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On the hull number of some graph classes. *

J. Araujo † ‡ , V. Campos ‡ , F. Giroire † , N. Nisse † , L. Sampaio † § , R. Soares † ‡

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Abstract:

In this paper, we study the geodetic convexity of graphs focusing on the problem of the complexity to compute inclusion-minimum hull set of a graph in several graph classes.

For any two vertices $u, v \in V$ of a connected graph $G = (V, E)$, the *closed interval* $I[u, v]$ of u and v is the the set of vertices that belong to some shortest (u, v)-path. For any $S \subseteq V$, let $I[S] = \bigcup_{u,v \in S} I[u, v]$. A subset $S \subseteq V$ is *geodesically convex* if $I[S] = S$. In other words, a subset S is convex if, for any $u, v \in S$ and for any shortest (u, v)-path P, $V(P) \subseteq S$. Given a subset $S \subseteq V$, the *convex hull* $I_h[S]$ of S is the smallest convex set that contains S. We say that S is a *hull set* of G if $I_h[S] = V$. The size of a minimum hull set of G is the *hull number* of G, denoted by $hn(G)$. The HULL NUMBER problem is to decide whether $hn(G) \leq k$, for a given graph G and an integer k. Dourado *et al.* showed that this problem is NP-complete in general graphs.

In this paper, we answer an open question of Dourado et al. [12] by showing that the HULL NUMBER problem is NP-hard even when restricted to the class of bipartite graphs. Then, we design polynomial time algorithms to solve the HULL NUMBER problem in several graph classes. First, we deal with the class of complements of bipartite graphs. Then, we generalize some results in [1] to the class of $(q, q-4)$-graphs and to the class of cacti. Finally, we prove tight upper bounds on the hull numbers. In particular, we show that the hull number of an n-node graph G without simplicial vertices is at most $1 + \lceil \frac{3(n-1)}{5} \rceil$ in general, at most $1 + \lceil \frac{n-1}{2} \rceil$ if G is regular or has no triangle, and at most $1 + \lceil \frac{n-1}{3} \rceil$ if G has girth at least 6.

Key-words: graph convexity, hull number, bipartite graph, cobipartite graph, cactus graph, $(q, q-4)$-graph.

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† {julio.araujo, frederic.giroire, nicolas.nisse, leonardo.sampaio_rocha, ronan.pardo_soares}@inria.fr - MASCOTTE Project, I3S (CNRS & UNS) and INRIA, INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 Sophia-Antipolis Cedex France.

‡ {campos@lia.ufc.br} ParGO Research Group - Universidade Federal do Ceará - UFC - Campus do Pici, Bloco 910. 60455-760 - Fortaleza, CE - Brazil.

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Centre de recherche INRIA Sophia Antipolis – Méditerranée
2004, route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex
Téléphone : +33 4 92 38 77 77 — Télécopie : +33 4 92 38 77 65

Le nombre enveloppe de quelques classes de graphes

Résumé : Dans cet article nous étudions une notion de convexité dans les graphes. Nous nous concentrons sur la question de la compléxité du calcul de l'enveloppe minimum d'un graphe dans le cas de diverses classes de graphes.

Étant donné un graphe $G = (V,E)$, l'intervalle $I[u,v]$ entre deux sommets $u,v \in V$ est l'ensemble des sommets qui appartiennent à un plus court chemin entre u et v. Pour un ensemble $S \subseteq V$, on note $I[S]$ l'ensemble $\bigcup_{u,v\in S} I[u,v]$. Un ensemble $S \subseteq V$ de sommets est dit *convexe* si $I[S] = S$. L'*enveloppe convexe* $I_h[S]$ d'un sous-ensemble $S \subseteq V$ de G est défini comme le plus petit ensemble convexe qui contient S. $S \subseteq V$ est une *enveloppe* de G si $I_h[S] = V$. Le *nombre enveloppe* de G, noté $hn(G)$, est la cardinalité minimum d'une enveloppe de graphe G.

Nous montrons que décider si $hn(G) \leq k$ est un problème NP-complet dans la classe des graphes bipartis et nous prouvons que $hn(G)$ peut être calculé en temps polynomial pour les cobipartis, $(q,q-4)$-graphes et cactus. Nous montrons aussi des bornes supérieures du nombre enveloppe des graphes en général, des graphes sans triangles et des graphes réguliers.

Mots-clés : convexité des graphes, nombre enveloppe, graphes bipartis, graphes cobipartis, graphes cactus, $(q,q-4)$-graphes

1 Introduction

A classical example of convexity is the one defined in Euclidean spaces. In an Euclidean space E, a set $S \subseteq E$ is said *convex* if for any two points x and y of S, $[x,y] \subseteq S$, i.e., the set of points lying in the straight line segment between x and y also belongs to S. Note that if two convex sets $X, Y \subseteq E$ contain a given set $S \subseteq E$ of points, then their intersection $X \cap Y$ is also a convex set of E containing S. Hence, we can define the *convex hull* of S as the inclusion-minimum convex set that contains S. Reciprocally, given a convex set S of E, a *hull set* of S is any subset S' of S such that S is the convex hull of S'. A naive way to compute the convex hull H of a set S consists in starting with $H = S$ and, while it is possible, adding $[x,y]$ to H for any $x, y \in H$. However there exist more efficient algorithms. For instance, for any set S of a d-dimensional euclidean space, the *gift wrapping algorithm* computes the convex hull and a minimum-inclusion hull set of S in polynomial-time in the size of S (d being fixed). For more results concerning the convexity in Euclidean spaces, we refer to [19].

In order to capture the abstract notion of convexity, [16] defines an *alignment* over a set X as a family $\mathcal{C}$ of subsets of X that is closed under intersection and that contains both X and the empty set. The members of $\mathcal{C}$ are called the *convex sets* of X. The pair $(X, \mathcal{C})$ is then called an *aligned space*. An example of aligned space $(E, \mathcal{C})$ is the one where E is an euclidean space and $\mathcal{C} = \{H \subseteq E \ : \ \forall x, y \in H, \ [x,y] \subseteq H\}$. Given an aligned space $(X, \mathcal{C})$, the definitions of convex hull and hull set are generalized as follows. For any $S \subseteq X$, the *convex hull* of S is the smallest member of $\mathcal{C}$ containing S. For any $S \in \mathcal{C}$, a *hull set* of S is a set $S' \subseteq S$ such that S is the convex hull of S'.

Various notions of convexity can be defined in graphs as specific alignments over the set of vertices. This paper is devoted to the study of the *geodetic convexity* of graphs. Let $G = (V, E)$ be a connected undirected graph. For any $u, v \in V$, let the *closed interval* $I[u,v]$ of u and v be the the set of vertices that belong to some shortest (u,v)-path. The closed interval of a set of vertices can be seen as an analog to segments in Euclidian spaces. For any $S \subseteq V$, let $I[S] = \bigcup_{u,v \in S} I[u,v]$. A subset $S \subseteq V$ is *geodesically convex* if $I[S] = S$. In this paper convexity refers to the geodesical variant. In other words, a subset S is convex if, for any $u, v \in S$ and for any shortest (u,v)-path P, $V(P) \subseteq S$. That is, the geodetic convexity can be defined as the alignment $\mathcal{C}$ over V where $\mathcal{C} = \{S \subseteq V \ : \ I[S] = S\}$.

Given a subset $S \subseteq V$, the *convex hull* $I_h[S]$ of S is the smallest convex set that contains S. We say that S is a *hull set* of G if $I_h[S] = V$. That is, S is a hull set of G if, starting from the vertices of S and successively adding in S the vertices in some shortest path between two vertices in S, we eventually obtain V. The size of a minimum hull set of G is the *hull number* of G, denoted by $hn(G)$. The HULL NUMBER problem is to decide whether $hn(G) \leq k$, for a given graph G and an integer k [15]. This problem is known to be NP-complete in general graphs [12]. In this paper, we consider the problem of the complexity to compute inclusion-minimum hull set of a graph in several graph classes.

Our results. We first answer an open question of Dourado et al. [12] by showing that the HULL NUMBER problem is NP-hard even when restricted to the class of bipartite graphs (Section 3). Then, we design polynomial time algorithms to solve the HULL NUMBER problem in several graphs' classes. In Section 4, we deal with the class of complements of bipartite graphs. In Section 5 we generalize some results in [1] to the class of $(q, q-4)$-graphs. Section 6 is devoted to the class of cacti. Finally, we prove tight upper bounds on the hull number of graphs in Section 7. In particular, we show that the hull number of an n-node graph G without simplicial vertices is at most $1 + \lceil \frac{3(n-1)}{5} \rceil$ in general, at most $1 + \lceil \frac{n-1}{2} \rceil$ if G is regular or has no triangle, and at most $1 + \lceil \frac{n-1}{3} \rceil$ if G has girth at least 6.

Related work. In the seminal work [15], the authors present some upper and lower bounds on the hull number of general graphs and characterize the hull number of some particular graphs. The corresponding minimization problem has been shown to be NP-complete [12]. Dourado *et al.* also proved that the hull number of unit interval graphs, cographs and split graphs can be computed in polynomial time [12]. Bounds on the hull number of triangle-free graphs are shown in [13]. The hull number of the cartesian and the strong product of two connected graphs is studied in [5, 11]. In [18], the authors have studied the relationship between the *Steiner number* and the hull number of a given graph. An oriented version of the HULL NUMBER problem is studied in [8, 17].

Other parameters related to the geodetic convexity have been studied in [9, 10]. Variations of graph convexity have been further proposed and studied. For instance, the *monophonic convexity* that deals with induced paths instead of shortest paths is studied in [14, 16]. Another example is the P_3-*convexity* where just paths of order three are considered [6, 16]. Other variants of graph convexity and other parameters are mentioned in [7].

