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# A new family of second-order absorbing boundary conditions for the acoustic wave equation - Part I: Construction and mathematical analysis

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**Abstract:** This paper adresses the question of constructing and analysing a one-parameter family of absorbing boundary conditions for the acoustic wave equation. We study the well-posedness of the corresponding boundary value problem and the long-time behavior of the solution.

**Key-words:** Absorbing boundary conditions, acoustic wave equation, well-posedness, long-time behavior

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# Une nouvelle famille de conditions aux limites absorbantes pour l'équation des ondes acoustiques -Partie I : Construction et analyse mathématique

**Résumé :** Ce travail porte sur la construction et l'analyse d'une famille de conditions aux limites absorbantes dépendant d'un paramètre pour l'équation des ondes acoustiques. On étudie le caractère bie posé du problème aux limites correspondant et le comportement en temps long de la solution.

**Mots-clés :** Conditions aux limites absorbantes, équation des ondes acoustiques, problème bien posé, comportement en temps long

# 1 Introduction

The numerical simulation of acoustic waves is often carried out after truncating the propagation domain. The size of the computational domain is then reduced and the spatial approximation can be done with finite element spaces. The corresponding boundary value problem is then defined by the coupling of the initial system of equations with a boundary condition set on the external boundary limiting the computational domain. This boundary condition is used to close the problem and it represents the behavior of any wave impinging the external boundary. A "good" condition should be the one for which the numerical waves are not reflected by the external boundary. The computed field would thus be exactly the restriction of the wave that propagates inside the larger domain for which the exteranl boundary has been pushed away. The construction of "good" conditions, that are generally called Absorbing Boundary Conditions (ABCs) has been consideed in a lot of works. Regarding time-dependent problems, Engquist and Majda [7] have shown that the micro-diagonalization of the wave equation, and more generally of hyperbolic systems, is an interesting way to construct ABCs on arbitrarily-shaped surfaces. Their work has been followed by a series of papers in which the issue was to construct high-order ABCs which can be used to improve the accuracy of the numerical waves by diminishing the strength of the reflected waves. For instance, Higdon [11] has constructed high-order ABCs which are given as the combination of low-order ABCs. Grote and Keller [8] have proposed non reflecting ABCs which are obtained from the approximation of the DtN operator of the sphere. There are many other interesting papers on the subject and we refer to the review article [15] for an exhaustive list of references.

In this paper, we tackle the question of constructing and analyzing a one-parameter family of ABCs for the acoustic wave equation. The results that are presented in this report are part of a work devoted to the construction of enriched ABCs that are able to attenuate the reflections generated by the complete wave field. The resulting ABCs are then deduced from the approximation of the full DtN operator which is defined both in the hyperbolic and the elliptic regions and that can reproduced also the behavior of glancing waves.

Herein, we aim at selecting the "best" ABC for propagating waves. Our criteria are the following:

1. the ABC can be applied on arbitrarily shaped boundaries;
2. the ABC must be local and it involves low-order differential operators;
3. the corresponding boundary value problem (BVP) is well-posed;
4. the BVP is associated with a Lyapunov functional (an energy) which decreases to zero, except possibly for a given set of data.

The first criterion allows us to consider computational domains which size can be optimized by using well-adapted external boundaries. For instance, in case of a scattering problem, the external boundary can be chosen in keeping with the boundary of the scatterer or at least of its convex hull. This property is particulary interesting when the scatterer is elongated, as a submarine for instance.

The second criterion ensures that the sparsity of the finite element matrices is

kept and that the ABC is easy to include inside the numerical scheme.
The third criterion gives a stability result with respect to the data, which is essential for numerical simulations.
The last criterion implies that the solution of the BVP is long-time stable, which is not obvious for each existing ABC.
Our work is motivated by the question of constructing an enriched ABC which is able to attenuate the full acoustic reflected wave. The condition must then be active for propagating, evanescent and creepy waves. We therefore believe to derive a full condition by combining terms representing each of the waves, as it was formerly suggested by [10]. Hence in this report, we show that it is possible to derive a family of low-order ABCs for acoustic propagating waves which satisfies all the criteria 1-4.

# 2 A new family of second-order ABCs for the acoustic wave equation

In this section, we are interested in constructing a new absorbing boundary condition using the micro-diagonalization method developed by M.E. Taylor [14]. We want this condition to be written for all regular convex domains and to take into account propagating waves. First, we present the main steps of the micro-diagonalization method. For the sake of simplicity, the velocity $c$ is supposed to be equal to 1.

## 2.1 The micro-diagonalization method applied to the wave equation

In this section, we tackle the construction of ABCs by applying the factorization theorem initially established by M.E. Taylor [14] to study the propagation of singularities of strictly hyperbolic systems. Because we want to build low-order conditions, we limit our work to the application of the first step of factorization. We are thus only dealing with the diagonalization of the principal symbol of the wave equation. Indeed, the mathematical analysis in [2] shows that the following steps necessarily involve differential operators with order higher than two.
The ABCs that we consider are derived from the micro-local approximation of the Dirichlet-to-Neumann operator related to the artificial surface $\Sigma$. We thus begin with rewriting the acoustic wave equation in a local coordinate system $(r, s)$. The couple $(r, s)$ describes a point in the neighborhood of $\Sigma$ in such a way that $\Sigma = \{r = 0\}$. We use the same coordinate system as in [3] and the acoustic wave equation reads then as

$$\partial_t^2 u - \partial_r^2 u - \kappa_r \partial_r u - h^{-1}\partial_s \left(h^{-1}\partial_s u\right) = 0, \tag{2.1}$$

where $\kappa$ is the curvature of $\Sigma$, $h = 1 + r\kappa(s)$ and $\kappa_r = h^{-1}\kappa$.
Next, to apply Taylor's method, we rewrite (2.1) as a first-order system. We thus introduce an auxiliary unknown $v$ which satisfies $\partial_t v + \partial_r u = 0$ in a neighborhood of $\Sigma$ and if $\mathbf{U}$ denotes the field $\mathbf{U} = (v, u)$, $\mathbf{U}$ is solution to the first-order system

$$\partial_r \mathbf{U} = L\mathbf{U}.$$

The entries of $L$ are first-order pseudodifferential operators and $\sigma(L) = \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_0$ is given by

$$\mathcal{L}_1 = \begin{pmatrix} 0 & -\frac{h^{-2}\xi^2-\omega^2}{i\omega} \\ -i\omega & 0 \end{pmatrix} \in S^1 \quad \text{and} \quad \mathcal{L}_0 = \begin{pmatrix} -\kappa_r & -\frac{h^{-3}\partial_s(h)\xi}{\omega} \\ 0 & 0 \end{pmatrix} \in S^0,$$

where $\xi$ and $\omega$ are the dual variables associated respectively to $s$ and $t$. We use standard notations such as $\sigma(L)$ for the symbol of $L$. Thereafter, as in [14], we more concisely note by $S^m$ the symbol class $S^m_{1,0}$ introduced by [12]. We also denote by $\tau_{-m}(\sigma(L))$ the truncation of the asymtpotic expansion at the order $m$ of the symbol $\sigma(L)$.

Let $\lambda_1$ denote the symbol $\lambda_1 = \left(h^{-2}\xi^2 - \omega^2\right)^{1/2}$. Then, when $\lambda_1 \neq 0$, the principal symbol of $\mathcal{L}$ admits two single eigenvalues $\lambda_1$ and $-\lambda_1$. When $h^{-2}\xi^2 - \omega^2 > 0$, $\lambda_1$ is real and when $h^{-2}\xi^2 - \omega^2 < 0$, $\lambda_1$ is imaginary. Following [2, 7], we introduce

$$\mathcal{V}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{i\omega}{\lambda_1} & 1 \\ 1 & -\frac{\lambda_1}{i\omega} \end{pmatrix} \tag{2.2}$$

and then we have

$$\mathcal{V}_0 \mathcal{L}_1 \mathcal{V}_0^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix}. \tag{2.3}$$

Now, a first-order ABC can be derived from the approximation and the localization of the global boundary condition

$$\left(\left(I + K_{-1}\right) V_0 \mathbf{U}\right)_2 = 0 \text{ on } \Sigma, \tag{2.4}$$

where $K_{-1}$ is a regularizing operator which principal symbol $\mathcal{K}_{-1}$ is given by

$$\mathcal{K}_{-1} = \begin{pmatrix} 0 & -\frac{i\kappa\omega^3}{4\lambda_1^4} \\ -\frac{i\kappa\omega}{4\lambda_1^2} & 0 \end{pmatrix}. \tag{2.5}$$

**Remark** When $h^{-2}\xi^2 - \omega^2 > 0$ , the frequencies $(\omega, \xi)$ cover the elliptic region. If not, $(\omega, \xi)$ lie in the hyperbolic region.

**Remark** The matrix $\mathcal{K}_{-1}$ is actually not unique because its diagonal coefficients are not fixed by the diagonalization process. Here we have chosen zero but we could have used any functions in $S^{-1}$.

This approach has been used by several authors, following Engquist and Majda [7]. Each of them only considers the case where the diagonal coefficients of $\mathcal{K}_{-1}$ are zero and $\lambda_1$ is imaginary corresponding to frequencies in the hyperbolic region. In this configuration, (2.4) can be rewritten, after using a truncated Taylor expansion on $\lambda_1$ for $\omega^2 >> h^{-2}\xi^2$, as:

$$\partial_t u + \partial_n u + \frac{\kappa}{2} u = 0 \text{ on } \Sigma.$$

The resulting condition involves differential operators but it should be called micro-differential since it is justified in the propagating cone $\omega^2 >> h^{-2}\xi^2$.This condition is widely used in case of curved surfaces. We will refer to this condition as the C-ABC for curvature ABC.

In the following, we show how to modify this ABC by using (2.4) again.

