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Abstract. In 2008, Groth and Sahai proposed a general methodology for constructing non-interactive zero-knowledge (and witness-indistinguishable) proofs in bilinear groups. While avoiding expensive NP-reductions, these proof systems are still inefficient due to a number of pairing computations required for verification. We apply recent techniques of *batch verification* to the Groth-Sahai proof systems and manage to improve significantly the complexity of proof verification. We give explicit batch verification formulas for generic Groth-Sahai equations (whose cost is less than a tenth of the original) and also for specific popular protocols relying on their methodology (namely Groth's group signatures and Belenkiy-Chase-Kohlweiss-Lysyanskaya's P-signatures).

Keywords. Pairing-based cryptography, Batch verification, Groth-Sahai proof system.

1 Introduction

In a zero-knowledge proof system, a prover convinces a verifier *via* an interactive protocol that a mathematical statement is true, without revealing anything else than the validity of the assertion. In 1988, Blum, Feldman, and Micali [BFM90] showed that the use of a common random string shared between the prover and the verifier permits to design a zero-knowledge proof system for all NP-languages without requiring interaction. These proofs, called non-interactive zero-knowledge (NIZK), turned out to be a particularly useful tool in constructing cryptographic primitives. Unfortunately, their work (as well as subsequent results) did not yield efficient proofs. Until recently, the only way to construct efficient proofs was to rely on the random-oracle model (ROM) [BR93], which has been subject to a series of criticisms starting with [CGH98].

In 2008, Groth and Sahai [GS08] proposed a way to produce efficient and practical NIZK and non-interactive witness-indistinguishable (NIWI) proofs for (algebraic) statements related to groups equipped with a bilinear map. In particular, they give proofs for the simultaneous satisfiability of a set of equations. They proposed three instantiations of their system based on different (mild) computational assumptions: the subgroup decision problem (SD), the symmetric external Diffie-Hellman problem (SXDH) and the decision linear problem (DLIN). Each one of these has already given rise to many applications (e.g. [BW06,BW07,CGS07,Gro07,GL07,BCKL08,BCC+09]). While much more efficient than all previous proposals, their proof system still lacks in practicality compared to the ROM, since the verification of a single equation requires the computation of dozens of bilinear map evaluations by the verifier.

The aim of this paper is to optimize the verification procedure at the expense of slightly weakening the soundness of the proof system.

Prior Work. In the last twenty years, there has been a lot of work in cryptography in which expensive tasks are processed in batch rather than individually to achieve better efficiency. Batch cryptography was first introduced by Fiat [Fia90], who proposed an algorithm to compute several private RSA key operations (with different exponents) through one full exponentiation and several small exponentiations. Batch cryptography is particularly relevant in settings where many exponentiations need to be verified together: many schemes were proposed to achieve batch verification of digital signatures (e.g. [NMVR94]

	Naive computation	Batch computation
SXDH		
Pairing-product equation	$5m + 3n + 16$	$m + 2n + 8$
Multi-scalar multiplication equation in $\mathbb{G}_1$	$8m + 2n + 14$	$\min(2n + 9, 2m + n + 7)$
Multi-scalar multiplication equation in $\mathbb{G}_2$	$8n + 2m + 14$	$\min(2m + 9, 2n + m + 7)$
Quadratic equation	$8m + 8n + 12$	$2\min(m, n) + 8$
DLIN		
Pairing-product equation	$12n + 27$	$3n + 6$
Multi-scalar multiplication equation	$9n + 12m + 27$	$3n + 3m + 6$
Quadratic equation	$18n + 24$	$3n + 6$

Table 1. Number of pairings per proof verification, where n and m stand for the number of different types of variables.

for DSA signatures) and it seems natural to apply such techniques to the verification of Groth-Sahai proofs, which require expensive evaluations of pairings. In 1998, Bellare, Garay and Rabin [BGR98] took the first systematic look at batch verification and described several techniques for conducting batch verification of exponentiations with high confidence. They proposed three generic methods called the *random subset test*, the *small exponents test* and the *bucket test*. More recently, Ferrara, Green, Hohenberger and Pedersen [FGHP09], presented a detailed study on how to securely batch verify a set of pairing-based equations and some applications on existing signatures schemes.

Our Results. The main result of the paper is a significant reduction of the cost of Groth-Sahai proof systems by using batch verification techniques. In particular, we give efficient explicit verification procedures for the three[1] instantiations proposed in [GS08]. The essence of our approach is a trade-off between soundness and efficiency: if the verification algorithm returns valid, the verifier is assured that all proved statements are indeed valid with overwhelming probability. The best improvements are for the proofs based on SXDH and DLIN, which are the ones with most practical relevance (see Sections 5 and 6). Table 1 summarizes the number of (dominant) pairing operations required to verify the different algebraic statements in Groth-Sahai terminology (see Section 3 for details).

In [CHP07], Camenisch *et al.* explicitly mentioned as an "exciting" open problem the development of fast batching schemes for various forms of anonymous authentication (such as group signatures and anonymous credentials). This paper is the first to address this issue in the standard security model by considering two schemes based on Groth-Sahai's methodology.

The first scheme we consider was proposed by Groth in 2007 [Gro07]. It is a constant-size group signature scheme whose security can be proved in the standard model (i.e. without relying on the random oracle heuristic). For illustrative purposes, we concentrate on the (simpler) variant of the scheme that provides CPA anonymity only. Even this variant does not achieve satisfactory efficiency—the verification of a signature requires the computation of 68 expensive pairing operations. In Section 7, we propose an improved verification procedure in which the total number of bilinear map evaluations drops to 11. In addition, if $n \geq 2$ signatures (for the same group) have to be verified at once, we manage to decrease this number from $11\,n$ to $4\,n + 7$.

In Section 8, we study the *P-signature* scheme[2] proposed by Belenkiy, Chase, Kohlweiss and Lysyanskaya [BCKL08]. Since anonymous credentials are an immediate consequence of P-signatures, we thereby apply our techniques to privacy-preserving authentication mechanisms. Belenkiy *et al.* proposed two instantiations of their protocol (based on SXDH and DLIN). They evaluated that the verification of the proof of possession of a signature would involve respectively 68 and 128 pairing evaluations. We show that this can be reduced to 15 and 12, respectively. Moreover, the number of pairing operations required to verify $n \geq 2$ signatures is reduced to $2\,n + 13$ and $3\,n + 9$, respectively, by using our techniques

[1] The results for the (least practical) instantiation based on the subgroup decision problem are postponed to Appendix A.
[2] A *P-signature* scheme is a digital signature scheme with an additional non-interactive proof of signature possession.

2 Preliminaries

2.1 Bilinear Groups

Since Groth-Sahai proof systems are for group-dependent languages, we summarize the basics of bilinear groups and pairing-based assumptions. In the sequel, we consider an algorithm $\mathcal{G}$ that, on input a security parameter λ, outputs a tuple $(N, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, e, g_1, g_2)$, where $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T$ are cyclic groups of order N, g_1 and g_2 generate $\mathbb{G}_1$ and $\mathbb{G}_2$ respectively, and e is an admissible bilinear map $e\colon \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$, which means that it is efficiently computable, $e(g_1, g_2)$ generates $\mathbb{G}_T$, and that $e(u^a, v^b) = e(u, v)^{ab}$ holds for all $u \in \mathbb{G}_1, v \in \mathbb{G}_2$ and $a, b \in \mathbb{Z}_N$.

Definition 1. *Let $\mathbb{G}$ be a cyclic group of order N. The* decisional Diffie-Hellman (DDH) assumption *states that the distributions of (u, u^x, u^y, u^z) and $(u, u^x, u^y, u^{x \cdot y})$ are computationally indistinguishable for a random group element $u \in \mathbb{G}$ and random scalars $x, y, z \in \mathbb{Z}_N$.*

Definition 2. *Consider a bilinear group $(N, \mathbb{G}, \mathbb{G}_T, e, g)$, where $N = p \cdot q$ is the product of two primes (and we have $\mathbb{G}_1 = \mathbb{G}_2 = \mathbb{G}$). The* subgroup decision (SD) assumption [BGN05] *in $\mathbb{G}$ states that given a random element $u \in \mathbb{G}$, it is computationally hard to decide whether u is in a subgroup of $\mathbb{G}$.*

Definition 3. *Consider a bilinear group $(p, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, e, g_1, g_2)$, where p is prime, output by $\mathcal{G}(\lambda)$. The* symmetric external decision Diffie-Hellman (SXDH) assumption [ACHdM05] *states that the DDH assumption holds in both $\mathbb{G}_1$ and $\mathbb{G}_2$.*

Definition 4. *Consider a bilinear group $(p, \mathbb{G}, \mathbb{G}_T, e, g)$, where p is prime (and we have $\mathbb{G}_1 = \mathbb{G}_2 = \mathbb{G}$). The* decision linear (DLIN) assumption [BBS04] *states that the two distributions (u, v, w, u^a, v^b, w^c) and $(u, v, w, u^a, v^b, w^{a+b})$ are computationally indistinguishable for random group elements $u, v, w \in \mathbb{G}$ and random scalars $a, b, c \in \mathbb{Z}_p$.*

2.2 Notation

We let "$\cdot$" denote the product of two elements either in $\mathbb{Z}_N$, in $\mathbb{G}$ or in $\mathbb{G}_T$. For equal-dimension vectors or matrices A and B of group elements, $A \odot B$ stands for their entry-wise product (*i.e.* it denotes their Hadamard product). For a vector or a matrix $A = \left(a_{i,j}\right)_{i,j}$ of group elements and an integer x, we denote by A^x the matrix $\left(a_{i,j}^x\right)_{i,j}$.

We will use $\langle \cdot, \cdot \rangle$ for bilinear products between vectors of either scalars or group elements. Let $\vec{a}, \vec{b} \in \mathbb{Z}_N^n$ and $\vec{\mathcal{A}}, \vec{\mathcal{B}} \in \mathbb{G}^n$. We define

$$\langle \vec{a}, \vec{b} \rangle := \textstyle\sum_{i=1}^n a_i \cdot b_i \qquad \langle \vec{a}, \vec{\mathcal{B}} \rangle := \prod_{i=1}^n \mathcal{B}_i^{a_i} \qquad \langle \vec{\mathcal{A}}, \vec{\mathcal{B}} \rangle := \prod_{i=1}^n e(\mathcal{A}_i, \mathcal{B}_i)$$

We employ Groth and Sahai's notation of a bilinear product $\bullet\colon \mathbb{G}_1^{n \times k} \times \mathbb{G}_2^{n \times k} \to \mathbb{G}_T^{k \times k}$ (for $k = 2, 3$) defined as follows:

$$\vec{\mathbf{c}} \bullet \vec{\mathbf{d}} := \begin{pmatrix} \prod_{i=1}^n e(c_{i,1}, d_{i,1}) & \cdots & \prod_{i=1}^n e(c_{i,1}, d_{i,k}) \\ \vdots & & \vdots \\ \prod_{i=1}^n e(c_{i,k}, d_{i,1}) & \cdots & \prod_{i=1}^n e(c_{i,k}, d_{i,k}) \end{pmatrix}$$

For the case $\mathbb{G}_1 = \mathbb{G}_2$ and $k = 3$ we define a symmetric variant[3] $\overset{s}{\bullet}\colon \mathbb{G}^{n \times 3} \times \mathbb{G}^{n \times 3} \to \mathbb{G}_T^{3 \times 3}$

$$\vec{\mathbf{c}} \overset{s}{\bullet} \vec{\mathbf{d}} := \begin{pmatrix} \prod_{i=1}^n e(c_{i,1}, d_{i,1}) & \prod_{i=1}^n e(c_{i,1}, d_{i,2})^{\frac{1}{2}} e(c_{i,2}, d_{i,1})^{\frac{1}{2}} & \prod_{i=1}^n e(c_{i,1}, d_{i,3})^{\frac{1}{2}} e(c_{i,3}, d_{i,1})^{\frac{1}{2}} \\ \prod_{i=1}^n e(c_{i,2}, d_{i,1})^{\frac{1}{2}} e(c_{i,1}, d_{i,2})^{\frac{1}{2}} & \prod_{i=1}^n e(c_{i,2}, d_{i,2}) & \prod_{i=1}^n e(c_{i,2}, d_{i,3})^{\frac{1}{2}} e(c_{i,3}, d_{i,2})^{\frac{1}{2}} \\ \prod_{i=1}^n e(c_{i,3}, d_{i,1})^{\frac{1}{2}} e(c_{i,1}, d_{i,3})^{\frac{1}{2}} & \prod_{i=1}^n e(c_{i,3}, d_{i,2})^{\frac{1}{2}} e(c_{i,2}, d_{i,3})^{\frac{1}{2}} & \prod_{i=1}^n e(c_{i,3}, d_{i,3}) \end{pmatrix}$$

[3] Note that in their DLIN instantiation, Groth and Sahai use $\tilde{\bullet}$ for the asymmetric map and $\bullet$ for the symmetric variant.

3 Groth-Sahai Proof Systems

We sketch the results of Groth and Sahai [GS08] on proofs of satisfiability of sets of equations over a bilinear group $(N, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, e, g_1, g_2)$. Due to the complexity of their methodology, we merely give what is needed for our results and refer to the full version of [GS08] for any additional details.

The three types of equations are the following:

A *pairing-product equation* over variables $\vec{\mathcal{X}} = (\mathcal{X}_1, \ldots, \mathcal{X}_m) \in \mathbb{G}_1$ and $\vec{\mathcal{Y}} = (\mathcal{Y}_1, \ldots, \mathcal{Y}_n) \in \mathbb{G}_2$ is of the form

$$\langle \vec{\mathcal{A}}, \vec{\mathcal{Y}} \rangle \cdot \langle \vec{\mathcal{X}}, \vec{\mathcal{B}} \rangle \cdot \langle \vec{\mathcal{X}}, \Gamma \vec{\mathcal{Y}} \rangle = t_T \ , \tag{1}$$

defined by constants $\vec{\mathcal{A}} \in \mathbb{G}_1^n$, $\vec{\mathcal{B}} \in \mathbb{G}_2^m$, $\Gamma = (\gamma_{i,j})_{\substack{1 \le i \le m \\ 1 \le j \le n}} \in \mathbb{Z}_N^{m \times n}$ and $t_T \in \mathbb{G}_T$.

A *multi-scalar multiplication equation* over variables $\vec{y} \in \mathbb{Z}_N^n$ and $\vec{\mathcal{X}} \in \mathbb{G}_1^m$ is of the form

$$\langle \vec{y}, \vec{\mathcal{A}} \rangle \cdot \langle \vec{b}, \vec{\mathcal{X}} \rangle \cdot \langle \vec{y}, \Gamma \vec{\mathcal{X}} \rangle = T, \tag{2}$$

defined by the constants $\vec{\mathcal{A}} \in \mathbb{G}_1^n$, $\vec{b} \in \mathbb{Z}_N^m$, $\Gamma \in \mathbb{Z}_N^{m \times n}$ and $T \in \mathbb{G}_1$.

A multi-scalar multiplication equation in group $\mathbb{G}_2$ is defined analogously.

A *quadratic equation in* $\mathbb{Z}_N$ over variables $\vec{x} \in \mathbb{Z}_N^m$ and $\vec{y} \in \mathbb{Z}_N^n$ is of the form

$$\langle \vec{a}, \vec{y} \rangle + \langle \vec{x}, \vec{b} \rangle + \langle \vec{x}, \Gamma \vec{y} \rangle = t, \tag{3}$$

defined by the constants $\vec{a} \in \mathbb{Z}_N^n$, $\vec{b} \in \mathbb{Z}_N^m$, $\Gamma \in \mathbb{Z}_N^{m \times n}$ and $t \in \mathbb{Z}_N$.

The common reference string for the proof system is a key to make commitments to the variables of the different types. A proof of satisfiability is constructed by first committing to the variables of the respective equation and then constructing a "proof" for each equation. The latter proves that the committed values indeed satisfy the equation.

There are three instantiations based on the SD, the SXDH and the DLIN assumption, respectively. We present the last two (the instantiation based on the subgroup decision assumption is deferred to the appendix.

