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Characterization of a local quadratic growth of the Hamiltonian for control constrained optimal control problems

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Abstract: We consider an optimal control problem with inequality control constraints given by smooth functions satisfying the hypothesis of linear independence of gradients of active constraints. For this problem, we formulate a generalization of strengthened Legendre condition and prove that this generalization is equivalent to the condition of a local quadratic growth of the Hamiltonian subject to control constraints.

Key-words: Pontryagin's principle, Legendre condition, Hamiltonian, control constraints, quadratic growth, sufficient optimality conditions

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Caractérisation de la croissance quadratique locale du hamiltonien pour des problèmes de commande optimale avec contraintes sur la commande

Résumé : Nous considérons un problème de commande optimale avec inégalités sur la commande définies par des fonctions lisses satisfaisant l'hypothèse d'indépendance linéaire des gradients des contraintes actives. Pour ce problème, nous formulons une généralisation de la condition de Legendre forte, et prouvons que cette généralisation est équivalente à la croissance quadratique du hamiltonien soumise aux contraintes sur la commande.

Mots-clés : Principe de Pontryagine, condition de Legendre, Hamiltonien, contraintes sur la commande, croissance quadratique, conditions suffisantes d'optimalité.

1 Introduction

In the calculus of variations, the second-order sufficient condition of a weak minimum (namely, the strengthened Jacobi condition) presuppose the strengthened Legendre condition; the latter is equivalent to a local quadratic growth of the Hamiltonian. So, the condition of a local quadratic growth of the Hamiltonian is a part of sufficient second-order condition. In optimal control, the situation is quite similar: the condition of a local quadratic growth of the Hamiltonian subject to control constraints is a part of sufficient second-order optimality conditions, cf. Bonnans and Osmolovskii [1], Osmolovskii [3, p.155-156], [7, 8]. Now, a question arises: what is a characterization of a local quadratic growth condition of the Hamiltonian at the presence of control constraints? Is it possible to formulate this characterization as a modification of the strengthened Legendre condition? The aim of this paper is to answer these questions. Note that the strict growth of the Hamiltonian implies the continuity of the control [1].

We note that, when the data of the optimal control problem are autonomous, the value of the "minimized" Hamiltonian is constant over time. Therefore the problem of minimizing the Hamiltonian

The paper is organized as follows. In section 2 we formulate an optimal control problem and Pontryagin's principle. Section 3 contains a notion of a local quadratic growth condition of the Hamiltonian at the presence of control constraints and a formulation of generalized strengthened Legendre condition such that it holds if and only if a local quadratic growth condition of the Hamiltonian holds (main theorem). In section 4 we prove a sufficient condition for a local quadratic growth of the Hamiltonian which is stronger than the generalized strengthened Legendre condition of section 3. Section 5 gives a proof of the main theorem. In section 6 we give an example of an optimal control problem with continuous control such that for any t the sufficient second order condition is fulfilled in the (local) problem of minimization of the Hamiltonian w.r.t. control under the control constraints, but the local quadratic growth condition of the Hamiltonian at the presence of control constraints does not hold, and therefore, the generalized strengthened Legendre condition of section 3 does not hold too.

2 Statement of the problem, Pontryagin's principle

Consider the following optimal control problem on a fixed interval $[0, T]$:

$$\dot{y}(t) = f(t, u(t), y(t)) \quad \text{for a.a. } t \in [0, T], \tag{1}$$

$$g_j(t, u(t)) \leq 0, \quad \text{for a.a. } t \in [0, T], \quad j = 1, \dots, q, \tag{2}$$

$$\phi_i(y(0), y(T)) \leq 0, \ i = 1, \dots, r_1, \tag{3}$$

$$\phi_i(y(0), y(T)) = 0, \ i = r_1 + 1, \dots, r, \tag{4}$$

$$J(u, y) := \phi_0(y(0), y(T)) \to \min, \tag{5}$$

where $f : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^q$ and $\phi_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $i = 0, \dots, r$, are C^2 (twice continuously differentiable) mappings. Denote by

$\mathcal{U} := L^\infty(0,T;\mathbb{R}^m)$ and $\mathcal{Y} := W^{1,\infty}(0,T;\mathbb{R}^n)$ the control and state space. We consider problem (1)-(5) in the space $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$, and refer to this problem as problem (P). Define the norm of element $w = (u, y) \in \mathcal{W}$ by $\|w\|_{\mathcal{W}} := \|u\|_\infty + \|y\|_{1,\infty}$. Elements of $\mathcal{W}$ satisfying (1)-(4) are said to be *feasible*. The set of feasible points is denoted by $F(P)$. We shall use abbreviations $y(0) = y_0$, $y(T) = y_T$, $(y_0, y_T) = \eta$.

Let us recall the formulation of Pontryagin's principle at the point $w \in F(P)$. Denote by $\mathbb{R}^{n*}$ the dual to $\mathbb{R}^n$ identified with the set of n dimensional row vectors. Set

$$\varphi^\mu(y_0, y_T) = \varphi(y_0, y_T, \mu) := \sum_{i=0}^{r} \mu_i \phi_i(y_0, y_T), \tag{6}$$

where $y_0 = y(0)$, $y_T = y(T)$, $\mu = (\mu_0, \dots \mu_r) \in \mathbb{R}^{(r+1)*}$. Consider the *Hamiltonian function* $H : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n*}$ defined by

$$H(t, u, y, p) = pf(t, u, y). \tag{7}$$

We call *costate associated with* $\mu \in \mathbb{R}^{(r+1)*}$ the solution $p = p^\mu$ (whenever it exists) of

$$\begin{array}{ll} -\dot{p}(t) & = H_y(t, u(t), y(t), p(t)), \quad \text{a.a. } t \in [0, T]; \\ p(0) & = -\varphi^\mu_{y_0}(y(0), y(T)); \;\; p(T) = \varphi^\mu_{y_T}(y(0), y(T)). \end{array} \tag{8}$$

