Local regularity-based interpolation

Pierrick Legrand, Jacques Lévy Véhel
Projet Fractales, IRCCyN
1 Rue de la Noé, B.P. , 44000 Nantes, France
{Pierrick.Legrand, Jacques.Levy-Vehel}@irccyn.ec-nantes.fr

Description of the work and motivation

A ubiquitous problem in signal processing is to obtain data sampled with the best possible resolution. At the acquisition step, the resolution is limited by various factors such as the physical properties of the captors or the cost. It is therefore desirable to seek methods which would allow to increase the resolution after acquisition. This is useful for instance in medical imaging or target recognition. In some applications, one dispose of several low resolution overlapping signals [1]. In more general situations, a single signal is available for superresolution. Interpolation then requires that the available data be supplemented by some *a priori* information. Two types of methods have been explored: In the first, "class-based" one, the signal is assumed to belong to some class, with conditions expressed mainly in the time or frequency domain [3, 5, 10]. This puts constraints on the interpolation, which is usually obtained as the minimum of a cost-function. The second type of approaches hypothesizes that the information needed to improve the resolution is local and is present in a class of similar signals [2, 4]. This type of approach could be called "contextual". Both "class-based" and "contextual" approaches use a "model" for interpolation: The "class-based model" is that the signal belongs to an abstract class characterized by a certain mathematical property. The "contextual model" is that the signal will behave locally under a change of resolution in way "similar" to other signals in a given set, for which a high resolution version is known. Both types of techniques have some drawbacks. Roughly speaking, class-based methods generally lead to overly smooth signals, while contextual-based ones, on the contrary, tend to generate spurious details.

Our motivation is to find a way to interpolate in such a way that smooth regions as well as irregular ones (e.g. sharp edges) remain so after zooming. We interpret this as the constraint that interpolation should preserve the local regularity. We measure this regularity through a notion of Hölder exponent. Hölder exponents have been shown to correspond to an intuitive notion of regularity in both images and 1D signals [6]. In order to control the interpolation and obtain a simple implementation, we need to make some assumptions on the signal, to the effect that (a) this Hölder exponent can be easily estimated from wavelet coefficients (b) the exponent allows to predict the finer scales coefficients. Technically, this requires that the signal is not *oscillatory*. Our scheme allows to control both the reconstruction error and the regularity of the interpolated signal, i.e. the visual appearance of the added information.

1 The method

The method is best understood in terms of wavelet coefficients: Let X denoted the input signal and let $d_{j,k}$ be its wavelet coefficients, where, as usual, j corresponds to scale and k to location. Roughly speaking, if a signal has regularity α at point t, then its wavelet coefficient $d_{j,k(j,t)}$ "above" t are bounded by $C2^{-j\alpha}$ for some constant C: $\forall j = 1 \dots n, |d_{j,k(j,t)}| \leq C2^{-j\alpha}$. The exponent α corresponds to an intuitive notion of regularity: A large α translates in a smooth signal, while $\alpha \in (0, 1)$ means that the signal is continuous and non differentiable at t. If the signal is discontinuous at t but bounded, then $\alpha = 0$. Now if we are willing to conserve the regularity, we should prescribe the wavelet coefficient above t at the superresolved scale $n + 1$ in such a way that $|d_{n+1,k(n+1,t)}| \leq C2^{-(n+1)\alpha}$.

For concreteness, let us explain schematically how our method would act on a uniform region and on a step edge. On a uniform region, all the wavelet coefficients are close to zero. The bound on the wavelet coefficients then holds with arbitrarily large α, since $0 \leq C2^{-j\alpha}$ for all $\alpha > 0$. As a consequence, the predicted coefficient $d_{n+1,k}$ will be zero, since it must satisfy the same inequality: Smooth regions will remain smooth, because no detail will be added. On the other hand, above a step edge, the wavelet coefficients $d_{j,k}$ do not decay in scale. This imply that $\alpha = 0$. The predicted coefficient $d_{n+1,k}$ will then be of the same order as $d_{n,k}$. As a consequence, the local regularity of the interpolated image will be again equal to 0 at this point, and the edge will not be blurred.

Our approach has similarities with both class-based and contextual methods. As in class-based methods, we do not rely on information learnt from a database, and require that the signal belongs to a definite class. A distinctive feature is that our class is based on local features. The similarity with contextual approaches is that the information is learnt from the image instead of being *a priori*, as in class-based ones. Also, this information is local in space and scale, and is versatile enough to handle a wide variety of situations.