## 2.2 Improving the C-ABC in the hyperbolic region

A family of ABCs can be derived when $\mathcal{K}_{-1}$ is modified by introducing a non-zero diagonal term. This idea has been formerly applied in [4] for the 2D Maxwell system but only from a theoretical point of view. To the best of our knowledge, the numerical impact of using these conditions has never been investigated. Herein, we propose to modify $\mathcal{K}_{-1}$ as follows

$$\mathcal{K}_{-1} = \begin{pmatrix} 0 & -\dfrac{i\kappa\omega^3}{4\lambda_1^4} \\ -\dfrac{i\kappa\omega}{4\lambda_1^2} & \dfrac{\gamma(s)}{\lambda_1} \end{pmatrix}, \tag{2.6}$$

where $\gamma$ is a parameter depending only on the curvilinear abscissa $s$. Let us note that we do not change the first diagonal entry since it is not involved in condition (2.4). We then get

**Theorem 2.1.** *A family of first-order condition depending on a parameter is*

$$\partial_t \left(\partial_n u + \partial_t u\right) = \left(\frac{\kappa}{4} - \gamma\right) \partial_n u - \left(\frac{\kappa}{4} + \gamma\right) \partial_t u \text{ on } \Sigma. \tag{2.7}$$

*Proof.* We recall that the first-order boundary condition is given by

$$\left(\left(I + K_{-1}\right) V_0 \mathbf{U}\right)_2 = 0 \text{ on } \Sigma. \tag{2.8}$$

The symbol of the corresponding operator reads as

$$\sigma\left(\left(I + K_{-1}\right) V_0\right) = \left(\left(\mathcal{I}_2 + \mathcal{K}_{-1}\right) \mathcal{V}_0\right) + \mathcal{R}_{-2},$$

where $\mathcal{R}_{-2} \in S^{-2}$. Therefore, the truncation of $\sigma\left(\left(I + K_{-1}\right) V_0\right)$ in $S^{-1}$ is given by

$$\begin{aligned} \tau_{-1}\left(\left(I + K_{-1}\right) V_0\right) &= \left(\left(\mathcal{I}_2 + \mathcal{K}_{-1}\right) \mathcal{V}_0\right) \\ &= \tfrac{1}{\sqrt{2}} \begin{pmatrix} \dfrac{i\omega}{\lambda_1} - \dfrac{i\kappa\omega^3}{4\lambda_1^4} & 1 + \dfrac{\kappa\omega^2}{4\lambda_1^3} \\ \dfrac{\kappa\omega^2}{4\lambda_1^3} + 1 + \dfrac{\gamma(s)}{\lambda_1} & -\dfrac{i\kappa\omega}{4\lambda_1^2} - \dfrac{\lambda_1}{i\omega} - \dfrac{\gamma(s)}{i\omega} \end{pmatrix}. \end{aligned} \tag{2.9}$$

Using a first-order Taylor expansion for $\omega >> h^{-1}\xi$, we then obtain

$$\left(\tau_{-1}\left(\mathcal{I}_2 + \mathcal{K}_{-1}\right) \mathcal{V}_0\right)_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 - \dfrac{i\kappa}{4\omega} & 1 + \dfrac{i\kappa}{4\omega} \\ 1 + \dfrac{i\kappa}{4\omega} + \dfrac{\gamma}{i\omega} & \dfrac{i\kappa}{4\omega} - 1 - \dfrac{\gamma}{i\omega} \end{pmatrix}. \tag{2.10}$$

Then, we get

$$\partial_t \left(\partial_n u + \partial_t u\right) = \left(\frac{\kappa}{4} - \gamma\right) \partial_n u - \left(\frac{\kappa}{4} + \gamma\right) \partial_t u \text{ on } \Sigma,$$

recalling that $U = (v, u)$ with $\partial_t v + \partial_n u = 0$ on $\Sigma$. □

When $\gamma = \frac{\kappa}{4}$, we have

$$\left(\tau_{-1}\left(\mathcal{I}_2 + \mathcal{K}_{-1}\right)\mathcal{V}_0\right)_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 - \dfrac{i\kappa}{4\omega} & 1 + \dfrac{i\kappa}{4\omega} \\ 1 & \dfrac{i\kappa}{2\omega} - 1 \end{pmatrix}$$

and the boundary condition simplyfies to :

$$\partial_n u + \partial_t u + \frac{\kappa}{2} u = 0 \text{ on } \Sigma.$$

We then obtain the C-ABC when $\gamma = \frac{\kappa}{4}$.
Hence, except for the case $\gamma = \frac{\kappa}{4}$, we have constructed a family of second-order conditions.
In the next section, we show that the corresponding BVP is well-posed. It reads as:

$$\begin{cases} \partial_t^2 u - \triangle u = 0 \text{ in } \Omega \times (0, +\infty)\,; \\ u(0, x) = u_0(x),\ \partial_t u(0, x) = u_1(x) \text{ in } \Omega; \\ u = 0 \text{ or } \partial_n u = 0 \text{ on } \Gamma \times (0, +\infty)\,; \\ \partial_t\left(\partial_n u + \partial_t u\right) = \left(\dfrac{\kappa}{4} - \gamma\right)\partial_n u - \left(\dfrac{\kappa}{4} + \gamma\right)\partial_t u \text{ on } \Sigma \times (0, +\infty)\,. \end{cases} \tag{2.11}$$

# 3 Mathematical study

Regarding the well-posedness of the BVP (2.11), we consider the more general mixed problem: find $u$ solution to

$$\begin{cases} \partial_t^2 u - \triangle u = 0 \text{ in } \Omega \times (0, +\infty)\,; \\ u(0, x) = u_0(x),\ \partial_t u(0, x) = u_1(x) \text{ in } \Omega; \\ u = 0 \text{ or } \partial_n u = 0 \text{ on } \Gamma \times (0, +\infty)\,; \\ \partial_t\left(\partial_n u + \partial_t u\right) = \alpha(x)\partial_n u - \beta(x)\partial_t u \text{ on } \Sigma \times (0, +\infty)\,. \end{cases} \tag{3.1}$$

Let $\Omega$ be a bounded domain whose boundary $\partial\Omega = \Gamma \cup \Sigma$ is regular, with $\Gamma \cap \Sigma = \varnothing$. The parameters $\alpha$ and $\beta$ are regular functions defined on $\Sigma$. We recall that in the case of the ABC constructed in Section (2), we have

$$\alpha(x) = \frac{\kappa(x)}{4} - \gamma(x),\ \beta(x) = \frac{\kappa(x)}{4} + \gamma(x)$$

where $\kappa$ is the curvature on $\Sigma$ and $\gamma$ is a regular parameter on $\Sigma$.
We propose to study the problem (3.1) by using the Hille-Yosida theory without coercing the sign of $\alpha$ and $\beta$ for the moment. We thus consider an equivalent problem in which (3.1) is transformed into a first order system in time. We introduce an auxiliary unknown $v$ defined by $v = \partial_t u$. The pair $U = (u, v)$ is thus solution in $\Omega \times (0, +\infty)$ to

$$\frac{dU}{dt} = AU,\ A = \begin{pmatrix} 0 & Id \\ \triangle & 0 \end{pmatrix} \tag{3.2}$$

with the following boundary conditions

$$u = 0 \text{ or } \partial_n u = 0 \text{ on } \Gamma \times (0, +\infty)$$

and

$$\partial_t (\partial_n u + v) = \alpha(x)\partial_n u - \beta(x)v \text{ on } \Sigma \times (0, +\infty).$$

Next, to control the variations of the functions $\alpha$ and $\beta$, we introduce another auxiliary unknown $U^\sharp$ defined by $U^\sharp = e^{-\delta t}U$ and we suppose in the following that $\delta$ checks

$$\delta > \max(0, \max_{x\in\Sigma} \alpha(x), \ -\min_{x\in\Sigma} \beta(x)). \tag{3.3}$$

We can see that $U^\sharp = (u^\sharp, v^\sharp)$ is solution to

$$\left\{\begin{array}{l} \dfrac{dU^\sharp}{dt} = A_\delta U^\sharp \text{ in } \Omega \times (0, +\infty) \\ u^\sharp = 0 \text{ or } \partial_n u^\sharp = 0 \text{ on } \Gamma \times (0, +\infty) \\ \partial_t \left(\partial_n u^\sharp + v^\sharp\right) = (\alpha(x) - \delta)\, \partial_n u^\sharp - (\beta(x) + \delta)\, v^\sharp \text{ on } \Sigma \times (0, +\infty). \end{array}\right. \tag{3.4}$$

with

$$A_\delta = -\delta\, Id + A. \tag{3.5}$$

Under hypothesis (3.3), we always have on $\Sigma$

$$\alpha(x) - \delta < 0 \text{ and } \beta(x) + \delta > 0. \tag{3.6}$$

In the following, we will focus on the problem (3.5) and we will interest ourselves on the solution of (3.5) in suitable Hilbert spaces.
Since the cases of a Dirichlet boundary condition on $\Gamma$ and a Neumann boundary condition on $\Gamma$ are quite different, we present the two cases in two different subsections.

## 3.1 A Dirichlet boundary condition on $\Gamma$

Let us first introduce $H$ the product space defined by

$$H = H_1 \times H_2$$

with

$$H_1 = \{h_1 \in H^1(\Omega), \triangle h_1 \in L^2(\Omega), h_1 = 0 \text{ on } \Gamma, \partial_n h_1 \in L^2(\Sigma)\}$$

and

$$H_2 = \{h_2 \in H^1(\Omega), h_2 = 0 \text{ on } \Gamma\}.$$

We equip $H$ with the Hilbertian graph norm

$$\|(h_1, h_2)\|_H = \left(\|h_1\|^2_{L^2} + \|\nabla h_1\|^2_{L^2} + \|\triangle h_1\|^2_{L^2} + \|\partial_n h_1\|^2_{L^2(\Sigma)} + \|h_2\|^2_{L^2} + \|\nabla h_2\|^2_{L^2}\right)^{1/2}.$$

Let $V_\delta$ be the product space defined by

$$V_\delta = \{(v_1, v_2) \in H, A_\delta(v_1, v_2) \in H, \partial_n v_2 + \triangle v_1 = \alpha(x)\partial_n v_1 - \beta(x)v_2 \text{ on } \Sigma\}.$$

The space $V_\delta$ corresponds to the domain of the operator $A_\delta$. By enforcing $A_\delta(v_1, v_2) \in H$, we have $v_1$ and $v_2$ regular enough such that the condition on $\Sigma$ is well-defined. Indeed, $A_\delta(v_1, v_2) \in H$ implies, since $A_\delta = -\delta\, Id + A$, $v_2 \in H^1(\Omega)$ and $\triangle v_2 \in L^2(\Omega)$ and $\triangle v_1 \in H^1(\Omega)$. We can give a sense to $\partial_n v_{2|_\Sigma}$ in $H^{-1/2}(\Sigma)$ and to $\triangle v_{1|_\Sigma}$ in $H^{1/2}(\Sigma)$. As for the two other traces, they are well-defined since $(v_1, v_2) \in H$.
First of all we recall the Green formula we will use: for all $(u, v) \in H^1(\Omega)\times H^1(\Omega)$ such that $\triangle u \in L^2(\Omega)$, we have

$$\int_\Omega \triangle u v\, dx = -\int_\Omega \nabla u \cdot \nabla v\, dx + < \partial_n u, v >_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} . \tag{3.7}$$

**Lemma 3.1.** *For all $h \in H$, the quantity*

$$\|h\|_\delta = \left( \int_\Omega \left( |\nabla (h_2 - \delta h_1)|^2 + |\triangle h_1 - \delta v_2|^2 \right) dx \right.$$
$$\left. + \int_\Sigma \frac{1}{2\delta - \alpha(x) + \beta(x)} |(\alpha - \delta)\, \partial_n h_1 - (\beta + \delta)\, h_2|^2\, d\sigma \right)^{1/2}$$

*is a norm on $H$ equivalent to the norm $\|h\|_H$.*

*Proof.* First of all, we remark that $\|h\|_\delta \geq 0$ for all $h$ in $H$ since we have chosen $\delta$ such that $2\delta - \alpha(x) + \beta(x) > 0$ for all $x$ on $\Sigma$.
To prove that $\|\cdot\|_\delta$ is a norm on $H$, we just have to show that

$$\|h\|_\delta = 0 \Rightarrow h = 0.$$

Let $h = (h_1, h_2) \in H$ such that $\|h\|_\delta = 0$. Then $h$ is solution to

$$\left\{ \begin{array}{l} \nabla (h_2 - \delta h_1) = 0 \text{ in } \Omega \\ \triangle h_1 - \delta v_2 = 0 \text{ in } \Omega \\ (h_1 = 0 \text{ and } h_2 = 0) \text{ on } \Gamma \\ (\alpha - \delta)\, \partial_n h_1 - (\beta + \delta)\, h_2 = 0 \text{ on } \Sigma \end{array} \right.$$

