

1 Introduction

The quasi-geostrophic equations are very familiar to oceanographers and meteorologists for they have been extensively used for modeling oceanic and atmospheric circulations ([Cha71, BK75]). These equations are obtained from the 3D primitive (hydrostatic) equations thanks to an asymptotic expansion with respect to the Rossby number. The model is also very familiar to applied mathematicians, and several studies establishing the well-posedness of the corresponding boundary-value problem have been published (see *e.g.*, [BB94, GM97, Mas00]). The primitive equations from which the traditional QG equations are obtained rely on a few well-known hypotheses, among which the so-called *traditional approximation*, which consists in neglecting the rotating terms involving $2\Omega\cos\theta$ that appear in the zonal and vertical components (1a) and (1c) of the momentum equation. This approximation has also been widely discussed in the literature (see [Eck60] and the correspondence in [Phi66, Ver68, Phi68, Wan70]). One may choose not to do this approximation, as in [PMHA97, WHRS05]: in this case, the primitive equations are called quasi-hydrostatic (see [LPR10] for

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a mathematical study).
Since the QG equations are obtained in the zero-limit of the Rossby number (large rotational effects), one could think of retaining all the rotating terms in the original equations, before performing the asymptotic analysis. This is the main purpose of this paper. We will see in (16) that the modified QG equations, that we will call quasi-hydrostatic quasi-geostrophic (QHQG) equations, are very similar to the traditional QG ones, except that they raise a new vertical direction (denoted Z hereafter), which differs from the traditional vertical direction z. The tilt between z and Z is proportional to the nondimensional parameter λ introduced in (2.1), which mesures the ratio between traditional and nontraditional Coriolis terms. Experimental and numerical evidences of this tilted vertical direction can be found in [She04, WB06, GZMvH08, SKR02]: hereafter we provide a first mathematical illustration.

The paper is organized as follows: in Section 2 we perform the rigorous derivation of the QHQG equations, starting from the QH primitive equations. Then, in Section 3, we adapt previous results of [BB94] for the QG equations to obtain local and global existence of solutions to the QHQG model. Last, we present in Section 4 some simple physical properties of the QHQG model.

2 Derivation of the QHQG model

In this section we present the derivation of the quasi-hydrostatic quasi-geostrophic (QHQG) equations. The derivation follows classical principles (as in [BB94]): scaling, asymptotic expansion with respect to a small parameter, equations satisfied at order zero and one. Here, the small parameter (denoted ε in the sequel) is the Rossby number, so that we underline the effect of rotating terms. In order to account for the complete Coriolis force (see *e.g.*, [LPR10] and references therein), we retain all the rotating terms in the original equations, including the terms that are usually neglected in the so-called *traditionnal approximation*. This denomination was introduced by Carl Eckart [Eck60], see also the discussion in [Phi66, Ver68, Phi68, Wan70].

2.1 Scaling Parameters and Scaled Equations

We consider a three-dimensional domain with periodic boundary conditions in the horizontal directions, rigid lid and flat bottom in the vertical. The governing non-hydrostatic equations (sometimes called the incompressible Boussinesq equations), including the complete Coriolis force, read:

$$\frac{Du}{Dt} - fv + f^*w = -\frac{\partial\varphi}{\partial x}, \tag{1a}$$

$$\frac{Dv}{Dt} + fu = -\frac{\partial\varphi}{\partial y}, \tag{1b}$$

$$\frac{Dw}{Dt} - f^*u + \frac{g\,\varrho}{\varrho_0} = -\frac{\partial\varphi}{\partial z}, \tag{1c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1d}$$

$$\frac{D\varrho}{Dt} = 0. \tag{1e}$$

Here (u, v, w) and ϱ are respectively the three-dimensional velocity and density of the fluid, ϱ_0 stands for the averaged density of the fluid, and φ is the renormalized pressure, $\varphi = p/\varrho_0$. The scalars $f = 2\Omega\sin(\theta)$ and $f^* = 2\Omega\cos(\theta)$ are the Coriolis parameters where Ω stands for the angular velocity of the earth and θ is the latitude; g is the universal gravity constant.
Equations (1a) to (1c) describe the conservation of momentum, where D/Dt is the material derivative $D/Dt = \partial/\partial_t + u\partial_x + v\partial_y + w\partial_z$, and Equation (1d) corresponds to the conservation of mass. Finally, Equation (1e) describes the advection of tracers (here the density ϱ). The density and the pressure may be classically decomposed as