Let X denote the original signal, and $X_n = (x_1^n, \dots x_{2^n}^n)$ its regular sampling over the 2^n points $(t_1^n, \dots t_{2^n}^n)$. Let ψ denote a wavelet such that the set $\{\psi_{j,k}\}_{j,k}$ forms an orthonormal basis of L^2. Let $d_{j,k}$ be the wavelet coefficients of X.

For $p = 1 \dots 2^n$, we consider the point $t = t_p^n$ and the wavelet coefficients which are located "above" it, i.e. $d_{j,k(j,t)}$ with $k(j, t) = \lfloor (t - 1)/(2^{n+1-j}) \rfloor + 1$. Let $\alpha_n(t)$ denote the slope of the liminf regression (see [7] for an account on liminf regressions) of the vector $(\log(d_{1,k(1,t)}, \dots d_{n,k(n,t)})$ versus $(-1, \dots, -n)$. When n tends to infinity, $\alpha_n(t)$ tends to liminf $\frac{\log d_{n,k(n,t)}}{-n}$. This number has been considered in the literature [9] under the name of *weak scaling* exponent, denoted β_w. It is a measure of the local regularity in the following sense. The weak scaling exponent of the signal X at t_0 is defined as: $\beta_w = \sup\{s : \exists n, \;\; X^{(-n)} \in C_{t_0}^{s+n}\}$ where $X^{(-l)}$ denotes a primitive of order l of X and $C_{t_0}^s$ is the usual pointwise Hölder space at t_0. When the local Hölder exponent α_l and the pointwise Hölder exponent α_p of X at t coincide, then β_w is also equal to their common value. See [8] for more on this topic. In the following we will always assume that this is the case. In other words, we consider that our signals belong to the class $\mathcal{S}$ defined as follows:

$$\mathcal{S} = \{X \in L^2(I\!R), \forall t \in I\!R, \alpha_p(t) = \alpha_l(t)\}$$

The class $\mathcal{S}$ may appear somewhat abstract to the reader. Here are a few clues. $\mathcal{S}$ contains all C^∞ signals and all signals of the type $\sum_{n\in I\!N} |t - t_n|^{\gamma_n}$, with $t_n \in I\!R, \gamma_n \in I\!R^+$. Many everywhere irregular signals are also in $\mathcal{S}$, such as the continuous nowhere differentiable Weierstrass function $\sum_{n\in I\!N} 2^{-nh} \sin(2^n t)$, $h \in (0,1)$. On the other hand, "chirp" signals as $|t|^\gamma \sin(1/|t|^\beta)$, $\gamma > 0, \beta > 0$ do not belong to $\mathcal{S}$. For X in $\mathcal{S}$, at each point t, the "largest" coefficients are located above t in the following sense. Take any sequence $d_{j,k}$ of coefficients such that $k2^{-j}$ tends to t. Then, if X belongs to $\mathcal{S}$,

$$\text{liminf}_{\ j\to\infty} \frac{\log|d_{j,k}|}{-j} \geq \text{liminf}_{\ j\to\infty} \frac{\log|d_{j,k(j,t)}|}{-j} \tag{1}$$

Such signals are called "non-oscillatory" in the literature. To perform the interpolation, we compute above each point t the regression of the wavelet coefficients *vs* scale (figure 1). The parameters of the regression allow to build the extrapolated coefficient. These coefficients in turn determine the "superresolved" signal. For a sampled signal X_n, we denote $\tilde{X}_n$ the signal interpolated on infinitely many levels.

2 Regularity and Asymptotic properties

Proposition 1 *Let X be in C^η for some $\eta > 0$. Then, for all $\varepsilon > 0$, there exists N such that for all $n > N$, all p and all $t \in (t_p^n, t_{p+1}^n)$, $\alpha_p^{\tilde{X}_n}(t) = \alpha_l^{\tilde{X}_n}(t) \in [\beta - \varepsilon, \beta + \varepsilon]$, where $\beta = \min(\beta_w(t_l^n), \beta_w(t_{l+1}^n))$.*

Proposition 2 *Let X belong to $\mathcal{S}$, and assume $X \in C^\alpha$. Then, $\forall \varepsilon > 0, \exists N : n > N \Rightarrow ||X - \tilde{X}_n||_2 = \mathcal{O}(2^{-n(\alpha-\varepsilon)})$. In addition, $||X - \tilde{X}_n||_{B^s_{p,q}} = \mathcal{O}(2^{-n(\alpha-s-\varepsilon)})$ for all $s < \alpha - \varepsilon$.*

3 Numerical experiments

We provide experiments on a step edge (figure 2), a generalized Weierstrass function (figure 3), and a real-world signal, namely a road profile (figure 4). In each case, we compare our method with a simple linear interpolation.