Definition 2.1. We say that $w = (u, y) \in F(P)$ satisfies *Pontryaguin's principle* if there exists a nonzero $\mu \in \mathbb{R}^{(r+1)*}$ and $p \in W^{1,\infty}(0, T, \mathbb{R}^{n*})$ such that (8) holds and

$$\mu_i \geq 0, \quad i = 0, \dots, r_1, \quad \mu_i \phi_i(y(0), y(T)) = 0, \quad i = 1, \dots, r_1, \tag{9}$$

$$\left\{ \begin{array}{l} H(t, u(t), y(t), p(t)) \leq H(t, v, y(t), p(t)), \\ \text{ for all } v \in I\!\!R^m \text{ such that } g(t, v) \leq 0, \text{ for a.a. } t \in (0, T). \end{array} \right. \tag{10}$$

As it is known [5], Pontryagin's principle is a first order necessary condition of a Pontryagin minimum in problem (P).

Let us denote the set of active inequality constraints as

$$I_g(t, u) = \big\{ j \in \{1, \dots, q\} \mid g_j(t, u) = 0 \big\}. \tag{11}$$

Let us denote by g', g'' the partial first or second derivative of g w.r.t. its second argument. We also assume that the following *qualification hypothesis of linear independence* holds:

$$\left\{ \begin{array}{l} \text{the gradients } g_i'(t, u),\ i \in I_g(t, u), \text{ are linearly independent} \\ \text{at each point } u \in \mathbb{R}^m \text{ such that } g(t, u) \leq 0. \end{array} \right. \tag{12}$$

Let us recall a first order necessary condition of a weak minimum, which is a local minimum in $\mathcal{W}$. To this end, define the *augmented Hamiltonian function* $H^a : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n*} \times \mathbb{R}^{q*} \to \mathbb{R}$ by

$$H^a(t, u, y, p, \lambda) = H(t, u, y, p) + \lambda g(t, u). \tag{13}$$

For $w = (u, y) \in F(P)$, denote by Λ_0 the set of all tuples $(\mu, p, \lambda) \in \mathbb{R}^{(r+1)*} \times W^{1,\infty}(0, T; \mathbb{R}^{n*}) \times L^\infty(0, T; \mathbb{R}^{q*})$ of Lagrange multipliers such that the following relations hold

$$\begin{array}{l} \mu_i \geq 0,\ i = 0, \dots, r_1,\ \mu_i \phi_i(\eta) = 0,\ i = 1, \dots, r_1, \\ \lambda(t) \geq 0,\ \lambda(t) g(t, u(t)) = 0,\ \text{a.a. } t \in (0, T), \\ -\dot{p}(t) = H_y(t, w(t), p(t)),\ \text{a.a. } t \in (0, T), \\ p(0) = -\varphi^\mu_{y_0}(\eta),\ p(T) = \varphi^\mu_{y_T}(\eta), \\ H^a_u(t, w(t), p(t), \lambda(t)) = 0,\ \text{a.a. } t \in (0, T);\ |\mu| = 1\}, \end{array} \tag{14}$$

where we recall that $\eta := (y(0), y(T))$. The following result is well-known, [2], [4], [5].

Theorem 2.2. *Let w be a solution of (1)-(5), such that the control is continuous. Then the set Λ_0 is nonempty and bounded, and if $(\mu, p, \lambda) \in \Lambda_0$, then the multiplier $\lambda(t)$ is a continuous function. Moreover, the projector $(\mu, p, \lambda) \to \mu$ is injective on Λ_0.*

Proof. The nonemptiness of Λ_0 is a consequence of Pontryagin's principle and of the fact that there is no singular multiplier in view of (12). The latter implies that the equation

$$H_u(t, u(t), y(t), p(t)) + \lambda g'(t, u(t)) = 0; \quad \lambda_i = 0, \quad i \notin I_g(t, u(t)) \tag{15}$$

has a unique solution $\lambda(t) = ML(t, u(t), y(t), p(t), I_g(t, u(t))$ where ML is a continuous function of its three first arguments. We prove by contradiction that λ is a continuous function. Otherwise there exists a sequence $t_k \to \bar{t}$ such that $\lambda(t_k)$ does not converge to $\lambda(\bar{t})$. Extracting if necessary a subsequence we may assume that $I = I_g(t_k, u(t_k))$ is constant along the sequence and that $\lambda(t_k)$ converges to some $\bar{\lambda}$. Passing to the limit we obtain that $\bar{\lambda}$ satisfies (15) at $t = \bar{t}$, and therefore $\bar{\lambda} = \lambda(\bar{t})$ since (15) has a unique solution.

We finally check the injectivity of $(\mu, p, \lambda) \to \mu$ over Λ_0. Let (μ, p', λ') and (μ, p'', λ'') belong to Λ_0. Then $p := p'' - p'$ is solution of

$$-\dot{p}(t) = p(t) f_y(t, u(t), y(t)), \quad t \in (0, T) \tag{16}$$

with zero initial and final conditions, implying $p = 0$ identically, and $\lambda := \lambda'' - \lambda'$ is solution of (15) with $p(t) = 0$, whose unique solution is zero, as was to be proved. $\square$

Denote by M_0 the set of all $(\mu, p, \lambda) \in \Lambda_0$ such that inequality (10) of Pontryagin's principle is satisfied. Obviously, $M_0 \subset \Lambda_0$, and the condition $M_0 \neq \emptyset$ is equivalent to Pontryagin's principle.