2 Preliminaries

In this paper, we adopt the graph terminology defined in [4]. Otherwise stated, all graphs considered in this work are simple, undirected and connected. Let $G = (V,E)$ be a graph. Given a vertex $v \in V$, $N(v)$ denotes the (open) neighborhood of v, i.e., the set of neighbors of v. Let $N[v] = N(v) \cup \{v\}$ be the closed neighborhood of v. A vertex v is *universal* if $N[v] = V$. A vertex is *simplicial* if $N[v]$ induces a complete subgraph in G. Finally, a subgraph H of G is *isometric* if, for any $u, v \in V(H)$, the distance $dist_H(u,v)$ between u and v in H equals $dist_G(u,v)$.

This section is devoted to basic lemmas on hull sets. These lemmas will serve as cornerstone of most of the results presented in this paper.

Lemma 1 ([15])**.** *For any hull set S of a graph G, S contains all simplicial vertices of G.*

Lemma 2 ([12])**.** *Let G be a graph which is not complete. No hull set of G with cardinality $hn(G)$ contains a universal vertex.*

Lemma 3 ([12])**.** *Let G be a graph, H be an isometric subgraph of G and S be any hull set of H. Then, the convex hull of S in G contains $V(H)$.*

Lemma 4 ([12])**.** *Let G be a graph and S a proper and non-empty subset of $V(G)$. If $V(G)\backslash S$ is convex, then every hull set of G contains at least one vertex of S.*

3 Bipartite graphs

In this section, we answer an open question of Dourado et al. [12] by showing that the Hull Number Problem is NP-complete in the class of bipartite graphs. Since the Hull Number Problem is in NP, as proved in [12], it only remains to prove the following theorem:

Theorem 1. *The* HULL NUMBER *problem is NP-hard in the class of bipartite graphs.*

Proof. To prove this theorem, we adapt the proof presented in [12]. We reduce the 3-SATisfiability Problem to the HULL NUMBER problem in bipartite graphs. Let us consider the following instance of 3-SAT. Given a formula in the conjunctive normal form, let $\mathcal{F} = \{C_1, C_2, \dots, C_m\}$ be the set of its 3-clauses and $X = \{x_1, x_2, \dots, x_n\}$ the set of its boolean variables. We may assume that $m = 2^p$, for a positive integer $p \geq 1$, since it is possible to add dummy variables and clauses without changing the satisfiability of $\mathcal{F}$ and such that the size of the instance is at most twice the size of the initial instance. Moreover, we also assume, without loss of generality, that each variable x_i and its negation appear at least once in $\mathcal{F}$ (otherwise the clauses where x_i appeared could always be satisfied).

Let us construct the bipartite graph $G(\mathcal{F})$ as follows. First, let T be a full binary tree of height p rooted in r with $m = 2^p$ leaves, and let $L = \{c_1, c_2, \dots, c_m\}$ be the set of leaves of T. We then construct a graph H as follows. First, let us add a vertex u that is adjacent to every vertex in L. Then, any edge $\{u, v\} \in E(T)$ with u the parent of v is replaced by a path with $2^{h(v)}$ edges, where $h(v)$ is the distance between v and any of its descendent leaves. Note that, in H, the distance between r and any leaf is $\sum_{i=0}^{p-1} 2^i = 2^p - 1 = m - 1$. Moreover, it is easy to see that $|V(H)| = O(m \cdot \log m)$.

The following claims are proved in [12].

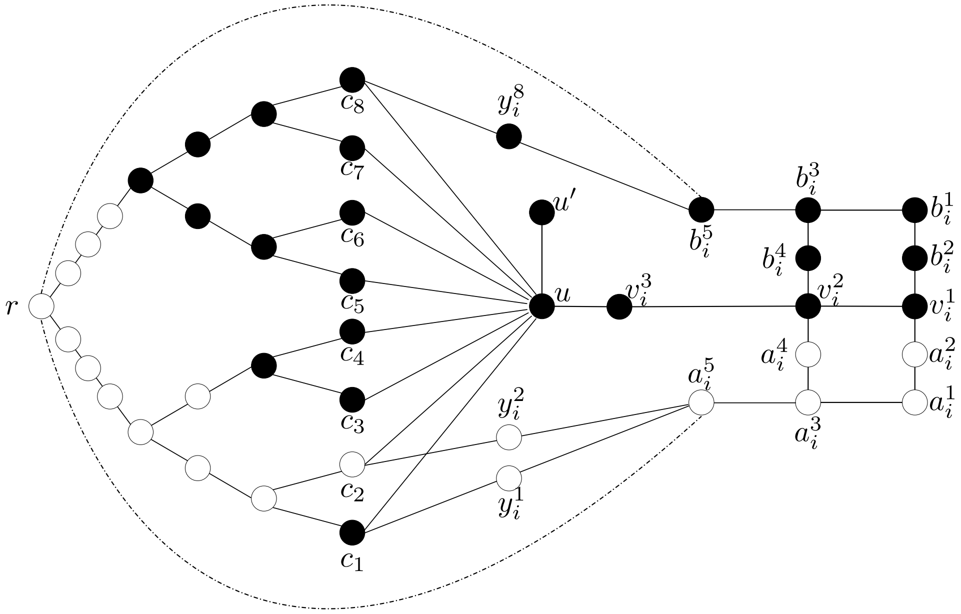

Figure 1: Subgraph of the bipartite instance $G(\mathcal{F})$ containing the gadget of a variable x_i that appears positively in clauses C_1 and C_2, and negatively in C_8. If x_i appears positively in C_j, link a_i^5 to c_j through y_i^j. If it appears negatively, we use b_i^5 instead of a_i^5.

Claim 1. *Let $v,w \in V(T)\setminus\{r\}$. The closed interval of v,w in H contains the parents of v in T if and only if v and w are siblings in T.*

Claim 2. *The set L is a* minimal *hull set of H.*

Then, let H' be obtained by adding a one degree vertex u' adjacent to u in H. Finally, we build a graph $G(\mathcal{F})$ from H' by adding, for any variable x_i, $i \leq n$, the gadget defined as follows.

Let us start with a cycle $\{a_i^1, a_i^2, v_i^1, b_i^2, b_i^1, b_i^3, b_i^4, v_i^2, a_i^4, a_i^3\}$ plus the edge $\{v_i^2, v_i^1\}$. Then, add the vertex v_i^3 as common neighbor of v_i^2 and u. Add a neighbor b_i^5 (resp., a_i^5) adjacent to b_i^3 (resp., a_i^3) and a path of length $2^{h(r)} - 3 = m - 3$ edges between b_i^5 (resp., a_i^5) and r. Let D be the set of internal vertices of all these $2n$ paths between a_i^5, resp., b_i^5, and r, $i \leq n$. Finally, for any clause C_j in which x_i appears, if x_i appears positively (resp., negatively) in C_j then add a common neighbor y_i^j between c_j and a_i^5 (resp., b_i^5). See an example of such a gadget in Figure 1. Note that $|V(G(\mathcal{F}))| = O(m \cdot (n + \log m))$.

Lemma 5. *$G(\mathcal{F})$ is a bipartite graph.*

Proof. Let us present a proper 2-coloring c of $G(\mathcal{F})$. Let $c(r) = 1$, and for each vertex w in $V(H)$, define $c(w)$ as 1 if w is in an even distance from r, and 2 otherwise. Clearly, c is a partial *proper* coloring of $G(\mathcal{F})$ and moreover we have $c(u) = 1$ and $c(c_j) = 2$, for any $j \in \{1, \ldots, m\}$ (Indeed, any c_i is at distance $m-1$ (odd) of r in H). Let $c(u') = 2$. For every $i \in \{1, \ldots, n\}$ and for any j such that $x_i \in C_j$, let $c(y_i^j) = 1$. For any $i \leq n$, for any $x \in \{b_i^5, a_i^5, v_i^3, b_i^4, a_i^4, b_i^1, v_i^1, a_i^1\}$, $c(x) = 2$.

$c(b_i^5) = c(a_i^5) = c(v_i^3) = 2$. Again, this partial coloring of $G(\mathcal{F})$ is proper. One can easily verify that this coloring can be extended to $\{a_i^1, a_i^2, v_i^1, b_i^2, b_i^1, b_i^3, b_i^4, v_i^2, a_i^4, a_i^3\}$ for any $i \leq n$. Moreover, since $c(r) = 1$ and $c(a_i^5) = 2$ ($c(b_i^5) = 2$), for every $i \in \{1, \ldots, n\}$, and since the path that we add in $G(\mathcal{F})$ between r and a_i^5 (b_i^5) is of odd length $m-3$, one can completely extend c in order to get a proper 2-coloring of $G(\mathcal{F})$. $\diamond$

Claim 3. *The set $V(G(\mathcal{F}))\setminus\{a_i^1, a_i^2, v_i^1, b_i^1, b_i^2\}$ is convex, for any $i \in \{1,\dots,n\}$.*

Proof. Denote $W_i = \{a_i^1, a_i^2, v_i^1, b_i^1, b_i^2\}$, for some $i \in \{1,\dots,n\}$, and $W_i' = \{a_i^3, b_i^3, v_i^2\}$. By contradiction, suppose that there exists an (x,y)-shortest path containing a vertex of W_i, for some $x,y \in V(G(\mathcal{F}))\setminus W_i$. Observe that it implies that that there are $x',y' \in W_i'$ such that $I[x',y']$ contains a vertex of W_i, since W_i' contains all the neighbors of W_i in $V(G(\mathcal{F}))\setminus W_i$. However, it is easy to verify that for any pair $x,y \in W_i'$, $I[x,y]$ contains no vertex of W_i. This is a contradiction. $\diamond$

Lemma 6. $hn(G(\mathcal{F})) \geq n+1$.