From the fact that $\triangle h_1 - \delta v_2 = 0$ in $\Omega$, we deduce that

$$\int_\Omega (\triangle h_1 - \delta v_2)\, h_2\, dx = 0.$$

Using the Green formula (3.7) and that $h_2 = 0$ on $\Gamma$, we obtain

$$-\int_\Omega \nabla h_1 \cdot \nabla h_2\, dx + \int_\Sigma \partial_n h_1 h_2\, d\sigma - \delta \int_\Omega |h_2|^2\, dx = 0.$$

We know that $\nabla (h_2 - \delta h_1) = 0$ in $\Omega$ and that $(\alpha - \delta)\, \partial_n h_1 - (\beta + \delta)\, h_2 = 0$ on $\Sigma$, we then get

$$-\delta \int_\Omega \left( |\nabla h_1|^2 + |h_2|^2 \right)\, dx + \int_\Sigma \frac{\beta + \delta}{\alpha - \delta} |h_2|^2\, d\sigma = 0.$$

Since $\beta + \delta > 0$ and $\alpha - \delta > 0$, we get $\nabla h_1 = 0$ in $\Omega$ and $h_2 = 0$ in $\Omega$. From $\nabla h_1 = 0$ in $\Omega$ we deduce that $h_1$ is constant in $\Omega$. Since $h_1 = 0$ on $\Gamma$, we finally

get $h_1 = 0$ in $\Sigma$.
Consequently $h = 0$ in $\Omega$ which proves that $\|\cdot\|_\delta$ is a norm on $H$.
Now, we prove that $\|\cdot\|_\delta$ equivalent to the norm $\|\cdot\|_H$ on $H$.
Since the function $\alpha$ and $\beta$ are regular, we easily obtain that there exists a positive constant $C_1$ such that

$$\forall h \in H,\, ||h||_\delta \leq C_1\, ||h||_H.$$

Reciprocally, we have to prove that there exists a positive constant $C_2$ such that

$$\forall h \in H,\, ||h||_H \leq C_2\, ||h||_\delta.$$

Since $h_1 = 0$ on $\Gamma$ and $mes(\Gamma) \neq 0$, we have

$$\exists C > 0, ||h_1||_{L^2} + ||\nabla h_1||_{L^2} \leq C||\nabla h_1||_{L^2}$$

and we finally get the existence of a positive constant $C_2$ such that

$$\forall h \in H,\, ||h||_H \leq C_2\, ||h||_\delta.$$

which ends the proof of the equivalence of $\|\cdot\|_\delta$ and $\|\cdot\|_H$ on $H$. $\square$

In the following, we denote by $(\ ,\ )_\delta$ the scalar product derived to the norm $\|\ \|_\delta$.

**Lemma 3.2.** *For all $v \in V_\delta$, we have*

$$(A_\delta v, v)_\delta \leq 0.$$

*Proof.* By definition of $A_\delta$, $A_\delta v = (-\delta v_1 + v_2, -\delta v_2 + \triangle v_1)$. We set

$$\begin{cases} w_1 = v_2 - \delta v_1; \\ w_2 = \triangle v_1 - \delta v_2. \end{cases}$$

We then get

$$(A_\delta v, v)_\delta = \int_\Omega \nabla\left(w_2 - \delta w_1 +\right) \cdot \nabla w_1\, dx + \int_\Omega w_2 \left(\triangle w_1 - \delta w_2\right)\, dx +$$
$$\int_\Sigma \frac{1}{2\delta - \alpha + \beta} \left(\left(\alpha - \delta\right) \partial_n w_1 - \left(\beta + \delta\right) w_2\right) \left(\left(\alpha - \delta\right) \partial_n v_1 - \left(\beta + \delta\right) v_2\right) d\sigma.$$

We develop and we use the Green formula (3.7)

$$(A_\delta v, v)_\delta = -\delta \int_\Omega \left(|\nabla w_1|^2 + |w_2|^2\right)\, dx + \int_\Omega \nabla w_1 \cdot \nabla w_2\, dx$$
$$- \int_\Omega \nabla w_1 \cdot \nabla w_2\, dx + \int_{\partial\Omega} \partial_n w_1\, w_2\, d\sigma$$
$$+ \int_\Sigma \frac{1}{2\delta - \alpha + \beta} \left(\left(\alpha - \delta\right) \partial_n w_1 - \left(\beta + \delta\right) w_2\right) \left(\left(\alpha - \delta\right) \partial_n v_1 - \left(\beta + \delta\right) v_2\right) d\sigma.$$

Moreover, in $V_\delta$, we have $v_{1|_\Gamma} = v_{2|_\Gamma} = w_{1|_\Gamma} = w_{2|_\Gamma} = 0$ and on $\Sigma$

$$\begin{aligned}(\alpha-\delta)\,\partial_n v_1 - (\beta+\delta)\,v_2 &= \alpha\partial_n v_1 - \beta v_2 - \delta\partial_n v_1 - \delta v_2;\\ &= \partial_n v_2 + \triangle v_1 - \delta\partial_n v_1 - \delta v_2;\\ &= \partial_n w_1 + w_2.\end{aligned}$$

Then,

$$\begin{aligned}&(A_\delta v, v)_\delta = -\delta\int_\Omega \left(|\nabla w_1|^2 + |w_2|^2\right)\,dx + \int_\Sigma \partial_n w_1\,w_2\,d\sigma\\ &\int_\Sigma \frac{1}{2\delta-\alpha+\beta}\left((\alpha-\delta)\,\partial_n w_1 - (\beta+\delta)\,w_2\right)(\partial_n w_1 + w_2)\,d\sigma.\end{aligned}$$

Now, we develop the integral on $\Sigma$

$$\begin{aligned}&(A_\delta v, v)_\delta = -\delta\int_\Omega \left(|\nabla w_1|^2 + |w_2|^2\right)\,dx + \int_\Sigma \partial_n w_1\,w_2\,d\sigma + \int_\Sigma \frac{\alpha-\delta}{2\delta-\alpha+\beta}|\partial_n w_1|^2\,d\sigma\\ &-\int_\Sigma \frac{\beta+\delta}{2\delta-\alpha+\beta}|w_2|^2\,d\sigma + \int_\Sigma \frac{1}{2\delta-\alpha+\beta}\left[(\alpha-\delta)\,\partial_n w_1 w_2 - (\beta+\delta)\,\partial_n w_1 w_2\right]\,d\sigma.\end{aligned}$$

Finally, we get that for all $v$ in $V_\delta$,

$$(A_\delta v, v)_\delta = -\delta\int_\Omega \left(|\nabla w_1|^2 + |w_2|^2\right)\,dx + \int_\Sigma \frac{\alpha-\delta}{2\delta-\alpha+\beta}|\partial_n w_1|^2\,d\sigma - \int_\Sigma \frac{\beta+\delta}{2\delta-\alpha+\beta}|w_2|^2\,d\sigma.$$

Finally, by definition of $\delta$, $\delta > 0$, $2\delta-\alpha+\beta > 0$, $\alpha-\delta < 0$ and $\beta+\delta > 0$. $(A_\delta v, v)_\delta$ is then the sum of negative terms, which ends the proof of the result. □

We continue by proving the following lemma

**Lemma 3.3.** *The operator $A_\delta$ is maximal on its domain $V_\delta$.*

*Proof.* Given $f = (f_1, f_2)$ in $H$, we study the following mixed problem: find $v \in V_\delta$ such that $(A_\delta - I)\,v = f$.
By defintion of $A_\delta$, we seek $v = (v_1, v_2) \in V_\delta$ such that

$$\left\{\begin{array}{l} -\delta v_1 + v_2 - v_1 = f_1 \text{ in } \Omega;\\ -\delta v_2 + \triangle v_1 - v_2 = f_2 \text{ in } \Omega;\\ v_1 = v_2 = 0 \text{ on } \Gamma;\\ \partial_n v_2 + \triangle v_1 = \alpha\partial_n v_1 - \beta v_2 \text{ on } \Sigma.\end{array}\right. \tag{3.8}$$

First of all, we assume that the problem (3.8) has a solution in $V_\delta$. Then, by removing $v_2$ thanks to the equation

$$v_2 = f_1 + (\delta+1)\,v_1 \text{ in } \Omega,$$

we obtain that $v_1$ is solution to the problem

$$\begin{cases} -\triangle v_1 + (\delta+1)^2 v_1 = \tilde{f} \text{ in } \Omega; \\ v_1 = 0 \text{ on } \Gamma; \\ \partial_n v_1 = \tilde{g} - \dfrac{\beta+\delta+1}{\delta-\alpha+1}(\delta+1) v_1 \text{ on } \Sigma. \end{cases} \tag{3.9}$$

with

$$\tilde{f} := -\left(f_2 + (\delta+1) f_1\right) \text{ in } H^1(\Omega)$$

and

$$\tilde{g} := -\frac{1}{\delta-\alpha+1}\left(\partial_n f_1 + f_2 + (\delta+\beta+1) f_1\right) \text{ in } L^2(\Sigma).$$

Reciprocally, let us prove that the problem (3.9) admits a solution $v_1$, for all $\tilde{f}$ given in $H^1(\Omega)$ and $\tilde{g}$ given in $L^2(\Sigma)$. To prove this result, we use a variational approach.
Let $\mathcal{H}_1(\Omega)$ be the space defined by

$$\mathcal{H}_1(\Omega) = H^1_\Gamma(\Omega) = \{v \in H^1(\Omega),\, v_{|\Gamma} = 0\}$$

We equip $\mathcal{H}_1(\Omega)$ with the norm $\|\ \|$ defined by

$$\forall v \in \mathcal{H}_1(\Omega), \|v\| = \left(\int_\Omega \left(|\nabla v|^2 + (\delta+1)^2 |v|^2\right) dx + (\delta+1)\int_\Sigma \frac{\delta+\beta+1}{\delta-\alpha+1}|v|^2\, d\sigma\right)^{1/2},$$

equivalent to the usual norm on $H^1(\Omega)$ because $(\delta+1)$, $(\delta+\beta+1)$ and $(\delta-\alpha+1)$ are positive quantities by definition of $\delta$.
Let $\mathcal{T}\left(\overline{\Omega}\right)$ be the space of test functions defined by

$$\mathcal{T}\left(\overline{\Omega}\right) = \{\varphi \in \mathcal{D}\left(\overline{\Omega}\right),\, \varphi_{|\Gamma} = 0\}$$

$\mathcal{T}\left(\overline{\Omega}\right)$ is dense in $\mathcal{H}_1\left(\overline{\Omega}\right)$ and if we assume that the problem (3.9) has a solution, we have

$$\forall \varphi \in \mathcal{T}\left(\overline{\Omega}\right), -\int_\Omega \triangle v_1 \varphi\, dx + (\delta+1)^2 \int_\Omega v_1 \varphi\, dx = \int_\Omega \tilde{f}\varphi\, dx.$$