$$\varrho(x, y, z, t) = \overline{\rho}(z) + \rho(x, y, z, t)$$

and

$$\varphi(x, y, z, t) = \overline{\varphi}(z) + \phi(x, y, z, t),$$

where $\overline{\rho}$ and $\overline{\varphi}$ are the (known) background density and potential, depending only on the vertical variable. The functions $\overline{\rho}$ and $\overline{\varphi}$ are in hydrostatic balance. We also denote by $N^2(z) = -\frac{d\overline{\rho}}{dz}(z)$ the buoyancy frequency, assuming that $\frac{d\overline{\rho}}{dz}$ is bounded away from zero.

Before going further in the derivation of the corresponding QG model, let us insist on the fact that we keep in Equations (1a) and (1c) the Coriolis terms f^*w and f^*u. This is actually the novelty of this study and will finally lead to a slightly modified QG model (see (14)). We think that it is a relevant modification, since the QG approximation aims at underlying the earth's rotation effects: one should thus include every rotation term in the primitive equations prior to an asymptotic expansion with respect to the Rossby number.
In the context of the β-plane approximation (see [Del11] for example), as suggested in [GZMvH08] we have, with θ_0 the average latitude and R_e the mean radius of the Earth:

$$f = 2\Omega\sin\theta_0 + \frac{1}{R_e}2\Omega\cos\theta_0 y,$$

$$f^* = f_0^* = 2\Omega\cos\theta_0.$$

We now introduce the following dimensionless variables, as it is classically done in QG modeling:

$$
\begin{aligned}
(x,y) &= L\,(x',y'), & z &= H\,z', & t &= \frac{L}{U}t', \\
u &= U\,u', & v &= U\,v', & w &= \frac{UH}{L}w', \\
\overline{\rho} &= P\overline{\rho}', & \rho &= \tfrac{\varrho_0 f_0 UL}{gH}\rho', & \phi &= f_0 UL\phi'.
\end{aligned}
$$

The Rossby number $\varepsilon = U/f_0L$ is the fundamental ordering parameter in the following asymptotic expansion. A secondary ordering parameter is the scale ratio of ρ to ϱ: this ratio is assumed to be ε, that is, we assume:

$$\frac{\varrho_0 f_0 UL}{gH} = P\varepsilon.$$

Finally, the density may be expressed in terms of nondimensionnal quantities:

$$\varrho = P\Big(\overline{\rho}'(z) + \varepsilon\rho'\Big),$$

as well as the Coriolis parameter:

$$f = f_0(1+\varepsilon\beta_0 y') \quad \text{with } f_0 = 2\Omega\sin\theta_0, \text{ and } \beta_0 = \frac{2\Omega L^2\cos\theta_0}{UR_e}.$$

Another usual nondimentional number is the aspect ratio $\delta = H/L$; δ also appears in the ratio between the two Coriolis terms in the zonal momentum equation (1a), which scales as

$$\lambda = \delta\cot\theta_0.$$

When we considered the scaling numbers introduced above, we have implicitely assumed that the leading term at the left-hand-side of Equation (1a) was fv, which means that $\delta\cot\theta_0$ should not be too large:

$$\lambda \lesssim 1. \tag{2}$$

Fortunately, because the aspect ratio $\delta = H/L$ is rather small in large ocean models, the condition (2) is easily satisfied. However, the objective of the present work is to draw the reader's attention on the fact that λ is not *necessarily* small, and that it may have some physical repercussions (see Section 4).