3 Main result

Let $w \in \mathcal{W}$ be such that constraints (1)-(4) are satisfied, $u(t)$ is a continuous function, and that Λ_0 is non empty. Fix an arbitrary tuple $(\mu, p, \lambda) \in \Lambda_0$. We set

$$\delta H[t, v] := H(t, u(t) + v, y(t), p(t)) - H(t, u(t), y(t), p(t)). \tag{17}$$

Definition 3.1. We say that, at the point w, the Hamiltonian satisfies a *local quadratic growth condition* if there exist $\varepsilon > 0$ and $\alpha > 0$ such that for all $t \in [0,T]$ the following inequality holds:

$$\delta H[t,v] \geq \alpha |v|^2 \quad \text{if } v \in \mathbb{R}^m,\ g(t,u(t)+v) \leq 0,\ |v| < \varepsilon. \tag{18}$$

Reminding the definition of H^a in (13), let us denote by

$$\begin{cases} H^a_u(t) & := H^a_u(t,u(t),y(t),p(t),\lambda(t)) \\ H^a_{uu}(t) & := H^a_{uu}(t,u(t),y(t),p(t),\lambda(t)) \end{cases} \tag{19}$$

the first and second derivative w.r.t. u of the augmented Hamiltonian, and adopt a similar notation for the Hamiltonian function H. We shall formulate a generalization of the strengthened Legendre condition using the quadratic form $\langle H^a_{uu}(t)v,v\rangle$ complemented by some special nonnegative term $\varrho(t,v)$ which will be not quadratic, but will be homogeneous of the second degree w.r.t. v. Let us define this additional term.

For any number a, we set $a^+ = \max\{a,0\}$ and $a^- = \max\{-a,0\}$, so that $a^+ \geq 0$, $a^- \geq 0$, and $a = a^+ - a^-$. Denote by

$$\chi_\ell(t) := \chi_{\{g_\ell(\tau,u(\tau))<0\}}(t) \tag{20}$$

the characteristic function of the set $\{\tau \mid g_\ell(\tau,u(\tau)) < 0\}$, $\ell = 1,\dots,q$. If $r > 1$, then, for any $t \in [0,T]$ and any $v \in \mathbb{R}^m$, we set

$$\varrho(t,v) = \sum_{j=1}^{q} \max_{1\leq \ell \leq q} \left\{ \frac{\lambda_j(t)}{|g_\ell(t,u(t))|} \chi_\ell(t) \big(g'_j(t,u(t))v\big)^- \big(g'_\ell(t,u(t))v\big)^+ \right\}. \tag{21}$$

Here, by definition,

$$\frac{\lambda_j(t)}{|g_\ell(t,u(t))|}\chi_\ell(t) = 0 \quad \text{if} \quad g_\ell(t,u(t)) = 0, \quad \ell, j = 1,\dots,q.$$

Particularly, for $q = 2$ the function ϱ has the form

$$\begin{aligned} \varrho(t,v) \;&=\; \tfrac{\lambda_1(t)}{|g_2(t,u(t))|}\chi_2(t)\big(g'_1(t,u(t))v\big)^-\big(g'_2(t,u(t))v\big)^+ \\ &+\; \tfrac{\lambda_2(t)}{|g_1(t,u(t))|}\chi_1(t)\big(g'_2(t,u(t))v\big)^-\big(g'_1(t,u(t))v\big)^+. \end{aligned} \tag{22}$$

In the case $q = 1$, we set $\varrho(t,v) \equiv 0$. For any $\Delta > 0$ and any $t \in [0,T]$, denote by $\mathcal{C}_t(\Delta)$ the set of all vectors $v \in \mathbb{R}^m$ satisfying, for all $j = 1,\dots,q$:

$$\begin{cases} g'_j(t,u(t))v & \leq 0, \quad \text{if } g_j(t,u(t)) = 0, \\ g'_j(t,u(t))v & = 0, \quad \text{if } \lambda_j(t) > \Delta. \end{cases} \tag{23}$$

Definition 3.2. We say that the Hamiltonian satisfies the *generalized strengthened Legendre condition* if

$$\begin{cases} \text{There exist } \alpha > 0 \text{ and } \Delta > 0 \text{ such that for all } t \in [0,T]\text{:} \\ \tfrac{1}{2}\langle H^a_{uu}(t)v,v\rangle + \varrho(t,v) \geq \alpha|v|^2, \quad \text{holds for all } v \in \mathcal{C}_t(\Delta). \end{cases} \tag{24}$$

Theorem 3.3. *A local quadratic growth condition for the Hamiltonian is equivalent to the generalized strengthened Legendre condition.*

We note that $\mathcal{C}_t(\Delta)$ is in general a larger set than the local cone C_t of critical directions for the Hamiltonian, i.e. the directions $v \in \mathbb{R}^m$, such that, for all $j = 1, \dots, q$:

$$\begin{cases} g_j'(t, u(t))v & \leq 0, \quad \text{if } g_j(t, u(t)) = 0, \\ g_j'(t, u(t))v & = 0, \quad \text{if } \lambda_j(t) > 0. \end{cases} \tag{25}$$

We shall give a proof of Theorem 3.3, but at first we shall prove a stronger sufficient condition for a local quadratic growth of the Hamiltonian than the generalized strengthened Legendre condition.

4 A simple sufficient condition for local quadratic growth of the Hamiltonian

Consider the following second-order condition for the Hamiltonian:

$$\begin{cases} \text{There exist } \alpha > 0 \text{ and } \Delta > 0 \text{ such that, for all } t \in [0, T]\text{:} \\ \frac{1}{2}\langle H^a_{uu}(t)v, v\rangle \geq \alpha |v|^2, \quad \text{for all } v \in \mathcal{C}_t(\Delta). \end{cases} \tag{26}$$

Let us note that this inequality is stronger than (24), since the function $\varrho(t, v)$ is nonnegative.