SXDH. The language is over a bilinear group $(p, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, e, g_1, g_2)$ with p being prime. The commitment key consists of $\mathbf{u}_1 = (u_{1,1}, u_{1,2}), \mathbf{u}_2 = (u_{2,1}, u_{2,2}) \in \mathbb{G}_1^2$ and $\mathbf{v}_1 = (v_{1,1}, v_{1,2}), \mathbf{v}_2 = (v_{2,1}, v_{2,2}) \in \mathbb{G}_2^2$; we write $\vec{\mathbf{u}} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix}$ and $\vec{\mathbf{v}} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix}$.

Let $X \in \mathbb{G}_1$, $Y \in \mathbb{G}_2$ and $x \in \mathbb{Z}_p$. We define $\iota_1(X) := (1, X)$, $\iota_2(Y) := (1, Y)$, $\iota_1'(x) := (u_{2,1}^x, (u_{2,2} g_1)^x)$ and $\iota_2'(x) := (v_{2,1}^x, (v_{2,2} g_2)^x)$.

To commit to $X \in \mathbb{G}_1$, one chooses randomness $s_1, s_2 \in \mathbb{Z}_p$ and sets $\mathbf{c}_X := \iota_1(X) \odot \mathbf{u}_1^{s_1} \odot \mathbf{u}_2^{s_2}$, a commitment to $Y \in \mathbb{G}_2$ is defined as $\mathbf{d}_Y := \iota_2(Y) \odot \mathbf{v}_1^{s_1} \odot \mathbf{v}_2^{s_2}$. To make a commitment to $x \in \mathbb{Z}_p$, which is in $\mathbb{G}_1^2$ one chooses $s \in \mathbb{Z}_p$ and sets $\mathbf{c}_x := \iota_1'(x) \odot \mathbf{u}_1^s$, a commitment in $\mathbb{G}_2^2$ is defined as $\mathbf{d}_x := \iota_2'(x) \odot \mathbf{v}_1^s$.

To show satisfiability of a set of equations of the form (1), (2) or (3), one first makes commitments to a satisfying witness (i.e., an assignment to the variables of each equation) and then adds a "proof" per equation. Groth and Sahai describe how to construct these; for Type (1), they are in $\mathbb{G}_2^{2\times 2} \times \mathbb{G}_1^{2\times 2}$, for Type (2) they are in $\mathbb{G}_2^{2\times 2} \times \mathbb{G}_1^2$ and for Type (3) in $\mathbb{G}_2^2 \times \mathbb{G}_1^2$.

The verification relations for the proofs are given in Section 5, where we also discuss how to optimize them. For convenience we define some notations. Let $t \in \mathbb{Z}_p$, $T_1 \in \mathbb{G}_1$, $T_2 \in \mathbb{G}_2$ and $t_T \in \mathbb{G}_T$. Then we let[4]

$$\iota_T(t_T) := \begin{pmatrix} 1 & 1 \\ 1 & t_T \end{pmatrix}, \ \hat{\iota}_T(T_1) := \begin{pmatrix} 1 & 1 \\ e(T_1, v_{2,1}) & e(T_1, v_{2,2} g_2) \end{pmatrix}, \ \hat{\iota}_T(T_2) := \begin{pmatrix} 1 & e(u_{2,1}, T_2) \\ 1 & e(u_{2,2} g_1, T_2) \end{pmatrix},$$

and $\iota_T'(t) := \left[(u_{2,1}, u_{2,2} g_1) \bullet (v_{2,1}, v_{2,2} g_2) \right]^t = \begin{pmatrix} e(u_{2,1}, v_{2,1})^t & e(u_{2,1}, v_{2,2} g_2)^t \\ e(u_{2,2} g_1, v_{2,1})^t & e(u_{2,2} g_1, v_{2,2} g_2)^t \end{pmatrix}$. For the sake of consistency with [GS08], for $\mathbf{c} \in \mathbb{G}_1^{1\times 2}$ and $\mathbf{d} \in \mathbb{G}_2^{1\times 2}$ we denote $F(\mathbf{c}, \mathbf{d}) := [\mathbf{c} \bullet \mathbf{d}]$.

[4] We use the rectifications of $\hat{\iota}_T$ and ι_T' by [GSW09].

DLIN. In this instantiation, the language is over a bilinear group $(p, \mathbb{G}, \mathbb{G}_T, e, g)$ with p being prime. The commitment key $\vec{\mathbf{u}} \in \mathbb{G}^{3\times 3}$ is of the form $\mathbf{u}_1 = (u_{1,1}, 1, g)$, $\mathbf{u}_2 = (1, u_{2,1}, g)$, $\mathbf{u}_3 = (u_{3,1}, u_{3,2}, u_{3,3})$. Let $X \in \mathbb{G}$ and $x \in \mathbb{Z}_p$. We define $\iota(X) := (1, 1, X)$ and $\iota'(x) := (u_{3,1}^x, u_{3,2}^x, (u_{3,3}g)^x)$.

To commit to $X \in \mathbb{G}$, one chooses randomness $s_1, s_2, s_3 \in \mathbb{Z}_p$ and sets $\mathbf{c}_X := \iota(X) \odot \mathbf{u}_1^{s_1} \odot \mathbf{u}_2^{s_2} \odot \mathbf{u}_3^{s_3}$. To commit to $x \in \mathbb{Z}_p$, one chooses $s_1, s_2 \in \mathbb{Z}_p$ and computes $\mathbf{c}_x := \iota'(x) \odot \mathbf{u}_1^{s_1} \odot \mathbf{u}_3^{s_2}$.

Due to the fact that $\mathbb{G}_1 = \mathbb{G}_2 = \mathbb{G}$ in this setting, the equations (1), (2) and (3) simplify to the following equations respectively:

$$\langle \vec{\mathcal{A}}, \vec{\mathcal{Y}} \rangle \cdot \langle \vec{\mathcal{Y}}, \Gamma \vec{\mathcal{Y}} \rangle = t_T \tag{1'}$$

$$\langle \vec{a}, \vec{\mathcal{Y}} \rangle \cdot \langle \vec{x}, \vec{\mathcal{B}} \rangle \cdot \langle \vec{x}, \Gamma \vec{\mathcal{Y}} \rangle = T \tag{2'}$$

$$\langle \vec{x}, \vec{b} \rangle + \langle \vec{x}, \Gamma \vec{x} \rangle = t \tag{3'}$$

Groth and Sahai show how to construct "proofs" for each type of equation, where for Types (1') and (2'), the proof is in $\mathbb{G}^{3\times 3}$, whereas for Type (3') it is in $\mathbb{G}^{2\times 3}$. The verification relations for the proofs are given in Section 6.

We define the following notations. Let $t \in \mathbb{Z}_p$, $T \in \mathbb{G}$ and $t_T \in \mathbb{G}_T$. Then we let[5]

$$\iota_T(t_T) := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & t_T \end{pmatrix}, \quad \hat{\iota}_T(T) := \begin{pmatrix} 1 & 1 & e(u_{3,1}, T)^{\frac{1}{2}} \\ 1 & 1 & e(u_{3,2}, T)^{\frac{1}{2}} \\ e(u_{3,1}, T)^{\frac{1}{2}} & e(u_{3,2}, T)^{\frac{1}{2}} & e(u_{3,3}g, T) \end{pmatrix},$$

and $\iota'_T(t) := \left[(u_{3,1}, u_{3,1}, u_{3,3}g) \overset{s}{\bullet} (u_{3,1}, u_{3,1}, u_{3,3}g)\right]^t = \begin{pmatrix} e(u_{3,1}, u_{3,1})^t & e(u_{3,1}, u_{3,2})^t & e(u_{3,1}, u_{3,3}g)^t \\ e(u_{3,2}, u_{3,1})^t & e(u_{3,2}, u_{3,2})^t & e(u_{3,2}, u_{3,3}g)^t \\ e(u_{3,3}g, u_{3,1})^t & e(u_{3,3}g, u_{3,2})^t & e(u_{3,3}g, u_{3,3}g)^t \end{pmatrix}$.

4 Batch Verification of Pairing Equations

We address the problem to securely batch the verification of (potentially many) Groth-Sahai proofs. We achieve a trade-off between soundness and efficiency: if the verification algorithm returns valid, the verifier is assured that all proved statements are indeed valid with overwhelming probability. Ferrara, Green, Hohenberger and Pedersen [FGHP09] presented a detailed study on how to securely batch-verify a set of pairing-based equations, which we briefly recall here (see the full version of [FGHP09] for any additional details).

Given a bilinear structure $(N, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, e, g_1, g_2)$, a *pairing-based verification equation* is a Boolean relation of the form: $\prod_{i=1}^{k} e(f_i, h_i)^{c_i} \stackrel{?}{=} A$ for $k \in \mathbb{N}$, $(f_i, h_i, c_i) \in \mathbb{G}_1 \times \mathbb{G}_2 \times \mathbb{Z}_N$ for $i \in \{1, \ldots, k\}$ and $A \in \mathbb{G}_T$. A *pairing-based verifier* is an algorithm which given a pairing-based verification equation outputs *yes* if the Boolean relation holds, and *no* otherwise (except with negligible probability).

In order to design a pairing-based verifier for m pairing-based verification equations, one has to find a way to combine all equations. The technique proposed in [FGHP09] consists in using the *small exponents test* proposed by Bellare et al. [BGR98], which here amounts to pick small random exponents $\delta_1, \ldots, \delta_m$ and checking whether $\prod_{j=1}^{m} \prod_{i=1}^{k_j} e(f_{i,j}, h_{i,j})^{c_{i,j}\delta_j} = \prod_{j=1}^{m} A_j^{\delta_j}$ holds. In order to further reduce the computational needs, three main techniques may be used:

1. **Move the exponent into the pairing:** Since, in practice, exponentiation in $\mathbb{G}_T$ is more expensive than in $\mathbb{G}_1$ and $\mathbb{G}_2$[6], this gives a first speed up. As we are working on pairings, we can also do the opposite if it allows another technique to apply: $e(f_i, h_i)^{\delta_i} \to e(f_i^{\delta_i}, h_i)$
2. **Move the product into the pairing:** When two pairings have a common element, they can be combined to reduce the number of pairing computations: $\prod_{j=1}^{m} e(f_j^{\delta_j}, h_i) \to e\left(\prod_{j=1}^{m} f_j^{\delta_j}, h_i\right)$

[5] We use the rectifications of $\hat{\iota}_T$ and ι'_T by [GSW09].

[6] Note that, for Type 2 pairings, exponentiation in $\mathbb{G}_2$ is more expensive than in $\mathbb{G}_T$ (see [GPS08] for details).

3. **Switch two products:** Sometimes improvements can be made by moving a product from the first to the second component of a pairing (or vice-versa). $\prod_{i=1}^{k} e\left(\prod_{j=1}^{m} f_j^{\delta_{i,j}}, h_i\right) \leftrightarrow \prod_{j=1}^{m} e\left(f_j, \prod_{i=1}^{k} h_i^{\delta_{i,j}}\right)$

The soundness of the pairing-based verifier based on the small exponents test is quantified in the following theorem [FGHP09, Theorem 3.2]:

Theorem 1. *Given m pairing-based verification equation, the small-exponents verifier described above with random exponents $\delta_1, \ldots, \delta_m$ of ℓ bit is a pairing-based batch verifier that accepts an invalid batch with probability at most $2^{-\ell}$.*

Handling Invalid Proofs. In the case of verification of multiple proofs (as in Sections 7 and 8) if there is an invalid proof in the batch, then the verifier will reject the entire batch with high probability. A simple technique for finding invalid proofs in a batch consists in using a recursive *divide-and-conquer* approach [PMPS00]. Recently, more efficient techniques were proposed for pairing-based signatures (see e.g. [Mat09] and references therein) and they apply as well to our setting.

5 Instantiation 2: SXDH

5.1 Pairing-Product Equation

A proof $(\vec{\mathbf{c}}, \vec{\mathbf{d}}, \vec{\pi}, \vec{\theta}) \in \mathbb{G}_1^{m\times 2} \times \mathbb{G}_2^{n\times 2} \times \mathbb{G}_2^{2\times 2} \times \mathbb{G}_1^{2\times 2}$ of satisfiability of an equation of Type (1) is verified by checking the following equation.

$$\left[\iota_1(\vec{\mathcal{A}}) \bullet \vec{\mathbf{d}}\right] \odot \left[\vec{\mathbf{c}} \bullet \iota_2(\vec{\mathcal{B}})\right] \odot \left[\vec{\mathbf{c}} \bullet \Gamma \vec{\mathbf{d}}\right] = \iota_T(t_T) \odot \left[\vec{\mathbf{u}} \bullet \vec{\pi}\right] \odot \left[\vec{\theta} \bullet \vec{\mathbf{v}}\right] .$$

Let us denote $\vec{\mathbf{u}} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} u_{1,1}\ u_{1,2} \\ u_{2,1}\ u_{2,2} \end{pmatrix}$, $\vec{\mathbf{v}} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} v_{1,1}\ v_{1,2} \\ v_{2,1}\ v_{2,2} \end{pmatrix}$, $\vec{\mathcal{A}} = (\mathcal{A}_j)_{1\le j\le n} \in \mathbb{G}_1^{n\times 1}$,

$\vec{\mathcal{B}} = (\mathcal{B}_i)_{1\le i\le m} \in \mathbb{G}_2^{m\times 1}$, $\vec{\mathbf{c}} = (c_{i,k})_{\substack{1\le i\le m \\ 1\le k\le 2}} \in \mathbb{G}_1^{m\times 2}$, $\vec{\mathbf{d}} = (d_{j,k})_{\substack{1\le j\le n \\ 1\le k\le 2}} \in \mathbb{G}_2^{n\times 2}$, and $\Gamma = (\gamma_{i,j})_{\substack{1\le i\le m \\ 1\le j\le n}} \in \mathbb{Z}_p^{m\times n}$.

Plugging in the definitions from Section 3, the left hand side is equal to

$$\begin{pmatrix} \prod_{i=1}^{m} e\left(c_{i,1}, \prod_{j=1}^{n} d_{j,1}^{\gamma_{i,j}}\right) & \prod_{i=1}^{m} e\left(c_{i,1}, \mathcal{B}_i \prod_{j=1}^{n} d_{j,2}^{\gamma_{i,j}}\right) \\ \prod_{j=1}^{n} e\left(\mathcal{A}_j \prod_{i=1}^{m} c_{i,2}^{\gamma_{i,j}}, d_{j,1}\right) & \prod_{j=1}^{n} e(\mathcal{A}_j, d_{j,2}) \prod_{i=1}^{m} e\left(c_{i,2}, \mathcal{B}_i \prod_{j=1}^{n} d_{j,2}^{\gamma_{i,j}}\right) \end{pmatrix}.$$

While, if we denote $\vec{\pi} = \begin{pmatrix} \pi_{1,1}\ \pi_{1,2} \\ \pi_{2,1}\ \pi_{2,2} \end{pmatrix}$, $\vec{\theta} = \begin{pmatrix} \theta_{1,1}\ \theta_{1,2} \\ \theta_{2,1}\ \theta_{2,2} \end{pmatrix}$, the right hand side is equal to

$$\begin{pmatrix} e(u_{1,1},\pi_{1,1})e(u_{2,1},\pi_{2,1})e(\theta_{1,1},v_{1,1})e(\theta_{2,1},v_{2,1}) & e(u_{1,1},\pi_{1,2})e(u_{2,1},\pi_{2,2})e(\theta_{1,1},v_{1,2})e(\theta_{2,1},v_{2,2}) \\ e(u_{1,2},\pi_{1,1})e(u_{2,2},\pi_{2,1})e(\theta_{1,2},v_{1,1})e(\theta_{2,2},v_{2,1}) & t_T e(u_{1,2},\pi_{1,2})e(u_{2,2},\pi_{2,2})e(\theta_{1,2},v_{1,2})e(\theta_{2,2},v_{2,2}) \end{pmatrix}.$$