By using the Green formula (3.7), we get

$$\forall \varphi \in \mathcal{T}\left(\overline{\Omega}\right), \int_\Omega \nabla v_1 \nabla \varphi\, dx - \int_{\partial\Omega} \partial_n v_1 \varphi\, d\sigma + (\delta+1)^2 \int_\Omega v_1 \varphi\, dx = \int_\Omega \tilde{f}\varphi\, dx.$$

Moreover, $\varphi_{|\Gamma} = 0$. Therefore,

$$\forall \varphi \in \mathcal{T}\left(\overline{\Omega}\right), \int_\Omega \nabla v_1 \nabla \varphi\, dx - \int_{\Sigma} \partial_n v_1 \varphi\, d\sigma + (\delta+1)^2 \int_\Omega v_1 \varphi\, dx = \int_\Omega \tilde{f}\varphi\, dx.$$

Then we use that $\partial_n v_1 = \tilde{g} - \dfrac{\delta+\beta+1}{\delta-\alpha+1}(\delta+1) v_1$ on $\Sigma$ and we get $\forall \varphi \in \mathcal{T}\left(\overline{\Omega}\right)$,

$$\int_\Omega \nabla v_1 \nabla \varphi\, dx + (\delta+1)\int_\Sigma \frac{\delta+\beta+1}{\delta-\alpha+1} v_1 \varphi\, d\sigma + (\delta+1)^2 \int_\Omega v_1 \varphi\, dx = \int_\Omega \tilde{f}\varphi\, dx + \int_\Sigma \tilde{g}\varphi\, d\sigma. \tag{3.10}$$

Let $a(\cdot,\cdot)$ the bilinear form defined by

$$a(v_1,\varphi)=\int_\Omega \nabla v_1 \nabla\varphi\,dx+(\delta+1)\int_\Sigma \frac{\delta+\beta+1}{\delta-\alpha+1}v_1\varphi\,d\sigma+(\delta+1)^2\int_\Omega v_1\varphi\,dx.$$

$a(\cdot,\cdot)$ is continuous on $\mathcal{H}_1(\Omega)\times\mathcal{H}_1(\Omega)$, $\mathcal{H}_1(\Omega)$-coercive, since we have equipped $\mathcal{H}_1(\Omega)$ with the norm $||\ ||$.
Let $l(\cdot)$ be the linear form defined by

$$l(\varphi)=\int_\Omega \tilde{f}\varphi\,dx+\int_\Sigma \tilde{g}\varphi\,d\sigma.$$

Since the pair $(\tilde{f},\tilde{g})$ belongs to $H^1(\Omega)\times L^2(\Sigma)$, $l(\cdot)$ is continuous in $\mathcal{H}_1(\Omega)$. $\mathcal{T}\left(\overline{\Omega}\right)$ is dense in $\mathcal{H}_1(\Omega)$, therefore the formulation (3.10) can be extended to $\mathcal{H}_1(\Omega)$. Every solution $v_1$ of (3.9) checks $\forall\varphi\in\mathcal{H}_1\left(\Omega\right),$

$$\int_\Omega \nabla v_1\nabla\varphi\,dx+(\delta+1)\int_\Sigma \frac{\delta+\beta+1}{\delta-\alpha+1}v_1\varphi\,d\sigma+(\delta+1)^2\int_\Omega v_1\varphi\,dx=\int_\Omega \tilde{f}\varphi\,dx+\int_\Sigma \tilde{g}\varphi\,d\sigma.$$

Reciprocally, according to Lax-Milgram theorem, the problem

$$\forall\varphi\in\mathcal{H}_1\left(\Omega\right),a(v_1,\varphi)=l(\varphi)$$

has a unique solution $v_1$ in $\mathcal{H}_1\left(\Omega\right)$. In particular,

$$\forall\varphi\in\mathcal{D}(\Omega)\subset\mathcal{H}_1(\Omega),\ \int_\Omega \nabla v_1\nabla\varphi\,dx+(\delta+1)^2\int_\Omega v_1\varphi\,dx=\int_\Omega \tilde{f}\varphi\,dx.$$

We then deduce that

$$\forall\varphi\in\mathcal{D}(\Omega),<(\delta+1)^2\,v_1-\triangle v_1-\tilde{f},\varphi>=0,$$

which means that

$$(\delta+1)^2\,v_1-\triangle v_1=\tilde{f}\text{ in }\mathcal{D}'(\Omega).$$

This identity allows us to give a sense to $\triangle v_1$ in $H^1(\Omega)$. Therefore, $\partial_n v_{1|_{\partial\Omega}}\in H^{-1/2}(\partial\Omega)$ and we also have $\forall\varphi\in\mathcal{T}\left(\overline{\Omega}\right),$

$$\int_\Omega \nabla v_1\nabla\varphi\,dx+(\delta+1)\int_\Sigma \frac{\delta+\beta+1}{\delta-\alpha+1}v_1\varphi\,d\sigma+(\delta+1)^2\int_\Omega v_1\varphi\,dx=\int_\Omega \tilde{f}\varphi\,dx+\int_\Sigma \tilde{g}\varphi\,d\sigma.$$

Using the Green formula (3.7), we get $\forall\varphi\in\mathcal{T}\left(\overline{\Omega}\right),$

$$\int_\Omega \triangle v_1\varphi\,dx+(\delta+1)\int_\Sigma \frac{\delta+\beta+1}{\delta-\alpha+1}v_1\varphi\,d\sigma+(\delta+1)^2\int_\Omega v_1\varphi\,dx+<\partial_n v_1,\varphi>_{-1/2,\,1/2}=\int_\Omega \tilde{f}\varphi\,dx+\int_\Sigma \tilde{g}\varphi\,d\sigma$$

*ie*

$$\forall\varphi\in\mathcal{T}\left(\overline{\Omega}\right),\ (\delta+1)\int_\Sigma \frac{\delta+\beta+1}{\delta-\alpha+1}v_1\varphi\,d\sigma+<\partial_n v_1,\varphi>_{-1/2,\,1/2}=\int_\Sigma \tilde{g}\varphi\,d\sigma.$$

Now, we have $\varphi_{|_\Gamma}=0$ and we get on $\Sigma$

$$<\partial_n v_1+(\delta+1)\frac{\delta+\beta+1}{\delta-\alpha+1}v_1-\tilde{g},\varphi>_{-1/2,\,1/2}=0,$$

which means that

$$\partial_n v_1 + (\delta + 1)\,\frac{\delta + \beta + 1}{\delta - \alpha + 1} v_1 = \tilde{g} \text{ on } \Sigma.$$

$v_1$ is then solution of (3.9) since the condition $v_{1|_\Gamma} = 0$ is given in $\mathcal{H}_1(\Omega)$.
Since $v_1$ and $f_1$ are in $H^1(\Omega)$, we deduce the existence of $v_2$ in $H^1(\Omega)$.
Now, we have to check that the pair $(v_1, v_2)$ is in $V_\delta$. To show this result, we have to prove that $(v_1, v_2)$ is in $H$ then $A_\delta(v_1, v_2)$ is in $H$ too and finally that $\partial_n v_2 + \triangle v_1 = \alpha(x)\partial_n v_1 - \beta(x) v_2$ on $\Sigma$.
First we see that the condition which links $v_1$ to $v_2$ on $\Sigma$ is checked.
To prove that $(v_1, v_2) \in H$, we just have to check that $\partial_n v_1 \in L^2(\Sigma)$ and that $v_{2|_\Gamma} = 0$. To prove this result, we use that

$$\partial_n v_1 + (\delta + 1)\,\frac{\delta + \beta + 1}{\delta - \alpha + 1} v_1 = \tilde{g} \text{ on } \Sigma.$$

Since $v_1$ is in $H^1(\Omega)$, $v_{1|_\Sigma} \in H^{1/2}(\Omega) \subset L^2(\Sigma)$. Moreover, since $(f_1, f_2) \in H$, $\tilde{g} \in L^2(\Sigma)$. Then, we get that $\partial_n v_{1|_\Sigma} \in L^2(\Sigma)$. Finally, $f_{1|_\Gamma} = 0$ and $v_{1|_\Gamma} = 0$ which gives that $v_{2|_\Gamma} = 0$. We have shown that $(v_1, v_2) \in H$.
To prove that $A_\delta(v_1, v_2)$ is also in $H$, we have to check that $\triangle v_2 \in L^2(\Omega)$, $\partial_n v_{2|_\Sigma} \in L^2(\Sigma)$ and $\triangle v_1 = 0$ on $\Gamma$. Using that

$$\partial_n v_2 + \triangle v_1 = \alpha(x)\partial_n v_1 - \beta(x) v_2 \text{ sur } \Sigma.$$

and $\partial_n v_{1|_\Sigma} \in L^2(\Sigma)$, $v_2 \in H^1(\Omega)$ and $\triangle v_1 \in H^1(\Omega)$, we get $\partial_n v_{2|_\Sigma}$ in $L^2(\Sigma)$. Moreover, we know that $v_2 = f_1 + (\delta + 1)\, v_1$ in $\Omega$ and that $\triangle v_1$ as $\triangle f_1$ are in $L^2(\Omega)$. We can conclude that $\triangle v_2$ is in $L^2(\Omega)$. Finally, we know that $\triangle v_1 = f_2 + (\delta + 1) v_2$ in $\Omega$ and $v_{2|_\Gamma} = v_{2|_\Gamma} = 0$ so $\triangle v_{1|_\Gamma} = 0$.
Consequently, we have shown that the pair $(v_1, v_2)$ is in $V_\delta$ which ends the proof. □

So we have the following theorem on the well-posedness of the problem.