Theorem 4.1. *Condition (26) implies a local quadratic growth of the Hamiltonian.*

Proof. Assume that (26) holds, whereas the condition of local quadratic growth of the Hamiltonian does not. Then there exist sequences $\{t_k\}$ and $\{v_k\}$ such that $v_k \to 0$ and

$$\delta H[t_k, v_k] \leq o(|v_k|^2), \tag{27}$$

$$g(t_k, u(t_k) + v_k) \leq 0, \tag{28}$$

where $\delta H[t, v]$ was defined in (17). Taking, if necessary, a subsequence, we obtain that

1) $t_k \to t'$, where $t' \in [0, T]$;

2) there exist four sets of indices I_0, I_-, I_{00}, I_{0+} such that

$$\begin{cases} I_0 \cup I_- = \{1, \dots, q\}, \quad I_0 \cap I_- = \emptyset, \\ g_i(t_k, u(t_k)) = 0, \text{for all } i \in I_0, \\ g_i(t_k, u(t_k)) < 0, \text{for all } i \in I_-, \end{cases} \tag{29}$$

and

$$\begin{cases} I_{00} & = \{i \in I_0 \mid \lambda_i(t') = 0\}, \\ I_{0+} & = \{i \in I_0 \mid \lambda_i(t') > 0\}. \end{cases} \tag{30}$$

From conditions (27),(28), and the definition of the set I_0 it follows that

$$H_u(t_k)v_k + \tfrac{1}{2}\langle H_{uu}(t_k)v_k, v_k\rangle \leq o(|v_k|^2), \tag{31}$$

$$g_i'(t_k, u(t_k))v_k + \tfrac{1}{2}\langle g_i''(t_k, u(t_k))v_k, v_k\rangle \leq o(|v_k|^2), \text{ for all } i \in I_0. \tag{32}$$

The last inequality implies

$$(g_i'(t_k, u(t_k))v_k)^+ = O(|v_k|^2), \quad \text{for all } i \in I_0. \tag{33}$$

Moreover, we have

$$0 = H^a_u(t_k) = H_u(t_k) + \sum_{i=1}^{q} \lambda_i(t_k) g'_i(t_k, u(t_k)). \tag{34}$$

It follows with (31) and (33) that

$$\begin{array}{rcl} \sum_{i\in I_0} \lambda_i(t_k)(g'_i(t_k,u(t_k))v_k)^- & = & \sum_{i\in I_0} \lambda_i(t_k)(g'_i(t_k,u(t_k))v_k)^+ \\ & & -\sum_{i\in I_0} \lambda_i(t_k) g'_i(t_k,u(t_k))v_k \\ & \le & H_u(t_k)v_k + O(|v_k|^2) \le O(|v_k|^2) \end{array} \tag{35}$$

We deduce that

$$(g'_i(t_k,u(t_k))v_k)^- = O(|v_k|^2), \quad \text{for all } i \in I_{0+}. \tag{36}$$

This and (33) imply the existence of a sequence $\{\hat{v}_k\}$ such that

$$\left\{ \begin{array}{rll} |\hat{v}_k| & = O(|v_k|^2), & \\ g'_i(t_k,u(t_k))(v_k+\hat{v}_k) & \le 0, & i \in I_0, \\ g'_i(t_k,u(t_k))(v_k+\hat{v}_k) & = 0, & i \in I_{0+}. \\ g_i(t_k,u(t_k)) + g'_i(t_k,u(t_k))(v_k+\hat{v}_k) & \le 0, & i \in I_-. \end{array} \right. \tag{37}$$

Setting $v'_k := v_k + \hat{v}_k$, we obviously have

$$|v'_k|^2 = |v_k|^2 + o(|v_k|^2), \tag{38}$$

$$\langle H^a_{uu}(t_k)v'_k, v'_k\rangle = \langle H^a_{uu}(t_k)v_k, v_k\rangle + o(|v_k|^2). \tag{39}$$

By the definition of I_{0+} we have that $v'_k \in \mathcal{C}_t(\Delta)$. Using (26), we deduce that $\frac{1}{2}\langle H^a_{uu}(t_k)v'_k, v'_k\rangle \ge \alpha|v'_k|^2$, which combined with (38) and (39) implies

$$\tfrac{1}{2}\langle H^a_{uu}(t_k)v_k, v_k\rangle \ge \alpha|v_k|^2 + o(|v_k|^2). \tag{40}$$

On the other hand, multiplying (32) by $\lambda_i(t_k)$, summing over i and adding the result to (31), we obtain that $\langle H^a_{uu}(t_k)v_k, v_k\rangle \le o(|v_k|^2)$, which gives the needed contradiction. □

5 Proof of theorem 3.3

Sufficiency of the generalized strengthened Legendre condition for a local quadratic growth of the Hamiltonian. Here we shall prove that the generalized strengthened Legendre condition (24) implies a local quadratic growth condition of the Hamiltonian. At the beginning of the proof the arguments will be similar to those in the previous section.

Indeed, let us assume that condition (24) holds, whereas the local quadratic growth condition of the Hamiltonian does not. All the beginning of the proof of theorem 4.1 is valid, until equation (39). Since $\lambda_i(t_k) \to \lambda_i(t') = 0$, for all $i \in I_{00}$, in view of (37), we have that, for large enough k:

$$\begin{array}{rl} & \lambda(t_k)\left(g'(t_k,u(t_k))v_k + \frac{1}{2}\langle g''(t_k,u(t_k))v_k, v_k\rangle\right)^- \\ \ge & \sum_{i\in I_{00}} \lambda_i(t_k)\left(g'_i(t_k,u(t_k))v_k + \frac{1}{2}\langle g''_i(t_k,u(t_k))v_k, v_k\rangle\right)^- \\ = & \sum_{i\in I_{00}} \lambda_i(t_k)\left(g'_i(t_k,u(t_k))v_k\right)^- + o(|v_k|^2) \\ = & \sum_{i\in I_{00}} \lambda_i(t_k)\left(g'_i(t_k,u(t_k))(v'_k)\right)^- + o(|v_k|^2) \\ = & \sum_{i\in I_0} \lambda_i(t_k)\left(g'_i(t_k,u(t_k))(v'_k)\right)^- + o(|v_k|^2). \end{array}$$