Proof. Let S be any hull set of $G(\mathcal{F})$. Clearly $u' \in S$, because u' is a simplicial vertex of $G(\mathcal{F})$ (Lemma 1). Furthermore, Claim 3 and Lemma 4 imply that S must contain at least one vertex w_i of the set $\{a_i^1, a_i^2, v_i^1, b_i^1, b_i^2\}$, for every $i \in \{1,\dots,n\}$. Hence, $|S| \geq n+1$. $\diamond$

The main part of the proof consists in showing:

Lemma 7. *$\mathcal{F}$ is satisfiable if and only if $hn(G(\mathcal{F})) = n+1$.*

First, consider that $\mathcal{F}$ is satisfiable. Given an assignment A that turns $\mathcal{F}$ true, define a set S as follows. For $1 \leq i \leq n$, if x_i is true in A add a_i^1 to S, otherwise add b_i^1 to S. Finally, add u' to S. Note that $|S| = n+1$. We show that S is a hull set of $G(\mathcal{F})$. First note that $a_i^5, c_j \in I[a_i^1, u']$, for every clause C_j containing the positive literal of x_i. Similarly, observe that $b_i^5, c_j \in I[b_i^1, u']$, for every clause C_j containing the negative literal of x_i. Since A satisfies $\mathcal{F}$, it follows $L \subseteq I_h[S]$. Therefore, H being an isometric subgraph of $G(\mathcal{F})$, Lemma 3 and Claim 3 imply that $V(H) \subseteq I_h[S]$. Furthermore, the shortest paths between r and u have length m, which implies that all vertices a_i^5, b_i^5, y_i^j $(i \leq n)$ and all vertices in D are included in $I_h[S]$. It remains to observe that $I_h[a_i^5, b_i^5, w, u']$, where $w \in \{a_i^1, b_i^1\}$, contains the variable subgraph of x_i. Therefore we have that S is a hull set of $G(\mathcal{F})$.

We prove the sufficiency by contradiction. Suppose that $G(\mathcal{F})$ contains a hull set S with $n+1$ vertices and that $\mathcal{F}$ is not satisfiable.

Recall that, by Lemma 1, $u' \in S$. For any $i \leq n$, let W_i as defined in Claim 3. Recall also that there must be a vertex $w_i \in W_i \cap S$, for any $i \leq n$. Since $v_i^1 \in I[u', a_i^1]$, $v_i^1 \in I[u', b_i^1]$, $a_i^2 \in I[u', a_i^1]$ and $b_i^2 \in I[u', b_i^1]$, we can assume, without loss of generality, that $w_i \in \{a_i^1, b_i^1\}$, for every $i \in \{1,\dots,n\}$ (indeed, if $w_i \in \{v_i^1, a_i^2\}$, it can be replaced by a_i^1, and if $w_i = b_i^2$, it can be replaced by b_i^1). Therefore S defines the following truth assignment $\mathcal{A}$ to $\mathcal{F}$. If $w_i = a_i^1$ set x_i to true, otherwise set x_i to false. As $\mathcal{F}$ is not satisfiable, there exists at least one clause C_j not satisfied by $\mathcal{A}$.

Using the hypothesis that $\mathcal{F}$ is not satisfiable, we complete the proof by showing that there is a non empty set U such that $V(G(\mathcal{F}))\setminus U$ is a convex set and $U \cap S = \emptyset$. That is, we show that $I_h[S] \subseteq V(G(\mathcal{F}))\setminus U$ for some $U \neq \emptyset$, contradicting the fact that S is a hull set.

For any clause C_j, let us define the subset U_j of vertices as follows. Let P_j be the path in T between c_j and r, let X_j be the p vertices in $V(T) \setminus V(P_j)$ that are adjacent to some vertex in P_j. Then, U_j is the union of the vertices that are either in P_j or that are internal vertices of the paths resulting of the subdivision of the edges $\{x,y\}$ where $x,y \in P_j \cup X_j$. Another way to build the set U_j is to start with the set of vertices of the (unique) shortest path between c_j and r in H and then add successively to this set, the vertices of $V(H) \setminus (V(T) \cup \{u\})$ that are adjacent to some vertex of the current set.

Now, let $U' = \cup_{j \in J} U_j$ where J is the (non empty) set of clauses that are not satisfied by $\mathcal{A}$. Note that $r \in U'$.

For any $i \leq n$, let W_i be defined as follows. If $w_i = a_i^1$ (x_i assigned to true by $\mathcal{A}$), then W_i is the union of $\{b_i^\ell \,:\, \ell \leq 5\}$ with the set of the y_i^k that are adjacent to b_i^5. Otherwise, $w_i = b_i^1$ (x_i assigned to false by $\mathcal{A}$), then W_i is the union of $\{a_i^\ell \,:\, \ell \leq 5\}$ with the set of the y_i^k that are adjacent to a_i^5.

Finally, let $U = U' \cup (\bigcup_{i \leq n} W_i) \cup D$. In Figure 1, U is depicted by the white vertices, assuming that clause C_2 is false and that x_i is set to false by $\mathcal{A}$. Observe that $U \cap S = \emptyset$.

It remains to prove that $V(G(\mathcal{F}))\setminus U$ is a convex set. Consider the partition $\{A_1, A_2, A_3\}$ of $V(G(\mathcal{F}))\setminus U$ where $A_1 = V(H)\setminus(U \cup \{u\})$, $A_2 = \{u, u'\}$ and $A_3 = V(G(\mathcal{F}))\setminus(U \cup A_1 \cup A_2)$. To prove that $V(G(\mathcal{F}))\setminus U$ is convex, let

$w \in A_i$ and $w' \in A_j$ for some $i, j \in \{1,2,3\}$. We show that $I[w,w'] \cap U = \emptyset$ considering different cases according to the values of i and j. Recall that $V(H) \setminus \{u\}$ induces a tree T' rooted in r and that, if a vertex of T' is in A_1, then, by definition of U', all its descendants in T' are also in A_1 (i.e., if $v \in U \cap V(T')$, then all ancestors of v in T' are in U). It is important to note that, for any vertex v in A_1, the shortest path in $G(\mathcal{F})$ from v to any leaf ℓ of T' is the path from v to ℓ in T' (in particular, such a shortest path does not pass through r and any vertices in D).

- The case $i = j = 2$, i.e., $m, m' \in \{u, u'\}$, is trivial;

- First, let us assume that $w \in A_1 = V(H) \setminus (U \cup \{u\})$ and $w' \in A_2 = \{u, u'\}$. If $w' = u$ (resp., if $w' = u'$) then $I_h[w,w']$ consists of the subtree of T' rooted in w union u (resp., union u and u'). Hence, $I_h[w,w'] \cap U = \emptyset$ because no descendants of w in T' are in U.

- Second, let $w, w' \in A_1$. If one of them, say w, is an ancestor of the other in T', then $I_h[w,w']$ consists of the path between them in T' (remember that $r \in U$ so $w \neq r$). Since no descendants of w in T' are in U, $I_h[w,w'] \cap U = \emptyset$. Otherwise, there are three cases: (1) either $I_h[w,w']$ consists of the path P between w and w' in T', or (2) $I_h[w,w']$ consists of the union of the subtree R of T' rooted in w, the subtree R' of T' rooted in w' and u, or (3) $I_h[w,w'] = R \cup R' \cup P \cup \{u\}$. Again, $(R \cup R' \cup \{u\}) \cap U = \emptyset$ because no descendants of w and w' in T' are in U. Hence, it only remains to prove that when $P \subseteq I_h[w,w']$ then $P \cap U = \emptyset$. It is easy to check that $P \subseteq I_h[w,w']$ only in the following case: there exist $x, y, z \in V(T)$ such that x is the parent of y and z in T, and w (resp., w') is a vertex of the path resulting from the subdivision of $\{x,y\}$ (resp., $\{x,z\}$). In this case, it means that all clause-vertices that are descendants of y and z are not in U. Therefore $x \notin U$ and hence no descendants of x are in U. In particular, $P \cap U = \emptyset$.

- Assume now that $w \in A_3$. Let $i \leq n$ such that w belongs to the gadget G_i corresponding to variable x_i. Let us assume that $w_i = b_i^1$. The case $w_i = a_i^1$ can be handled in a similar way by symmetry. Then, by definition, U contains $\{a_i^1, \cdots, a_i^5\}$ and the y_i^j's adjacent to a_i^5. With this setting, x_i is set to false in the assignment $\mathcal{A}$. If there is a vertex y_i^j adjacent to b_i^5, let C_j be the other neighbor of j_i^j. By definition, it means that clause C_j contains the negation of variable x_i. Since x_i is set to false, it means that clause C_j is satisfied and so $C_j \notin U$.

  Let $x \in V(G_i) \setminus U$. Then, any shortest path P from w to x either passes through $V(G_i) \setminus U$ or, there is y_i^j adjacent to b_i^5 such that P passes through y_i^j, C_j, u and v_i^3 (the latter case may occur if $a \in \{y_i^j, b_i^5\}$ and $b = v_i^3$, or $a = y_i^j$ and $b \in \{v_i^3, v_i^2\}$ where $\{a,b\} = \{x,w\}$). Hence, such a path P avoid U, and the result holds if $x = w' \in A_3 \cap G_i$.

  Similarly, if $x \in \{u,u'\}$, then, any shortest path P from w to x either passes through $V(G_i) \setminus U$ or through y_i^j, C_j, u with y_i^j adjacent to b_i^5. In particular, if $x = w' \in \{u,u'\} = A_2$, then the result holds.