By grouping pairings, we reduced the number of pairings on the left-hand side of the equation from $5m + 3n$ to $3m + 2n$, while the right-hand side remains at 16 pairings. Using the techniques explained in Section 4, i.e., taking each element $M_{i,j}$ of the equation to a random power $r_{i,j}$, multiplying all the components, and regrouping pairings, we get the following equation:

$$\begin{aligned} &\prod_{j=1}^{n} e\left(\left(\prod_{i=1}^{m} c_{i,1}^{\gamma_{i,j}}\right)^{r_{1,1}} \left(\mathcal{A}_j \prod_{i=1}^{m} c_{i,2}^{\gamma_{i,j}}\right)^{r_{2,1}}, d_{j,1}\right) \prod_{j=1}^{n} e\left(\left(\prod_{i=1}^{m} c_{i,1}^{\gamma_{i,j}}\right)^{r_{1,2}} \left(\mathcal{A}_j \prod_{i=1}^{m} c_{i,2}^{\gamma_{i,j}}\right)^{r_{2,2}}, d_{j,2}\right) \cdot \prod_{i=1}^{m} e\left(c_{i,1}^{r_{1,2}} c_{i,2}^{r_{2,2}}, \mathcal{B}_i\right) \\ &= e(u_{1,1}^{r_{1,1}} u_{1,2}^{r_{2,1}}, \pi_{1,1}) e(u_{2,1}^{r_{1,1}} u_{2,2}^{r_{2,1}}, \pi_{2,1}) e(\theta_{1,1}^{r_{1,1}} \theta_{1,2}^{r_{2,1}}, v_{1,1}) e(\theta_{2,1}^{r_{1,1}} \theta_{2,2}^{r_{2,1}}, v_{2,1}) \\ &\qquad \cdot e(u_{1,1}^{r_{1,2}} u_{1,2}^{r_{2,2}}, \pi_{1,2}) e(u_{2,1}^{r_{1,2}} u_{2,2}^{r_{2,2}}, \pi_{2,2}) e(\theta_{1,1}^{r_{1,2}} \theta_{1,2}^{r_{2,2}}, v_{1,2}) e(\theta_{2,1}^{r_{1,2}} \theta_{2,2}^{r_{2,2}}, v_{2,2}) \cdot t_T^{r_{2,2}} \end{aligned} \tag{4}$$

which requires $m + 2n$ pairings and $2mn + 2m + 4n$ exponentiations in $\mathbb{G}_1$ for the left part and 8 pairing computations and 16 exponentiations in $\mathbb{G}_1$ and one exponentiation in $\mathbb{G}_T$ for the right side of the equation. The alternative expression

$$\prod_{j=1}^{n} e\left(\mathcal{A}_j, d_{j,1}^{r_{2,1}} d_{j,2}^{r_{2,2}}\right) \cdot \prod_{i=1}^{m} e\left(c_{i,1}, \left(\prod_{j=1}^{n} d_{j,1}^{\gamma_{i,j}}\right)^{r_{1,1}} \left(\mathcal{B}_i \prod_{j=1}^{n} d_{j,2}^{\gamma_{i,j}}\right)^{r_{1,2}}\right) \cdot \prod_{i=1}^{m} e\left(c_{i,2}, \left(\prod_{j=1}^{n} d_{j,1}^{\gamma_{i,j}}\right)^{r_{2,1}} \left(\mathcal{B}_i \prod_{j=1}^{n} d_{j,2}^{\gamma_{i,j}}\right)^{r_{2,2}}\right)$$

for the left side of the equation requires $2m + n$ pairings and $2mn + 4m + 2n$ exponentiations in $\mathbb{G}_2$.

5.2 Multi-Scalar Multiplication Equation in $\mathbb{G}_1$

Here, we consider equations of Type (2) in $\mathbb{G}_1$ (the case of equations in $\mathbb{G}_2$, which work analogously, is treated in Appendix B). The verification of a proof $(\vec{\mathbf{c}}, \vec{\mathbf{d}}', \vec{\pi}, \theta) \in \mathbb{G}_1^{m\times 2} \times \mathbb{G}_2^{n\times 2} \times \mathbb{G}_2^{2\times 2} \times \mathbb{G}_1^{1\times 2}$ consists in checking the following:

$$\left[\iota_1(\vec{\mathcal{A}}) \bullet \vec{\mathbf{d}}'\right] \odot \left[\vec{\mathbf{c}} \bullet \iota_2'(\vec{b})\right] \odot \left[\vec{\mathbf{c}} \bullet \Gamma \vec{\mathbf{d}}'\right] = \hat{\iota_T}(\mathcal{T}_1) \odot \left[\vec{\mathbf{u}} \bullet \vec{\pi}\right] \odot F(\theta, \mathbf{v}_1)$$

Let us denote $\vec{\mathcal{A}} = (\mathcal{A}_j)_{1\leq j\leq n} \in \mathbb{G}_1^{n\times 1}$, $\vec{\mathbf{d}}' = (d'_{j,k})_{\substack{1\leq j\leq n\\ 1\leq k\leq 2}} \in \mathbb{G}_2^{n\times 2}$, $\vec{\mathbf{c}} = (c_{i,k})_{\substack{1\leq i\leq m\\ 1\leq k\leq 2}} \in \mathbb{G}_1^{m\times 2}$, $\vec{b} = (b_i)_{1\leq i\leq m} \in \mathbb{Z}_p^{m\times 1}$ and $\Gamma = (\gamma_{i,j})_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}$. The left hand-side is equal to

$$\begin{pmatrix} \prod_{i=1}^{m} e(c_{i,1}, v_{2,1}^{b_i} \prod_{j=1}^{n} {d'_{j,1}}^{\gamma_{i,j}}) & \prod_{i=1}^{m} e(c_{i,1}, (v_{2,2}g_2)^{b_i} \prod_{j=1}^{n} {d'_{j,2}}^{\gamma_{i,j}}) \\ \prod_{i=1}^{m} e(c_{i,2}, v_{2,1}^{b_i} \prod_{j=1}^{n} {d'_{j,1}}^{\gamma_{i,j}}) \prod_{j=1}^{n} e(\mathcal{A}_j, d'_{j,1}) & \prod_{i=1}^{m} e(c_{i,2}, (v_{2,2}g_2)^{b_i} \prod_{j=1}^{n} {d'_{j,2}}^{\gamma_{i,j}}) \prod_{j=1}^{n} e(\mathcal{A}_j, d'_{j,2}) \end{pmatrix}$$

while the right-hand side is equal to

$$\begin{pmatrix} e(\theta_1, v_{1,1})e(u_{1,1}, \pi_{1,1})e(u_{2,1}, \pi_{2,1}) & e(\theta_1, v_{1,2})e(u_{1,1}, \pi_{1,2})e(u_{2,1}, \pi_{2,2}) \\ e(\theta_2, v_{1,1})e(u_{1,2}, \pi_{1,1})e(u_{2,2}, \pi_{2,1})e(\mathcal{T}_1, v_{2,1}) & e(\theta_2, v_{1,2})e(u_{1,2}, \pi_{1,2})e(u_{2,2}, \pi_{2,2})e(\mathcal{T}_1, g_2 v_{2,2}) \end{pmatrix}$$

By grouping the pairings, the number of pairings on the left-hand side member of the equation has already been reduced from $8m + 2n$ to $4m + 2n$. Now, by using the batch technique, i.e., multiplying each member by a random value and multiplying all the members, we obtain on the left-hand side

$$\begin{aligned} \prod_{j=1}^{n} e\left(\left(\textstyle\prod_{i=1}^{m} c_{i,1}^{\gamma_{i,j}}\right)^{r_{1,1}} \left(\mathcal{A}_j \prod_{i=1}^{m} c_{i,2}^{\gamma_{i,j}}\right)^{r_{2,1}}, d'_{j,1}\right) \cdot \prod_{j=1}^{n} e\left(\left(\textstyle\prod_{i=1}^{m} c_{i,1}^{\gamma_{i,j}}\right)^{r_{1,2}} \left(\mathcal{A}_j \prod_{i=1}^{m} c_{i,2}^{\gamma_{i,j}}\right)^{r_{2,2}}, d'_{j,2}\right) \\ \cdot e\left((\textstyle\prod_{i=1}^{m} c_{i,1}^{b_i})^{r_{1,1}} (\prod_{i=1}^{m} c_{i,2}^{b_i})^{r_{2,1}}, v_{2,1}\right) \cdot e\left((\textstyle\prod_{i=1}^{m} c_{i,1}^{b_i})^{r_{1,2}} (\prod_{i=1}^{m} c_{i,2}^{b_i})^{r_{2,2}}, v_{2,2}g_2\right) \end{aligned}$$

which requires $2mn+2m+4n+4$ exponentiations in $\mathbb{G}_1$ and $2n+2$ pairing computations. The alternative expression

$$\begin{aligned} \prod_{j=1}^{n} e\left(\mathcal{A}_j, {d'_{j,1}}^{r_{2,1}} {d'_{j,2}}^{r_{2,2}}\right) \cdot \prod_{i=1}^{m} e\left(c_{i,1}, \left(v_{2,1}^{b_i} \textstyle\prod_{j=1}^{n} {d'_{j,1}}^{\gamma_{i,j}}\right)^{r_{1,1}} \left((v_{2,2}g_2)^{b_i} \prod_{j=1}^{n} {d'_{j,2}}^{\gamma_{i,j}}\right)^{r_{1,2}}\right) \\ \cdot \prod_{i=1}^{m} e\left(c_{i,2}, \left(v_{2,1}^{b_i} \textstyle\prod_{j=1}^{n} {d'_{j,1}}^{\gamma_{i,j}}\right)^{r_{2,1}} \left((v_{2,2}g_2)^{b_i} \prod_{j=1}^{n} {d'_{j,2}}^{\gamma_{i,j}}\right)^{r_{2,2}}\right) \end{aligned}$$

for the left side of the equation requires $2mn + 6m + 2n$ exponentiations in $\mathbb{G}_2$ and $2m + n$ pairing computations.

On the right-hand side, the same technique achieves a reduction from 14 to 7 pairings:

$$e(\theta_1^{r_{1,1}}\theta_2^{r_{2,1}}, v_{1,1})e(\theta_1^{r_{2,1}}\theta_2^{r_{2,2}}, v_{1,2})e(u_{1,1}^{r_{1,1}}u_{1,2}^{r_{2,1}}, \pi_{1,1})e(u_{2,1}^{r_{1,1}}u_{2,2}^{r_{2,1}}, \pi_{2,1})e(u_{2,1}^{r_{1,2}}u_{2,2}^{r_{2,2}}, \pi_{1,2})e(u_{2,1}^{r_{2,1}}u_{2,2}^{r_{2,2}}, \pi_{2,2})e(\mathcal{T}_1, v_{2,1}^{r_{2,1}}(g_2v_{2,2})^{r_{2,2}})$$

5.3 Quadratic Equation

The verification of $(\vec{\mathbf{c}}', \vec{\mathbf{d}}', \pi, \theta) \in \mathbb{G}_1^{m\times 2} \times \mathbb{G}_2^{n\times 2} \times \mathbb{G}_2^{1\times 2} \times \mathbb{G}_1^{1\times 2}$ for an equation of Type (3) consists in checking

$$\left[\iota_1'(\vec{a}) \bullet \vec{\mathbf{d}}'\right] \odot \left[\vec{\mathbf{c}'} \bullet \iota_2'(\vec{b})\right] \odot \left[\vec{\mathbf{c}'} \bullet \Gamma \vec{\mathbf{d}}'\right] = \iota_T'(t) \odot F(\mathbf{u}_1, \pi) \odot F(\theta, \mathbf{v}_1) \quad .$$

Let $\vec{a} = (a_j)_{1\leq j\leq n} \in \mathbb{Z}_p^{n\times 1}$, $\vec{b} = (b_i)_{1\leq i\leq m} \in \mathbb{Z}_p^{m\times 1}$, $\vec{\mathbf{c}'} = (c'_{i,k})_{\substack{1\leq i\leq m\\ 1\leq k\leq 2}} \in \mathbb{G}_1^{n\times 2}$, $\vec{\mathbf{d}}' = (d'_{j,k})_{\substack{1\leq j\leq n\\ 1\leq k\leq 2}} \in \mathbb{G}_2^{n\times 2}$, and $\Gamma = (\gamma_{i,j})_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} \in \mathbb{Z}_p^{m\times n}$. The left hand side is equal to

$$\begin{pmatrix} \prod_{j=1}^{n} e(u_{2,1}^{a_j}, d'_{j,1}) \prod_{i=1}^{m} e(c'_{i,1}, v_{2,1})^{b_i} & \prod_{j=1}^{n} e(u_{2,1}^{a_j}, d'_{j,2}) \prod_{i=1}^{m} e(c'_{i,1}, v_{2,2}g_2)^{b_i} \\ \prod_{j=1}^{n} e((u_{2,2}g_1)^{a_j}, d'_{j,1}) \prod_{i=1}^{m} e(c'_{i,2}, v_{2,1})^{b_i} & \prod_{j=1}^{n} e((u_{2,2}g_1)^{a_j}, d'_{j,2}) \prod_{i=1}^{m} e(c'_{i,2}, v_{2,2}g_2)^{b_i} \end{pmatrix}$$

$$\odot \begin{pmatrix} \prod_{j=1}^{n} e\left(\prod_{i=1}^{m} {c'_{i,1}}^{\gamma_{i,j}}, d'_{j,1}\right) & \prod_{j=1}^{n} e\left(\prod_{i=1}^{m} {c'_{i,1}}^{\gamma_{i,j}}, d'_{j,2}\right) \\ \prod_{j=1}^{n} e\left(\prod_{i=1}^{m} {c'_{i,2}}^{\gamma_{i,j}}, d'_{j,1}\right) & \prod_{j=1}^{n} e\left(\prod_{i=1}^{m} {c'_{i,2}}^{\gamma_{i,j}}, d'_{j,2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \prod_{i=1}^{m} e(c'_{i,1}, v_{2,1}^{b_i}) \prod_{j=1}^{n} e\left(u_{2,1}^{a_j} \prod_{i=1}^{m} {c'_{i,1}}^{\gamma_{i,j}}, d'_{j,1}\right) & \prod_{i=1}^{m} e(c'_{i,1}, (v_{2,2}g_2)^{b_i}) \prod_{j=1}^{n} e\left(u_{2,1}^{a_j} \prod_{i=1}^{m} {c'_{i,1}}^{\gamma_{i,j}}, d'_{j,2}\right) \\ \prod_{i=1}^{m} e(c'_{i,2}, v_{2,1}^{b_i}) \prod_{j=1}^{n} e\left((u_{2,2}g_1)^{a_j} \prod_{i=1}^{m} {c'_{i,2}}^{\gamma_{i,j}}, d'_{j,1}\right) & \prod_{i=1}^{m} e(c'_{i,2}, (v_{2,2}g_2)^{b_i}) \prod_{j=1}^{n} e\left((u_{2,2}g_1)^{a_j} \prod_{i=1}^{m} {c'_{i,2}}^{\gamma_{i,j}}, d'_{j,2}\right) \end{pmatrix}$$

Denoting $\pi = (\pi_1, \pi_2)$ and $\theta = (\theta_1, \theta_2)$, for the right-hand side we have

$$\iota'_T(t) \odot F(\mathbf{u}_1, \pi) \odot F(\theta, \mathbf{v}_1) = \begin{pmatrix} e(u_{1,1}, \pi_1)e(\theta_1, v_{1,1})e(u_{2,1}, v_{2,1})^t & e(u_{1,1}, \pi_2)e(\theta_1, v_{1,2})e(u_{2,1}, v_{2,2}g_2)^t \\ e(u_{1,2}, \pi_1)e(\theta_2, v_{1,1})e(u_{2,2}g_1, v_{2,1})^t & e(u_{1,2}, \pi_2)e(\theta_2, v_{1,2})e(u_{2,2}g_1, v_{2,2}g_2)^t \end{pmatrix}$$

and therefore

$$\begin{pmatrix} \prod_{i=1}^{m} e(c'_{i,1}, v_{2,1}^{b_i}) \prod_{j=1}^{n} e\left(u_{2,1}^{a_j} \prod_{i=1}^{m} {c'_{i,1}}^{\gamma_{i,j}}, d'_{j,1}\right) & \prod_{i=1}^{m} e(c'_{i,1}, (v_{2,2}g_2)^{b_i}) \prod_{j=1}^{n} e\left(u_{2,1}^{a_j} \prod_{i=1}^{m} {c'_{i,1}}^{\gamma_{i,j}}, d'_{j,2}\right) \\ \prod_{i=1}^{m} e(c'_{i,2}, v_{2,1}^{b_i}) \prod_{j=1}^{n} e\left((u_{2,2}g_1)^{a_j} \prod_{i=1}^{m} {c'_{i,2}}^{\gamma_{i,j}}, d'_{j,1}\right) & \prod_{i=1}^{m} e(c'_{i,2}, (v_{2,2}g_2)^{b_i}) \prod_{j=1}^{n} e\left((u_{2,2}g_1)^{a_j} \prod_{i=1}^{m} {c'_{i,2}}^{\gamma_{i,j}}, d'_{j,2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} e(u_{1,1}, \pi_1)e(\theta_1, v_{1,1})e(u_{2,1}, v_{2,1})^t & e(u_{1,1}, \pi_2)e(\theta_1, v_{1,2})e(u_{2,1}, v_{2,2}g_2)^t \\ e(u_{1,2}, \pi_1)e(\theta_2, v_{1,1})e(u_{2,2}g_1, v_{2,1})^t & e(u_{1,2}, \pi_2)e(\theta_2, v_{1,2})e(u_{2,2}g_1, v_{2,2}g_2)^t \end{pmatrix}$$