We also have

$$\sum_{i\in I_0} \lambda_i(t_k)\left(g_i'(t_k,u(t_k))(v_k')\right)^- = \lambda(t_k)\left(g'(t_k,u(t_k))v_k'\right)^- .$$

Consequently,

$$\begin{array}{rl} \lambda(t_k)\left(g(t_k,u(t_k)+v_k)\right)^- & \\ = & \lambda(t_k)\left(g'(t_k,u(t_k))v_k + \frac{1}{2}\langle g''(t_k,u(t_k))v_k, v_k\rangle\right)^- + o(|v_k|^2) \\ \geq & \lambda(t_k)\left(g'(t_k,u(t_k))(v_k')\right)^- + o(|v_k|^2). \end{array} \tag{41}$$

Then setting

$$\delta H^a[t,v] := H^a(t,u(t)+v,y(t),p(t)) - H^a(t,u(t),y(t),p(t)), \tag{42}$$

we have using (27)-(28) and (38)-(39) that

$$\begin{array}{rcl} o(|v_k|^2) & \geq & \delta H[t_k,v_k] \\ & = & \delta H^a[t_k,v_k] + \lambda(t_k)\left(g(t_k,u(t_k)+v_k)\right)^- \\ & \geq & \langle H^a_{uu}(t_k)v_k', v_k'\rangle + \lambda(t_k)\left(g'(t_k,u(t_k))v_k'\right)^- + o(|v_k|^2), \end{array} \tag{43}$$

and by (38) again, we deduce that

$$H^a_{uu}(t_k)v_k', v_k'\rangle + \lambda(t_k)\left(g'(t_k,u(t_k))v_k'\right)^- \leq o(|v_k'|^2). \tag{44}$$

It follows from relation the last relation in (37) that

$$(g_\ell(t_k,u(t_k))v_k')^+ \leq |g_\ell(t_k,u(t_k))|, \quad \ell\in I_-. \tag{45}$$

Consequently,

$$\frac{(g_\ell(t_k,u(t_k))v_k')^+}{|g_\ell(t_k,u(t_k))|} \leq 1, \quad \ell\in I_-. \tag{46}$$

This implies that

$$\max_{\ell\in I_-} \frac{(g_\ell(t_k,u(t_k))v_k')^+}{|g_\ell(t_k,u(t_k))|} \leq 1. \tag{47}$$

Multiplying this inequality by $\lambda_j(t_k)(g_j(t_k,u(t_k))v_k')^-$ and summing on j, we get

$$\begin{array}{l} \displaystyle\sum_j \max_{\ell\in I_-} \frac{\lambda_j(t_k)}{|g_\ell(t_k,u(t_k))|}(g_j'(t_k,u(t_k))v_k')^-(g_\ell(t_k,u(t_k))v_k')^+ \\ \qquad\qquad \leq \sum_j \lambda_j(t_k)(g_j(t_k,u(t_k))v_k')^-, \end{array}$$

that is

$$\varrho(t_k,v_k') \leq \sum_j \lambda_j(t_k)(g_j'(t_k,u(t_k))v_k')^-. \tag{48}$$

Relations (44) and (48) imply

$$\tfrac{1}{2}\langle H^a_{uu}(t_k)v_k', v_k'\rangle + \varrho(t_k,v_k') \leq o(v_k'^2). \tag{49}$$

Moreover, by (37), for any $\Delta>0$, we have $v_k'\in\mathcal{C}_{t_k}(\Delta)$ for all sufficiently large k. This along with relation (49) contradicts condition (24). $\square$

Necessity of the generalized strengthened Legendre condition for a local quadratic growth of the Hamiltonian. Now we must prove that a local quadratic growth of the Hamiltonian implies condition (24). Assume the contrary: a local quadratic growth of the Hamiltonian holds, but condition (24) does not hold, i.e., for any $\Delta > 0$ and any $\alpha > 0$, there exist $t \in [0,T]$ and $v \in \mathcal{C}_t(\Delta)$ such that

$$\langle H^a_{uu}(t)v, v\rangle + \varrho(t,v) < \alpha|v|^2.$$

Equivalently, there exist sequences $t_k \in [0,T]$, $\Delta_k \to +0$, $\alpha_k \to +0$, $v_k \in \mathcal{C}_{t_k}(\Delta_k)$, $v_k \neq 0$, such that

$$\langle H^a_{uu}(t_k)v_k, v_k\rangle + \varrho(t_k,v_k) < \alpha_k|v_k|^2. \tag{50}$$

Again, we can take a subsequences such that $t_k \to t' \in [0,T]$ and then there are two sets of indices I_0 and I_- such that

$$I_0 \cup I_- = \{1,\dots,q\}, \quad I_0 \cap I_- = \emptyset,$$

and if $i \in I_0$, then $g_i(t_k,u(t_k)) = 0\ \forall k$; if $i \in I_-$, then $g_i(t_k,u(t_k)) < 0\ \forall k$. Since $v_k \in \mathcal{C}_{t_k}(\Delta_k)$, we have

$$g'_j(t_k,u(t_k))v_k \le 0, \quad i \in I_0,$$

$$g'_j(t_k,u(t_k))v_k = 0 \text{ if } \lambda_i(t_k) > \Delta_k, \quad i \in I_0.$$