We end this section with the non-hydrostatic scaled equations (we naturally drop the primes):

$$\begin{aligned}
\varepsilon \frac{Du}{Dt} - (1+\varepsilon\beta_0 y)v + \lambda w &= -\frac{\partial \phi}{\partial x}, && (3a)\\
\varepsilon \frac{Dv}{Dt} + (1+\varepsilon\beta_0 y)u &= -\frac{\partial \phi}{\partial y}, && (3b)\\
\varepsilon\delta^2 \frac{Dw}{Dt} - \lambda u + \rho &= -\frac{\partial \phi}{\partial z}, && (3c)\\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, && (3d)\\
\varepsilon \frac{D\rho}{Dt} + w\overline{\rho}_z &= 0, && (3e)
\end{aligned}$$

where we recall that ε is the Rossby number (meant to go to zero), $\delta = H/L$ is the domain aspect ratio, and $\lambda = \delta \cot\theta_0$.

Remark 1. *Actually, the term $\varepsilon\delta^2 Dw/Dt$ could be set to zero in* (3c) *above with no modification in the sequel: indeed the reader will see below that $w^{(0)} = 0$, hence $\varepsilon\delta^2 Dw/Dt = O(\varepsilon^2)$ can be neglected prior to the QG approximation.*
The new model is thus called QHQG since the differences between the new QG model and the traditional one rely only on the terms related to the Coriolis force.

2.2 Geostrophic Balance

We now consider an asymptotic expansion of all variables with respect to the Rossby number: for every unknown function γ, we write the formal asymptotic expansion

$$\gamma = \gamma^{(0)} + \varepsilon\gamma^{(1)} + \varepsilon^2\gamma^{(2)} + ...$$

where $(\gamma^{(j)})_{j\geq 0}$ behave as $O(1)$ as ε goes to zero. Equations (3a)-(3c) give, keeping only the order zero terms in ε:

$$\begin{aligned}
-v^{(0)} + \lambda w^{(0)} &= -\phi_x^{(0)}, && (4a)\\
u^{(0)} &= -\phi_y^{(0)}, && (4b)\\
-\lambda u^{(0)} + \rho^{(0)} &= -\phi_z^{(0)}. && (4c)
\end{aligned}$$

The incompressibility condition reads $w_z^{(0)} = -u_x^{(0)} - v_y^{(0)}$ and this *traditionally* leads to $w^{(0)} = 0$, thanks to Equations (4a), (4b) and boundary conditions on w (see [BB94]). Here, the incompressibility condition does not provide $w_z^{(0)} = 0$, but we have, denoting $\partial_Z = \partial_z + \lambda\partial_y$:

$$\begin{aligned}
w_Z^{(0)} &= w_z^{(0)} + \lambda w_y^{(0)}\\
&= -u_x^{(0)} - v_y^{(0)} + \lambda w_y^{(0)}\\
&= \mathrm{curl}(\phi_y, \phi_x)\\
w_Z^{(0)} &= 0. && (5)
\end{aligned}$$

Thanks to (5) and to homogeneous boundary conditions on $w^{(0)}$, we finally obtain that $w^{(0)} = 0$. Alternatively, we have Equation (3e) which (written to the order zero and since $\overline{\rho}_z$ never vanishes) leads to $w^{(0)} = 0$.

The geostrophic equations read:

$$\begin{aligned} -v^{(0)} &= -\phi_x^{(0)}, && (6a)\\ u^{(0)} &= -\phi_y^{(0)}, && (6b)\\ \rho^{(0)} &= -\phi_z^{(0)} - \lambda\phi_y^{(0)} = -\phi_Z^0, && (6c)\\ w^{(0)} &= 0. && (6d)\end{aligned}$$

2.3 Quasi-Geostrophic Equations

Now we need the first order equations in order to determine the evolution of $\phi^{(0)}$. We denote by d_g the zero-order material derivative:

$$d_g = \partial_t + u^{(0)}\partial_x + v^{(0)}\partial_y.$$

The first order equations are:

$$\begin{aligned} d_g u^{(0)} - \beta_0 y v^{(0)} - v^{(1)} + \lambda w^{(1)} &= -\phi_x^{(1)}, && (7a)\\ d_g v^{(0)} + \beta_0 y u^{(0)} + u^{(1)} &= -\phi_y^{(1)}, && (7b)\\ -\lambda u^{(1)} + \rho^{(1)} &= -\phi_z^{(1)}, && (7c)\\ u_x^{(1)} + v_y^{(1)} + w_z^{(1)} &= 0, && (7d)\\ d_g\rho^{(0)} + w^{(1)}\overline{\rho}_z = 0. & && (7e)\end{aligned}$$