References

[1] S. Baker, T. Kanade, *Super-resolution: Reconstruction of Recognition*, Proc. of IEEE-Eurasip Workshop on Nonlinear Signal and Image Processing, **2001**, pp. 349-385.

[2] F.M. Candocia, J.C. Principe, *Superresolutions of images based on local correlations*, IEEE Tr. Neural Networks, **1999**, pp. .

[3] D. M. Donoho, I. M. Johnstone,J.C. Hoch, A.S. Stern, *Maximum Entropy and the Nearly Balck Object*, J.R. Statist. Soc. B., 54 **1 (1992)**, pp. 41–81 .

[4] W.T. Freeman, T.R. Jones, E.C. Pazstor, *Example-Based Super-Resolution*, MERL Tech. Rep. 2001-30, **2001**.

[5] K.-L Kueh, T. Olson, D. Rockmore, K.-S Tan, *Non-linear Approximation Theory on Finite Group*, 1999.

[6] J. Lévy Véhel, *Fractal Approaches in Signal Processing*, Fractals, **3(4) (1995)**, pp. 755-775.

[7] P. Legrand, J. Lévy Véhel, *Signal Processing with FracLab* Preprint

[8] J. Lévy Véhel, S. Seuret *The 2-microlocal formalism*, preprint.

[9] Y. Meyer, *Wavelets, Vibrations and Scalings*, American Mathematical Society, **9 (1997)**, CRM Monograph Series.

[10] D.C. Youla, H. Webb, *Image restoration by the method of convex projections: Part I - theory*, IEEE Tr. Med. Imag. **1**(2), 1982, pp. 81–94.

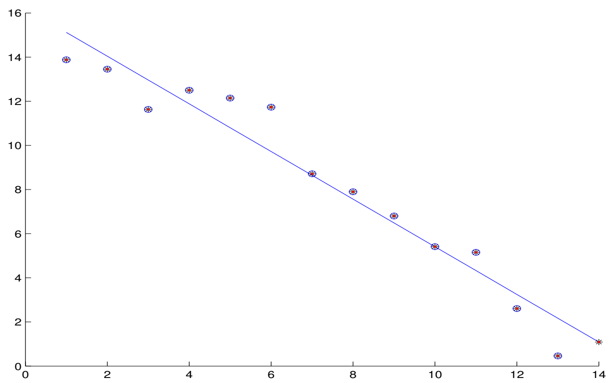

Figure 1: *Regression above a point of the signal. We find scales in abscissa and the log of the wavelet coefficients in ordinate. The extrapolated coefficient is the rightmost point.*

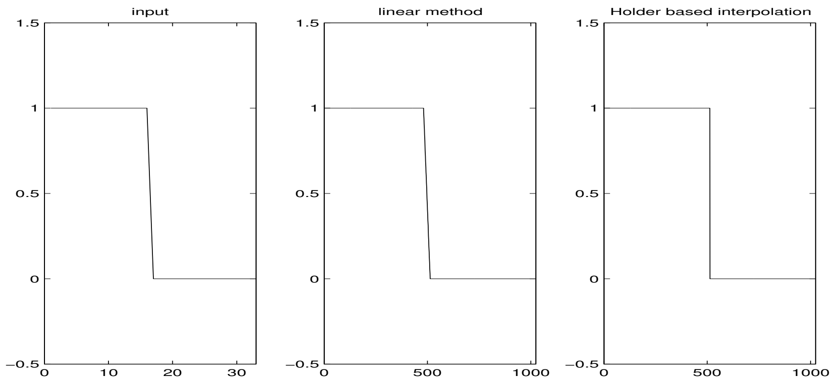

Figure 2: *Interpolation of a step edge (32 points to 1024 points). Notice how the edge is not blurred with the Hölder based interpolation.*