  Now, let $x = C_{j'}$ be a leaf of T' that is not in U. Then, any shortest path P from w to x either passes through u or through y_i^j, C_j and, if $j \neq j'$, through u. In any case, P avoids U. If $w' \in A_3 \setminus G_i$, any path between w and w' passes through u or through one or two leaves that are not in U. Finally, if $w' \in A_1$, let R be the subtree of T' rooted in w'. $V(R) \subseteq I_h[w,w']$. Moreover, any shortest path from w to w' path through a leaf of R, i.e., a leaf not in U. By previous remarks, in all these cases, the shortest paths between w and w' avoid u, and $I_h[w,w']$ are disjoint from U.

□

We conclude this section by showing one *approximability result.* Let $IG(G)$ be the *incidence graph* of G, obtained from G by subdividing each edge once. That is, let us add one vertex s_{uv}, for each edge $uv \in E(G)$, and replace the edge uv by the edges $us_{uv}, s_{uv}v$.

Proposition 2. $hn(IG(G)) \leq hn(G) \leq 2hn(IG(G))$.

Proof. Let $IG(G)$ be the incidence graph of G. Observe that any hull set of G is a hull set of $IG(G)$, since for any shortest path, $P = \{v_1, \ldots, v_k\}$ in G there is a shortest path $P' = \{v_1, s_{v_1v_2}, v_2, \ldots, s_{v_{k-1}v_k}, v_k\}$ in $IG(G)$ (the edges were subdivided). Consequently, $hn(IG(G)) \leq hn(G)$. However, given a hull set S_h of $IG(G)$, one may find a hull set of G by simply replacing each vertex of S_h that represents an edge of G by its neighbors (vertices of G). Thus, $hn(G) \leq 2hn(IG(G))$.

□

Corollary 1. *If there exists a k-approximation algorithm B to compute the hull number of bipartite graphs, then B is a* $2k$*-approximation algorithm for any graph.*

4 Complement of bipartite graphs

A graph $G = (V,E)$ is a complement of a bipartite graph if there is a partition $V = A \cup B$ such that A and B are cliques. In this section, we give a polynomial-time algorithm to compute a hull set of G with size $hn(G)$. We start with some notations.

Given the partition (A,B) of V, we say that an edge $uv \in E$ is a *crossing-edge* if $u \in A$ and $v \in B$. Denote by S the set of simplicial vertices of G, by $S_A = S \cap A$ and by $S_B = S \cap B$. Let U be the set of universal vertices of G. Note that, if G is not a clique, $U \cap S = \emptyset$. Let H be the graph obtained from G by removing the vertices in S and U, and removing the edges intra-clique, i.e., $V(H) = V \setminus (U \cup S)$ and $E(H) = \{\{u,v\} \in E \ : \ u \in A \cap V(H) \textit{ and } v \in B \cap V(H)\}$. Let $\mathcal{C} = \{C_1, \cdots, C_r\}$ $(r \geq 1)$ denote the set of connected components C_i of H. Observe that, if G is neither one clique nor the disjoint union of A and B, H is not empty and each connected component C_i has at least two vertices, for every $i \in \{1, \ldots, r\}$. Indeed, any vertex in $A \setminus S_A$ (resp., in $B \setminus S_B$) has a neighbor in $B \cap V(H)$ (resp. in $A \cap V(H)$).

Theorem 3. *Let* $G = (A \cup B, E)$ *be the complement of a bipartite n-node graph. There is an algorithm that computes* $hn(G)$ *and a hull set of this size in time* $O(n^7)$.

Proof. We use the notations defined above. Recall that, by Lemma 1, S is contained in any hull set of G. In particular, if G is a clique or G is the disjoint union of two cliques A and B, then $hn(G) = n$. From now on, we assume it is not the case. By Lemma 2, no vertices in U belong to any minimal hull set of G. Now, several cases have to be considered.

Claim 4. *If* $U = \emptyset$*,* $S_A \neq \emptyset$ *and* $S_B \neq \emptyset$*, then S is a minimum hull set of G and thus* $hn(G) = |S|$.

Proof. Since G has no universal vertex, a simplicial vertex in S_A (in S_B) has no neighbor in B (resp., in A). Since G is not the disjoint union of two cliques, every vertex $u \in A \backslash S_A$ has a neighbor $v \in B \backslash S_B$ and vice-versa. Thus, $s_a uvs_b$ is a shortest (s_a, s_b)-path, for any $s_a \in A$ and $s_b \in B$, and then $u, v \in I_h[S]$. □

Hence, from now on, let us assume that $U \neq \emptyset$ or, w.l.o.g., $S_B = \emptyset$.

Again, if there is some simplicial vertex in G, i.e., if $S_A \neq \emptyset$, all the vertices of S belong to any hull set of G and thus $hn(G) \geq |S|$. In fact, for each connected component of H, we prove that it is necessary to choose at least one of its vertices to be part of any hull set of G.

Claim 5. *If* $U \neq \emptyset$ *or* $S_B = \emptyset$ *or* $S_A = \emptyset$*, then* $hn(G) \geq |S| + r$.

Proof. Again, all vertices of S belong to any hull set of G. We show that, for any $1 \leq i \leq r$, $V \backslash C_i$ is a convex set. Thus, by Lemma 4, any hull set of G contains at least one vertex of C_i for any $i \leq r$.

It is sufficient to show that no pair $u, v \in V(G) \backslash C_i$ can generate a vertex v_i of C_i. By contradiction, suppose that there exists a pair of vertices $u, v \in V(G) \backslash C_i$ such that there is a shortest (u,v)-path P containing a vertex v_i of C_i. Consequently, u and v must not be adjacent and we consider that $u \in A$ and $v \in B$. If $U = \emptyset$, then, w.l.o.g.,

$S_B = \emptyset$ and v is not simplicial and has at least one neighbor in A. Hence, since $U \neq \emptyset$ or $S_b = \emptyset$, u and v are at distance two. Consequently, $P = uv_iv$. However, if $v_i \in A$, v belongs to C_i, because of the crossing edge v_iv, otherwise, $u \in C_i$. In both cases we reach a contradiction. □

Now, two cases remain to be considered. We recall that $U \neq \emptyset$ or $S_B = \emptyset$.

1. If $r \geq 2$, then $hn(G) = |S| + r$, and we can build a minimum convex hull by taking the vertices in S, one arbitrary vertex in $A \cap C_i$ for all $i < r$ and one arbitrary vertex in $B \cap C_r$.

   Let $R = \{v_1, \dots, v_r\}$ such that $v_i \in C_i \cap A$ for any $i < r$ and $v_r \in C_r \cap B$.

   Claim 6. *$S \cup R$ is a hull set of G.*

   Proof. Since all vertices in U are generated by v_1 and v_r (that are not adjacent, since they are in different components), it is sufficient to show that $S \cup R$ generates all the vertices in C_i, for any $i \in \{1, \dots, r\}$. Actually, we show that R generates all the vertices in C_i.

   By contradiction, suppose that there is a vertex $z \notin I_h[R]$. Let $i \leq r$ such that $z \in C_i$. Because C_i contains one vertex in R and is connected, we can choose z and $w \in C_i \cap I_h[R]$ linked by a crossing edge. We will show that $z \in I_h[R]$ (a contradiction), hence, w.l.o.g., we may assume that $z \in A$. If $i = r$, then v_1zw is a shortest (v_1, w)-path and $z \in I_h[R]$.

   Otherwise, recall that $N(v_r) \cap A \cap C_r \neq \emptyset$ and, for any $i < r$, $N(v_i) \cap B \cap C_i \neq \emptyset$ because v_i is not simplicial for any $i \leq r$. Let $x \in N(v_r) \cap A \cap C_r$ and $y_i \in N(v_i) \cap B \cap C_i$. Note that $x \in I_h[R]$ because v_1xv_r is a shortest (v_r, v_1)-path, and $y_i \in I_h[R]$ because $v_iy_iv_r$ is a shortest (v_r, v_i)-path. Hence, since xzy_i is a shortest (x, y_i)-path, we have $z \in I_h[R]$. □

   As $|R| = r$, we conclude by Claim 5 that $hn(G) = |S| + r$.

2. If $r = 1$, then $hn(G) \leq |S| + 4$, and any minimum convex hull contains at most 4 vertices not in S.

   Again, S is included in any hull set of G by Lemma 1, and no vertices in U belong to some hull set by Lemma 2. In this case, when H has just one connected component $C_1 = C$, one vertex of C may not suffice to generate this component, as in the previous case. However, we prove that at most 4 vertices in C are needed.

   (a) If $S_A \neq \emptyset$ and $S_B \neq \emptyset$ (and thus $U \neq \emptyset$ because Claim 4 applies otherwise), then $hn(G) = |S| + 1$.

   By Claim 5, we know that $hn(G) \geq |S| + 1$. Let v be an arbitrary vertex of C. We claim that $S \cup \{v\}$ is a minimum hull set of G. By contradiction, let $z \notin I_h[S \cup \{v\}]$. Since C is a connected component of H, we may choose z such that there is $w \in N(z) \cap C \cap I_h[S \cup \{v\}]$. Moreover, we may assume w.l.o.g. that $z \in A$, and thus $w \in B$. In that case, since $S_A \neq \emptyset$, there is $v_A \in S_A$ and as $v_Aw \notin E(G)$ (indeed, any vertex in $N(v_A) \cap B$ must be universal because v_A is simplicial, which is not the case since w is not universal because it belongs to C), z is generated by v_A and w.

   (b) If $S_A \neq \emptyset$ and $S_B = \emptyset$, then $hn(G) \leq |S| + 2$.

   Let $v_A \in A \cap C$ be such that $|N(v_A) \cap B \cap C|$ is maximum. Since v_A is not universal in G, there exists $x \in B$ such that $v_Ax \notin E(G)$. Note that $x \in C$ since x is not universal and $S_B = \emptyset$. Let $R = \{v_A, x\}$. Observe that $N(v_A) \cap B \cap C \subseteq I_h[R \cup S]$ since $v_Ax \notin E$.