We now take the curl of Equations (7a)-(7b) to obtain, thanks to Equation (7d)

$$d_g(v_x^{(0)} - u_y^{(0)}) - w_z^{(1)} - \lambda w_y^{(1)} + \beta_0 v^{(0)} = 0. \tag{8}$$

We notice, as for the traditional QG equations, that $\beta_0 v^{(0)} = d_g(\beta_0 y)$. We thus try to express $-w_z^{(1)} - \lambda w_y^{(1)} = -w_Z^{(1)}$ as $d_g(\Gamma)$ where Γ is a function to be defined. To this aim, we will extensively make use of Equation (7e) that we reformulate:

$$w^{(1)} = N^{-2} d_g\rho^{(0)} = d_g\left(N^{-2}\rho^{(0)}\right). \tag{9}$$

Given (9), we may compute the required quantity

$$w_z^{(1)} + \lambda w_y^{(1)} = \left(d_g\left(N^{-2}\rho^{(0)}\right)\right)_z + \lambda\left(d_g\left(N^{-2}2\rho^{(0)}\right)\right)_y. \tag{10}$$

We remark that for any function γ and any variable $*$ we have the identity

$$\left(d_g(\gamma)\right)_* = d_g\left(\gamma_*\right) + u_*^{(0)}\partial_x\gamma + v_*^{(0)}\partial_y\gamma,$$

so that we can write Equation (10)

$$w_z^{(1)} + \lambda w_y^{(1)} = d_g\left(\left(N^{-2}\rho^{(0)}\right)_z\right) + \lambda d_g\left(\left(N^{-2}\rho^{(0)}\right)_y\right) + R, \tag{11}$$

where the remainder R, according to the remark above, is

$$\begin{aligned} R &= u_z^{(0)}(N^{-2}\rho^{(0)})_x + v_z^{(0)}(N^{-2}\rho^{(0)})_y \\ &\quad + \lambda u_y^{(0)}(N^{-2}\rho^{(0)})_x + \lambda v_y^{(0)}(N^{-2}\rho^{(0)})_y \\ &= (N^{-2}\rho^{(0)})_x\,(u_z^{(0)} + \lambda u_y^{(0)}) + (N^{-2}\rho^{(0)})_y\,(v_z^{(0)} + \lambda v_y^{(0)}). \end{aligned}$$

Using (6a)-(6c) again, we have

$$R = N^{-2}\rho_x^{(0)}\rho_y^{(0)} - N^{-2}\rho_y^{(0)}\rho_x^{(0)} = 0,$$

which simplifies Equation (11) as follows:

$$w_Z^{(1)} = w_z^{(1)} + \lambda w_y^{(1)} = d_g\left(\left(N^{-2}\rho^{(0)}\right)_z + \lambda\left(N^{-2}\rho^{(0)}\right)_y\right). \tag{12}$$

Back to Equation (8), we obtain the quasi-hydrostatic quasi-geostrophic equation:

$$d_g\Big(v_x^{(0)} - u_y^{(0)} - (N^{-2}\rho^{(0)})_z - \lambda(N^{-2}\rho^{(0)})_y + \beta_0 y\Big) = 0. \tag{13}$$

The potential vorticity

$$v_x^{(0)} - u_y^{(0)} - (N^{-2}\rho^{(0)})_z - \lambda(N^{-2}\rho^{(0)})_y + \beta_0 y$$

is thus conserved along material paths.