**Theorem 3.4.** *Let $(u_0, u_1)$ in $V_0$. The problem (3.1) admits a unique solution $u$ such that*

$$(u,\, \partial_t u) \in C^1([0, +\infty[; V_0) \cap C^0([0, +\infty[; H). \tag{3.11}$$

*Proof.* The proof of this theorem lies on the Hille-Yosida theory.
The two previous lemmas show that the operator $A_\delta$ is a maximal dissipative operator on its domain $V_\delta$. We conclude that the problem (3.4) has one and only one solution $U^\sharp = (u^\sharp,\, v^\sharp)$ such that

$$(u^\sharp,\, v^\sharp) \in C^1([0, +\infty[; V_\delta) \cap C^0([0, +\infty[; H). \tag{3.12}$$

$A_\delta$ is the infinitesimal generator of a semi-group of contraction $Z_\delta(t)$. Thus, we can define the finite energy solution of (3.4) with initial data $(u_0, u_1)$ in $V_\delta$ and

$$(u^\sharp,\, v^\sharp) = Z_\delta(t)(u_0, u_1) \in C^1([0, +\infty[; V_\delta) \cap C^0([0, +\infty[; H).$$

For $\delta = 0$ we recover the operator $A$, associated to the reference problem (3.1). Thus, $A$ is the infinitesimal generator of the continuous semi-group $Z(t) =$

$e^{\delta t} Z_\delta(t)$.
Moreover, we recall that the problems (3.1) with $(u_0, u_1) \in V_0$ and (3.4) with $(u_0, u_1)$ in $V_\delta$ are equivalent. Indeed, $(u, \partial_t u)$ is solution of (3.1) if and only if $(u^\sharp,\, v^\sharp)$ is solution of (3.4). Therefore, we have shown the existence of a unique solution $(u, \partial_t u)$ of problem (3.1). More, the application $t \longmapsto e^{-\delta t}$ being in $C^\infty([0, +\infty[)$, we have

$$\begin{array}{l}(u,\, \partial_t u) \in C^1([0, +\infty[; V_0) \cap C^0([0, +\infty[; H) \Leftrightarrow \\ (u^\sharp,\, v^\sharp) \in C^1([0, +\infty[; V_\delta) \cap C^0([0, +\infty[; H),\end{array}$$

which ends the proof of the theorem. □

## 3.2 A Neumann boundary condition on $\Gamma$

In this case, we have to assume that

$$\left\{\begin{array}{l}\alpha(x) \neq 0,\, \forall x \in \Sigma; \\ \beta(x) \neq 0,\, \forall x \in \Sigma; \\ \alpha \text{ and } \beta \text{ are continuous and bounded on } \Sigma\end{array}\right.$$

Regarding the continuity of $\alpha$ and $\beta$, it implies that $\alpha$ and $\beta$ are necessarily positive or negative since we have supposed that they nerver vanish.
To ensure the uniqueness of a solution to the problem (3.1), we first establish an invariance property for the solution to (3.1). We first remark that for any regular $u$ solution to (3.1)

$$\int_\Omega \left(\partial_t^2 u - \triangle u\right) 1\, dx = 0$$

which implies that

$$\int_\Omega \partial_t^2 u\, dx = \int_\Sigma \partial_n u\, d\sigma$$

because $\partial_n u = 0$ on $\Gamma$.
Moreover, on $\Sigma$, $\alpha \partial_n u = \partial_t \left(\partial_t u + \partial_n u\right) + \beta \partial_t u$. Since $\alpha(x) \neq 0$ for all $x$ on $\Sigma$, we thus obtain

$$\int_\Omega \partial_t^2 u\, dx = \int_\Sigma \alpha^{-1} \left(\partial_t \left(\partial_t u + \partial_n u\right) + \beta \partial_t u\right)\, d\sigma,$$

which is equivalent to

$$\partial_t \left[\int_\Omega \partial_t u\, dx - \int_\Sigma \alpha^{-1} \left(\partial_t u + \partial_n u + \beta u\right)\, d\sigma\right] = 0.$$

Therefore, for any $t$,

$$\int_\Omega \partial_t u\, dx - \int_\Sigma \alpha^{-1} \left(\partial_t u + \partial_n u + \beta u\right)\, d\sigma = \int_\Omega u_1\, dx - \int_\Sigma \alpha^{-1} \left(u_1 + \partial_n u_0 + \beta u_0\right)\, d\sigma. \tag{3.13}$$

Let $X$ be the Hilbert space defined by:

$$X = \left\{(h_1, h_2) \in \left(H^1(\Omega)\right)^2,\, \triangle h_1 \in L^2(\Omega),\, \partial_n h_1 = 0 \text{ on } \Gamma,\, \partial_n h_1 \in L^2(\Sigma)\right\}$$

Then, $X$ is equipped with the Hilbertian graph norm:

$$\| (h_1, h_2) \|_X = \left( \|h_1\|^2_{L^2} + \|\nabla h_1\|^2_{L^2} + \| \triangle h_1\|^2_{L^2} + \|\partial_n h_1\|^2_{L^2(\Sigma)} + \|h_2\|^2_{L^2} + \|\nabla h_2\|^2_{L^2} \right)^{1/2}.$$

Next, let $H \subset X$ defined by

$$H = \left\{ (h_1, h_2) \in X, \int_\Omega h_2 dx - \int_\Sigma \alpha^{-1} (h_2 + \partial_n h_1 + \beta h_1) \, d\sigma = 0 \right\}.$$

**Proposition 3.5.** *$H$ is a closed subspace of $X$ and the semi-norm on $X$ defined by*

$$| (h_1, h_2) |_X = \left( \|\nabla h_1\|^2_{L^2} + \| \triangle h_1\|^2_{L^2} + \|\partial_n h_1\|^2_{L^2(\Sigma)} + \|h_2\|^2_{L^2} + \|\nabla h_2\|^2_{L^2} \right)^{1/2}.$$

*defines a norm in $H$ equivalent to the norm $\| \cdot \|_X$.*

*Proof.* Let $\Psi : (h_1, h_2) \in X \longmapsto \int_\Omega h_2 dx - \int_\Sigma \alpha^{-1} (h_2 + \partial_n h_1 + \beta h_1) \, d\sigma$. Then $\Psi$ is a continuous function on $X$ and $H = \Psi^{-1}\{0\}$. Hence $\overline{H}^{\| \, \|_X} = H$.
Now, let $(h_1, h_2) \in H$ such that $|(h_1, h_2)|_X = 0$. Then we have

$$\begin{cases} \nabla h_1 = 0, \, \triangle h_1 = 0 \text{ in } \Omega, \, \partial_n h_1 = 0 \text{ on } \partial\Omega\,; \\ \nabla h_2 = 0 \text{ in } \Omega, \, h_2 = 0 \text{ on } \Sigma \end{cases}$$

which implies that $h_2 = 0$ in $\Omega$ and $h_1$ is constant in the connex open set $\Omega$.
Moreover, $\Psi(h_1, h_2) = 0$ implies that

$$\int_\Sigma \alpha^{-1} \beta h_1 d\sigma = 0 = h_1 \int_\Sigma \alpha^{-1} \beta d\sigma$$

since $h_1$ is constant. Now, $\alpha^{-1}\beta$ never vanishes on $\Sigma$ and $\alpha^{-1}\beta$ keeps the same sign on $\Sigma$. We thus have $h_1 = 0$ on $\Sigma$ which implies that $h_1 = 0$ in $\Omega$. Hence the semi-norm $| \cdot |_X$ defines a norm on $H$. To prove that $| \, |_X$ is equivalent to the norm $\| \, \|_X$ in $H$, we just have to show that there exists a constant $C > 0$ such that for any $(h_1, h_2) \in H$,

$$\|h_1\|_{L^2(\Omega)} \leq C |(h_1, h_2)|_X \tag{3.14}$$

since it is straightforward that

$$|(h_1, h_2)|_X \leq \|(h_1, h_2)\|_X.$$

Suppose that (3.14) is false. Then there exists a sequence $(h_{1,k}, h_{2,k})$ in $H$ such that

$$\|h_{1,k}\|_{L^2(\Omega)} > k |(h_{1,k}, h_{2,k})|_X.$$

Then (3.2) implies that $\|h_{1,k}\|_{L^2(\Omega)} \neq 0$ and thus, we can suppose that

$$\|(h_{1,k}, h_{2,k})\|_X = 1$$

only to divide by $\|h_{1,k}\|_{L^2(\Omega)}$.
We can thus extract a subsequence that we denote by $(h_{1,k}, h_{2,k})$ too and which converges to $(h_1, h_2)$ in $X$ weakly. Moreover, (3.14) implies that $|(h_{1,k}, h_{2,k})|_X \underset{k \to +\infty}{\longrightarrow}$

0.
Hence we have

$$\begin{cases} \nabla h_{1,k} \to 0 & \text{in } L^2(\Omega); \\ \triangle h_{1,k} \to 0 & \text{in } L^2(\Omega); \\ \partial_n h_{1,k_{|\Sigma}} \to 0 & \text{in } L^2(\Sigma); \end{cases} \tag{3.15}$$

and

$$\begin{cases} \nabla h_{2,k} \to 0 & \text{in } L^2(\Omega); \\ h_{2,k_{|\Sigma}} \to 0 & \text{in } L^2(\Sigma). \end{cases}$$

We can thus deduce that

$$\begin{cases} \nabla h_1 = 0 \text{ in } \Omega,\ \triangle h_1 = 0 \text{ in } \Omega \text{ and } \partial_n h_{1,k_{|\Sigma}} = 0 \\ \nabla h_2 = 0 \text{ in } \Omega,\ h_{2,k_{|\Sigma}} = 0. \end{cases}$$

We thus have $(h_1, h_2) = (cste, 0)$.
Moreover, $H$ is closed. Hence $(h_1, h_2) \in H$ which implies that necessarily $h_1 = h_2 = 0$ in $\Omega$.
To conclude, $(h_{1,k}, h_{2,k})$ weakly converges to $(0, 0)$ in $X$ and since $X \subset H^{3/2}(\Omega)\times H^1(\Omega)$, $X$ is compactly embedded in $L^2(\Omega)\times L^2(\Omega)$. We thus have $h_{1,k}$ converges strongly to 0 in $L^2(\Omega)$, which contradicts (3.2) since it implies that $(h_{1,k}, h_{2,k})$ strongly converges to $(0, 0)$ in $X$. □

In the following, we adopt the notation $\|h\|_H = |h|_X$.
Now, let $V_\delta$ be the product space defined by

$$V_\delta = \left\{(v_1, v_2) \in H,\ A_\delta(v_1, v_2) \in H,\ \partial_n v_2 + \triangle v_1 = \alpha(x)\partial_n v_1 - \beta(x) v_2 \text{ on } \Sigma\right\}.$$

The space $V_\delta$ corresponds to the domain of the operator $A_\delta$. As in the case of a Dirichlet boundary condition on $\Gamma$, $v_1$ and $v_2$ are regular enough such that the condition on $\Sigma$ is well-defined.