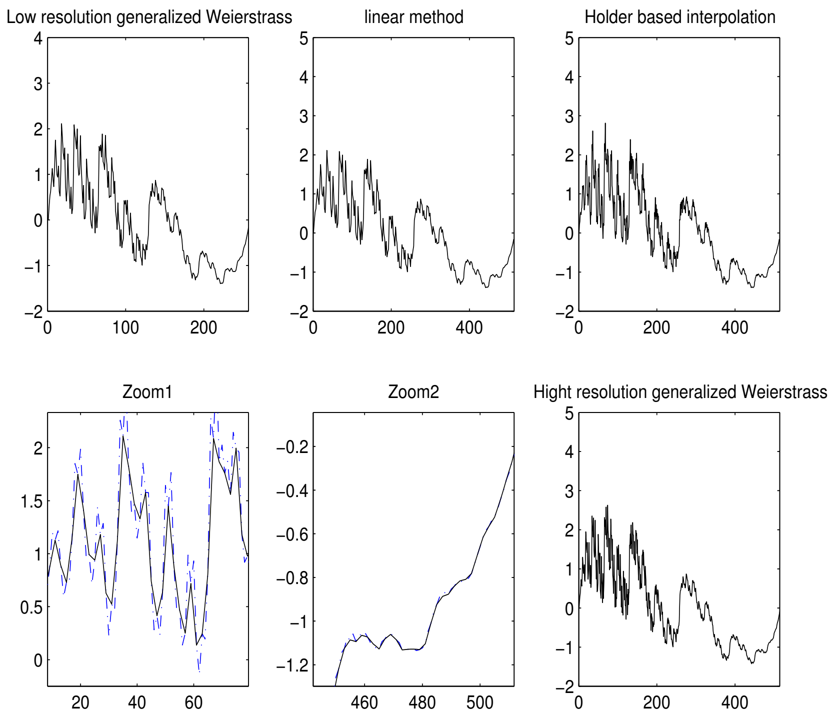

Figure 3: *Generalized Weierstrass* ($\sum_{n\in\mathbb{N}} 2^{-nt}\sin(2^n t)$) *with zooms and comparison with linear interpolation. Notice how the Hölder based method preserves the irregular as well as regular parts of the signal. In the zoomed figures the Hölder based interpolation is the dashed one.*

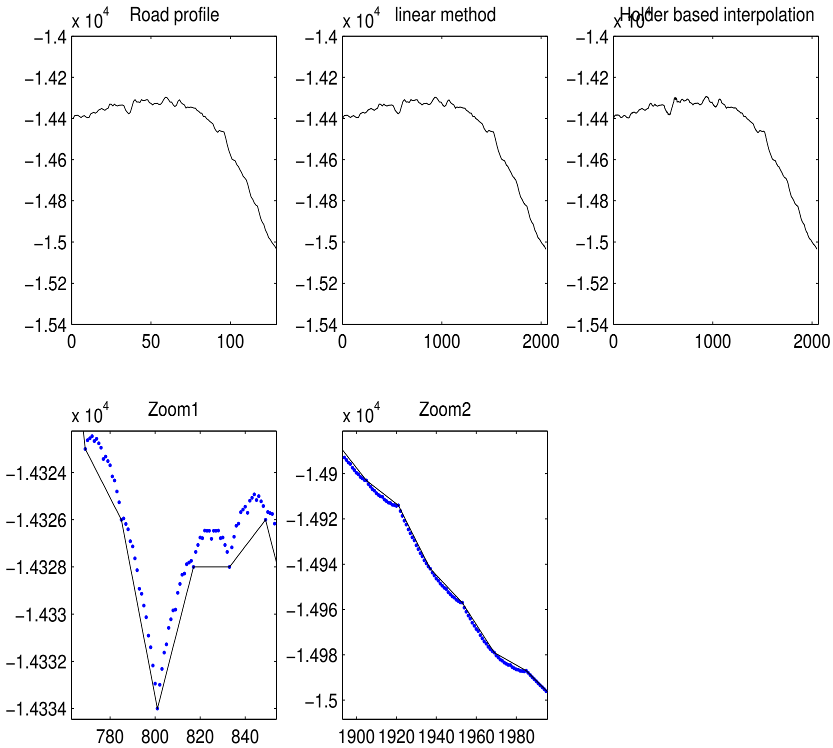

Figure 4: *Experiment on a road profile. Same comments that in the figure 3.*