Let us now rewrite Equation (13), expressing everything in terms of $\phi^{(0)}$. We have

$$\Big(\partial_t - \phi_y^{(0)}\partial_x + \phi_x^{(0)}\partial_y\Big)\zeta = 0, \tag{14}$$

$$\zeta = \Delta\phi^{(0)} + N^{-2}(\partial_z + \lambda\partial_y)^2\phi^{(0)} + N_z^{-2}(\partial_z + \lambda\partial_y)\phi^{(0)} + \beta_0 y, \tag{15}$$

where Δ is horizontal Laplacian operator. We could write this more compactly with $\partial_Z = \partial_z + \lambda\partial_y$,

$$\Big(\partial_t - \phi_y^{(0)}\partial_x + \phi_x^{(0)}\partial_y\Big)\Big(\Delta\phi^{(0)} + (N^{-2}\phi_Z^{(0)})_Z + \beta_0 y\Big) = 0. \tag{16}$$

One can thus easily recognize the traditional QG equation (see Equation (2.23) in [BB94]), except that the differential operator ∂_z is replaced by $\partial_Z = \partial_z + \lambda\partial_y$. We recall here that $\lambda = \delta\cot\theta_0$ is proportional to the domain aspect ratio. In particular, we recover the traditional QG equation when setting $\delta = 0$ in Equations (13), (14) or (16).

3 Existence of solutions

In this section, we are interested in existence results for the quasi-hydrostatic quasi-geostrophic equation (14). We consider a periodic domain in the horizontal variables with rigid boundaries at the top and bottom. More precisely, we denote by $B = \Sigma \times (0,h) = \left(-\frac{1}{2},\frac{1}{2}\right) \times \left(-\frac{1}{2},\frac{1}{2}\right) \times (0,h)$ the spatial domain where we establish existence of solutions, with $t \in [0,T]$ (where $T > 0$) the time interval.

We assume the solutions to be horizontally periodic with period 1. The pressure potential $\phi^{(0)}$ is defined by Equation (16) up to a constant at each time; then we add the following condition: $\int_B \phi^{(0)} = 0$. We define $\mathbf{u} = (u^{(0)}, v^{(0)})$ the horizontal velocity at the leading order, we omit the (0) superscripts and rewrite Equation (14) as:

$$
\begin{aligned}
v &= \phi_x, && (17)\\
u &= -\phi_y, && (18)\\
\rho &= -\phi_z - \lambda\phi_y, && (19)\\
\omega &:= v_x - u_y - \left(N^{-2}\rho\right)_z - \lambda\left(N^{-2}\rho\right)_y \quad \text{in } B, && (20)\\
\omega_t + \mathbf{u}\cdot\nabla\omega &= -\beta_0 v \quad \text{in } B\times[0,T], && (21)
\end{aligned}
$$

with the initial condition

$$\omega(x,y,z,0) = \omega_0(x,y,z), \tag{22}$$

and the boundary conditions

$$\rho = 0 \qquad \text{at } z = 0 \text{ and } z = h. \tag{23}$$

Existence results are based on the following property:

Lemma 1. *The partial differential operator $\mathcal{L}$ defined by*

$$\mathcal{L}\phi := \Delta\phi + \left(N^{-2}\left(\phi_z + \lambda\phi_y\right)\right)_z + \lambda\left(N^{-2}\left(\phi_z + \lambda\phi_y\right)\right)_y \ \ \textit{in } B$$

is uniformly elliptic.

Proof. The function $N^{-2}(z)$ is assumed to be bounded away from 0 for $z \in [0,h]$. We expand $\mathcal{L}$ as:

$$
\begin{aligned}
\mathcal{L}\phi &= \phi_{xx} + \left(1+\lambda^2N^{-2}\right)\phi_{yy} + N^{-2}\phi_{zz} + 2\lambda N^{-2}\phi_{yz} + \lambda\left(N^{-2}\right)_z\phi_y + \left(N^{-2}\right)_z\phi_z\\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\partial^2_{i,j}\phi + \sum_{i=1}^{3} b_i\partial_i\phi,
\end{aligned}
$$

where:

$$A = \left(a_{ij}\right)_{i,j=1\dots3} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1+\lambda^2N^{-2} & \lambda N^{-2}\\ 0 & \lambda N^{-2} & N^{-2}\end{pmatrix}.$$

For all $\xi = (\xi_1, \xi_2, \xi_3)^\intercal$, we have, for almost every z in $[0, h]$:

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\xi_i\xi_j &= \xi_1^2 + \left(1+\lambda^2 N^{-2}\right)\xi_2^2 + N^{-2}\xi_3^2 + 2\lambda N^{-2}\xi_2\xi_3 \\ &\geq \xi_1^2 + \frac{1}{2}\xi_2^2 + \frac{1}{2\lambda^2+N^2}\xi_3^2 \quad \text{(thanks to Young inequality)} \\ &\geq \theta|\xi| \quad \text{with } \theta := \min\left(\frac{1}{2}, \frac{1}{2\lambda^2+N^2}\right) > 0.\end{aligned}$$

□

The condition $\theta > 0$ is satisfied as soon as $\lambda > 0$. In the case where $\lambda = 0$ (traditional equation), one needs to assume that N^{-2} is bounded away from zero (see [BB94]).
Lemma 1 generalizes the ellipticity result of [BB94] for any $\lambda \in \mathbb{R}$ and we can now obtain the same existence results. Indeed, System (17)–(21) only differs from the corresponding system in [BB94] in Equations (19) and (20).

We first obtain the local existence of solutions:

Theorem 1 (Short-time existence of solutions to the QHQG model)**.** *If the initial vorticity ω_0 is in $H^s(B)$ for some $s \geq 3$ with $|\omega_0|_s \leq M$, then there exists a time $T^* > 0$ and a solution ω in $\mathcal{C}([0,T^*], H^s(B))$ to the QHQG model, where T^* depends only on M, B, λ and β_0. The vorticity ω satisfies the estimate $\|\omega\|_{s,T^*} \leq 2M$.*

Proof. The proof follows the lines of [BB94], considering an iterative process:

- starting from $\xi^0(x,y,z,t) = \omega_0(x,y,z)$ given,
- for $k \geq 0$ compute ϕ^k thanks to the relation:

$$\begin{cases} \phi^k_{xx} + \phi^k_{yy} + \left(\dfrac{\phi^k_z + \lambda\phi^k_y}{N^2}\right)_z + \dfrac{\lambda}{N^2}\left(\phi^k_z + \lambda\phi^k_y\right)_y = \xi^k \text{ in } B, \\ \phi^k_z + \lambda\phi^k_y = 0 \quad \text{for } z = 0 \text{ and } z = h, \\ \displaystyle\int_B \phi^k = 0, \end{cases} \tag{24}$$

- then set $u^k = -\phi^k_y$ and $v^k = \phi^k_x$,
- for every $z \in [0, h]$, compute ξ^{k+1} solution of

$$\xi^{k+1}_t + \left(u^k, v^k\right)^\intercal \cdot \nabla\xi^{k+1} = -\beta_0 v^k,$$

  with $\xi^{k+1}(t=0) = \omega_0$, with periodic boundary conditions in x and y.

The elliptic result (Lemma 1) still gives the estimate:

$$|\phi^k|_{s+2} \leq C_0|\xi^k|_s \quad \text{with } C_0 = C_0(B, N(z), \lambda) > 0$$

such that, Equations (17), (18) and (21) being unchanged, the short-time existence of the k-th vorticity iterate ξ^k follows, as well as the upper bound on its H^s norm.

The end of the proof, for the convergence of the iterates $\{\xi^k(t)\}_{k\geq 0}$, is not modified by the λ coefficient. □

We are also able to adapt the proof of global solutions for the QG model by [BB94] to get:

Theorem 2 (Global existence of solutions to the QHQG model)**.** *If ω_0 is in $H^s(B)$ for some $s \geq 3$, then given any time $T > 0$, there exists a solution ω in $\mathcal{C}([0,T], H^s(B))$ to the QHQG model.*