Optimizing the verification. By grouping the pairings, the number of pairings on the left-hand side member of the equation has been reduced from $8m + 8n$ to $4m + 4n$. By using the batch technique, i.e., multiplying each member by a random value and multiplying all the members, we obtain on the left-hand side:

$$e\left(\left(\prod_{i=1}^{m} {c'_{i,1}}^{b_i}\right)^{r_{1,1}} \left(\prod_{i=1}^{m} {c'_{i,2}}^{b_i}\right)^{r_{2,1}}, v_{2,1}\right) \cdot e\left(\left(\prod_{i=1}^{m} {c'_{i,1}}^{b_i}\right)^{r_{1,2}} \left(\prod_{i=1}^{m} {c'_{i,2}}^{b_i}\right)^{r_{2,2}}, v_{2,2}g_2\right)$$
$$\cdot \prod_{j=1}^{n} e\left(\left(u_{2,1}^{a_j} \prod_{i=1}^{m} {c'_{i,1}}^{\gamma_{i,j}}\right)^{r_{1,1}} \left((u_{2,2}g_1)^{a_j} \prod_{i=1}^{m} {c'_{i,2}}^{\gamma_{i,j}}\right)^{r_{2,1}}, d'_{j,1}\right)$$
$$\cdot \prod_{j=1}^{n} e\left(\left(u_{2,1}^{a_j} \prod_{i=1}^{m} {c'_{i,1}}^{\gamma_{i,j}}\right)^{r_{1,2}} \left((u_{2,2}g_1)^{a_j} \prod_{i=1}^{m} {c'_{i,2}}^{\gamma_{i,j}}\right)^{r_{2,2}}, d'_{j,2}\right)$$

which requires $2mn+2m+6n+4$ exponentiations in $\mathbb{G}_1$ and $2n+2$ pairing computations. Alternatively, the left-hand side is also equal to

$$e\left(u_{2,1}, \left(\prod_{j=1}^{n} {d'_{j,1}}^{a_j}\right)^{r_{1,1}} \left(\prod_{j=1}^{n} {d'_{j,2}}^{a_j}\right)^{r_{1,2}}\right) \cdot e\left(u_{2,2}g_2, \left(\prod_{j=1}^{n} {d'_{j,1}}^{a_j}\right)^{r_{2,1}} \left(\prod_{j=1}^{n} {d'_{j,2}}^{a_j}\right)^{r_{2,2}}\right)$$
$$\cdot \prod_{i=1}^{m} e\left(c'_{i,1}, \left(v_{2,1}^{b_i} \prod_{j=1}^{n} {d'_{j,1}}^{\gamma_{i,j}}\right)^{r_{1,1}} \left((v_{2,2}g_2)^{b_i} \prod_{j=1}^{n} {d'_{j,2}}^{\gamma_{i,j}}\right)^{r_{1,2}}\right)$$
$$\cdot \prod_{i=1}^{m} e\left(c'_{i,2}, \left(v_{2,1}^{b_i} \prod_{j=1}^{n} {d'_{j,1}}^{\gamma_{i,j}}\right)^{r_{2,1}} \left((v_{2,2}g_2)^{b_i} \prod_{j=1}^{n} {d'_{j,2}}^{\gamma_{i,j}}\right)^{r_{2,2}}\right)$$

which requires $2mn + 6m + 2n + 4$ exponentiations in $\mathbb{G}_2$ and $2m + 2$ pairing computations. On the right-hand side, the same technique achieves a reduction from 12 to 6 pairings:

$$e(u_{1,1}^{r_{1,1}} u_{1,2}^{r_{2,1}}, \pi_1)e(u_{1,1}^{r_{1,2}} u_{1,2}^{r_{2,2}}, \pi_2)e(\theta_1^{r_{1,1}} \theta_2^{r_{2,1}}, v_{1,1})e(\theta_1^{r_{1,2}} \theta_2^{r_{2,2}}, v_{1,2})$$
$$e(u_{2,1}^{r_{1,1}t}(u_{2,2}g_1)^{r_{2,1}t}, v_{2,1})e(u_{2,1}^{r_{1,2}t}(u_{2,2}g_1)^{r_{2,2}t}, v_{2,2}g_2)$$

6 Instantiation 3: DLIN

6.1 Pairing-Product Equation

The verification relation of a proof $(\vec{\mathbf{d}}, \phi) \in \mathbb{G}^{n\times 3} \times \mathbb{G}^{3\times 3}$ for equation Type $(1')$ is the following:

$$\left[\iota(\vec{\mathcal{A}}) \overset{s}{\bullet} \vec{\mathbf{d}}\right] \odot \left[\vec{\mathbf{d}} \overset{s}{\bullet} \Gamma\vec{\mathbf{d}}\right] = \iota_T(t_T) \odot \left[\vec{\mathbf{u}} \overset{s}{\bullet} \vec{\phi}\right]$$

For simplicity, we consider the squares of all $\mathbb{G}_T$ elements in both sides of the equation. Writing $\Gamma\vec{\mathbf{d}}$ as $\left(\prod_{k=1}^{n} d_{k,j}^{\gamma_{i,k}}\right)_{\substack{1\le i\le n\\ 1\le j\le 3}}$ and replacing the bilinear product $\overset{s}{\bullet}$ by its definition, we get for the left-hand side

$$\begin{pmatrix} \prod_{i=1}^{n} e(d_{i,1}, \prod d_{k,1}^{\gamma_{i,k}})^2 & \prod_{i=1}^{n} e(d_{i,1}, \prod d_{k,2}^{\gamma_{i,k}}) e(d_{i,2}, \prod d_{k,1}^{\gamma_{i,k}}) & \prod_{i=1}^{n} e(\mathcal{A}_i, d_{i,1}) e(d_{i,1}, \prod d_{k,3}^{\gamma_{i,k}}) e(d_{i,3}, \prod d_{k,1}^{\gamma_{i,k}}) \\ \prod_{i=1}^{n} e(d_{i,2}, \prod d_{k,1}^{\gamma_{i,k}}) e(d_{i,1}, \prod d_{k,2}^{\gamma_{i,k}}) & \prod_{i=1}^{n} e(d_{i,2}, \prod d_{k,2}^{\gamma_{i,k}})^2 & \prod_{i=1}^{n} e(\mathcal{A}_i, d_{i,2}) e(d_{i,2}, \prod d_{k,3}^{\gamma_{i,k}}) e(d_{i,3}, \prod d_{k,2}^{\gamma_{i,k}}) \\ \prod_{i=1}^{n} e(\mathcal{A}_i, d_{i,1}) e(d_{i,3}, \prod d_{k,1}^{\gamma_{i,k}}) \cdot e(d_{i,1}, \prod d_{k,3}^{\gamma_{i,k}}) & \prod_{i=1}^{n} e(\mathcal{A}_i, d_{i,2}) e(d_{i,3}, \prod d_{k,2}^{\gamma_{i,k}}) \cdot e(d_{i,2}, \prod d_{k,3}^{\gamma_{i,k}}) & \prod_{i=1}^{n} e(\mathcal{A}_i, d_{i,3})^2 e(d_{i,3}, \prod d_{k,3}^{\gamma_{i,k}})^2 \end{pmatrix}$$

and for the right-hand side:

$$\begin{pmatrix} \prod_{i=1}^{3} e(u_{i1}, \phi_{i1})^2 & \prod_{i=1}^{3} e(u_{i1}, \phi_{i2}) e(u_{i2}, \phi_{i1}) & \prod_{i=1}^{3} e(u_{i1}, \phi_{i3}) e(u_{i3}, \phi_{i1}) \\ \prod_{i=1}^{3} e(u_{i2}, \phi_{i1}) e(u_{i1}, \phi_{i2}) & \prod_{i=1}^{3} e(u_{i2}, \phi_{i2})^2 & \prod_{i=1}^{3} e(u_{i2}, \phi_{i3}) e(u_{i3}, \phi_{i2}) \\ \prod_{i=1}^{3} e(u_{i3}, \phi_{i1}) e(u_{i1}, \phi_{i3}) & \prod_{i=1}^{3} e(u_{i3}, \phi_{i2}) e(u_{i2}, \phi_{i3}) & t_T^2 \prod_{i=1}^{3} e(u_{i3}, \phi_{i3})^2 \end{pmatrix}$$

Taking each element $M_{i,j}$ of the equation to the power of $r_{i,j}$, multiplying everything, and regrouping pairings, we get the following for the left-hand side:

$$\begin{aligned} \prod_{i=1}^{n} e\big(d_{i,1},\ \mathcal{A}_i^{r_{1,3}+r_{3,1}} \prod d_{k,1}^{\gamma_{i,k} 2\cdot r_{1,1}} d_{k,2}^{\gamma_{i,k}(r_{1,2}+r_{2,1})} d_{k,3}^{\gamma_{i,k}(r_{1,3}+r_{3,1})}\big) \cdot \\ e\big(d_{i,2},\ \mathcal{A}_i^{r_{2,3}+r_{3,2}} \prod d_{k,1}^{\gamma_{i,k}(r_{1,2}+r_{2,1})} d_{k,2}^{\gamma_{i,k} 2\cdot r_{2,2}} d_{k,3}^{\gamma_{i,k}(r_{2,3}+r_{3,2})}\big) \cdot \\ e\big(d_{i,3},\ \mathcal{A}_i^{2\cdot r_{3,3}} \prod d_{k,1}^{\gamma_{i,k}(r_{1,3}+r_{3,1})} d_{k,2}^{\gamma_{i,k}(r_{2,3}+r_{3,2})} d_{k,3}^{\gamma_{i,k} 2 r_{3,3}}\big) \end{aligned} \quad (5)$$

and for the right-hand side:

$$\begin{aligned} \prod_{i=1}^{3} e\big(u_{i,1},\ \phi_{i,1}^{2\cdot r_{1,1}} \phi_{i,2}^{r_{1,2}+r_{2,1}} \phi_{i,3}^{r_{1,3}+r_{3,1}}\big) \cdot e\big(u_{i,2},\ \phi_{i,1}^{r_{1,2}+r_{2,1}} \phi_{i,2}^{2\cdot r_{2,2}} \phi_{i,3}^{r_{2,3}+r_{3,2}}\big) \\ \cdot e\big(u_{i,3},\ \phi_{i,1}^{r_{1,3}+r_{3,1}} \phi_{i,2}^{r_{2,3}+r_{3,2}} \phi_{i,3}^{2\cdot r_{3,3}}\big) \cdot t_T^{2 r_{3,3}} \end{aligned}$$

Due to the fact that $u_{1,2} = u_{2,1} = 1$, and $u_{1,3} = u_{2,3}$ this simplifies to:

$$\begin{aligned} & e\big(u_{1,1},\ \phi_{1,1}^{2\cdot r_{1,1}} \phi_{1,2}^{r_{1,2}+r_{2,1}} \phi_{1,3}^{r_{1,3}+r_{3,1}}\big) \cdot e\big(u_{1,3},\ (\phi_{1,1}\phi_{2,1})^{r_{1,3}+r_{3,1}} (\phi_{1,2}\phi_{2,2})^{r_{2,3}+r_{3,2}} (\phi_{1,3}\phi_{2,3})^{2\cdot r_{3,3}}\big) \\ & e\big(u_{2,2},\ \phi_{2,1}^{r_{1,2}+r_{2,1}} \phi_{2,2}^{2\cdot r_{2,2}} \phi_{2,3}^{r_{2,3}+r_{3,2}}\big) \cdot e\big(u_{3,1},\ \phi_{3,1}^{2\cdot r_{1,1}} \phi_{3,2}^{r_{1,2}+r_{2,1}} \phi_{3,3}^{r_{1,3}+r_{3,1}}\big) \cdot e\big(u_{3,2},\ \phi_{3,1}^{r_{1,2}+r_{2,1}} \phi_{3,2}^{2\cdot r_{2,2}} \phi_{3,3}^{r_{2,3}+r_{3,2}}\big) \cdot \\ & \qquad e\big(u_{3,3},\ \phi_{3,1}^{r_{1,3}+r_{3,1}} \phi_{3,2}^{r_{2,3}+r_{3,2}} \phi_{3,3}^{2\cdot r_{3,3}}\big) \cdot t_T^{2 r_{3,3}} \end{aligned}$$

In total we reduced the number of pairings from $12n + 27$ to $3n + 6$ pairings at the expense of adding $9n^2 + 3n$ exponentiations in $\mathbb{G}$ and one exponentiation in $\mathbb{G}_T$.

6.2 Multi-Scalar Multiplication Equation

The verification equation of a proof $(\vec{\mathbf{c}}, \vec{\mathbf{d}}, \phi)$, with $\phi \in \mathbb{G}^{3\times 3}$, of an equation of Type $(2')$ is the following:

$$\left[\iota'(\vec{a}) \overset{s}{\bullet} \vec{\mathbf{d}}\right] \odot \left[\vec{\mathbf{c}} \overset{s}{\bullet} \iota(\vec{\mathcal{B}})\right] \odot \left[\vec{\mathbf{c}} \overset{s}{\bullet} \Gamma\vec{\mathbf{d}}\right] = \hat{\iota}_T(\mathcal{T}) \odot \left[\vec{\mathbf{u}} \overset{s}{\bullet} \vec{\phi}\right]$$

$\Gamma\vec{\mathbf{d}}$ can be written as $\left(\prod_{k=1}^{n} d_{k,j}^{\gamma_{i,k}}\right)_{\substack{1\le i\le m\\ 1\le j\le 3}}$. The left-hand side of the equation is thus

$$\begin{aligned}\Big[\big(u_{3,1}^{a_i}, u_{3,2}^{a_i}, (u_{3,3}g)^{a_i}\big)_{1\le i\le n} \overset{s}{\bullet} \big(d_{i,1}, d_{i,2}, d_{i,3}g\big)_{1\le i\le n}\Big] \odot \Big[\big(c_{i,1}, c_{i,2}, c_{i,3}\big)_{1\le i\le m} \overset{s}{\bullet} \big(1, 1, \mathcal{B}_i\big)_{1\le i\le m}\Big]\\ \odot \Big[\big(c_{i,1}, c_{i,2}, c_{i,3}\big)_{1\le i\le m} \overset{s}{\bullet} \big((\Gamma\vec{\mathbf{d}})_{i,1}, (\Gamma\vec{\mathbf{d}})_{i,2}, (\Gamma\vec{\mathbf{d}})_{i,3}\big)_{1\le i\le m}\Big]\end{aligned}$$