   By contradiction, assume $V(G) \backslash I_h[R \cup S] \neq \emptyset$. Let $z \in V(G) \backslash I_h[R \cup S]$. First, suppose that $z \in A$. Since C is connected in H, we may assume that z has a neighbor $w \in I_h[R \cup S] \cap B \cap C$. As $S_A \neq \emptyset$, there is $v \in S_A$ and as $vw \notin E(G)$ (because otherwise w would be universal in G and not in C), z is generated by v and w. Now suppose that $z \in B$, and now it has a neighbor $w \in I_h[R \cup S] \cap A \cap C$. Observe that $I_h[R \cup S] \cap B \subseteq N(w)$, otherwise z would be in $I_h[R \cup S]$. However, since $N(v_A) \cap B \cap$

$C \subset (N(v_A) \cap B \cap C) \cup \{x\} \subseteq I_h[R \cup S] \cap B$, we get that $N(v_A) \cap B \cap C \subset N(w) \cap B \cap C$, contradicting the maximality of $|N(v_A) \cap B \cap C|$.

(c) If $S_A = \emptyset$ and $S_B = \emptyset$, then $hn(G) \leq 4$.

Let $v_A \in A \cap C$ be such that $|N(v_A) \cap B \cap C|$ is maximum and $v_B \in B \cap C$ be such that $|N(v_B) \cap A \cap C|$ is maximum. Since v_A is not universal in G and $S_B = \emptyset$, there exists $y \in C \cap B \setminus N(v_a)$, and similarly there exists $x \in C \cap A \setminus N(v_B)$. Let $R = \{v_A, v_B, x, y\}$. Observe that $N(v_A) \cap B \subseteq I_h[R]$ and $N(v_B) \cap A \subseteq I_h[R]$, since $v_A y \notin E$ and $v_B x \notin E$.

By contradiction, assume $V(G) \backslash I_h[R] \neq \emptyset$. Let $z \in V(G) \backslash I_h[R]$. First, suppose that $z \in A$. As in the previous case, since C is connected in H, we may assume that z has a neighbor $w \in I_h[R] \cap B \cap C$. Observe that $I_h[R] \cap A \cap C \subseteq N(w)$, otherwise z would be in $I_h[R]$. However, since $N(v_B) \cap A \cap C \subset (N(v_B) \cap A \cap C) \cup \{x\} \subseteq I_h[R] \cap A \cap C$, we get that $N(v_B) \cap A \cap C \subset N(w) \cap A \cap C$, contradicting the maximality of $|N(v_B) \cap A \cap C|$.

Whenever $z \in B$, one can use the same arguments to reach a contradiction on the maximality of $|N(v_A) \cap B \cap C|$.

Since $|S| + 1 \leq hn(G) \leq |S| + 4$, S is included in any hull set of G and no vertices in U belong to some hull set, there exist a subset R of at most 4 vertices in C such that $S \cup R$ is a minimum hull set of G. There are $O(|V|^4)$ subsets to be tested and, for each one, its convex hull can be computed in $O(|V||E|)$ time [12]. This leads to the announced result.

□

5 Graphs with few P_4's

A graph $G = (V, E)$ is a $(q, q-4)$-graph, for a fixed $q \geq 4$, if for any $S \subseteq V$, $|S| \leq q$, S induces at most $q-4$ paths on 4 vertices [2]. Observe that cographs and P_4-sparse graphs are the $(q, q-4)$-graphs for $q = 4$ and $q = 5$, respectively. The hull number of a cograph can be computed in polynomial time [12]. This result is improved in [1] to the class of P_4-sparse graphs. In this section, we generalize these results by proving that for any fixed $q \geq 4$, computing the hull number of a $(q, q-4)$-graph can be done in polynomial time. Our algorithm runs in time $O(2^q n^2)$ and is therefore a Fixed Parameter Tractable for any graph G, where the number of induced P_4's of G is the parameter.

5.1 Definitions and brief description of the algorithm

The algorithm that we present in this section uses the canonical decomposition of $(q, q-4)$-graphs, called *Primeval Decomposition*. For a survey on Primeval Decomposition, the reader is referred to [3]. In order to present this decomposition of $(q, q-4)$-graphs, we need the following definitions.

Let G_1 and G_2 be two graphs. $G_1 \cup G_2$ denotes the disjoint union of G_1 and G_2. $G_1 \oplus G_2$ denotes the join of G_1 and G_2, i.e., the graph obtained from $G_1 \cup G_2$ by adding an edge between any two vertices $v \in V(G_1)$ and $w \in V(G_2)$. A *spider* $G = (S, K, R, E)$ is a graph with vertex set $V = S \cup K \cup R$ and edge set E such that

1. (S, K, R) is a partition of V and R may be empty;
2. the subgraph $G[K \cup R]$ induced by K and R is the join $K \oplus R$, and K separates S and R, i.e., any path from a vertex in S to a vertex in R contains a vertex in K;
3. S is a stable set, K is a clique, $|S| = |K| \geq 2$, and there exists a bijection $f : S \rightarrow K$ such that, either $N(s) \cap K = K - \{f(s)\}$ for all vertices $s \in S$, or $N(s) \cap K = \{f(s)\}$ for all vertices $s \in S$. In the latter case or if $|S| = |K| = 2$, G is called *thin*, otherwise G is *thick*.

A graph $G = (S,K,R,E)$ is a *pseudo-spider* if it satisfies only the first two properties of a spider. A graph $G = (S,K,R,E)$ is a *q-pseudo-spider* if it is a pseudo-spider and, moreover, $|S \cup K| \leq q$. Note that q-pseudo-spiders and spiders are pseudo-spiders.

We now describe the decomposition of $(q,q-4)$-graphs.

Theorem 4 ([2])**.** *Let $q \geq 0$ and let G be a $(q,q-4)$-graph. Then, one of the following holds:*

1. *G is a single vertex, or*
2. *$G = G_1 \cup G_2$ is the disjoint union of two $(q,q-4)$-graphs G_1 and G_2, or*
3. *$G = G_1 \oplus G_2$ is the join of two $(q,q-4)$-graphs G_1 and G_2, or*
4. *G is a spider (S,K,R,E) where $G[R]$ is a $(q,q-4)$-graph if $R \neq \emptyset$, or*
5. *G is a q-pseudo-spider (H_2,H_1,R,E) where $G[R]$ is a $(q,q-4)$-graph if $R \neq \emptyset$.*

Theorem 4 leads to a tree-like structure $T(G)$ (the *primeval tree*) which represents the Primeval Decomposition of a $(q,q-4)$-graph G. $T(G)$ is a rooted binary tree where any vertex v corresponds to an induced $(q,q-4)$-subgraph G_v of G and the root corresponds to G itself. Moreover, the vertices of subgraphs corresponding to the leaves of $T(G)$ form a partition of $V(G)$, i.e., $\{V(G_\ell)\}_{\ell \text{ } leaf \text{ } of \text{ } T(G)}$ is a partition of $V(G)$.

For any leaf ℓ of $T(G)$, G_ℓ is either a spider $(S,K,\emptyset,E)$, or has at most q vertices. Moreover, any internal vertex v has its label following one of the four cases in Theorem 4 corresponds to G_v. More precisely, let v be an internal vertex of $T(G)$ and let u and w be its two children. v is a *parallel node* if $G_v = G_u \cup G_w$. v is a *series node* if $G_v = G_u \oplus G_w$. v is a *spider node* if u is a leaf with G_u is a spider $(S,K,\emptyset,F)$ and G_v is the spider (S,K,R,E) where $G_v[R] = G_w$ and $G_v[S \cup K] = G_u$. Finally, v is a *small node* if u is a leaf with $|V(G_u)| \leq q$ and G_v is the q-pseudo-spider (S,K,R,E) where $G_v[R] = G_w$ and $G_v[S \cup K] = G_u$.

This tree can be obtained in linear-time [3].

We compute $hn(G)$ by a post-order traversal in $T(G)$. More precisely, given $v \in V(T(G))$, let H_v be an optimal hull set of G_v and let H^*_v be an optimal hull set of G^*_v, the graph obtained by adding a universal vertex to G_v. We show in next subsection that we can compute (H_ℓ, H^*_ℓ) for any leaf ℓ of $T(G)$ in time $O(2^q n)$. Moreover, for any internal vertex v of $T(G)$, we show that we can compute (H_v, H^*_v) in time $O(2^q n)$, using the information that was computed for the children and grand children of v in $T(G)$.

Theorem 5. *Let $q \geq 0$ and let G be a n-node $(q,q-4)$-graph. An optimal hull set of G can be computed in time $O(2^q n^2)$.*

Before going into the details of the algorithm in next subsection, we prove some useful lemmas.

Lemma 8 ([1])**.** *Let $G = (S,K,R,E)$ be a pseudo-spider with R neither empty nor a clique. Then any minimum hull set of G contains a minimum hull set of the subgraph $G[K \cup R]$.*

Proof. Let H be a minimum hull set of G. Let $H_S = H \cap S$ and $H_R = H \setminus H_S$. We prove that H_R is a minimum hull set of $G[K \cup R]$.

Let H' be any minimum hull set of $G[K \cup R]$. Note that $H' \subseteq R$ because K is a set of universal vertices in $G[K \cup R]$ and by Lemma 2. Moreover, By Lemma 3, because $G[K \cup R]$ is an isometric subgraph of G, the convex hull of H' in G contains $G[K \cup R]$. Hence, $H_S \cup H'$ is a hull set of G and $hn(G) \leq |H_S| + hn(G[K \cup R])$.