Proof. From Lemma 1, relations (17), (18), (19) and from

$$\begin{cases} \phi_{xx} + \phi_{yy} + \left(\dfrac{\phi_z + \lambda\phi_y}{N^2}\right)_z + \dfrac{\lambda}{N^2}\left(\phi_z + \lambda\phi_y\right)_y = \omega \text{ in } B, \\ \phi_z + \lambda\phi_y = 0 \quad \text{ for } z = 0 \text{ and } z = h, \\ \displaystyle\int_B \phi = 0, \end{cases}$$

we have the elliptic estimate:

$$|\mathbf{u}|_{s+1} + |\rho|_{s+1} \leq C|\omega|_s,$$

such that we also have a global estimate on the QHQG vorticity ω of the form

$$|\omega(t)|_s \leq K(t) \quad \forall t \in [0,T],$$

where the function K only depends on the H^s norm of the initial condition ω_0. Thanks to the short-time existence, the global existence can be proved, using the global bound

$$|\omega(t)|_s \leq \max_{t\in[0,T]} K(t) \quad \forall t \in [0,T]$$

to reach any time T thanks to an iterative process. □

4 Simple physical properties of the QHQG model

We detail here some basic physical properties that can be obtain for the QHQG model and we perform comparisons with the well-known QG system.

We reformulate the QHQG equation (16) as follows:

$$D[q + \beta y] = 0, \tag{25}$$

$$\Big(\Delta + \partial_Z[\frac{1}{N^2}\partial_Z]\Big)\phi = q, \tag{26}$$

where $D = \partial_t + J[\phi,\ ]$, $J[a,b] = a_x b_y - a_y b_x$, $\Delta = \partial_x^2 + \partial_y^2$, $\partial_Z = \partial_z + \lambda\partial_y$, and $\lambda = \dfrac{H}{L}\cot\theta_0$.

4.1 Coordinate transformation

We define a coordinate transform from $(\mathbf{x}, t) = (x, y, z, t)$ to $(\mathbf{X}, T) = (X, Y, Z, T)$ by

$$X = x, \quad Y = y - \lambda z, \quad Z = z, \quad T = t. \tag{27}$$

The transformed QHQG system is isomorphic to (26) because $D = \partial_T + J[\phi, \ ]$, $\Delta = \partial_X^2 + \partial_Y^2$, and $J[a, b] = a_X b_Y - a_Y b_X$ have identical functional forms in the transformed coordinates; $N^2(z) = N^2(Z)$; and $\beta y = \beta(Y + \lambda Z)$ can be replaced by βY because $D[\lambda Z] = 0$. Thus, all QG solutions $\phi(\mathbf{x}, t)$ are also QHQG solutions $\phi(\mathbf{X}, T)$ if the initial and boundary conditions are consistent. Initial conditions are equivalently specified at $t = 0$ or $T = 0$. Planar vertical boundary surfaces at $z = z_0$ are also planar at $Z = z_0$. The same is true in x and X. Only in y is a planar surface in y no longer planar in Y; therefore, the simple equivalence is for solutions that have an unbounded y domain, or are y-periodic, or decay away in y before the boundary.

4.2 Separable solutions and vertical modes

QHQG has vertically separable solutions when linearized by neglecting $J[\phi, q]$, even for general $N(z)$, $\phi = \Phi(X, Y, T)F(Z)$. (QG has an analogous property.) Such QHQG solutions have a vertically upward phase tilt in the (y, z) plane with a slope of $dz/dy = \lambda^{-1}$ relative to the z axis. This direction is aligned with the full rotation vector. Many phenomena influenced by the non-traditional approximation (*e.g.*, convection and centrifugal instability as reviewed in [GZMvH08]) are known to exhibit this type of phase tilt. If we assume constant density at the vertical boundaries, F is determined by the 1D eigenvalue problem,

$$(N^{-2} F_Z)_Z + \mathcal{R}^{-2} F = 0 \,, \qquad F_Z = 0 \text{ at } Z = Z_0, Z_1 \,. \tag{28}$$

The eigenvalue $\mathcal{R}$ is the horizontal radius of deformation. With N constant, the eigenmodes F are cosine functions in Z.

4.3 Thermal wind balance

The geostrophic equations (6a)-(6c) have a steady solution for a zonal flow. This can be expressed as $u = U(Y, Z)$, $\varrho = -\int^Z N^2(Z')\, dZ' + B(Y, Z)$ with $U_Z = -B_Y$, which is thermal wind balance.