**Lemma 3.6.** *For all $h \in H$, the quantity*

$$\|h\|_\delta = \left(\int_\Omega \left(|\nabla\left(h_2 - \delta h_1\right)|^2 + |\triangle h_1 - \delta v_2|^2\right) dx \right.$$
$$\left. + \int_\Sigma \frac{1}{2\delta - \alpha(x) + \beta(x)} |\left(\alpha - \delta\right)\partial_n h_1 - \left(\beta + \delta\right) h_2|^2\, d\sigma\right)^{1/2}$$

*is a norm on $H$ equivalent to the norm $\|h\|_H$ and thus to $\|h\|_X$.*

*Proof.* First of all, we remark that $\|h\|_\delta \geq 0$ for all $h$ in $H$ since we have chosen $\delta$ such that $2\delta - \alpha(x) + \beta(x) > 0$ for all $x$ on $\Sigma$.
To prove that $\|\cdot\|_\delta$ is a norm on $H$, we just have to show that

$$\|h\|_\delta = 0 \Rightarrow h = 0.$$

Indeed, $\|h\|_\delta$ is defined as a linear combination of norms in $L^2(\Omega)$ and $L^2(\Sigma)$.
Let $h = (h_1, h_2) \in H$ such that $\|h\|_\delta = 0$. Then $h$ is solution to

$$\begin{cases} \nabla (h_2 - \delta h_1) = 0 \text{ in } \Omega \\ \triangle h_1 - \delta v_2 = 0 \text{ in } \Omega \\ \partial_n h_1 = 0 \text{ on } \Gamma \\ (\alpha - \delta)\, \partial_n h_1 - (\beta + \delta)\, h_2 = 0 \text{ on } \Sigma \end{cases}$$

From the fact that $\triangle h_1 - \delta v_2 = 0$ in $\Omega$, we deduce that

$$\int_\Omega (\triangle h_1 - \delta v_2)\, h_2\, dx = 0.$$

Using the Green formula (3.7) and $\partial_n h_1 = 0$ on $\Gamma$, we obtain

$$-\int_\Omega \nabla h_1 \cdot \nabla h_2\, dx + \int_\Sigma \partial_n h_1 h_2\, d\sigma - \delta \int_\Omega |h_2|^2\, dx = 0.$$

Given that $\nabla (h_2 - \delta h_1) = 0$ in $\Omega$ and that $(\alpha - \delta)\, \partial_n h_1 - (\beta + \delta)\, h_2 = 0$ on $\Sigma$, we then get

$$-\delta \int_\Omega \left(|\nabla h_1|^2 + |h_2|^2\right)\, dx + \int_\Sigma \frac{\beta + \delta}{\alpha - \delta} |h_2|^2\, d\sigma = 0.$$

Since $\beta + \delta > 0$ and $\alpha - \delta < 0$, we get $\nabla h_1 = 0$ in $\Omega$ and $h_2 = 0$ in $\Omega$. From $\nabla h_1 = 0$ in $\Omega$, we conclude that $h_1$ is constant in $\Omega$. Next (3.13) implies that

$$\int_\Sigma \alpha^{-1} \beta h_1 d\sigma = 0.$$

Since $h_1$ is constant in $\Omega$ and the sign of $\alpha^{-1}\beta$ is constant according to (3.2) on $\Sigma$, $h_1 = 0$ on $\Sigma$. Consequently $h_1 = 0$ in $\Omega$ which prove that $\|\cdot\|_\delta$ is a norm on $H$.
Now, we have to prove that $\|\cdot\|_\delta$ is equivalent to $\|\cdot\|_H$ on $H$. This is a straightforward consequence of the boundedness of $\alpha$ and $\beta$. □

In the following, we denote by $(\ ,\ )_\delta$ the scalar product derived to the norm $\|\ \|_\delta$.

**Lemma 3.7.** *For all $v \in V_\delta$, we have*

$$(A_\delta v, v)_\delta \leq 0.$$

*Proof.* The proof of this lemma is exactly the same as for Lemma 3.2. The only difference comes from the fact that

$$\int_\Gamma \partial_n w_1\, w_2 d\sigma$$

disappears since $(w_1, w_2) \in H$ and so $\partial_n w_1 = 0$ on $\Gamma$. Consequently we get that for all $v$ in $V_\delta$, $(A_\delta v, v)_\delta$ is the sum of negative terms, which ends the proof of the result. □

We continue by proving the following lemma

**Lemma 3.8.** *The operator $A_\delta$ is maximal on its domain $V_\delta$.*

*Proof.* Given $f = (f_1, f_2)$ in $H$, we study the following mixed problem: find $v \in V_\delta$ such that $(A_\delta - I)\, v = f$.
By defintion of $A_\delta$, we seek $v = (v_1, v_2) \in V_\delta$ such that

$$\begin{cases} -\delta v_1 + v_2 - v_1 = f_1 \text{ in } \Omega; \\ -\delta v_2 + \triangle v_1 - v_2 = f_2 \text{ in } \Omega; \\ \partial_n v_1 = 0 \text{ on } \Gamma; \\ \partial_n v_2 + \triangle v_1 = \alpha \partial_n v_1 - \beta v_2 \text{ on } \Sigma. \end{cases} \tag{3.16}$$

First of all, we assume that the problem (3.16) has a solution in $V_\delta$. Then, by removing $v_2$ thanks to the equation

$$v_2 = f_1 + (\delta + 1)\, v_1 \text{ in } \Omega,$$

we obtain that $v_1$ is solution to the problem

$$\begin{cases} -\triangle v_1 + (\delta+1)^2\, v_1 = -\left(f_2 + (\delta+1)\, f_1\right) \text{ in } \Omega; \\ \partial_n v_1 = 0 \text{ on } \Gamma; \\ (\delta - \alpha + 1)\, \partial_n v_1 = -\partial_n f_1 - f_2 - (\delta + \beta + 1)\left(f_1 + (\delta+1)\, v_1\right) \text{ on } \Sigma. \end{cases} \tag{3.17}$$

Reciprocally, let us prove that the problem (3.17) admits a solution $v_1$, for all $\tilde{f} := -\left(f_2 + (\delta+1)\, f_1\right)$ given in $H^1(\Omega)$ and $\tilde{g} := -\frac{1}{\delta-\alpha+1}\left(\partial_n f_1 + f_2 + (\delta+\beta+1)\, f_1\right)$ given in $L^2(\Sigma)$. To prove this result, we use a variational approach.
We equip $H^1(\Omega)$ with the norm $\|\ \|$ defined by

$$\forall v \in H^1(\Omega), \|v\| = \left(\int_\Omega \left(|\nabla f|^2 + (\delta+1)^2 |f|^2\right) dx + (\delta+1) \int_\Sigma \frac{\delta+\beta+1}{\delta-\alpha+1} |f|^2\, d\sigma\right)^{1/2},$$

equivalent to the usual norm on $H^1(\Omega)$ because $(\delta+1)$, $(\delta+\beta+1)$ and $(\delta-\alpha+1)$ are positive quantities by definition of $\delta$.
$\mathcal{D}\left(\overline{\Omega}\right)$ is dense in $H^1\left(\overline{\Omega}\right)$ and if we assume that the problem (3.17) has a solution, we have

$$\forall \varphi \in \mathcal{D}\left(\overline{\Omega}\right), -\int_\Omega \triangle v_1 \varphi\, dx + (\delta+1)^2 \int_\Omega v_1 \varphi\, dx = \int_\Omega \tilde{f} \varphi\, dx,$$

as in the proof of Lemma 3.3. We finally obtain $\forall \varphi \in \mathcal{D}\left(\overline{\Omega}\right)$,

$$\int_\Omega \nabla v_1 \nabla \varphi\, dx + (\delta+1) \int_\Sigma \frac{\delta+\beta+1}{\delta-\alpha+1} v_1 \varphi\, d\sigma + (\delta+1)^2 \int_\Omega v_1 \varphi\, dx = \int_\Omega \tilde{f} \varphi\, dx + \int_\Sigma \tilde{g} \varphi\, d\sigma, \tag{3.18}$$

using the Green formula (3.7) and that $\partial_n v_{1|_\Gamma} = 0$. Let $a(\cdot, \cdot)$ the same bilinear form as in the proof of Lemma 3.3 which is also is continuous on $H^1(\Omega) \times H^1(\Omega)$,

$H^1(\Omega)$-coercive, since we have equipped $H^1(\Omega)$ with the norm $|| \, ||$.
Let $l(\cdot)$ be the linear form defined by

$$l(\varphi) = \int_\Omega \tilde{f}\varphi \, dx + \int_\Sigma \tilde{g}\varphi \, d\sigma.$$

Since the pair $(\tilde{f}, \tilde{g})$ belongs to $H^1(\Omega) \times L^2(\Sigma)$, $l(\cdot)$ is continuous in $H^1(\Omega)$.
$\mathcal{D}\left(\overline{\Omega}\right)$ is dense in $H^1(\Omega)$, therefore the formulation (3.18) can be extended to $H^1(\Omega)$. Every solution $v_1$ of (3.17) checks

$$\forall \varphi \in H^1\left(\Omega\right), \, a(v_1, \varphi) = l(\varphi).$$

Reciprocally, according to Lax-Milgram theorem, the problem

$$\forall \varphi \in H^1\left(\Omega\right), a(v_1, \varphi) = l(\varphi)$$

has a unique solution $v_1$ in $H^1\left(\Omega\right)$. In particular, since $\mathcal{D}(\Omega) \subset H^1(\Omega)$, we deduce that

$$\forall \varphi \in \mathcal{D}(\Omega), < \left(\delta+1\right)^2 v_1 - \triangle v_1 - \tilde{f}, \varphi >= 0,$$

which means that

$$\left(\delta+1\right)^2 v_1 - \triangle v_1 = \tilde{f} \text{ in } \mathcal{D}'(\Omega).$$

Thanks to this result, we can give a sense to $\triangle v_1$ in $H^1(\Omega)$ and therefore, $\partial_n v_{1|\partial\Omega} \in H^{-1/2}(\partial\Omega)$. We also have $\forall \varphi \in \mathcal{D}\left(\overline{\Omega}\right)$,

$$\int_\Omega \nabla v_1 \nabla \varphi \, dx + (\delta+1) \int_\Sigma \frac{\delta+\beta+1}{\delta-\alpha+1} v_1 \varphi \, d\sigma + (\delta+1)^2 \int_\Omega v_1 \varphi \, dx = \int_\Omega \tilde{f}\varphi \, dx + \int_\Sigma \tilde{g}\varphi \, d\sigma.$$

Using the Green formula (3.7), we get $\forall \varphi \in \mathcal{D}\left(\overline{\Omega}\right)$,

$$\begin{aligned} &\int_\Omega \triangle v_1 \varphi \, dx + (\delta+1) \int_\Sigma \frac{\delta+\beta+1}{\delta-\alpha+1} v_1 \varphi \, d\sigma + (\delta+1)^2 \int_\Omega v_1 \varphi \, dx \\ &+ < \partial_n v_1, \varphi >_{-1/2, \, 1/2} = \int_\Omega \tilde{f}\varphi \, dx + \int_\Sigma \tilde{g}\varphi \, d\sigma \end{aligned}$$

*ie*

$$\forall \varphi \in \mathcal{T}\left(\overline{\Omega}\right), \, (\delta+1) \int_\Sigma \frac{\delta+\beta+1}{\delta-\alpha+1} v_1 \varphi \, d\sigma + < \partial_n v_1, \varphi >_{-1/2, \, 1/2} = \int_\Sigma \tilde{g}\varphi \, d\sigma.$$

Now, by chosing the functions $\varphi$ in $\mathcal{D}\left(\overline{\Omega}\right)$ such that $\varphi_{|\Gamma} = 0$, we get

$$\partial_n v_1 + (\delta+1) \frac{\delta+\beta+1}{\delta-\alpha+1} v_1 = \tilde{g} \text{ on } \Sigma.$$

and, taking $\varphi$ in $\mathcal{D}\left(\overline{\Omega}\right)$, we obtain

$$\partial_n v_1 = 0 \text{ on } \Gamma.$$

Therefore $v_1$ is solution of the problem (3.17).
Since $v_1$ and $f_1$ are in $H^1(\Omega)$, we deduce the existence of $v_2$ in $H^1(\Omega)$.
Now, we have to check that the pair $(v_1, v_2)$ is in $V_\delta$. Actually, we just have to prove that $(v_1, v_2)$ checks the condition on $\Gamma$ and the invariance property

because all the other properties have been shown in the proof of Lemma 3.3. Since $\partial_n v_1 = 0$ on $\Gamma$, we just have to prove that $\partial_n v_2 = 0$ on $\Gamma$. We know that $\partial_n v_1 = \partial_n f_1 = 0$ on $\Gamma$ and by using that $v_2 = f_1 + (\delta+1)v_1$ in $\Omega$, we get $\partial_n v_2 = 0$ on $\Gamma$. As for the invariance property, we have assume that $(f_1, f_2)$ is in $H$. Hence we have

$$\int_\Omega f_2\, dx - \int_\Sigma \alpha^{-1}\left(f_2 + \partial_n f_1 + \beta f_1\right) d\sigma = 0.$$