Considering the squares of all matrix elements, this is written out as

$$\begin{pmatrix} \prod_{i=1}^{n} e\big(u_{3,1}^{a_i}, d_{i,1}\big)^2 & \prod_{i=1}^{n} e\big(u_{3,1}^{a_i}, d_{i,2}\big)e\big(u_{3,2}^{a_i}, d_{i,1}\big) & \prod_{i=1}^{n} e\big(u_{3,1}^{a_i}, d_{i,3}\big)e\big((u_{3,3}g)^{a_i}, d_{i,1}g\big)\\ \prod_{i=1}^{n} e\big(u_{3,2}^{a_i}, d_{i,1}\big)e\big(u_{3,1}^{a_i}, d_{i,2}\big) & \prod_{i=1}^{n} e\big(u_{3,2}^{a_i}, d_{i,2}\big)^2 & \prod_{i=1}^{n} e\big(u_{3,2}^{a_i}, d_{i,3}\big)e\big((u_{3,3}g)^{a_i}, d_{i,2}g\big)\\ \prod_{i=1}^{n} e\big((u_{3,3}g)^{a_i}, d_{i,1}\big)e\big(u_{3,1}^{a_i}, d_{i,3}\big) & \prod_{i=1}^{n} e\big((u_{3,3}g)^{a_i}, d_{i,2}\big)e\big(u_{3,2}^{a_i}, d_{i,3}\big) & \prod_{i=1}^{n} e\big((u_{3,3}g)^{a_i}, d_{i,3}g\big)^2 \end{pmatrix}$$

$$\odot \begin{pmatrix} 1 & 1 & \prod_{i=1}^{m} e(c_{i,1}, \mathcal{B}_i)\\ 1 & 1 & \prod_{i=1}^{m} e(c_{i,2}, \mathcal{B}_i)\\ \prod_{i=1}^{m} e(c_{i,1}, \mathcal{B}_i) & \prod_{i=1}^{m} e(c_{i,2}, \mathcal{B}_i) & \prod_{i=1}^{m} e(c_{i,3}, \mathcal{B}_i) \end{pmatrix}$$

$$\odot \begin{pmatrix} \prod_{i=1}^{m} e(c_{i,1}, \prod d_{k,1}^{\gamma_{i,k}})^2 & \prod_{i=1}^{m} e(c_{i,1}, \prod d_{k,2}^{\gamma_{i,k}})e(c_{i,2}, \prod d_{k,1}^{\gamma_{i,k}}) & \prod_{i=1}^{m} e(c_{i,1}, \prod d_{k,3}^{\gamma_{i,k}})e(c_{i,3}, \prod d_{k,1}^{\gamma_{i,k}})\\ \prod_{i=1}^{m} e(c_{i,2}, \prod d_{k,1}^{\gamma_{i,k}})e(c_{i,1}, \prod d_{k,2}^{\gamma_{i,k}}) & \prod_{i=1}^{m} e(c_{i,2}, \prod d_{k,2}^{\gamma_{i,k}})^2 & \prod_{i=1}^{m} e(c_{i,2}, \prod d_{k,3}^{\gamma_{i,k}})e(c_{i,3}, \prod d_{k,2}^{\gamma_{i,k}})\\ \prod_{i=1}^{m} e(c_{i,3}, \prod d_{k,1}^{\gamma_{i,k}})e(c_{i,1}, \prod d_{k,3}^{\gamma_{i,k}}) & \prod_{i=1}^{m} e(c_{i,3}, \prod d_{k,2}^{\gamma_{i,k}})e(c_{i,2}, \prod d_{k,3}^{\gamma_{i,k}}) & \prod_{i=1}^{m} e(c_{i,3}, \prod d_{k,3}^{\gamma_{i,k}})^2 \end{pmatrix}$$

Using the batching technique, we get the following left-hand side:

$$\begin{aligned}\prod_{i=1}^{n} e\big(u_{3,1}^{2a_i r_{1,1}} u_{3,2}^{a_i(r_{1,2}+r_{2,1})}(u_{3,3}g)^{a_i\cdot(r_{1,3}+r_{3,1})},\, d_{i,1}\big) \cdot e\big(u_{3,1}^{a_i(r_{1,2}+r_{2,1})} u_{3,2}^{2a_i r_{2,2}}(u_{3,3}g)^{a_i\cdot(r_{2,3}+r_{3,2})},\, d_{i,2}\big)\\ \cdot e\big(u_{3,1}^{a_i(r_{1,3}+r_{3,1})} u_{3,2}^{a_i(r_{2,3}+r_{3,2})}(u_{3,3}g)^{a_i\cdot 2r_{3,3}},\, d_{i,3}\big)\\ \cdot \prod_{i=1}^{m} e\big(c_{i,1}, \mathcal{B}_i^{r_{1,3}+r_{3,1}}(\prod d_{k,1}^{\gamma_{i,k}})^{2r_{1,1}}(\prod d_{k,2}^{\gamma_{i,k}})^{r_{1,2}+r_{2,1}}(\prod d_{k,3}^{\gamma_{i,k}})^{r_{1,3}+r_{3,1}}\big)\\ \cdot e\big(c_{i,2}, \mathcal{B}_i^{r_{2,3}+r_{3,2}}(\prod d_{k,1}^{\gamma_{i,k}})^{r_{1,2}+r_{2,1}}(\prod d_{k,2}^{\gamma_{i,k}})^{2r_{2,2}}(\prod d_{k,3}^{\gamma_{i,k}})^{r_{2,3}+r_{3,2}}\big)\\ \cdot e\big(c_{i,3}, \mathcal{B}_i^{2r_{3,3}}(\prod d_{k,1}^{\gamma_{i,k}})^{r_{3,1}+r_{1,3}}(\prod d_{k,2}^{\gamma_{i,k}})^{r_{3,2}+r_{2,3}}(\prod d_{k,1}^{\gamma_{i,k}})^{2r_{3,3}}\ ,\end{aligned}$$

The right-hand side is similar to Section 6.1:

$$\begin{aligned}&e(g,T)^{2r_{3,3}} e\big(u_{1,1},\ \phi_{1,1}^{2\cdot r_{1,1}}\phi_{1,2}^{r_{1,2}+r_{2,1}}\phi_{1,3}^{r_{1,3}+r_{3,1}}\big)\cdot e\big(u_{1,3},\ (\phi_{1,1}\phi_{2,1})^{r_{1,3}+r_{3,1}}(\phi_{1,2}\phi_{2,2})^{r_{2,3}+r_{3,2}}(\phi_{1,3}\phi_{2,3})^{2\cdot r_{3,3}}\big)\\ &e\big(u_{2,2},\ \phi_{2,1}^{r_{1,2}+r_{2,1}}\phi_{2,2}^{2\cdot r_{2,2}}\phi_{2,3}^{r_{2,3}+r_{3,2}}\big)\cdot e\big(u_{3,1},\ \phi_{3,1}^{2\cdot r_{1,1}}\phi_{3,2}^{r_{1,2}+r_{2,1}}(\phi_{3,3}T)^{r_{1,3}+r_{3,1}}\big)\cdot e\big(u_{3,2},\ \phi_{3,1}^{r_{1,2}+r_{2,1}}\phi_{3,2}^{2\cdot r_{2,2}}(\phi_{3,3}T)^{r_{2,3}+r_{3,2}}\big)\cdot\\ &\qquad e\big(u_{3,3},\ \phi_{3,1}^{r_{1,3}+r_{3,1}}\phi_{3,2}^{r_{2,3}+r_{3,2}}(\phi_{3,3}T)^{2\cdot r_{3,3}}\big)\end{aligned}$$

In total, we reduced from $9n + 12m + 27$ to $3n + 3m + 6$ pairings.

6.3 Quadratic Equation

The verification of a proof $(\vec{\mathbf{c}}, \phi) \in \mathbb{G}^{n\times 3} \times \mathbb{G}^{2\times 3}$ for an equation of Type $(3')$ consists in checking the following:

$$\left[\vec{\mathbf{c}} \overset{s}{\bullet} (\iota'(\vec{b}))\right] \odot \left[\vec{\mathbf{c}} \overset{s}{\bullet} \Gamma\vec{\mathbf{c}}\right] = \iota'_T(t) \odot \left[\vec{\mathbf{v}} \overset{s}{\bullet} \phi\right]\ ,$$

where the commitment key $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ defines the function ι' as $\iota'(b) = (u_{3,1}^b, u_{3,2}^b, (u_{3,3}g)^b)$ and $\vec{\mathbf{v}}$ as $(\mathbf{u}_1, \mathbf{u}_2)$. We develop the left-hand side of the equation:

$$\left[\left(c_{i,1}, c_{i,2}, c_{i,3}\right)_{1\leq i\leq n} \overset{s}{\bullet} \left(u_{3,1}^{b_i}, u_{3,2}^{b_i}, \left(u_{3,3}g\right)^{b_i}\right)_{1\leq i\leq n}\right] \odot \left[\left(c_{i,1}, c_{i,2}, c_{i,3}\right)_{1\leq i\leq n} \overset{s}{\bullet} \left((\Gamma\vec{c})_{i,1}, (\Gamma\vec{c})_{i,2}, (\Gamma\vec{c})_{i,3}\right)_{1\leq i\leq n}\right]$$

which is written out as

$$\begin{pmatrix} \prod_{i=1}^n e(c_{i,1}, u_{3,1}^{b_i})^2 & \prod_{i=1}^n e(c_{i,1}, u_{3,2}^{b_i}) e(c_{i,2}, u_{3,1}^{b_i}) & \prod_{i=1}^n e(c_{i,1}, (u_{3,3}g)^{b_i}) e(c_{i,3}, u_{3,1}^{b_i}) \\ \prod_{i=1}^n e(c_{i,2}, u_{3,1}^{b_i}) e(c_{i,1}, u_{3,2}^{b_i}) & \prod_{i=1}^n e(c_{i,2}, u_{3,2}^{b_i})^2 & \prod_{i=1}^n e(c_{i,2}, (u_{3,3}g)^{b_i}) e(c_{i,3}, u_{3,2}^{b_i}) \\ \prod_{i=1}^n e(c_{i,3}, u_{3,1}^{b_i}) e(c_{i,1}, (u_{3,3}g)^{b_i}) & \prod_{i=1}^n e(c_{i,3}, u_{3,2}^{b_i}) e(c_{i,2}, (u_{3,3}g)^{b_i}) & \prod_{i=1}^n e(c_{i,3}, (u_{3,3}g)^{b_i})^2 \end{pmatrix}^{\frac{1}{2}}$$

$$\odot \begin{pmatrix} \prod_{i=1}^n e(c_{i,1}, \prod c_{k,1}^{\gamma_{i,k}})^2 & \prod_{i=1}^n e(c_{i,1}, \prod c_{k,2}^{\gamma_{i,k}}) e(c_{i,2}, \prod c_{k,1}^{\gamma_{i,k}}) & \prod_{i=1}^n e(c_{i,1}, \prod c_{k,3}^{\gamma_{i,k}}) e(c_{i,3}, \prod c_{k,1}^{\gamma_{i,k}}) \\ \prod_{i=1}^n e(c_{i,2}, \prod c_{k,1}^{\gamma_{i,k}}) e(c_{i,1}, \prod c_{k,2}^{\gamma_{i,k}}) & \prod_{i=1}^n e(c_{i,2}, \prod c_{k,2}^{\gamma_{i,k}})^2 & \prod_{i=1}^n e(c_{i,2}, \prod c_{k,3}^{\gamma_{i,k}}) e(c_{i,3}, \prod c_{k,2}^{\gamma_{i,k}}) \\ \prod_{i=1}^n e(c_{i,3}, \prod c_{k,1}^{\gamma_{i,k}}) e(c_{i,1}, \prod c_{k,3}^{\gamma_{i,k}}) & \prod_{i=1}^n e(c_{i,3}, \prod c_{k,2}^{\gamma_{i,k}}) e(c_{i,2}, \prod c_{k,3}^{\gamma_{i,k}}) & \prod_{i=1}^n e(c_{i,3}, \prod c_{k,3}^{\gamma_{i,k}})^2 \end{pmatrix}^{\frac{1}{2}}$$

Multiplying all matrix elements after taking them to a random power (and squaring everything for simplicity), we obtain the following batched version:

$$\prod_{i=1}^n e\Big(c_{i,1},\ u_{3,1}^{b_i\cdot 2r_{1,1}} u_{3,2}^{b_i\cdot(r_{1,2}+r_{2,1})} (u_{3,3}g)^{b_i\cdot(r_{1,3}+r_{3,1})} \prod \left(c_{k,1}^{\gamma_{i,k}\cdot 2r_{1,1}} c_{k,2}^{\gamma_{i,k}\cdot(r_{1,2}+r_{2,1})} c_{k,3}^{\gamma_{i,k}\cdot(r_{1,3}+r_{3,1})}\right)\Big)$$
$$e\Big(c_{i,2},\ u_{3,1}^{b_i\cdot(r_{1,2}+r_{2,1})} u_{3,2}^{b_i\cdot 2r_{2,2}\cdot} (u_{3,3}g)^{b_i\cdot(r_{2,3}+r_{3,2})} \prod \left(c_{k,1}^{\gamma_{i,k}\cdot(r_{1,2}+r_{2,1})} c_{k,2}^{\gamma_{i,k}\cdot 2r_{2,2}} c_{k,3}^{\gamma_{i,k}\cdot(r_{2,3}+r_{3,2})}\right)\Big)$$
$$e\Big(c_{i,3},\ u_{3,1}^{b_i\cdot(r_{1,3}+r_{3,1})} u_{3,2}^{b_i\cdot(r_{2,3}+r_{3,2})} (u_{3,3}g)^{b_i\cdot 2r_{3,3}} \prod \left(c_{k,1}^{\gamma_{i,k}\cdot(r_{1,3}+r_{3,1})} c_{k,2}^{\gamma_{i,k}\cdot(r_{2,3}+r_{3,2})} c_{k,3}^{\gamma_{i,k}\cdot 2r_{3,3}}\right)\Big)$$

On the right hand side we have that $\iota'_T(t) \odot \left[\vec{\mathbf{v}} \overset{s}{\bullet} \phi\right]$. Taking the square of each entry we get

$$\begin{pmatrix} e(u_{3,1}, u_{3,1})^{2t} & e(u_{3,1}, u_{3,2})^{2t} & e(u_{3,1}, u_{3,3}g)^{2t} \\ e(u_{3,2}, u_{3,1})^{2t} & e(u_{3,2}, u_{3,2})^{2t} & e(u_{3,2}, u_{3,3}g)^{2t} \\ e(u_{3,3}g, u_{3,1})^{2t} & e(u_{3,3}g, u_{3,2})^{2t} & e(u_{3,3}g, u_{3,3}g)^{2t} \end{pmatrix} \odot$$
$$\begin{pmatrix} \prod_{i=1}^2 e(u_{i1}, \phi_{i1})^2 & \prod_{i=1}^2 e(u_{i1}, \phi_{i2}) e(u_{i2}, \phi_{i1}) & \prod_{i=1}^2 e(u_{i1}, \phi_{i3}) e(u_{i3}, \phi_{i1}) \\ \prod_{i=1}^2 e(u_{i2}, \phi_{i1}) e(u_{i1}, \phi_{i2}) & \prod_{i=1}^2 e(u_{i2}, \phi_{i2})^2 & \prod_{i=1}^2 e(u_{i2}, \phi_{i3}) e(u_{i3}, \phi_{i2}) \\ \prod_{i=1}^2 e(u_{i3}, \phi_{i1}) e(u_{i1}, \phi_{i3}) & \prod_{i=1}^2 e(u_{i3}, \phi_{i2}) e(u_{i2}, \phi_{i3}) & \prod_{i=1}^2 e(u_{i3}, \phi_{i3})^2 \end{pmatrix},$$

which batches to

$$e\left(u_{1,1},\ \phi_{1,1}^{2\cdot r_{1,1}} \phi_{1,2}^{r_{1,2}+r_{2,1}} \phi_{1,3}^{r_{1,3}+r_{3,1}}\right) \cdot e\left(u_{2,2},\ \phi_{2,1}^{r_{1,2}+r_{2,1}} \phi_{2,2}^{2\cdot r_{2,2}} \phi_{2,3}^{r_{2,3}+r_{3,2}}\right)$$
$$\cdot e\left(u_{1,3},\ (\phi_{1,1}\phi_{2,1})^{r_{1,3}+r_{3,1}} (\phi_{1,2}\phi_{2,2})^{r_{2,3}+r_{3,2}} (\phi_{1,3}\phi_{2,3})^{2\cdot r_{3,3}}\right)$$
$$\cdot e(u_{3,1}, u_{3,1}^{2tr_{1,1}} u_{3,2}^{2tr_{1,2}} (u_{3,3}g)^{2tr_{1,3}}) e(u_{3,2}, u_{3,1}^{2tr_{2,1}} u_{3,2}^{2tr_{2,2}} (u_{3,3}g)^{2tr_{2,3}}) e(u_{3,3}g, u_{3,1}^{2tr_{3,1}} u_{3,2}^{2tr_{3,2}} (u_{3,3}g)^{2tr_{3,3}}).$$

which means a reduction from $18n + 24$ pairings to $3n + 6$ pairings.