4.4 Rossby wave modes

One class of simple solutions is Rossby wave modes. They satisfy a linearized

PDE usually justified by an assumption of small amplitude flow,

$$\frac{\partial q}{\partial t} + \beta \frac{\partial \phi}{\partial x} = 0. \tag{29}$$

With constant N in either a vertically bounded or unbounded domain, eigenmodes are

$$\phi \propto e^{i(kx+ly+mz-\omega t)}, \tag{30}$$

with a dispersion relation,

$$\omega = \frac{-\beta k}{K^2}, \quad K^2 = k^2 + l^2 + \frac{1}{N^2}(m + \lambda l)^2. \tag{31}$$

We can equivalently write this as

$$\phi \propto e^{i(kX+lY+MZ-\omega T)}, \tag{32}$$

with $M = m + \lambda l$.

Thus, $\lambda \neq 0$ implies lower frequency and slower phase speed compared to QG Rossby waves with the same wavenumber (k, l, m). We also consider the group velocity for wave energy propagation, $c_g = \partial_k \omega$. Its components are

$$\begin{aligned} \partial_k \omega &= \frac{\beta(2k^2 - K^2)}{K^4}, \\ \partial_l \omega &= \frac{2\beta k(l + \lambda(m + \lambda l)/N^2)}{K^4}, \\ \partial_m \omega &= \frac{2\beta k(m + \lambda l)/N^2}{K^4}. \end{aligned} \tag{33}$$

This usually has a more westward zonal propagation due to $\lambda \neq 0$, and its (y, z) propagation is altered as well. These effects also occur for the barotropic mode ($m = 0$) where the vertical propagation is nonzero because of the QH correction.

4.5 Vortex solutions

A simple vortex solution is a nonlinear stationary state when $\beta = 0$ and N is constant. In QG this occurs for any axisymmetric profile, $\phi(r, z)$, where $r = \sqrt{x^2 + y^2}$ is the radial coordinate, which is sufficient to make $J[\phi, q] = 0$ even when the arguments have large amplitude. Physical interest usually lies in profiles that are localized in r, *e.g.*, a monopole with $\phi(r)$ decaying away from a central extremum. In QHQG nonlinear stationary solutions exist for any profile $\phi(R, Z)$, where $R^2 = X^2 + Y^2$. Thus, QH vortices are meridionally tilted rather than verically aligned. This tilted structure was previously proposed for Meddy vortices by [SS03].

4.6 Fourier-space estimates for non-traditional effects

Assuming a wavenumber-space characterization of the solution, as commonly done for turbulent flows, in terms of vertical and horizontal wavenumbers, kv and kh, or equivalently local values of H and L, we can ask when the λ value is not small.

In QG the common view (sometimes called Charney's stretched isotropy) is that the Burger number, $Bu = NH/2\Omega L = kh/(2\Omega kv/N)$, is order one while $2\Omega/N$ is small. If $H/L \sim 2\Omega/N \ll 1$, then $\lambda \sim (2\Omega/N)\cot\theta_0$, which will be small except when θ_0 is very close to the Equator.

Alternatively, if we consider flow patterns with $2\Omega kv/Nkh \sim r \ll 1$, then $\lambda \sim (2\Omega/Nr)\cot\theta_0$. For small enough r, λ need not be small, and the QH correction to QG will be important. This happens when the aspect ratio H/L is large, the stratification is weak, and/or θ_0 is small. Large aspect ratio can be described from the QG perspective as atypically "tall" flows.

5 Summary

The quasi-hydrostatic equations have been used to derive a new quasi-geostrophic model for the simulation of geophysical fluid dynamics. The new QHQG model is derived in Section 2, in the same spirit than the usual derivation of the traditional QG model, such as in [BB94]. The resulting QHQG model (16) is slightly different from the traditional one, in the sense that the traditional operator ∂z is changed in $\partial Z = \partial z + \lambda\partial y$ where λ measures the effect of nontraditional terms of the Coriolis force, $\lambda = (H/L)\cot\theta_0$. Mathematical properties such as local or global existence of solutions have been proved, in Section 3, in the same way as for the traditional QG equation.

In Section 4, we discuss some physical properties of the QHQG model, and point the similarities / differences with the traditional QG model. This discussion can be seen as a first theoretical illustration of well-known phenomena that have already been mentioned in the literature.

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