Therefore,

$$\begin{aligned}\int_\Omega v_2\, dx &= \frac{1}{\delta+1}\int_\Omega \left(\triangle v_1 - f_2\right)\, dx \\ &= \frac{1}{\delta+1}\left[\int_{\partial\Omega} \partial_n v_1\, d\sigma - \int_\Sigma \alpha^{-1}\left(f_2 + \partial_n f_1 + \beta f_1\right)\, d\sigma\right]\end{aligned}$$

Hence, we get

$$\begin{aligned}\int_\Omega v_2\, dx &= \frac{1}{\delta+1}\left[\int_\Sigma \alpha^{-1}\left(\partial_n v_2 + \triangle v_1 + \beta v_2 - (\triangle v_1 - (\delta+1)v_2 \; + \right.\right. \\ &\qquad \left.\left. \partial_n v_2 - (\delta+1)\partial_n v_1 + \beta v_2 - (\delta+1)\beta v_1)\right)\, d\sigma\right]\end{aligned}$$

Finally, we have

$$\int_\Omega v_2\, dx - \int_\Sigma \alpha^{-1}\left(v_2 + \partial_n v_1 + \beta v_1\right) d\sigma = 0.$$

Consequently, we have shown that the pair $(v_1, v_2)$ is in $H$. Now, we have to check that

$$\int_\Omega \left(\triangle v_1 - \delta v_2\right)\, dx - \int_\Sigma \alpha^{-1}\left(\triangle v_1 - \delta v_2 + \partial_n v_2 - \delta\partial_n v_1 + \beta v_2 - \beta\delta v_1\right) d\sigma = 0.$$

To prove this result, we just use that $(f_1, f_2)$ and $(v_1, v_2)$ check the invariance property.
Finally, we have proved that $A_\delta(v_1, v_2)$ is in $H$ and since the condition on $\Sigma$ is satisfied, we have shown that $(v_1, v_2)$ is in $V_\delta$ which ends the proof. □

So we have the following theorem on the well-posedness of the problem.

**Theorem 3.9.** *Let $(u_0, u_1)$ in $V_0$. The problem (3.1) admits a unique solution $u$ such that*

$$(u,\, \partial_t u) \in C^1([0, +\infty[; V_0) \cap C^0([0, +\infty[; H). \tag{3.19}$$

The proof is the same as in Subsection 3.4.

# 4 Long time behavior

We can enrich the results of Section 3 by considering the energy functional defined on $H$ by

$$\mathcal{E}(h_1, h_2) = \frac{1}{2}\int_\Omega \left(|\triangle h_1|^2 + |\nabla h_2|^2\right) dx + \frac{1}{2}\int_\Sigma \frac{1}{\beta - \alpha}|\alpha\partial_n h_1 - \beta h_2|^2 d\sigma$$

which is well defined if

$$\beta(x) - \alpha(x) \neq 0 \text{ for all } x \in \Sigma. \tag{4.1}$$

Moreover, $\mathcal{E}(h_1, h_2) \geq 0$ if

$$\beta(x) - \alpha(x) > 0 \text{ for all } x \in \Sigma. \tag{4.2}$$

Then, $\mathcal{E}(u, \partial_t u)$ defines an energy on $H$ under the condition (4.2) which takes into account the condition (4.1).
We recall that in the case of the ABC we have constructed in Section 2, we have

$$\alpha(x) = \frac{\kappa(x)}{4} - \gamma(x),\ \beta(x) = \frac{\kappa(x)}{4} + \gamma(x).$$

and that the boundary $\Sigma$ has been chosen so that $\kappa(x) > 0$ for any $x \in \Sigma$.
In the following, we assume that $\alpha$ and $\beta$ check the condition

$$\left\{ \begin{array}{l} \alpha(x) < 0 \text{ and } \beta(x) > 0 \text{ for all } x \in \Sigma, \\ \beta(x) - \alpha(x) \neq 0 \text{ for all } x \in \Sigma. \end{array} \right. \tag{4.3}$$

This condition implies that the function $\gamma$ has to be positive on $\Sigma$ and that for all $x \in \Sigma$, $\gamma(x) > \frac{\kappa(x)}{4}$.

**Lemma 4.1.** *For all $(u_0, u_1) \in V_0$, $t \longmapsto \mathcal{E}(u, \partial_t u)$ is differentiable and is decreasing under the condition (4.3)*

*Proof.* In the previous section, we have seen that if the initial conditions $(u_0, u_1)$ are in $V_0$, $\mathcal{E}(u, \partial_t u) \in C^1([0, +\infty[)$ and we get

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(u, \partial_t u) &= \int_\Omega \triangle(\partial_t u) \triangle u \, dx + \int_\Omega \partial_t(\nabla u)\partial_t^2 \nabla u \, dx \\ &+ \int_\Sigma \frac{1}{\beta - \alpha} \left(\alpha \partial_t \partial_n u - \beta \partial_t^2 u\right) \left(\alpha \partial_n u - \beta \partial_t u\right) d\sigma. \end{aligned} \tag{4.4}$$

We use the Green formula (3.7) and that $\partial_t^2 u = \triangle u$

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(u, \partial_t u) = & \int_\Omega \triangle (\partial_t u)\, \partial_t^2 u \, dx - \int_\Omega \triangle (\partial_t u)\, \partial_t^2 u \, dx + \int_{\partial\Omega} \partial_n \partial_t u \, \partial_t^2 u \, d\sigma + \\ & \int_\Sigma \frac{1}{\beta - \alpha} \left(\alpha \partial_t \partial_n u - \beta \partial_t^2 u\right) \left(\alpha \partial_n u - \beta \partial_t u\right) d\sigma. \end{aligned}$$

Moreover, $\partial_t u_{|\Gamma} = 0$ in the case of a Dirichlet boundary condition on $\Gamma$ and $\partial_n u_{|\Gamma} = 0$ in the case of a Neumann boundary condition on $\Gamma$.
Then, using that $\partial_t \left(\partial_n u + \partial_t u\right) = \alpha(x)\partial_n u - \beta(x)\partial_t u$ on $\Sigma$, we have

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}(u, \partial_t u) = \int_\Sigma \partial_n \partial_t u \, \partial_t^2 u \, d\sigma \\ &+ \int_\Sigma \frac{1}{\beta - \alpha} \left[\alpha \partial_t^2 u \, \partial_n \partial_t u - \beta \left(\partial_t^2 u\right)^2 + \alpha \left(\partial_t \partial_n u\right)^2 - \beta \partial_t^2 u \, \partial_n \partial_t u\right] d\sigma. \end{aligned}$$

Consequently we get

$$\frac{d}{dt}\mathcal{E}(u, \partial_t u) = -\int_\Sigma \frac{1}{\beta - \alpha} \left[\beta \left(\partial_t^2 u\right)^2 - \alpha \left(\partial_t \partial_n u\right)^2\right] d\sigma.$$

Under the condition (4.3) ,

$$\frac{d}{dt}\mathcal{E}(u,\partial_t u) \leq 0,$$

which ends the proof of the result. □

**Theorem 4.2.** *If the functions $\alpha$ and $\beta$ check the condition (4.3), then for all $(u_0, u_1) \in V_0$,*

$$\lim_{t\to+\infty} \mathcal{E}(u,\partial_t u) = 0.$$

*Proof.* We have already seen that $A$ is the generator of the continuous semi-group $Z(t)$, of contraction if $\alpha$ and $\beta$ check the condition (4.3). As it is sufficient to prove the theorem on a dense subspace of $V_0 = D(A)$, we consider the initial data $(u_0, u_1)$ in $D(A^2)$, where

$$D(A^2) = \{(v_1, v_2) \in V_0, A(v_1, v_2) \in V_0\}$$

is equipped with the norm graph

$$\|(v_1, v_2)\|_{D(A^2)} = \|(v_1, v_2)\|_{V_0} + \|A(v_1, v_2)\|_{V_0} + \|A^2(v_1, v_2)\|_{V_0}.$$

For all solution of (3.1), we have

$$\begin{aligned}\|(u,\partial_t u)\|_{D(A^2)} &= \|Z(t)(u_0, u_1)\|_{D(A^2)} \\ &= \|Z(t)(u_0, u_1)\|_{V_0} + \|A(Z(t)(u_0, u_1))\|_{V_0} + \|A^2(Z(t)(u_0, u_1))\|_{V_0}.\end{aligned}$$

As we can switch $A$, $A^2$ and $Z(t)$ on $D(A^2)$, we have

$$\begin{aligned}\|(u,\partial_t u)\|_{D(A^2)} &= \|Z(t)(u_0, u_1)\|_{D(A^2)} \\ &= \|Z(t)(u_0, u_1)\|_{V_0} + \|Z(t)A(u_0, u_1)\|_{V_0} + \|Z(t)A^2(u_0, u_1)\|_{V_0}.\end{aligned}$$

Since $Z(t)$ is continuous in $V_0$, we deduce that there exists a positive constant $C$ such that

$$\|(u,\partial_t u)\|_{D(A^2)} \leq C\|(u_0, u_1)\|_{D(A^2)}.$$

But, since the injection $i : D(A^2) \to V_0$ is compact, we can extract a subsequence $Z(t_k)(u_0, u_1)$ such that

$$\lim_{t_k\to+\infty} Z(t_k)(u_0, u_1) = (u_\infty, v_\infty) \text{ in } V_0 \text{ strongly .}$$

Since $\mathcal{E}(u,\partial_t u)$ is continuous, we have

$$\lim_{t\to+\infty} \mathcal{E}(u,\partial_t u) = \lim_{t\to+\infty} \mathcal{E}(Z(t)(u_0, u_1)) = \lim_{t_k\to+\infty} \mathcal{E}(Z(t_k)(u_0, u_1)) = \mathcal{E}(u_\infty, v_\infty).$$

We also have, for all $s$ positive;

$$\begin{aligned}\lim_{t\to+\infty} \mathcal{E}(Z(t+s)(u_0, u_1)) &= \lim_{t\to+\infty} \mathcal{E}(u,\partial_t u) \\ &= \lim_{t_k\to+\infty} \mathcal{E}(Z(s)Z(t_k)(u_0, u_1)) \\ &= \mathcal{E}(Z(s)(u_\infty, v_\infty)).\end{aligned}$$