Now it remains to prove that H_R is a hull set of $G[K \cup R]$. Clearly, if H_R generate all vertices of R in $G[K \cup R]$ then H_R is a hull set of $G[K \cup R]$ since there are at least two non adjacent vertices in R and any vertex in K is adjacent to all vertices in R. For purpose of contradiction, assume H_R does not generate R in $G[K \cup R]$. This means that there is a vertex $v \in R$, that is generated in G by a vertex in $S \cup K$, i.e., $v \in R$ is an internal vertex of a shortest path between $s \in S \cup K$ and some other vertex, which is not possible, since we have all the edges between K and R. Hence, $hn(G[K \cup R]) \leq |H_R|$.

Therefore, $|H_S| + |H_R| = hn(G) \leq |H_S| + hn(G[K \cup R]) \leq |H_S| + |H_R|$. So, $hn(G[K \cup R]) = |H_R|$, i.e., H_R is a minimum hull set of $G[K \cup R]$ contained in H. $\square$

The next lemma is straightforward by the use of isometry.

Lemma 9. *Let G be a graph which is not complete and that has a universal vertex. Let H obtained from G by adding some new universal vertices. A set is a minimum hull set of G if, and only if, it is a minimum hull set of H.*

5.2 Dynamic programming and correctness

In this section, we detail the algorithm presented in previous section and we prove its correctness. Let $v \in V(T(G))$, which may therefore be either a leaf, a parallel node, a series node, a spider node or a small node. For each of these five cases, we describe how to compute (H_v, H_v^*), in time $O(2^q n)$.

Let us first consider the case when v is a leaf of $T(G)$.

If G_v is a singleton $\{w\}$, then $H_v = V(G_v) = \{w\}$ and $H_v^* = V(G_v^*)$. If G_v is a spider $(S, K, \emptyset, E)$ then $H_v = S$ since S is a set of simplicial vertices (so it has to be included in any hull set by Lemma 1) and it is sufficient to generate G_v. One may easily check that if G_v is a thick spider, S is also a minimum hull set of G_v^*, i.e., $S = H_v^*$. However, in case G_v is a thin spider, S does not suffice to generate G_v^* and in this case it is easy to see that this is done by taking any extra vertex $k \in K$, in which case we have $H_v^* = S \cup \{k\}$. Finally, if G_v has at most q vertices, H_v and H_v^* can be computed in time $O(2^q)$ by an exhaustive search.

Now, let v be an internal node of $T(G)$ with children u and w.

If v is a parallel node, then $G_v = G_u \cup G_w$. Then, (H_v, H_v^*) can be computed in time $O(1)$ from (H_u, H_u^*) and (H_w, H_w^*) thanks to Lemma 10.

Lemma 10 ([12]). *Let $G_v = G_u \cup G_w$. Then $(H_v, H_v^*) = (H_u \cup H_w, H_u^* \cup H_w^*)$.*

Proof. The fact that $H_u \cup H_w$ is an optimal hull set for G_v is trivial. The second part comes from the fact that H_u^* (resp., H_w^*) is an isometric subgraph of H_v^* and from Lemma 3. $\square$

Now, we consider the case when v is a series node.

Lemma 11. *If $G_v = G_u \oplus G_w$, then (H_v, H_v^*) can be computed from the sets (H_x, H_x^*) of the children or grand children x of v in $T(G)$, in time $O(2^q n)$.*

Proof. If G_u and G_w are both complete, then G_v is a clique and $(H_v, H_v^*) = (V(G_v), V(G_v^*))$.

If G_u and G_w are both not complete, let x, y be any two non adjacent vertices in G_u. Then, we claim that $H_v = H_v^* = \{x, y\}$. Indeed, in G_v, x and y generate all vertices in $V(G_w)$ (resp., of G_w^*). In particular, two non adjacent vertices $z, r \in V(G_w)$ are generated. Symmetrically, z, r generate all vertices in $V(G_u)$ (resp., in $V(G_u^*)$).

Without loss of generality, we suppose now that G_u is a complete graph and that G_w is a non-complete $(q, q-4)$-graph. First, observe that no vertex of G_u belongs to any minimum hull set of G_v, since they are universal (Lemma 2). Note also that, by Lemma 9 and since G_v is not a clique and has universal vertices, we can make $H_v = H_v^*$. Hence, in what follows, we consider only the computation of H_v. Let us consider all possible cases for w in $T(G)$.

- w is a series node. G_w is the join of two graphs. We claim that $H_v = H_w$.

  In this case, G_w is an isometric subgraph of G_v. Thus, by Lemma 3, any minimum hull set of G_w generates all vertices of $V(G_w)$ in G_v. Finally, since G_w has two non-adjacent vertices they generate all vertices of G_u in G_v.

- w is a parallel node. G_w is the disjoint union of two graphs. Let x and y the children of w in $T(G)$. Then $G_w = G_x \cup G_y$. Let $X = H_x^*$ if G_x is not a clique and $X = V(G_x)$, otherwise, let $Y = H_y^*$ if G_y is not a clique and $Y = V(G_y)$, otherwise. We claim that $H_v = X \cup Y$.

  Clearly, if G_x (resp., G_y) is a clique, all its vertices are simplicial in G_v and then must be contained in any hull set by Lemma 1. Moreover, recall that, by Lemma 2, no vertex of G_u belongs to any minimum hull set of G.

  Now, let $z \in \{x,y\}$ such that G_z is not complete. It remains to show that it is necessary and sufficient to also include any minimum hull set H_z^* of G_z^*, in any minimum hull set of G.

  The necessity can be easily proved by using Lemma 8 to every G_z that is not a complete graph.

  The sufficiency follows again from the fact that G_u is generated by two non adjacent vertices of G_w and since, in all cases, $X \cup Y$ contains at least one vertex in G_x and one vertex in G_y, all vertices in G_u will be generated.

- w is a spider node and G_w is a thin spider $(S,K,\emptyset,E')$. Then, $H_v = S \cup \{k\} = G_w^*$ where k is any vertex in K.

  All vertices in S are simplicial in G_v, hence any hull set of G_v must contain S by Lemma 1. Now, in G_v, the vertices in S are at distance two and no shortest path between two vertices in S passes through a vertex in K, since there is a join to a complete graph. Therefore, S is not a hull set of G_v. However, since $|S| \geq 2$, it is easy to check that adding any vertex $k \in K$ to S is sufficient to generate all vertices in G_v. So $S \cup \{k\}$ is a minimum hull set of G_v.

  Note that, in that way, $H_v = S \cup \{k\} = G_w^*$

- w is a spider node and G_w is a spider (S,K,R,E') that is either thick or $R \neq \emptyset$ and R induces a $(q,q-4)$-graph. Then, $H_v = H_w$.

  If $R = \emptyset$, then G_w is thick. In this case, it is easy to check that the only minimum hull set of G_w is S (because it consists of simplicial vertices) and it is also a minimum hull set for G_v. Hence, $H_v = H_w = S$.

  If $R \neq \emptyset$, then by Lemma 1 any minimum hull set of G_w contains S. Moreover, by Lemma 8 any minimum hull set of G_w contains a minimum hull set of $K \cup R$ which is composed by vertices of R.

  By the same lemmas, a minimum hull set of G_w is a minimum hull set of G_v since, by Lemma 2, no vertex of G_u belongs to any minimum hull set of G_v and G_u is generated by non-adjacent vertices of G_w.

- w is a small node. G_w is a q-pseudo-spider (H_2,H_1,R,E') and R induces a $(q,q-4)$-graph.

  If $R = \emptyset$, G_v is the join of a clique G_u with a graph G_w that has at most q vertices. No vertex of G_u belongs to any minimum hull set of G_v, since they are universal. Thus, H_v can be computed in time $O(2^q)$ by testing all the possible subsets of vertices of G_w.

  Similarly, if R is a clique, all vertices in R are simplicial in G_v so they must belong to any hull set of G_v. Moreover, no vertices in G_u belong to any minimum hull set of G_v. So H_v can be computed in time $O(2^q)$ by testing all the possible subsets of vertices of $H_1 \cup H_2$ and adding R to them.

  In case $R \neq \emptyset$ nor a clique, two cases must be considered. By definition of the decomposition, there exists a child r of w in $T(G)$ such that $V(G_r) = R$.

  - If $G[H_1]$ is a clique, then, $G_v = (H_2, H_1 \cup V(G_u), R, E)$ is a pseudo-spider that satisfies the conditions in Lemma 8. Hence, any minimum hull set of G_v contains a minimum hull set of $P = G[H_1 \cup V(G_u) \cup R]$. Let Z be a minimum hull set of G_v and let $Z' = Z \cap H_2$. By Lemma 8, we have $|Z'| \leq hn(G_v) - hn(P)$.

    By Lemma 9, H_r^* is a minimum hull set of $G[H_1 \cup V(G_u) \cup R]$. Now, $G[H_1 \cup V(G_u) \cup R]$ is an isometric subgraph of G_v. Hence, by Lemma 3, H_r^* generates all vertices of $G[H_1 \cup V(G_u) \cup R]$ in

G_v. Therefore, $H_r^* \cup Z'$ will generate all vertices of G_v. Since $|H_r^*| = hn(P)$, we get that $|H_r^* \cup Z'| \leq hn(G_v)$ and then $H_r^* \cup Z'$ is a minimum hull set of G_v.

So, we have shown that there exists a minimum hull set for G_v that can be obtained from H_r^* by adding some vertices in $H_1 \cup H_2$. Since $|H_1 \cup H_2| \leq q$, such a subset of $H_1 \cup H_2$ can be found in time $O(2^q)$.