Since all terms of inequality (50) are homogeneous functions (of the second degree), multiplying, if necessary, v_k by positive number, we obtain that the following three conditions hold for each k:

$$\left\{\begin{array}{ll} \text{(a)} & |v_k| \le \sqrt{\Delta_k}; \\ \text{(b)} & g(t_k,u(t_k)) + g'(t_k,u(t_k))v_k \le 0; \\ \text{(c)} & \text{If } |v_k| < \sqrt{\Delta_k}, \text{ then } g(t_k,u(t_k)) \neq 0 \text{ (i.e., } I_- \neq \emptyset) \text{ and} \\ & \max\limits_{\ell\in I_-}\Big\{g_\ell(t_k,u(t_k)) + g'_\ell(t_k,u(t_k))v_k\Big\} = 0. \end{array}\right. \tag{51}$$

The last relation is equivalent to

$$\max_{\ell\in I_-}\Big\{-|g_\ell(t_k,u(t_k))| + \Big(g'_\ell(t_k,u(t_k))v_k\Big)^+\Big\} = 0, \tag{52}$$

or

$$\max_{\ell\in I_-}\frac{\Big(g'_\ell(t_k,u(t_k))v_k\Big)^+}{|g_\ell(t_k,u(t_k))|} = 1. \tag{53}$$

So, if $|v_k| < \sqrt{\Delta_k}$, then

$$\begin{aligned} &\varrho(t_k,v_k) \\ &= \sum_{j=1}^q \max_{1\le\ell\le r}\left\{\frac{\lambda_j(t_k)}{|g_\ell(t_k,u(t_k)|}\chi_\ell(t_k)\big(g'_j(t_k,u(t_k))v_k\big)^-\big(g'_\ell(t_k,u(t_k))v_k\big)^+\right\} \\ &= \sum_{j=1}^q \lambda_j(t_k)\big(g'_j(t_k,u(t_k))v_k\big)^-\max_{\ell\in I_-}\left\{\frac{\Big(g'_\ell(t_k,u(t_k))v_k\Big)^+}{|g_\ell((t_k,u(t_k)))|}\right\} \\ &= \sum_{j=1}^q \lambda_j(t_k)\big(g'_j(t_k,u(t_k))v_k\big)^-. \end{aligned} \tag{54}$$

And if $|v_k| = \sqrt{\Delta_k}$, the following estimate holds

$$\begin{aligned}\sum_{j=1}^{q} \lambda_j(t_k)\big(g_j'(t_k,u(t_k))v_k\big)^- &\leq \Delta_k \sum_{j=1}^{q} \big(g_j'(t_k,u(t_k))v_k\big)^- \\ &\leq M\Delta_k\sqrt{\Delta_k} = M\sqrt{\Delta_k}|v_k|^2,\end{aligned} \tag{55}$$

were $M > 0$ does not depend on k. Thus, from (54) and (55) we get

$$\sum_{j=1}^{q} \lambda_j(t_k)\big(g_j'(t_k,u(t_k))v_k\big)^- \leq \varrho(t_k,v_k) + M\sqrt{\Delta_k}|v_k|^2 \quad \forall\, k. \tag{56}$$

Relations (50) and (56) imply that

$$\tfrac{1}{2}\langle H_{uu}^a(t_k)v_k, v_k\rangle + \sum_{j=1}^{q} \lambda_j(t_k)\big(g_j'(t_k,u(t_k))v_k\big)^- \leq \alpha_k|v_k|^2 + M\sqrt{\Delta_k}|v_k|^2. \tag{57}$$

Set I_{00}, I_{0+} as in (30). Then

$$\lim_{n\to\infty} \lambda_i(t_k) = 0, \forall i \in I_{00}; \quad \lim_{n\to\infty} \lambda_i(t_k) > 0, \forall i \in I_{0+}. \tag{58}$$

Therefore, for any $i \in I_{0+}$, we have

$$g_i'(t_k,u(t_k))v_k = 0 \quad \text{for all } k. \tag{59}$$

From condition (59) and condition (b) it follows that there exists a sequence of corrections $\{\tilde{v}_k\}$ such that

$$|\tilde{v}_k| = O(|v_k|^2), \tag{60}$$

$$g(t_k,u(t_k)+v_k+\tilde{v}_k) \leq 0 \;\text{ for all } k, \tag{61}$$

$$\text{for any } i \in I_{0+}, \quad g_i(t_k,u(t_k)+v_k+\tilde{v}_k) = 0 \;\text{ for all } k. \tag{62}$$

For the sequence $v_k' := v_k + \tilde{v}_k$, we obviously have

$$|v_k'|^2 = |v_k'|^2 + o(|v_k|^2), \tag{63}$$

$$\langle H_{uu}^a(t_k)v_k', v_k'\rangle = \langle H_{uu}^a(t_k)v_k, v_k\rangle + o(|v_k|^2). \tag{64}$$

Moreover, in view of (58) and (60), for any $i \in I_{00}$, we have

$$\begin{aligned}&\lambda_i(t_k)g_i^-(t_k,u(t_k)+v_k+\tilde{v}_k) \\ =\;& \lambda_i(t_k)(g_i'(t_k,u(t_k))v_k + g_i'(t_k,u(t_k))\tilde{v}_k + \\ &\tfrac{1}{2}\langle g_i''(t_k,u(t_k))v_k,v_k\rangle)^- + o(|v_k|^2) \\ =\;& \lambda_i(t_k)(g_i'(t_k,u(t_k))v_k)^- + o(|v_k|^2),\end{aligned} \tag{65}$$

and in view of (62), for any $i \in I_{0+}$, we have

$$0 = \lambda_i(t_k)g_i^-(t_k,u(t_k)+v_k+\tilde{v}_k) \leq \lambda_i(t_k)(g_i'(t_k,u(t_k))v_k)^-. \tag{66}$$

Finally, for any $i \notin I_0$, we have $\lambda_i(t_k) = 0$ for all k. Therefore,

$$\lambda(t_k)g^-(t_k,u(t_k)+v_k+\tilde{v}_k) \leq \lambda(t_k)(g'(t_k,u(t_k))v_k)^- + o(|v_k|^2). \tag{67}$$