The solution $(w, \partial_t w) = Z(t)(u_\infty, v_\infty)$ of problem (3.1), for initial data $(u_\infty, v_\infty)$ in $D(A)$, is thus such that

$$\mathcal{E}(w, \partial_t w) = \mathcal{E}(u_\infty, v_\infty) \text{ pour tout } t \text{ positif,}$$

and according to the proof of Lemma 4.1,

$$\frac{d}{dt}\mathcal{E}(, \partial_t w) = -\int_\Sigma \frac{1}{\beta - \alpha}\left[\beta\left(\partial_t^2 w\right)^2 - \alpha\left(\partial_t \partial_n w\right)^2\right] d\sigma,$$

we necessarily have $\partial_t^2 w = 0$ on $\Sigma$ and $\partial_t \partial_n w = 0$ on $\Sigma$. $w$ is then solution to the following problem

$$\begin{cases} \partial_t^2 w - \triangle w = 0 \text{ in } \Omega \times [0, +\infty[ \\ w(x,0) = u_\infty,\ \partial_t w(x,0) = v_\infty \text{ in } \Omega \\ w = 0 \text{ or } \partial_n w = 0 \text{ on } \Gamma \times [0, +\infty[ \\ \alpha(x)\partial_n w = \beta(x)\partial_t w \text{ on } \Sigma \times (0, +\infty) \end{cases}$$

and $z := \partial_t w$ is solution to

$$\begin{cases} \partial_t^2 z - \triangle z = 0 \text{ in } \Omega \times [0, +\infty[ \\ w(x,0) = v_\infty,\ \partial_t z(x,0) = \triangle u_\infty \text{ in } \Omega \\ z = 0 \text{ or } \partial_n z = 0 \text{ on } \Gamma \times [0, +\infty[ \\ \partial_n z = z = 0 \text{ on } \Sigma \times (0, +\infty)\,. \end{cases}$$

Since $\partial_n z = z = 0$ on $\Sigma$, we deduce that $z = 0$ in $\Omega \times [0, +\infty[$, as a consequence of the Holmgren theorem (see Lions [13]). Therefore, $w$ is solution to

$$\begin{cases} \triangle w = 0 \text{ in } \Omega \times (0, +\infty) \\ w = 0 \text{ or } \partial_n w = 0 \text{ on } \Gamma \times [0, +\infty[ \\ \alpha(x)\partial_n w = 0 \text{ on } \Sigma \times [0, +\infty[ \end{cases}$$

Since $\alpha$ never vanishes on $\Sigma$, we thus have $w = 0$ in $\Omega \times [0, +\infty[$. We then conclude that $\mathcal{E}(w, \partial_t w) = 0$ which implies that $\mathcal{E}(u_\infty, v_\infty) = 0$ and completes the proof of 4.2. □

**Lemma 4.3.** *$\mathcal{E}^{1/2}(h_1, h_2)$ is a norm on $H$.*

*Proof.* The proof of this lemma is divided in two parts. First we consider the case of a Dirichlet boundary condition on $\Gamma$ and in the second part the case of a Neumann boundary condition on $\Gamma$.
(a) *A Dirichlet boundary condition on* $\Gamma$
We consider $(h_1, h_2)$ such that $\mathcal{E}(h_1, h_2) = 0$. We have

$$\begin{cases} \triangle h_1 = 0 \text{ in } \Omega\,; \\ \nabla h_2 = 0 \text{ in } \Omega\,; \\ h_1 = h_2 = 0 \text{ on } \Gamma\,; \\ \alpha \partial_n h_1 - \beta h_2 = 0 \text{ on } \Sigma. \end{cases}$$

Since $\Omega$ is connexe, $\nabla h_2 = 0$ in $\Omega$ implies that $h_2$ is constant in $\Omega$. But we already know that $h_2 = 0$ on $\Gamma$ so $h_2 = 0$ in $\Omega$.
Therefore, $h_1$ is solution to

$$\left\{\begin{array}{l} \triangle h_1 = 0 \text{ in } \Omega\,; \\ h_1 = 0 \text{ on } \Gamma\,; \\ \alpha \partial_n h_1 = 0 \text{ on } \Sigma. \end{array}\right.$$

Since $\alpha \neq 0$ for all point in $\Sigma$, we have that $\partial_n h_1 = 0$ on $\Sigma$. Consequently, $(h_1, h_2) = 0$ in $\Omega$ which proves that $\mathcal{E}^{1/2}(h_1, h_2)$ is a norm on $H$.

(b) *A Neumann boundary condition on* $\Gamma$
Here again, we consider $(h_1, h_2)$ such that $\mathcal{E}(h_1, h_2) = 0$. We have

$$\left\{\begin{array}{l} \triangle h_1 = 0 \text{ in } \Omega\,; \\ \nabla h_2 = 0 \text{ in } \Omega\,; \\ \partial_n h_1 = 0 \text{ on } \Gamma\,; \\ \alpha \partial_n h_1 - \beta h_2 = 0 \text{ on } \Sigma. \end{array}\right.$$

and since $(h_1, h_2) \in H$

$$\int_\Omega h_2\, dx - \int_\Sigma \alpha^{-1} \left(h_2 + \partial_n h_1 + \beta h_1\right) d\sigma = 0.$$

$\nabla h_2 = 0$ in $\Omega$ implies that $h_2$ is constant in $\Omega$ because $\Omega$ is connexe. Since $\triangle h_1 = 0$ in $\Omega$ and $\nabla h_2 = 0$ in $\Omega$, we get using the Green formula (3.7) that

$$\int_{\partial\Omega} \partial_n h_1 h_2\, d\sigma = 0.$$

But we have $\partial_n h_1 = 0$ on $\Gamma$ and $\partial_n h_1 = \alpha^{-1}\beta h_2$ on $\Sigma$. Hence,

$$\int_{\partial\Omega} \alpha^{-1}\beta |h_2|^2\, d\sigma = 0$$

which implies that $h_2 = 0$ in $\Omega$ because $\alpha$ and $\beta$ check (4.3). We then get that $h_1$ is solution of

$$\left\{\begin{array}{l} \triangle h_1 = 0 \text{ in } \Omega\,; \\ \partial_n h_1 = 0 \text{ on } \partial\Omega\,; \end{array}\right.$$

which implies that $h_1$ is constant in $\Omega$.
Moreover, we have

$$\int_\Omega h_2\, dx - \int_\Sigma \alpha^{-1} \left(h_2 + \partial_n h_1 + \beta h_1\right)\, d\sigma = 0.$$

*ie*

$$\int_\Sigma \alpha^{-1}\beta h_1\, d\sigma = 0$$

Since $h_1$ is constant in $\Omega$ and $\alpha$ and $\beta$ check the condition (4.3) on $\Sigma$, we get $h_1 = 0$ in $\Omega$ which proves that $\mathcal{E}^{1/2}(h_1, h_2)$ is a norm on $H$. $\square$

We are now willing to state

**Theorem 4.4.** *Let $(u_0, u_1)$ in $H$. Then the solution $u$ of (3.1) is such that*

$$\lim_{t\to+\infty}(u, \partial_t u) = (0,0) \text{ in } H.$$

*Proof.* If the pair $(u_0, u_1)$ is in $H$, we know that (cf. Theorem 4.2)

$$\lim_{t\to+\infty}\mathcal{E}(u, \partial_t u) = 0$$

and that $\mathcal{E}$ is a norm on $H$ (cf. Lemma 4.3).
Therefore,

$$\lim_{t\to+\infty}(u, \partial_t u) = (0,0) \text{ in } H.$$

□

Now, in case of a Neumann boundary condition, general data are in $V_0$ and not in $H \cap V_0$. In that case, we have:

**Theorem 4.5.** *Let $(u_0, u_1)$ in $V_0$. Then if $\gamma > \frac{\kappa}{4}$ on $\Sigma$,*

$$\lim_{t\to+\infty}(u, \partial_t u) = (u_\infty, 0) \text{ in } X$$

*where*

$$u_\infty = \frac{\displaystyle\int_\Omega u_1 dx - \int_\Sigma \alpha^{-1}(u_1 + \beta u_0) d\sigma}{\displaystyle\int_\Sigma \alpha^{-1}\beta d\sigma}.$$

*Proof.* Let $(\tilde{u}, \partial_t \tilde{u})$ be the solution to (3.1) with initial data $(\tilde{u_0}, \tilde{u_1})$ defined by

$$\tilde{u_0} = u_0 - u_\infty,\ \tilde{u_1} = u_1.$$

Then $(\tilde{u_0}, \tilde{u_1}) \in V_0 \cap H$ and we can apply Theorem 4.4 to $(\tilde{u}, \partial_t \tilde{u})$ which completes the proof of Theorem 4.5. □

# 5 Conclusion

We have constructed a new family of ABCs depending on the parameter $\gamma \neq 0$ and which can be applied on arbitrarily-shaped regular boundaries. We have performed a mathematical analysis which shows that the corresponding boundary value problem is well-posed in Hilbert spaces of Sobolev class. Moreover, if $\gamma$ is greater than $\frac{\kappa}{4}$, where $\kappa$ denotes the curvature of the absorbing boundary $\Sigma$, there exists an energy defined as a combination of volume and surface terms. It is non conventional in the sense that it is given as the sum of the kinematic energy of $\partial_t u$ and the $L^2$-norm on $\Sigma$ of a linear combination of $\partial_n u$ and $\partial_t u$. We have been able to prove that if $\gamma \geq \frac{\kappa}{4}$, the energy is decreasing and is converging to zero. We have also proved that in the case of a sound-soft scatterer, the solution converges to zero. In case of a sound-hard scatterer, the solution can converge to a constant and we have established a necessary condition for the constant to be zero. Now it would be interesting to analyse if there is a value of

$\gamma$ that can improve the numerical results. For that purpose, we have considered different approaches and we now believe that the best way to implement the ABC (2.7) into a finite element formulation consists in defining an auxiliary unknown to obtain an ABC easier to introduce in the formulation. The ABC (2.7) is then rewritten as follows :

$$\partial_n u = -\partial_t u - \frac{\kappa}{2}\left(\partial_t - \frac{\kappa}{4} + \gamma\right)^{-1} \partial_t u \text{ on } \Sigma$$

and we define $\psi$ as the surface field satisfying

$$\left(\partial_t - \frac{\kappa}{4} + \gamma\right)\psi = \partial_t u \text{ on } \Sigma.$$

Then the solution $u$ satisfies

$$\partial_n u = -\partial_t u - \frac{\kappa}{2}\psi \text{ on } \Sigma,$$

which can be easily included into the variational formulation after introducing $\psi$ which is only defined on the boundary $\Sigma$. Then, the boundary value problem reads as: find $(u, \partial_t u, \psi)$ solution to

$$\begin{cases} \partial_t^2 u - \triangle u = 0 \text{ in } \Omega \times (0, +\infty)\,; \\ u(0,x) = u_0(x),\, \partial_t u(0,x) = u_1(x) \text{ in } \Omega; \\ \psi(0,x) = \psi_0(x) \text{ on } \Sigma; \\ u = 0 \text{ on } \Gamma \times (0, +\infty)\,; \\ \partial_n u + \partial_t u + \dfrac{\kappa}{2}\psi = 0 \text{ on } \Sigma \times (0, +\infty)\,; \\ \left(\partial_t - \dfrac{\kappa}{4} + \gamma\right)\psi = \partial_t u \text{ on } \Sigma \times (0, +\infty)\,. \end{cases} \tag{5.1}$$

and we suppose that $\psi(0,x) = 0$ on $\Sigma$. By this way, (5.1) is equivalent to (2.11). The mathematical and the numerical studies of (5.1) will be developed in another research report that will follow this work.

# Contents

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