– In case $G[H_1]$ is not a clique, let x and y be two non adjacent vertices of H_1. We claim in this case that there exists a minimum hull set of G_v containing at most one vertex of R. Let S be a minimum hull set of G_v containing at least two vertices in R. Observe that $S' = (S \backslash R) \cup \{x,y\}$ is also a hull set of G_v since x and y are sufficient to generate all vertices in R. Consequently, $|S'| \leq |S|$ and S' is minimum.

Since no hull set of G_v contains a vertex in $V(G_u)$, there always exists a minimum hull set of G_v that consists of only vertices in $H_1 \cup H_2$ plus at most one vertex in R. Therefore an exhaustive search can be performed in time $O(n2^q)$.

□

Now, we consider the case when v is a spider node or a small node. That is $G_v = (S,K,R,E)$. If $R \neq \emptyset$, let r be the child of v such that $V(G_r) = R$.

Lemma 12. *Let $G_v = (S,K,R,E)$ be a spider such that R induces a $(q,q-4)$-graph.*
Then, $H_v = H_v^* = S \cup H_r^*$ if $R \neq \emptyset$ and R is not a clique, and $H_v = H_v^* = S \cup R$, otherwise.

Proof. Since all the vertices in S are simplicial vertices in G_v and in G_v^*, we apply Lemma 1 to conclude that they are all contained in any hull set of G_v (resp., of G_v^*).

By the structure of a spider, every vertex of K (and the universal vertex in G_v^*) belongs to a shortest path between two vertices in S and are therefore generated by them in any minimum hull set of G_v (resp., of G_v^*). Consequently, if $R = \emptyset$, S is a minimum hull set of G_v (resp., of G_v^*). If R is a clique, $S \cup R$ is the set of simplicial vertices of G_v (resp., of G_v^*) and also a minimum hull set of G_v (resp., of G_v^*).

Finally, if $R \neq \emptyset$ and R is not a clique, then G_v is a pseudo-spider satisfying the conditions of Lemma 8. Similarly, G_v^* is a pseudo-spider (by including the universal vertex in K). Then, by Lemma 8, any hull set of G_v (resp., of G_v^*) contains a minimum hull set of $G[K \cup R]$ (resp., of $G_v^* \setminus S$. Moreover, any hull set contains all vertices in S since they are simplicial. Hence, $hn(G_v) = hn(G_v^*) = |S| + hn(G[K \cup R])$ (recall that, by Lemma 9, $hn(G[K \cup R]) = hn(G_v^* \setminus S)$). Finally, since $G[K \cup R])$ is an isometric subgraph of G_v, then H_r^* (which is a minimum hull set of $G[K \cup R]$ by Lemma 9) generates $G[K \cup R]$ in G_v (resp., in G_v^*).

Hence, $S \cup H_r^*$ is a hull set of G_v and G_v^*. Moreover, it has size $|S| + hn(G[K \cup R])$, so it is optimal.

□

Lemma 13. *Let $G_v = (H_2,H_1,R,E)$ be a q-pseudo-spider such that R is a $(q,q-4)$-graph. Then, H_v and H_v^* can be computed in time $O(2^q n)$.*

Proof. All the arguments to prove this lemma are in the proof of Lemma 11. Moreover, the following arguments hold both for G_v and G_v^*: they allow to compute both H_v and H_v^*.

If $R = \emptyset$, G_v has at most q vertices, for a fixed positive integer q. Thus, its hull number can be computed in $O(2^q)$-time.

Otherwise, if H_1 is a clique, by Lemma 8, any minimum hull set of G_v contains a minimum hull set of $G[H_1 \cup R]$. Moreover, by the same arguments as in Lemma 11, we can show that there is an optimal hull set for G_v that can be obtained from H_r^* (minimum hull set of $G[H_1 \cup R]$) and some vertices in H_2.

If H_1 is not a clique, two non-adjacent vertices of H_1 can generate R. Thus, we conclude that there exists a minimum hull set of G_v containing at most one vertex of R. Then, a minimum hull set of G_v can be found in $O(2^q n)$-time, where $n = |V(G_v)|$. □

6 Hull Number via 2-connected components

In this section, we introduce a generalized variant of the hull number of a graph. Let $G = (V,E)$ be a graph and $S \subseteq V$. Let $hn(G,S)$ denote the minimum size of a set $U \subseteq V \setminus S$ such that $U \cup S$ is a hull set for G. We prove that to compute the hull number of a graph, it is sufficient to compute the generalized hull number of its 2-connected components (or *blocks*). This extends a result in [15].

Theorem 6. *Let G be a graph and $G_1,\dots,G_n$ be its 2-connected components. For any $i \leq n$, let $S_i \subseteq V(G_i)$ be the set of cut-vertices of G in G_i. Then,*

$$hn(G) = \sum_{i \leq n} hn(G_i, S_i).$$

Proof. Clearly, the result holds if $n = 1$, so we assume $n > 1$.

A block G_i is called a *leaf-block* if $|S_i| = 1$. Note that, for any leaf-block G_i, $G[V \setminus (V(G_i) \setminus S_i)]$ is convex, so by Lemma 4, any hull set of G contains at least one vertex in $V(G_i) \setminus S_i$. Moreover,

Claim 7. *For any minimum hull set S of G, $S \cap (\cup_{i \leq n} S_i) = \emptyset$.*

Proof. For purpose of contradiction, let us assume that a minimum hull set S of G contains a vertex $v \in S_i$ for some $i \leq n$. Note that there exist two leaf-blocks G_1 and G_2 such that v is on a shortest path between vertices in $V(G_1)$ and $V(G_2)$ or $\{v\} = V(G_1) \cap V(G_2)$. By the remark above, there exist $x \in (V(G_1) \setminus S_1) \cap S$ and $y \in (V(G_2) \setminus S_2) \cap S$. Hence, v is on a shortest (x,y)-path, i.e., $v \in I[x,y] \subseteq I_h[S \setminus \{v\}]$. Hence, $V \subseteq I_h[S] \subseteq I_h[S \setminus \{v\}]$ and $S \setminus \{v\}$ is a hull set of G, contradicting the minimality of S. $\diamond$

Claim 8. *Let S be a hull set of G. Then $S' = (S \cap V(G_i)) \cup S_i$ is a hull set of G_i.*

Proof.

For purpose of contradiction, assume that $I_h[S'] = V(G_i) \setminus X$ for some $X \neq \emptyset$. Then, there is $v \in X \cap I[a,b]$ for some $a \in V(G) \setminus V(G_i)$ and $b \in V(G) \setminus X$. Then, there is a shortest (a,b)-path P containing v. Hence, there is $u \in S_i$ such that u is on the subpath of P between a and v. Moreover, let $w = b$ if $b \in G_i$, and else let w be a vertex of S_i on the subpath of P between v and b. Hence, $v \in I[u,w] \subseteq I_h[S']$, a contradiction.

$\diamond$

Let X be any minimum hull set of G. By Claim 7, $X \cap (\cup_{i \leq n} S_i) = \emptyset$, hence we can partition $X = \cup_{i \leq n} X_i$ such that $X_i \subseteq V(G_i) \setminus S_i$ and $X_i \cap X_j = \emptyset$ for any $i \neq j$. Moreover, by Claim 8, $X_i \cup S_i$ is a hull set of G_i, i.e., $|X_i| \geq hn(G_i,S_i)$. Hence, $hn(G) = |X| = \sum_{i \leq n} |X_i| \geq \sum_{i \leq n} hn(G_i,S_i)$.

It remains to prove the reverse inequality. For any $i \leq n$, let $X_i \subseteq V(G_i) \setminus S_i$ such that $X_i \cup S_i$ is a hull set of G_i and $|X_i| = hn(G_i,S_i)$. We prove that $S = \cup_{i \leq n} X_i$ is a hull set for G. Indeed, for any $v \in S_i$, there are two leaf-blocks G_1, G_2 such that v is on a shortest path between G_1 and G_2 or $\{v\} = V(G_1) \cap V(G_2)$. So, there exist $x \in X_1$ and $y \in X_2$ such that v is on a shortest (x,y)-path, i.e., $v \in I[x,y] \subseteq I_h[S]$. Hence, $\cup_{i \leq n} S_i \subseteq I_h[S]$ and therefore, $V = \cup_{i \leq n} I_h[X_i \cup S_i] \subseteq I_h[\cup_{i \leq n}(X_i \cup S_i)] \subseteq I_h[\cup_{i \leq n}(X_i)] = I_h[S]$. $\square$

A *cactus* G is a graph in which every pair of cycles have at most one common vertex. This definition implies that each block of G is either a cycle or an edge. By using the previous result, one may easily prove that:

Corollary 2 ([1]). *In the class of cactus graphs, the hull number can be computed in linear time.*

7 Bounds

In this section, we use the same techniques as presented in [12, 15] to prove new bounds on the hull number of several graphs classes. These techniques mainly rely on a greedy algorithm for computing a hull set of a graph and that consists of the following: given a connected graph $G = (V,E)$ and its set S of simplicial vertices, we start with $H = S$ or $H = \{v\}$ (v is any vertex of V) if $S = \emptyset$, and $C_0 = I_h[H]$. Then, at each step $i \geq 1$, if $C_{i-1} \subset V$, the algorithm greedily chooses a subset $X_i \subseteq V \setminus C_{i-1}$, add X_i to H and set $C_i = I_h[H]$. Finally, if $C_i = V$, the algorithm returns H which is a hull set of G.

Claim 9. *If for every $i \geq 1$, $|C_i \setminus (C_{i-1} \cup X_i)| \geq c \cdot |X_i|$, for some constant $c > 0$, then $|H| \leq \max\{1,|S|\} + \left\lceil \frac{|V|-\max\{1,|S|\}}{1+c} \right\rceil$.*

In the following, we keep the notation used to describe the algorithm.