7 Application 1: Groth's Group Signatures

7.1 Description

We demonstrate our techniques by applying them to one of the most practical group signature schemes in the standard model to date: Groth's construction [Gro07]. Groth proposed a methodology of transforming *certified signatures* [BFPW07] that respect a certain structure into group signatures using Groth-Sahai NIWI proofs:

- a member picks keys for a certified signature scheme and asks the issuer to certify her public verification key for the signature scheme;
- to produce a group signature, the member will make a certified signature, encrypt it and then use NIWI proofs to demonstrate that the ciphertext contains a valid certified signature.

Groth proposed an efficient certified signature scheme based on the so called q-**U** assumption (see [Gro07] for details). In the CPA-anonymous version[7] of the scheme, the issuer's public key is a triple

[7] Groth also proposes group signatures achieving CCA-anonymity [BSZ05], but for illustrative purposes we restrict ourselves to the basic CPA-anonymous scheme in this paper.

$(f, h, T) \in \mathbb{G}^2 \times \mathbb{G}_T$ (and its private key is $z \in \mathbb{G}$ such that $e(f, z) = T$) and the certificate of a group member with public key $v = g^x \in \mathbb{G}$ is a pair (a, b) satisfying $e(a, vh)\, e(f, b) = T$. To sign a message $m \in \mathbb{Z}_p$, the group member first computes a weak Boneh-Boyen signature [BB08] $\sigma = g^{1/(x+m)}$ using her private key x; then she forms Groth-Sahai commitments $\mathbf{d}_v$, $\mathbf{d}_b$ and $\mathbf{d}_\sigma$ to the group elements v, b and σ, resp., and makes a proof that they satisfy the following:

$$e(a, vh)\, e(f, b) = T \quad \text{and} \quad e(\sigma, g^m v) = e(g, g) \tag{6}$$

The fact that a is given in the clear is not a problem since the certificate is malleable, so the group member can unlinkably re-randomize it each time she signs a message. A group signature is thus of the form $(a, \mathbf{d}_b, \mathbf{d}_v, \mathbf{d}_\sigma, \psi, \phi)$, where ψ and ϕ denote the Groth-Sahai proofs for the two equations in (6), respectively.

We first instantiate our generic batch construction to verify a single signature more efficiently and then show how to verify multiple signatures at once. The first equation is of a particular form that allows for more efficient proofs and verification. We describe the verification relations and the batch verification in the next section.

7.2 Batching Linear Pairing-Product Equations

We consider a special case of pairing-product equations for which $\Gamma = 0$, called *linear equations*, i.e., the equation is of the following form: $\langle \vec{\mathcal{A}}, \vec{\mathcal{Y}} \rangle = t_T$, that is $\prod_{i=1}^n e(\mathcal{A}_i, \mathcal{Y}_i) = t_T$. In this case, the proof simplifies to three group elements and is verified as follows (taking into account that $u_{1,2} = u_{2,1} = 1$, and $u_{1,3} = u_{2,3}$):

$$\begin{aligned}
\textstyle\prod_{i=1}^n e(\mathcal{A}_i, d_{i,1}) &= e(u_{11}, \psi_1)\, e(u_{31}, \psi_3) \\
\textstyle\prod_{i=1}^n e(\mathcal{A}_i, d_{i,2}) &= e(u_{22}, \psi_2)\, e(u_{32}, \psi_3) \\
\textstyle\prod_{i=1}^n e(\mathcal{A}_i, d_{i,3}) &= t_T\, e(u_{13}, \psi_1\psi_2)\, e(u_{33}, \psi_3)
\end{aligned}$$

which can be batch-verified by checking[8]

$$\prod_{i=1}^n e(\mathcal{A}_i, d_{i,1}^{s_1} d_{i,2}^{s_2} d_{i,3}^{s_3}) = t_T^{s_3}\, e(u_{11}, \psi_1^{s_1})\, e(u_{13}, (\psi_1\psi_2)^{s_3})\, e(u_{22}, \psi_2^{s_2})\, e(u_{31}, \psi_3^{s_1})\, e(u_{32}, \psi_3^{s_2})\, e(u_{33}, \psi_3^{s_3})\ . \tag{7}$$

7.3 Batching the Equations for One Group Signature

1st Equation. Instantiating (7) for first equation in (6), we get, after some more optimization (shifting $e(a, h^{-1}))^{s_3}$ to the left-hand side of the equation)

$$\begin{aligned}
&e(d_{v,1}^{s_1} d_{v,2}^{s_2} (d_{v,3} h)^{s_3},\, a)\, e(d_{b,1}^{s_1} d_{b,2}^{s_2} d_{b,3}^{s_3},\, f) \ = \\
&\qquad = \ T^{s_3}\, e(u_{11}, \psi_1^{s_1})\, e(u_{13}, (\psi_1\psi_2)^{s_3})\, e(u_{22}, \psi_2^{s_2})\, e(u_{31}, \psi_3^{s_1})\, e(u_{32}, \psi_3^{s_2})\, e(u_{33}, \psi_3^{s_3})
\end{aligned}$$

2nd Equation. Setting $\vec{\mathcal{A}} := \begin{pmatrix} g^m \\ 1 \end{pmatrix}$, $\vec{\mathcal{Y}} := \begin{pmatrix} \sigma \\ v \end{pmatrix}$, $\Gamma := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $t_T := e(g, g)$, $\mathbf{d}_1 := \mathbf{d}_\sigma$ and $\mathbf{d}_2 := \mathbf{d}_v$ and substituting in (5), we get

$$\begin{aligned}
&e\big(d_{\sigma 1},\ (g^m d_{v3})^{(r_{13}+r_{31})} d_{v1}^{2\cdot r_{11}} d_{v2}^{(r_{12}+r_{21})}\big)\, e\big(d_{\sigma 2},\ (g^m d_{v3})^{(r_{23}+r_{32})} d_{v1}^{(r_{12}+r_{21})} d_{v2}^{2\cdot r_{22}}\big) \\
&\cdot e\big(d_{\sigma 3},\ (g^m d_{v3})^{2\cdot r_{33}} d_{v1}^{(r_{13}+r_{31})} d_{v2}^{(r_{23}+r_{32})}\big) \ = \\
&\qquad = \ e\big(u_{11},\ \phi_{11}^{2\cdot r_{11}} \phi_{12}^{r_{12}+r_{21}} \phi_{13}^{r_{13}+r_{31}}\big)\, e\big(u_{13},\ (\phi_{11}\phi_{21})^{r_{13}+r_{31}} (\phi_{12}\phi_{22})^{r_{23}+r_{32}} (\phi_{13}\phi_{23})^{2\cdot r_{33}}\big) \\
&\qquad\qquad \cdot e\big(u_{22},\ \phi_{21}^{r_{12}+r_{21}} \phi_{22}^{2\cdot r_{22}} \phi_{23}^{r_{23}+r_{32}}\big)\, e\big(u_{31},\ \phi_{31}^{2\cdot r_{11}} \phi_{32}^{r_{12}+r_{21}} \phi_{33}^{r_{13}+r_{31}}\big) \\
&\qquad\qquad \cdot e\big(u_{32},\ \phi_{31}^{r_{12}+r_{21}} \phi_{32}^{2\cdot r_{22}} \phi_{33}^{r_{23}+r_{32}}\big)\, e\big(u_{33},\ \phi_{31}^{r_{13}+r_{31}} \phi_{32}^{r_{23}+r_{32}} \phi_{33}^{2\cdot r_{33}}\big)\, e(g, g^{2 r_{33}})\ .
\end{aligned}$$

[8] If we considered a single set of equations then it would be more efficient to order the right-hand side by the ψ_i's and save 3 pairings. We order by the u_{ij} though, since this enables us to batch with other equations containing pairings of these constants.

Multiplying the two equations we get a single verification relation of the following form:

$$
\begin{aligned}
&e(d_{v,1}^{s_1} d_{v,2}^{s_2} (d_{v,3} h)^{s_3}, a)\, e(d_{b,1}^{s_1} d_{b,2}^{s_2} d_{b,3}^{s_3}, f) e\big(d_{\sigma 1},\ (g^m d_{v3})^{(r_{13}+r_{31})} d_{v1}^{2\cdot r_{11}} d_{v2}^{(r_{12}+r_{21})}\big) \\
&\cdot e\big(d_{\sigma 2},\ (g^m d_{v3})^{(r_{23}+r_{32})} d_{v1}^{(r_{12}+r_{21})} d_{v2}^{2\cdot r_{22}}\big)\, e\big(d_{\sigma 3},\ (g^m d_{v3})^{2\cdot r_{33}} d_{v1}^{(r_{13}+r_{31})} d_{v2}^{(r_{23}+r_{32})}\big) \ = \\
&\qquad = \ e\big(u_{11},\ \phi_{11}^{2\cdot r_{11}} \phi_{12}^{r_{12}+r_{21}} \phi_{13}^{r_{13}+r_{31}} \psi_1^{s_1}\big)\, e\big(u_{13},\ (\phi_{11}\phi_{21})^{r_{13}+r_{31}} (\phi_{12}\phi_{22})^{r_{23}+r_{32}} (\phi_{13}\phi_{23})^{2\cdot r_{33}} (\psi_1\psi_2)^{s_3}\big) \\
&\qquad\qquad \cdot e\big(u_{22},\ \phi_{21}^{r_{12}+r_{21}} \phi_{22}^{2\cdot r_{22}} \phi_{23}^{r_{23}+r_{32}} \psi_2^{s_2}\big)\, e\big(u_{31},\ \phi_{31}^{2\cdot r_{11}} \phi_{32}^{r_{12}+r_{21}} \phi_{33}^{r_{13}+r_{31}} \psi_3^{s_1}\big) \\
&\qquad\qquad \cdot e\big(u_{32},\ \phi_{31}^{r_{12}+r_{21}} \phi_{32}^{2\cdot r_{22}} \phi_{33}^{r_{23}+r_{32}} \psi_3^{s_2}\big)\, e\big(u_{33},\ \phi_{31}^{r_{13}+r_{31}} \phi_{32}^{r_{23}+r_{32}} \phi_{33}^{2\cdot r_{33}} \psi_3^{s_3}\big)\, \big(T^{s_3} e(g, g^{2 r_{33}})\big)
\end{aligned}
$$

Analysis. With no use of batching techniques, the verification of a single signature takes for the first equation 13 pairings and for the second 20 pairings for the left-hand side and 35 for its right-hand side. This is an overall of 68 pairing evaluations, compared to 11 for the batched verification.

7.4 Batching Several Group Signatures

Consider the situation where we want to verify multiple group signatures at once. That is given a group public key $(f, h, T, u_{11}, u_{13}, u_{22}, u_{31}, u_{32}, u_{33})$ and n group signatures

$$\left(a^{(k)}, \mathbf{d}_b^{(k)}, \mathbf{d}_v^{(k)}, \mathbf{d}_\sigma^{(k)}, (\psi_i^{(k)})_{1\le i\le 3}, (\phi_{ij}^{(k)})_{1\le i,j\le 3}\right) \ .$$

Using the same technique of taking each of the (new) equations to the power of some randomness and multiplying them, we can unify the pairings $e(\cdot, f)$ on the left-hand side and all pairings (which are of the form $e(u_{ij}, \cdot)$) on the right-hand side.

Instead of $11n$ pairings needed when checking each equation, the batched version only requires $4n+7$ pairings.

8 Application 2: Belenkiy-Chase-Kohlweiss-Lysyanskaya's P-signatures

8.1 Description

Belenkiy *et al.* formalized in [BCKL08] digital signature schemes with an additional non-interactive proof of signature possession that they called *P-signature schemes*. They proposed two constructions[9]: the first one relies on the weak Boneh-Boyen signature scheme [BB08] while the second one is inspired by its full version.

Since Belenkiy *et al.*'s first scheme relies on a rather strong assumption, we consider only their second proposal: a signature σ on a message $m \in \mathbb{Z}_p$ is a triple $\sigma = (C_1, C_2, C_3) \in \mathbb{G}_1 \times \mathbb{G}_2 \times \mathbb{G}_1$ such that $e(C_1, v h^m C_2) = e(g, h)$ and $e(f, C_2) = e(C_3, w)$, where f, g are (public) generators of $\mathbb{G}_1$, h is a (public) generator of $\mathbb{G}_2$ and $v, w \in \mathbb{G}_2$ are parts of the signer's public key. To prove the possession of such a signature, a prover forms the Groth-Sahai commitments $\mathbf{c_1}$, $\mathbf{c_2}$ and $\mathbf{c_3}$ for the group elements $C_1, M_1 = f^m, C_3$ in $\mathbb{G}_1$ and $\mathbf{d_1}$ and $\mathbf{d_2}$ for the group elements $M_2 = h^m$ and C_2 in $\mathbb{G}_2$ (respectively) and provides a proof that they satisfy:

$$e(C_1, v M_2 C_2) = e(g, h), \quad e(f, C_2) = e(C_3, w) \quad \text{and} \quad e(f, M_2) = e(M_1, h) \tag{8}$$

8.2 SXDH Instantiation

In [BCKL08], the authors evaluated that the verification of the proof in the SXDH instantiation requires the computation of 68 pairings. In Appendix C we show that it can be reduced to 15.

[9] An extended version of their scheme was recently proposed [BCKL09] but in this paper, for illustrative purposes, we restrict ourselves to the basic schemes from [BCKL08].

8.3 DLIN Instantiation

As in Section 7, the last two pairing-product equations from (8) are actually linear pairing-product equations. We denote the Groth-Sahai commitments for the group elements $C_1, C_2, C_3, M_1 = f^m, M_2 = h^m$ in $\mathbb{G}$ by $\mathbf{d_1}$,$\mathbf{d_2}$, $\mathbf{d_3}$, $\mathbf{d_4}$ and $\mathbf{d_5}$ (respectively) and ϕ, ψ and θ the proofs that they satisfy the first, the second and the third equation (respectively).

For the first equation, setting and substituting

$$\vec{\mathcal{A}} = \begin{pmatrix} v \\ 1 \\ 1 \end{pmatrix}, \ \vec{\mathbf{d}} = \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_5 \end{pmatrix}, \ \Gamma = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } t_T = e(g,g)$$

in (5), we get:

$$\begin{aligned} &e\big(d_{1,1},\ (v\,d_{2,3}\,d_{5,3})^{r_{1,3}+r_{3,1}}(d_{2,1}\,d_{5,1})^{2r_{1,1}}(d_{2,2}\,d_{5,2})^{(r_{1,2}+r_{2,1})}\big)\ e\big(d_{1,2},\ (v\,d_{2,3}\,d_{5,3})^{r_{2,3}+r_{3,2}}(d_{2,1}\,d_{5,1})^{r_{1,2}+r_{2,1}}(d_{2,2}\,d_{5,2})^{2r_{2,2}}\big) \\ &\qquad \cdot e\big(d_{1,3},\ (v\,d_{2,3}\,d_{5,3})^{2r_{3,3}}(d_{2,1}\,d_{5,1})^{r_{1,3}+r_{3,1}}(d_{2,2}\,d_{5,2})^{r_{2,3}+r_{3,2}}\big) \\ &= e\big(u_{1,1},\ \phi_{1,1}^{2\cdot r_{1,1}}\phi_{1,2}^{r_{1,2}+r_{2,1}}\phi_{1,3}^{r_{1,3}+r_{3,1}}\big)\cdot e\big(u_{1,3},\ (\phi_{1,1}\phi_{2,1})^{r_{1,3}+r_{3,1}}(\phi_{1,2}\phi_{2,2})^{r_{2,3}+r_{3,2}}(\phi_{1,3}\phi_{2,3})^{2\cdot r_{3,3}}\big) \\ &e\big(u_{2,2},\ \phi_{2,1}^{r_{1,2}+r_{2,1}}\phi_{2,2}^{2\cdot r_{2,2}}\phi_{2,3}^{r_{2,3}+r_{3,2}}\big)\cdot e\big(u_{3,1},\ \phi_{3,1}^{2\cdot r_{1,1}}\phi_{3,2}^{r_{1,2}+r_{2,1}}\phi_{3,3}^{r_{1,3}+r_{3,1}}\big)\cdot e\big(u_{3,2},\ \phi_{3,1}^{r_{1,2}+r_{2,1}}\phi_{3,2}^{2\cdot r_{2,2}}\phi_{3,3}^{r_{2,3}+r_{3,2}}\big)\cdot \\ &\qquad e\big(u_{3,3},\ \phi_{3,1}^{r_{1,3}+r_{3,1}}\phi_{3,2}^{r_{2,3}+r_{3,2}}\phi_{3,3}^{2\cdot r_{3,3}}\big)\cdot e(g,g)^{2r_{3,3}} \end{aligned}$$