From (57) and (67), we get

$$\tfrac{1}{2}\langle H^a_{uu}(t_k)v_k, v_k\rangle + \lambda(t_k)g^-(t_k, u(t_k) + v_k + \tilde{v}_k) \le o(|v_k|^2). \tag{68}$$

Set $v'_k = v_k + \tilde{v}_k$. Then, by virtue of (63) and (64), relation (68) implies

$$\tfrac{1}{2}\langle H^a_{uu}(t_k)v'_k, v'_k\rangle + \lambda(t_k)g^-(t_k, u(t_k) + v'_k) \le o(v'^2_k). \tag{69}$$

Moreover, by virtue of (60) and (61),

$$v'_k \to 0, \quad g(t_k, u(t_k) + v'_k) \le 0 \text{ for all } k. \tag{70}$$

From (61), (69) and (70) we get, using the notation in (17):

$$\begin{aligned}
\delta H[t_k, v'_k] &= \delta H[t_k, v'_k] + \lambda g(t_k, u(t_k) + v'_k) + \lambda g^-(t_k, u(t_k) + v'_k) \\
&= \delta H^a[t_k, v'_k] + \lambda g^-(t_k, u(t_k) + v'_k) \\
&= \tfrac{1}{2}\langle H^a_{uu}(t_k)v'_k, v'_k\rangle + \lambda g^-(t_k, u(t_k) + v'_k) + o(v'^2_k) \le o(v'^2_k).
\end{aligned} \tag{71}$$

Relations (70) and (71) mean that a local quadratic growth of the Hamiltonian does not hold. Thus, we come to a contradiction. The theorem is proved. □

6 Example

We shall give an example of a continuous control $\bar{u}(t)$ such that, for each $t \in [0, T]$, the second order derivative H^a_{uu} of the augmented Hamiltonian is positively definite on the local critical cone C_t, but the condition of a local quadratic growth of the Hamiltonian does not hold. We shall also show that, in this example, the generalized strengthened Legendre condition does not hold. Consider the following problem, where the parameter c has an arbitrary value:

$$\int_0^T \left((u_1(t) - t)^2 + \tfrac{1}{2}tu_2(t) - cu_2^2(t)\right) dt \to \min,$$

$$0 \le u_2(t) \le u_1(t), \quad t \in [0, T].$$

Let us rewrite it as an optimal control problem of the form (1)-(5):

$$\dot{y} = (u_1(t) - t)^2 + \tfrac{1}{2}tu_2(t) - cu_2^2(t), \tag{72}$$

$$y(0) = 0, \tag{73}$$

$$u_2(t) - u_1(t) \le 0, \quad -u_2(t) \le 0, \quad t \in [0, T], \tag{74}$$

$$y(T) \to \min. \tag{75}$$

So, here $u = (u_1, u_2)$, the control constraint $g(u) = (u_2 - u_1, -u_2)$ does not depend on time, and the dynamics is $f(t, u, y) = (u_1 - t)^2 + \frac{1}{2}tu_2 - cu_2^2$. The Hamiltonian function and the augmented Hamiltonian function have the form

$$\begin{aligned}
&H(t, u, y, p) = pf(t, u, y), \\
&H^a(t, u, y, p, a) = H(t, u, y, p) + \lambda_1(u_2 - u_1) - \lambda_2 u_2.
\end{aligned}$$

The adjoint system is:

$$-\dot{p} = 0, \quad p(T) = \mu_0 \ge 0. \tag{76}$$

The local Pontryagin's principle is:

$$H^a_{u_1} = pf_{u_1}(t, u, y) - \lambda_1 = 0; \tag{77}$$
$$H^a_{u_2} = pf_{u_2}(t, u, y) + \lambda_1 - \lambda_2 = 0, \tag{78}$$
$$\lambda_1 \geq 0,\ \lambda_2 \geq 0, \tag{79}$$
$$\lambda_1(u_2 - u_1) = 0,\ \lambda_2 u_2 = 0. \tag{80}$$

If $\mu_0 = 0$ in these conditions, then it follows from (76) that $p = 0$, and hence from (77) and (78) we get $\lambda_1 = \lambda_2 = 0$. Thus, in (76)-(80), we may put $\mu_0 = 1$.

The control $\bar{u}(t) = (\bar{u}_1(t), \bar{u}_2(t)) = (t, 0)$ is feasible in this problem. Set $\bar{y}(t) = 0$. Let us show that the point $(\bar{u}(t), \bar{y}(t))$ satisfies the first order optimality conditions (76)-(80) with $\mu_0 = 1$. From (76) we get $p(t) = 1$. Since

$$f_{u_1}(\bar{u}(t), \bar{y}(t)) = 2(\bar{u}_1(t) - t) = 0,$$

$$f_{u_2}(\bar{u}(t), \bar{y}(t)) = \tfrac{1}{2}t - 2c\bar{u}_2(t) = \tfrac{1}{2}t,$$

from (77) and (78) we obtain

$$\lambda_1(t) \equiv 0, \qquad \lambda_2(t) = \tfrac{1}{2}t,$$

and thus all conditions (76)-(80) are satisfied with

$$\mu_0 = 1,\ p(t) = 1,\ \lambda_1(t) = 0,\ \lambda_2(t) = \tfrac{1}{2}t.$$

For each $t \in [0, T]$, consider the local minimization problem

$$H(t, u, \bar{y}(t), \bar{p}(t)) \to \min_u$$

subject to the constraints

$$0 \leq u_2 \leq u_1$$

and the point $\bar{u}(t)$. The equivalent problem is:

$$f(t, u, \bar{y}(t)) \to \min_u, \quad 0 \leq u_2 \leq u_1.$$

Define the second order optimality conditions in this problem, at the point $\bar{u}(t)$. The quadratic form is:

$$\omega(v) := \tfrac{1}{2}\langle f_{uu}(\bar{u}(t), \bar{y}(t))v, v\rangle = v_1^2 - cv_2^2.$$

Define the critical cone C_t at the point $\bar{u}(t)$.