Claim 10. *Let G be a connected graph. Then, before each step $i \geq 1$ of the algorithm, for any $v \in V \setminus C_{i-1}$, $N(v) \cap C_{i-1}$ induces a clique. Moreover, any connected component induced by $V \setminus C_{i-1}$ has at least 2 vertices.*

Proof. Let $v \in V \setminus C_{i-1}$ and assume v has two neighbors u and w in C_{i-1} that are not adjacent. Then, $v \in I[u,w] \subseteq C_{i-1}$ because C_{i-1} is convex, a contradiction. Note that, at any step $i \geq 1$ of the algorithm, $V \setminus C_{i-1}$ contains no simplicial vertex. By previous remark, if v has only neighbors in C_{i-1}, then v is simplicial, a contradiction. □

Claim 11. *If G is a connected C_3-free graph, then, at every step $i \geq 1$ of the algorithm, a vertex in $V \setminus C_{i-1}$ has at most one neighbor in C_{i-1}.*

Proof. Assume that $v \in V \setminus C_{i-1}$ has two neighbors $u,w \in C_{i-1}$. $\{u,w\} \notin E$ because G is triangle-free. This contradicts Claim 10. □

Lemma 14. *For any C_3-free connected graph G and at step $i \geq 1$ of the algorithm, either $C_{i-1} = V$ or there exists $X_i \subset V \setminus C_{i-1}$ such that $|C_i \setminus (C_{i-1} \cup X_i)| \geq |X_i|$.*

Proof. If there is $v \in V \setminus C_{i-1}$ at distance at least 2 from C_{i-1}, let $X_i = \{v\}$ and the result clearly holds. Otherwise, let v be any vertex in $V \setminus C_{i-1}$. By Claim 10, v has a neighbor u in $V \setminus C_{i-1}$. Moreover, because no vertices of $V \setminus C_{i-1}$ are at distance at least 2 from C_{i-1}, v and u have some neighbors in C_{i-1}. Finally, u and v have no common neighbors because G is triangle-free. Hence, by taking $X_i = \{v\}$, we have $u \in C_i$ and the result holds. □

Recall that the *girth* of a graph is the length of its smallest cycle.

Lemma 15. *Let G connected with girth at least 6. Before any step $i \geq 1$ of the algorithm when $C_{i-1} \neq V$, there exists $X_i \subset V \setminus C_{i-1}$ such that $|C_i \setminus (C_{i-1} \cup X_i)| \geq 2|X_i|$.*

Proof. If there is $v \in V \setminus C_{i-1}$ at distance at least 3 from C_{i-1}, let $X_i = \{v\}$ and the result clearly holds. Otherwise, let v be a vertex in $V \setminus C_{i-1}$ at distance two from any vertex of C_{i-1}. Let $w \in V \setminus C_{i-1}$ be a neighbor of v that has a neighbor $z \in C_{i-1}$. Since v is not simplicial, v has another neighbor $u \neq w$ in $V \setminus C_{i-1}$. If u is at distance two from C_{i-1}, let $y \in V \setminus C_{i-1}$ be a neighbour of u that has a neighbor $x \in C_{i-1}$. In this case, since the girth of G is at least six, $z \neq x$ and, there is a shortest (v,z)-path containing w and a shortest (v,x)-path containing u and y. Consequently, by setting $X_i = \{v\}$ we obtain the desired result. The same happens in case u has a neighbor $x \in C_{i-1}$. One may use again the hypothesis that the girth of G is at least six to conclude that, by setting $X_i = \{v\}$ we obtain that $w,u \in C_i$.

Finally, we claim that no vertex remains in $V \setminus C_{i-1}$. By contradiction, suppose that it is the case and that there are in $V \setminus C_{i-1}$ and all these vertices have a neighbor in C_{i-1}. Let v be a vertex in $V \setminus C_{i-1}$ that has a neighbor z in C_{i-1}. Again, v has a neighbor $u \in V \setminus C_{i-1}$, since it is not simplicial. The vertex u must have a neighbor x in C_{i-1}.

Observe that x and z are at distance 3, since the girth of G is at least six. Consequently, v and u are in a shortest (x,z)-path should not be in $V \setminus C_{i-1}$, that is a contradiction.

□

Lemma 16. *Let G be a connected graph. Before any step $i \geq 1$ of the algorithm when $C_{i-1} \neq V$, there exist $X_i \subset V \setminus C_{i-1}$ such that $|C_i \setminus (C_{i-1} \cup X_i)| \geq 2|X_i|/3$.*

Moreover, if G is k-regular ($k \geq 1$), there exist $X_i \subset V \setminus C_{i-1}$ such that $|C_i \setminus (C_{i-1} \cup X_i)| \geq |X_i|$.

Proof. By Claim 10, all connected component of $V \setminus C_{i-1}$ contains at least one edge.

- If there is $v \in V \setminus C_{i-1}$ at distance at least 2 from C_{i-1}, let $X_i = \{v\}$ and $|C_i \setminus (C_{i-1} \cup X_i)| \geq |X_i|$.
- Now, assume all vertices in $V \setminus C_{i-1}$ are adjacent to some vertex in C_{i-1}. If there are two adjacent vertices u and v in $V \setminus C_{i-1}$ such that there is $z \in C_{i-1} \cap N(u) \setminus N(v)$, then let $X_i = \{v\}$. Therefore, $u \in C_i$ and $|C_i \setminus (C_{i-1} \cup X_i)| \geq |X_i|$. So, the result holds.
- Finally, assume that for any two adjacent vertices u and v in $V \setminus C_{i-1}$, $N(u) \cap C_{i-1} = N(v) \cap C_{i-1} \neq \emptyset$.

  We first prove that this case actually cannot occur if G is k-regular. Let $v \in V \setminus C_{i-1}$. By Claim 10, $K = N(v) \cap C_{i-1}$ induces a clique. Moreover, for any $u \in N(v) \setminus C_{i-1}$, $N(u) \cap C_{i-1} = K$. Note that $k = |K| + |N(v) \setminus C_{i-1}|$. Let $w \in K$. Then, $A = (K \cup N(v) \cup \{v\}) \setminus \{w\} \subseteq N(w)$ and since $|A| = k$, we get that $A = N(w)$. Moreover, $N[u]$ cannot induce a clique since $V \setminus C_{i-1}$ contains no simplicial vertices, $i \geq 1$. Hence, there are $x, y \in N(v) \setminus C_{i-1}$ such that $\{x,y\} \notin E$. Because G is k-regular, there is $z \in N(x) \setminus (N(v) \cup C_{i-1})$. However, $N(z) \cap C_{i-1} = N(x) \cap C_{i-1} = K$. Hence, $z \in N(w) \setminus A$, a contradiction.

  Now, assume that G is a general graph. Let v be a vertex of minimum degree in $V \setminus C_{i-1}$. Recall that, by Claim 10, $N(v) \cap C_{i-1}$ induces a clique. Because any neighbor $u \in V \setminus C_{i-1}$ of v has the same neighborhood as v in C_{i-1} and because v is not simplicial, then there must be $u, w \in N(v) \setminus C_{i-1}$ such that $\{u,w\} \notin E$. Now, by minimality of the degree of v, there exists $y \in N(u) \setminus (N(v) \cup C_{i-1}) \neq \emptyset$. Similarly, there exists $z \in N(w) \setminus (N(v) \cup C_{i-1}) \neq \emptyset$. Let us set $X_i = \{v, z, y\}$. Hence, $u, w \in C_i \setminus (C_{i-1} \cup X_i)$ and the result holds.

□

Theorem 7. *Let G be a connected n-node graph with s simplicial vertices. All bounds below are tight:*

- $hn(G) \leq \max\{1,s\} + \left\lceil \frac{3(n-\max\{1,s\})}{5} \right\rceil$;
- *If G is C_3-free or k-regular ($k \geq 1$), then* $hn(G) \leq \max\{1,s\} + \left\lceil \frac{n-\max\{1,s\}}{2} \right\rceil$;
- *If G has girth ≥ 6, then* $hn(G) \leq \max\{1,s\} + \left\lceil \frac{1(n-\max\{1,s\})}{3} \right\rceil$.

Proof. First statement follows from Claim 9 and first statement in Lemma 16. The second statement follows from Claim 9 and Lemma 14 (case C_3-free) and second part of Lemma 16 (case regular graphs) . Last statement follows from Claim 9 and Lemma 15.

All bounds are reached in case of complete graphs. In case with no simplicial vertices: the first bound is reached by the graph obtained by taking several disjoint C_5 and adding a universal vertex, the second bound is obtained for a C_5, and the third one is reached by a C_7.

□

The first statement of the previous theorem improves another result in [15]:

Corollary 3. *If G is a graph with no simplicial vertex, then:*

$$\limsup_{|V(G)|\to\infty} \frac{hn(G)}{|V(G)|} = \frac{3}{5}.$$

It it important to remark that the second statement of Theorem 7 is closely related to a bound of Everett and Seidman proved in Theorem 9 of [15]. However, the graphs they consider do not have simplicial vertices and, consequently, they do not have vertices of degree one, which is not a constraint for our result.

8 Conclusions

In this paper, we simplified the reduction of Dourado et al. [12] to answer a question they asked about the complexity of computing the hull number of bipartite graphs. We presented polynomial-time algorithms for computing the hull number of cobipartite graphs, $(q,q-4)$-graphs and cactus graphs. Finally, we presented upper bounds for general graphs and two particular graph classes.

The result in Section 5 provides an FPT algorithm where the parameter is the number of induced P_4's in the input graph. It would be nice to know about the paramerized complexity of HULL NUMBER when the parameter is the size of the hull set.

Another question of Dourado *et al.* [12], concerning to the complexity of this problem for interval graphs and chordal graphs, remains open. Up to the best of our knowledge, determining the complexity of the HULL NUMBER problem on planar graphs is also an open problem.

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