With the substitution $\vec{\mathcal{A}} = \begin{pmatrix} f \\ w^{-1} \end{pmatrix}, \vec{\mathbf{d}} = \begin{pmatrix} \mathbf{d}_2 \\ \mathbf{d}_3 \end{pmatrix}$, and $t_T = 1$ and $\vec{\mathcal{A}} = \begin{pmatrix} f \\ h^{-1} \end{pmatrix}, \vec{\mathbf{d}} = \begin{pmatrix} \mathbf{d}_5 \\ \mathbf{d}_4 \end{pmatrix}$, and $t_T = 1$ (respectively) in (7), we obtain the second and third equation. Once the three equations multiplied, we obtain:

$$\begin{aligned} &e\big(d_{1,1},\ (v\,d_{2,3}\,d_{5,3})^{r_{1,3}+r_{3,1}}(d_{2,1}\,d_{5,1})^{2r_{1,1}}(d_{2,2}\,d_{5,2})^{(r_{1,2}+r_{2,1})}\big)\ e\big(d_{1,2},\ (v\,d_{2,3}\,d_{5,3})^{r_{2,3}+r_{3,2}}(d_{2,1}\,d_{5,1})^{r_{1,2}+r_{2,1}}(d_{2,2}\,d_{5,2})^{2r_{2,2}}\big) \\ &e\big(d_{1,3},\ (v\,d_{2,3}\,d_{5,3})^{2r_{3,3}}(d_{2,1}\,d_{5,1})^{r_{1,3}+r_{3,1}}(d_{2,2}\,d_{5,2})^{r_{2,3}+r_{3,2}}\big)\ e(f, d_{2,1}^{s_1}d_{2,2}^{s_2}d_{2,3}^{s_3}d_{5,1}^{t_1}d_{5,2}^{t_2}d_{5,3}^{t_3})e(w^{-1}, d_{3,1}^{s_1}d_{3,2}^{s_2}d_{3,3}^{s_3})e(h^{-1}, d_{4,1}^{t_1}d_{4,2}^{t_2}d_{4,3}^{t_3}) \\ &= e\big(u_{1,1},\ \phi_{1,1}^{2\cdot r_{1,1}}\phi_{1,2}^{r_{1,2}+r_{2,1}}\phi_{1,3}^{r_{1,3}+r_{3,1}}\psi_1^{s_1}\theta_1^{t_1}\big)\cdot e\big(u_{1,3},\ (\phi_{1,1}\phi_{2,1})^{r_{1,3}+r_{3,1}}(\phi_{1,2}\phi_{2,2})^{r_{2,3}+r_{3,2}}(\phi_{1,3}\phi_{2,3})^{2\cdot r_{3,3}}(\psi_1\psi_2)^{s_3}(\theta_1\theta_2)^{t_3}\big) \\ &\qquad \cdot e\big(u_{2,2},\ \phi_{2,1}^{r_{1,2}+r_{2,1}}\phi_{2,2}^{2\cdot r_{2,2}}\phi_{2,3}^{r_{2,3}+r_{3,2}}\psi_2^{s_2}\theta_2^{t_2}\big)e\big(u_{3,1},\ \phi_{3,1}^{2\cdot r_{1,1}}\phi_{3,2}^{r_{1,2}+r_{2,1}}\phi_{3,3}^{r_{1,3}+r_{3,1}}\psi_3^{s_1}\theta_3^{t_1}\big) \\ &\qquad\qquad \cdot e\big(u_{3,2},\ \phi_{3,1}^{r_{1,2}+r_{2,1}}\phi_{3,2}^{2\cdot r_{2,2}}\phi_{3,3}^{r_{2,3}+r_{3,2}}\psi_3^{s_2}\theta_3^{t_2}\big)e\big(u_{3,3},\ \phi_{3,1}^{r_{1,3}+r_{3,1}}\phi_{3,2}^{r_{2,3}+r_{3,2}}\phi_{3,3}^{2\cdot r_{3,3}}\psi_3^{s_3}\theta_3^{t_3}\big)e(g,g)^{2r_{3,3}} \end{aligned}$$

In [BCKL08], the authors evaluated that the verification of the proof in the DLIN instantiation requires the computation of 126 pairings. With our result, we prove it can be reduced to 12.

Batching Several P-Signatures. As in the previous section, in the situation where we want to verify multiple P-signatures at once, we can unify the pairings containing f, h and w on the left-hand side and all pairings (which are of the form $e(u_{i,j}, \cdot)$) on the right-hand side. Instead of $15n$ (*resp.* $12n$) pairings needed when checking each equation, the batched version only requires $2n + 13$ (*resp.* $3n + 9$) pairings.

9 Conclusion

In this paper, we presented efficiency improvements for the verification of Groth-Sahai non-interactive (zero-knowledge and witness-indistinguishable) proofs and two privacy-preserving authentication schemes (with an improvement in efficiency of up to 90% for the number of (dominant) pairing operations). These results can be combined with known methods to compute the product of many pairing evaluations efficiently [GS06]. Our results notably provide the first algorithm to batch-verify a group signature scheme in the standard model (an open problem raised in [FGHP09]) and (surprisingly) demonstrate that thanks to batch verification techniques, the DLIN instantiation of the Groth-Sahai proof system may be the most efficient implementation for the verification of a single signature.

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A Instantiation 1: Subgroup Membership

A.1 Description of Groth-Sahai Proof System

For the Subgroup Decision instantiation, the language is over a bilinear group $(N, \mathbb{G}, \mathbb{G}_T, e, g)$ where $N = pq$ is a product of two primes and with generator g. The commitment key consists of one element u in $\mathbb{G}$, chosen to either generate $\mathbb{G}$ itself or to generate $\mathbb{G}_q$, its subgroup of order q. Under the SD assumption, the two choices are indistinguishable.

Let $X \in \mathbb{G}$ and $x \in \mathbb{Z}_N$. We define $\iota(X) = X$ and $\iota'(x) = g^x$. To commit to $X \in \mathbb{G}$, one chooses randomness $r \in \mathbb{Z}_N$ and sets $\mathbf{c}_X := \iota(X) \cdot u^r$. To make a commitment to $x \in \mathbb{Z}_N$, one chooses $r \in \mathbb{Z}_p$ and sets $\mathbf{c}_x := \iota'(x) \cdot u^r$.

To show satisfiability of a set of equations of the form $(1')$, $(2')$ or $(3')$, one first makes commitments to a satisfying witness (i.e., an assignment to the variables of each equation) and then adds a "proof" per equation. Groth and Sahai describe how to construct these; for all equations, they are a single element in $\mathbb{G}$. The verification relations for the proofs are given in the rest of the section, where we also discuss how to optimize them.

A.2 Pairing-Product Equation

A pairing-product equation is of the form $\langle \vec{\mathcal{A}}, \vec{\mathcal{Y}} \rangle \cdot \langle \vec{\mathcal{Y}}, \Gamma \vec{\mathcal{Y}} \rangle = t_T$, that is

$$\prod_{i=1}^{n} e(\mathcal{A}_i, \mathcal{Y}_i) \cdot \prod_{i=1}^{n} \cdot \prod_{j=1}^{n} e(\mathcal{Y}_i, \mathcal{Y}_j)^{\gamma_{ij}} = t_T \tag{9}$$

Given a vector of commitments $\vec{\mathcal{D}}$, a proof $\phi \in \mathbb{G}$ is verified by the following equation:

$$\prod_{i=1}^{n} e(\mathcal{A}_i, \mathcal{D}_i) \prod_{i=1}^{n} \prod_{j=1}^{n} e(\mathcal{D}_i, \mathcal{D}_j)^{\gamma_{ij}} = t_T \cdot e(u, \phi)$$

By applying the different techniques explained in 4, it becomes:

$$\prod_{i=1}^{n} e(\mathcal{A}_i, \mathcal{D}_i) \prod_{i=1}^{n} \prod_{j=1}^{n} e(\mathcal{D}_i, \mathcal{D}_j)^{\gamma_{ij}} = \prod_{i=1}^{n} \left(e(\mathcal{D}_i, \mathcal{A}_i)\, e(\mathcal{D}_i, \prod_{j=1}^{n} \mathcal{D}_j^{\gamma_{ij}}) \right) \prod_{i=1}^{n} e(\mathcal{D}_i, \mathcal{A}_i \prod_{j=1}^{n} \mathcal{D}_j^{\gamma_{ij}}) = t_T\, e(\mathcal{U}, \phi)$$

A.3 Multi-Scalar Multiplication

A multi-scalar multiplication is of the form $\langle \vec{a}, \vec{\mathcal{Y}} \rangle \cdot \langle \vec{x}, \vec{\mathcal{B}} \rangle \cdot \langle \vec{x}, \Gamma \vec{\mathcal{Y}} \rangle = \mathcal{T}$. A proof $\phi \in \mathbb{G}$ is verified by

$$\prod_{j=1}^{n} e(g^{a_i}, \mathcal{D}_j) \cdot \prod_{i=1}^{m} e(\mathcal{C}_i, \mathcal{B}_i) \cdot \prod_{i=1}^{m} \prod_{j=1}^{n} e(\mathcal{C}_i, \mathcal{D}_j)^{\gamma_{ij}} = e(g, \mathcal{T}) e(u, \phi)$$

By applying the different techniques to batch computation, we obtain:

$$e(g, \prod_{j=1}^{n} \mathcal{D}_j^{a_j}) \cdot \prod_{i=1}^{m} \left(e(\mathcal{C}_i, \mathcal{B}_i \cdot \prod_{j=1}^{n} \mathcal{D}_j^{\gamma_{i,j}}) \right) = e\left(g, \mathcal{T}\right) e\left(u, \phi\right)$$

A.4 Quadratic Equation

A quadratic equation is of the form $\langle \vec{x}, \vec{b} \rangle + \langle \vec{x}, \Gamma \vec{x} \rangle = t$. A proof $\phi \in \mathbb{G}$ is verified by

$$\prod_{i=1}^{m} e(\mathcal{C}_i, g^{b_i}) \cdot \prod_{i=1}^{m} \prod_{j=1}^{m} e(\mathcal{C}_i, \mathcal{C}_j)^{\gamma_{ij}} = e(g,g)^t e(u, \phi)$$

By applying the different techniques to batch computation:

$$\prod_{i=1}^{m} e(\mathcal{C}_i, g^{b_i} \cdot \prod_{j=1}^{m} \mathcal{C}_j^{\gamma_{ij}}) = e(g,g)^t e(u, \phi)$$

A.5 Batching in Composite Groups

When based on the SXDH or on the DLIN assumption, GS proof systems are instantiated in prime-order bilinear groups. In such groups, the small exponents test result can be trusted as the probability of accepting an invalid input is $2^{-\ell}$, with ℓ being the bit length of the random exponents [FGHP09].

However, in the case of the subgroup decision assumption, the setup is a bilinear group of composite order $N = pq$. Even though the verification equations for relations consist of a single equation per relation, the small exponent test can be used to verify several relations at once, or to perform a batch verification of the equations for a given relation for several executions of a protocol.

If one wants to verify the validity of multiple Groth-Sahai proofs, it is thus necessary to evaluate how much the small exponent test can be trusted in this setup:

Theorem 2. *Let* $\mathrm{PSetup}(1^k) \to (N, g_1, g_2, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, e)$*, with* $N = pq$ *composite. Consider a set of* m *pairing-based claims, with* $m \in \mathrm{poly}(k)$*, written as* $\prod_{i=1}^{k_j} e(f_{i_j}, h_{i_j})^{c_{i_j}} = A_j$ *in* $\mathbb{G}_T$*, for* $j \in [1, m]$*. For any random vector* $\Delta = (\delta_1, \ldots, \delta_m)$ *of* ℓ*-bit elements in* $\mathbb{Z}_n$*, a batch pairing-based verifier which tests the equation*

$$\prod_{j=1}^{m} \prod_{i=1}^{k_j} e(f_{i_j}, h_{i_j})^{c_{i_j} \delta_j} = \prod_{j=1}^{m} A_j^{\delta_j}$$

accepts an invalid set of claims with probability at most $\max(p,q) \cdot 2^{-\ell}$*.*

Proof. For every j in $[1, m]$, one can write $A_j = e(g_1, g_2)^{a(j)}$ and $\prod_{i=1}^{k_j} e(f_{i_j}, h_{i_j})^{c_{i_j}} = e(g_1, g_2)^{c(j)}$. With these notations, the batch equation becomes $\prod_{j=1}^{m} e(g_1, g_2)^{\delta_j (c(j) - a(j))} = 1$.

The batch equation yields a correct verification if and only if $c(j) - a(j) = 0$ for all j. Suppose now that verification succeeds with $c(1) - a(1)$ non-zero. Then, we have

$$\delta_1 (c(1) - a(1)) = -\sum_{j=2}^{m} \delta_j (c(j) - a(j)) \mod n.$$

If $c(1) - a(1)$ and n are coprime then $c(1) - a(1)$ is invertible in $\mathbb{Z}_n$, so there exists a single value of δ_1 that satisfies the equation. Suppose now that $c(1) - a(1)$ and n are not coprime. This is the case if and only if $c(1) - a(1)$ is a multiple of p or q. Suppose for instance that $c(1) - a(1)$ is a multiple of p. Then, the batch equation has a solution if and only if $-\sum_{j=2}^{m} \delta_j (c(j) - a(j))$ is a multiple of p, in which case p different values of $\delta_1 = \Delta + \mu q, \mu = 0, \ldots, p-1$ verify the equation.

Similarly, if $c(1) - a(1)$ is a multiple of q then either the batch equation has no solution in δ_1, or it has q solutions.

Depending on his role and capacities, an adversary may ensure that all the $c(j) - a(j)$ are multiples of p, with p being the biggest prime factor of n. In this worst case, the probability that the verification succeeds with wrong input is $P \leq p \cdot 2^{-\ell}$, which yields the result.

B Multi-Scalar Multiplication Equation in $\mathbb{G}_2$

We consider equations of Type (2) in $\mathbb{G}_2$. For each such multiplication, the verification of $(\pi, \vec{\theta}) \in \mathbb{G}_2^{1\times 2} \times \mathbb{G}_1^{2\times 2}$ consists in checking the following:

$$\left[\iota_1'(\vec{a}) \bullet \vec{\mathbf{d}}\right] \odot \left[\vec{\mathbf{c}'} \bullet \iota_2(\vec{\mathcal{B}})\right] \odot \left[\vec{\mathbf{c}'} \bullet \Gamma\vec{\mathbf{d}}\right] = \hat{\iota_T}(\mathcal{T}_2) \odot F(\mathbf{u}_1, \pi) \odot \left[\vec{\theta} \bullet \vec{\mathbf{v}}\right] .$$

We denote

$$\vec{a} = (a_j)_{1\leq j\leq n} \in \mathbb{Z}_p^{n\times 1}, \quad \vec{\mathbf{c}'} = \left(c'_{i,k}\right)_{\substack{1\leq i\leq m\\ 1\leq k\leq 2}} \in \mathbb{G}_1^{m\times 2}, \quad \vec{\mathbf{d}} = (d_{j,k})_{\substack{1\leq j\leq n\\ 1\leq k\leq 2}} \in \mathbb{G}_2^{n\times 2}, \quad \vec{\mathcal{B}} = (\mathcal{B}_i)_{1\leq i\leq m} \in \mathbb{G}_2^{m\times 1}$$

and $\Gamma = (\gamma_{i,j})_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} \in \mathbb{Z}_p^{n\times n}$. The left hand side is equal to

$$\begin{pmatrix} \prod_{j=1}^{n} e\left(u_{2,1}^{a_j} \prod_{i=1}^{m} {c'}_{i,1}^{\gamma_{i,j}}, d_{j,1}\right) & \prod_{i=1}^{m} e({c'}_{i,1}, \mathcal{B}_i) \prod_{j=1}^{n} e\left(u_{2,1}^{a_j} {c'}_{i,1}^{\gamma_{i,j}}, d_{j,2}\right) \\ \prod_{j=1}^{n} e\left((u_{2,2}g_1)^{a_j} \prod_{i=1}^{m} {c'}_{i,2}^{\gamma_{i,j}}, d_{j,1}\right) & \prod_{i=1}^{m} e(c'_{i,2}, \mathcal{B}_i) \prod_{j=1}^{n} e\left((u_{2,2}g_1)^{a_j} {c'}_{i,2}^{\gamma_{i,j}}, d_{j,2}\right) \end{pmatrix}$$

If we denote $\pi = (\pi_1, \pi_2)$, $\vec{\theta} = \begin{pmatrix} \theta_{1,1} & \theta_{1,2} \\ \theta_{2,1} & \theta_{2,2} \end{pmatrix}$, we have:

$$\begin{aligned} &\hat{\iota_T}(\mathcal{T}_2) \odot F(\mathbf{u}_1, \pi) \odot \left[\vec{\theta} \bullet \vec{\mathbf{v}}\right] \\ &\quad = \begin{pmatrix} e(u_{2,1}, \pi_1)e(\theta_{1,1}, v_{1,1})e(\theta_{2,1}, v_{2,1}) & e(u_{2,1}, \pi_2)e(\theta_{1,1}, v_{1,2})e(\theta_{2,1}, v_{2,2})e(u_{2,1}, \mathcal{T}_2) \\ e((u_{2,2}g_1), \pi_1)e(\theta_{1,2}, v_{1,1})e(\theta_{2,2}, v_{2,1}) & e((u_{2,2}g_1), \pi_2)e(\theta_{1,2}, v_{1,2})e(\theta_{2,2}, v_{2,2})e(u_{2,2}g_1, \mathcal{T}_2) \end{pmatrix} \end{aligned}$$

and therefore

$$\begin{aligned} &\begin{pmatrix} \prod_{j=1}^{n} e\left(u_{2,1}^{a_j} \prod_{i=1}^{m} (c'_{i,1})^{\gamma_{i,j}}, d_{j,1}\right) & \prod_{i=1}^{m} e(c'_{i,1}, \mathcal{B}_i) \prod_{j=1}^{n} e\left(u_{2,1}^{a_j} {c'}_{i,1}^{\gamma_{i,j}}, d_{j,2}\right) \\ \prod_{j=1}^{n} e\left((u_{2,1}g_1)^{a_j} \prod_{i=1}^{m} {c'}_{i,2}^{\gamma_{i,j}}, d_{j,1}\right) & \prod_{i=1}^{m} e(c'_{i,2}, \mathcal{B}_i) \prod_{j=1}^{n} e\left((u_{2,1}g_1)^{a_j} {c'}_{i,2}^{\gamma_{j,i}}, d_{j,2}\right) \end{pmatrix} \\ = &\begin{pmatrix} e(u_{2,1}, \pi_1)e(\theta_{1,1}, v_{1,1})e(\theta_{2,1}, v_{2,1}) & e(u_{2,1}, \pi_2)e(\theta_{1,1}, v_{1,2})e(\theta_{2,1}, v_{2,2})e(u_{2,1}, \mathcal{T}_2) \\ e((u_{2,2}g_1), \pi_1)e(\theta_{1,2}, v_{1,1})e(\theta_{2,2}, v_{2,1}) & e((u_{2,2}g_1), \pi_2)e(\theta_{1,2}, v_{1,2})e(\theta_{2,2}, v_{2,2})e(u_{2,2}g_1, \mathcal{T}_2) \end{pmatrix} \end{aligned}$$

Optimizing the verification. By grouping the pairings, the number of pairings on the left-hand side member of the equation is already reduced to $4n + 2m$ instead of $8n + 2m$. Now, by using the batch technique, i.e. multiplying each member by a random value and multiplying all the members, we obtain on the left-hand side

$$\begin{aligned} \prod_{i=1}^{m} e\left({c'_{i,1}}^{r_{1,2}} {c'_{i,2}}^{r_{2,2}}, \mathcal{B}_i\right) \cdot &\prod_{j=1}^{n} e\left(\left(u_{2,1}^{a_j} \prod_{i=1}^{m} {c'_{i,1}}^{\gamma_{i,j}}\right)^{r_{1,1}} \left((u_{2,2}g_1)^{a_j} \prod_{i=1}^{m} {c'_{i,2}}^{\gamma_{i,j}}\right)^{r_{2,1}}, d'_{j,1}\right) \\ &\cdot \prod_{j=1}^{n} e\left(\left(u_{2,1}^{a_j} \prod_{i=1}^{m} {c'_{i,1}}^{\gamma_{i,j}}\right)^{r_{1,2}} \left((u_{2,2}g_1)^{a_j} \prod_{i=1}^{m} {c'_{i,2}}^{\gamma_{i,j}}\right)^{r_{2,2}}, d'_{j,2}\right) \end{aligned}$$

which requires $2mn + 2m + 6n$ exponentiations in $\mathbb{G}_1$ and $m + 2n$ pairing computations.

An alternative expression of the left hand-side of the equation,

$$\prod_{i=1}^{m} e\left(c'_{i,1}, \left(\prod_{j=1}^{n} {d_{j,1}}^{\gamma_{i,j}}\right)^{r_{1,1}} \left(\mathcal{B}_i \prod_{j=1}^{n} {d_{j,2}}^{\gamma_{i,j}}\right)^{r_{1,2}}\right) \cdot \prod_{i=1}^{m} e\left(c'_{i,2}, \left(\prod_{j=1}^{n} {d_{j,1}}^{\gamma_{i,j}}\right)^{r_{2,1}} \left(\mathcal{B}_i \prod_{j=1}^{n} {d_{j,2}}^{\gamma_{i,j}}\right)^{r_{2,2}}\right)$$
$$\cdot e\left(u_{2,1}, \left(\prod_{j=1}^{n} {d_{j,1}}^{a_j}\right)^{r_{1,1}} \left(\prod_{j=1}^{n} {d_{j,2}}^{a_j}\right)^{r_{1,2}}\right) \cdot e\left(u_{2,2}g_1, \left(\prod_{j=1}^{n} {d_{j,1}}^{a_j}\right)^{r_{2,1}} \left(\prod_{j=1}^{n} {d_{j,2}}^{a_j}\right)^{r_{2,2}}\right) \quad ,$$

requires $2mn + 4m + 2n + 4$ exponentiations in $\mathbb{G}_2$ and $2m + 2$ pairing computations.

On the right-hand side, the same technique achieves a reduction from 14 to 7 pairings:

$$e(u_{2,1}^{r_{1,1}}(u_{2,2}g_1)^{r_{2,1}}, \pi_1)e(u_{2,1}^{r_{1,2}}(u_{2,2}g_1)^{r_{2,2}}, \pi_2)e(\theta_{1,1}^{r_{1,1}}\theta_{1,2}^{r_{2,1}}, v_{1,1})e(\theta_{1,1}^{r_{1,2}}\theta_{1,2}^{r_{2,2}}, v_{1,2})$$
$$\cdot e(\theta_{2,1}^{r_{1,1}}\theta_{2,2}^{r_{2,1}}, v_{2,1})e(\theta_{2,1}^{r_{1,2}}\theta_{2,2}^{r_{2,2}}, v_{2,2})e(u_{2,1}^{r_{1,2}}(g_1u_{2,2})^{r_{2,2}}, \mathcal{T}_2)$$

C SXDH Instantiation of Belenkiy-Chase-Kohlweiss-Lysyanskaya's P-signatures

Setting and substituting

$$\vec{\mathcal{A}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{d} = \begin{pmatrix} \mathbf{d_1} \\ \mathbf{d_2} \end{pmatrix}, \vec{c} = \begin{pmatrix} \mathbf{c_1} \\ \mathbf{1} \end{pmatrix}, \vec{\mathcal{B}} = \begin{pmatrix} v \\ 1 \end{pmatrix}, \Gamma = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } t_T = e(g, h)$$

in (4), we get for the first equation:

$$\begin{aligned} & e(c_{1,1}, (v\, d_{1,2}\, d_{2,2})^{r_{1,2}}(d_{1,1}\, d_{2,1})^{r_{1,1}})e(c_{1,2}, (v\, d_{1,2}\, d_{2,2})^{r_{2,2}}(d_{1,1}\, d_{2,1})^{r_{2,1}}) \\ = {} & e(u_{1,1}^{r_{1,1}} u_{1,2}^{r_{2,1}}, \pi_{1,1})\, e(u_{2,1}^{r_{1,1}} u_{2,2}^{r_{2,1}}, \pi_{2,1}) e(\theta_{1,1}^{r_{1,1}}\theta_{1,2}^{r_{2,1}}, v_{1,1})\, e(\theta_{2,1}^{r_{1,1}}\theta_{2,2}^{r_{2,1}}, v_{2,1}) \\ & \cdot e(u_{1,1}^{r_{1,2}} u_{1,2}^{r_{2,2}}, \pi_{1,2})\, e(u_{2,1}^{r_{1,2}} u_{2,2}^{r_{2,2}}, \pi_{2,2}) e(\theta_{1,1}^{r_{1,2}}\theta_{1,2}^{r_{2,2}}, v_{1,2})\, e(\theta_{2,1}^{r_{1,2}}\theta_{2,2}^{r_{2,2}}, v_{2,2})\, e(g^{r_{2,2}}, h) \end{aligned}$$

For the second and the third equation, setting and substituting in (4)

$$\vec{\mathcal{A}} = \begin{pmatrix} f \end{pmatrix}, \vec{d} = \begin{pmatrix} \mathbf{d_2} \end{pmatrix}, \vec{c} = \begin{pmatrix} \mathbf{c_3} \end{pmatrix}, \vec{\mathcal{B}} = \begin{pmatrix} w^{-1} \end{pmatrix}, \Gamma = \begin{pmatrix} 0 \end{pmatrix} \text{ and } t_T = 1, \text{ and}$$
$$\vec{\mathcal{A}} = \begin{pmatrix} f \end{pmatrix}, \vec{d} = \begin{pmatrix} \mathbf{d_1} \end{pmatrix}, \vec{c} = \begin{pmatrix} \mathbf{c_2} \end{pmatrix}, \vec{\mathcal{B}} = \begin{pmatrix} h^{-1} \end{pmatrix}, \Gamma = \begin{pmatrix} 0 \end{pmatrix} \text{ and } t_T = 1$$

we get:

$$e(c_{3,1}, w^{-s_{1,2}})e(c_{3,2}, w^{-s_{2,2}})e\left(f, d_{2,1}^{s_{2,1}} d_{2,2}^{s_{2,2}}\right) = e(u_{1,1}^{s_{1,1}} u_{1,2}^{s_{2,1}}, \phi_{1,1})\, e(u_{2,1}^{s_{1,1}} u_{2,2}^{s_{2,1}}, \phi_{2,1}) e(\psi_{1,1}^{s_{1,1}} \cdot \psi_{1,2}^{s_{2,1}}, v_{1,1})$$
$$\cdot e(\psi_{2,1}^{s_{1,1}}\psi_{2,2}^{s_{2,1}}, v_{2,1})e(u_{1,1}^{s_{1,2}} u_{1,2}^{s_{2,2}}, \phi_{1,2})\, e(u_{2,1}^{s_{1,2}} u_{2,2}^{s_{2,2}}, \phi_{2,2})e(\psi_{1,1}^{s_{1,2}}\psi_{1,2}^{s_{2,2}}, v_{1,2})\, e(\psi_{2,1}^{s_{1,2}}\psi_{2,2}^{s_{2,2}}, v_{2,2})$$

and

$$e(c_{2,1}, h^{-t_{1,2}})e(c_{2,2}, h^{-t_{2,2}})e\left(f, d_{1,1}^{t_{2,1}} d_{1,2}^{t_{2,2}}\right) = e(u_{1,1}^{t_{1,1}} u_{1,2}^{t_{2,1}}, \xi_{1,1})\, e(u_{2,1}^{t_{1,1}} u_{2,2}^{t_{2,1}}, \xi_{2,1})$$
$$\cdot e(\zeta_{1,1}^{t_{1,1}}\zeta_{1,2}^{t_{2,1}}, v_{1,1})\, e(\zeta_{2,1}^{t_{1,1}}\zeta_{2,2}^{t_{2,1}}, v_{2,1}) \cdot e(u_{1,1}^{t_{1,2}} u_{1,2}^{t_{2,2}}, \xi_{1,2})\, e(u_{2,1}^{t_{1,2}} u_{2,2}^{t_{2,2}}, \xi_{2,2}) e(\zeta_{1,1}^{t_{1,2}}\zeta_{1,2}^{t_{2,2}}, v_{1,2})\, e(\zeta_{2,1}^{t_{1,2}}\zeta_{2,2}^{t_{2,2}}, v_{2,2})$$

Multiplying the three equations we get a single verification relation of the following form:

$$\begin{aligned} & e(c_{1,1}, (v\, d_{1,2}\, d_{2,2})^{r_{1,2}}(d_{1,1}\, d_{2,1})^{r_{1,1}})e(c_{1,2}, (v\, d_{1,2}\, d_{2,2})^{r_{2,2}}(d_{1,1}\, d_{1,1})^{r_{2,1}})e(c_{3,1}, w^{-s_{1,2}})e(c_{3,2}, w^{-s_{2,2}}) \\ & \cdot e\left(f, d_{2,1}^{s_{2,1}} d_{2,2}^{s_{2,2}} d_{1,1}^{t_{2,1}} d_{1,2}^{t_{2,2}}\right) e(c_{2,1}, h^{-t_{1,2}})e(c_{2,2}, h^{-t_{2,2}}) \\ = {} & e(g, h)^{r_{2,2}} e(u_{1,1}, \pi_{1,1}^{r_{1,1}} \pi_{1,2}^{r_{1,2}} \phi_{1,1}^{s_{1,1}} \phi_{1,2}^{s_{1,2}} \xi_{1,1}^{t_{1,1}} \xi_{1,2}^{t_{1,2}})\, e(u_{1,2}, \pi_{1,1}^{r_{2,1}} \pi_{1,2}^{r_{2,2}} \phi_{1,1}^{s_{2,1}} \phi_{1,2}^{s_{2,2}} \xi_{1,1}^{t_{2,1}} \xi_{1,2}^{t_{2,2}})\, e(u_{2,1}, \pi_{2,1}^{r_{1,1}} \pi_{2,2}^{r_{1,2}} \phi_{2,1}^{s_{1,1}} \phi_{2,2}^{s_{1,2}} \xi_{2,1}^{t_{1,1}} \xi_{2,2}^{t_{1,2}}) \\ & \cdot e(u_{2,2}, \pi_{2,1}^{r_{2,1}} \pi_{2,2}^{r_{2,2}} \phi_{2,1}^{s_{2,1}} \phi_{2,2}^{s_{2,2}} \xi_{2,1}^{t_{2,1}} \xi_{2,2}^{t_{2,2}})\, e(\theta_{1,1}^{r_{1,1}}\theta_{1,2}^{r_{2,1}} \psi_{1,1}^{s_{1,1}} \psi_{1,2}^{s_{2,1}} \zeta_{1,1}^{t_{1,1}} \zeta_{1,2}^{t_{2,1}}, v_{1,1})\, e(\theta_{1,1}^{r_{1,2}}\theta_{1,2}^{r_{2,2}} \psi_{1,1}^{s_{1,2}} \psi_{1,2}^{s_{2,2}} \zeta_{1,1}^{t_{1,2}} \zeta_{1,2}^{t_{2,2}}, v_{1,2}) \\ & \cdot e(\theta_{2,1}^{r_{1,1}}\theta_{2,2}^{r_{2,1}} \psi_{2,1}^{s_{1,1}} \psi_{2,2}^{s_{2,1}} \zeta_{2,1}^{t_{1,1}} \zeta_{2,2}^{t_{2,1}}, v_{2,1})\, e(\theta_{2,1}^{r_{1,2}}\theta_{2,2}^{r_{2,2}} \psi_{2,1}^{s_{1,2}} \psi_{2,2}^{s_{2,2}}, \zeta_{2,1}^{t_{1,2}} \zeta_{2,2}^{t_{2,2}}, v_{2,2}) \end{aligned}$$

In [BCKL08], the authors evaluated that the verification of the proof in the SXDH instantiation requires the computation of 68 pairings. With our result, we prove it can be reduced to 15.