For $t \in (0, T]$, only the constraint $u_2 \geq 0$ is active, and the correspondent Lagrange multiplier $\lambda_2(t) = \frac{1}{2}t$ is positive. Therefore, for $t > 0$, the critical cone is:

$$C_t := \mathbb{R} \times \{0\},$$

the quadratic form on C_t is equal to v_1^2, and so the second-order sufficient conditions are satisfied uniformly.

For $t = 0$, we have $\bar{u}(0) = (0, 0)$, and both constraints $u_2 \leq u_1$ and $0 \leq u_2$ are active with $\lambda_1(0) = \lambda_2(0) = 0$. So, for $t = 0$, the critical cone is:

$$C_0 = \{v \in \mathbb{R}^2;\ v_1 \geq v_2,\ v_2 \geq 0\},$$

and the worst case being when $v_2 = v_1$:

$$\omega(v) \geq (1-c)v_1^2 \geq \tfrac{1}{2}(1-c)(v_1^2+v_2^2)\ \forall v \in C_0.$$

Consequently, for $c < 1$ the uniform second-order sufficient conditions are satisfied on $[0,T]$. Does it guarantee the uniform quadratic growth of the Hamiltonian under the control constraints?

The answer is no. Indeed, for $\tilde{u}(t) = (t,t)$ and corresponding $\tilde{y}(t) = (\frac{1}{2}-c)\frac{t^3}{3}$ we have $f(t,\tilde{u}(t),\tilde{y}(t)) = (\frac{1}{2}-c)t^2$, while $f(t,\bar{u}(t),\bar{y}(t)) = 0$. So, we see that the uniform quadratic growth does not hold when $c \geq \frac{1}{2}$.

Now, using Theorem 3.3, we will find more precisely the values of the parameter c, for which the quadratic growth condition of the Hamiltonian does not hold. According to relation (22),

$$\varrho(t,v) = \frac{\lambda_2(t)}{|g_1(\bar{u}(t))|}\chi_{(0,T]}(t)(v_2)^+(v_2-v_1)^+,$$

since $(-v_2)^- = (v_2)^+$. Using the relations $\lambda_2(t) = t/2$ and $|g_1(\bar{u}(t))| = |\bar{u}_2(t) - \bar{u}_1(t)| = |0-t| = t$, we obtain $\lambda_2(t)/|g_1(\bar{u}(t))| = 1/2$, and hence

$$\varrho(t,v) = \tfrac{1}{2}(v_2)^+(v_2-v_1)^+\chi_{(0,T]}(t).$$

Consequently,

$$\tfrac{1}{2}\langle H^a_{uu}v,v\rangle + \varrho(t,v) = v_1^2 - cv_2^2 + \tfrac{1}{2}(v_2)^+(v_2-v_1)^+\chi_{(0,T]}(t). \tag{81}$$

According to Theorem 3.3, the necessary and sufficient condition for quadratic growth of the Hamiltonian has the following form: there exist $\Delta > 0$ and $\alpha > 0$ such that for all $t \in [0,T]$ we have

$$\hat{\omega}(v) := v_1^2 - cv_2^2 + \tfrac{1}{2}(v_2)^+(v_2-v_1)^+\chi_{(0,T]}(t) \geq \alpha|v|^2 \quad \forall v \in \mathcal{C}_t(\Delta), \tag{82}$$

where the cone $\mathcal{C}_t(\Delta)$ is defined by the conditions

$$v_2 \geq 0 \quad \text{if} \quad 0 < t < 2\Delta; \quad v_2 = 0 \quad \text{if} \quad 2\Delta \leq t \leq T \tag{83}$$

(here, the condition $0 \leq t < 2\Delta$ corresponds to the condition $\lambda_2(t) < \Delta$), and

$$\mathcal{C}_0(\Delta) = \{v \in \mathbb{R}^2;\ v_1 \geq v_2,\ v_2 \geq 0\}.$$

Note that $(v_2-v_1)^+ = 0$ for any $v \in \mathcal{C}_0(\Delta)$, $\Delta > 0$. Therefore, (82) is equivalent to

$$\tilde{\omega}(v) := v_1^2 - cv_2^2 + \tfrac{1}{2}(v_2)^+(v_2-v_1)^+ \geq \alpha|v|^2 \quad \forall v \in \mathcal{C}_t(\Delta), \tag{84}$$

This form does not depend on t, therefore we must consider it on a widest cone $\mathcal{C}_t(\Delta)$ which corresponds to the case $0 < t < 2\Delta$. For this case the cone is given by the inequality $v_2 \geq 0$. Further, it is clear that the positive definiteness of the form $\tilde{\omega}$ on the half space $v_2 \geq 0$ is equivalent to its positiveness on all nonzero elements of this half space. If $v_2 = 0$ and $v \neq 0$, then $\tilde{\omega}$ is positive. Therefore, let us consider the case $v_2 > 0$. We may set $v_2 = 1$ and $v_1 = s$, where $s \in \mathbb{R}$. Then

$$\tilde{\omega} = s^2 - c + \tfrac{1}{2}(1-s)^+.$$

We have to minimize a nonsmooth function

$$\sigma(s) := s^2 - c + \tfrac{1}{2}(1-s)^+, \qquad s \in \mathbb{R}.$$

If $s \geq 1$, then $\sigma(s) = s^2 - c$, and the minimum of σ on $[1, \infty)$ is attained at the point $s = 1$, moreover,

$$\sigma(1) = 1 - c.$$

If $s \leq 1$, then $\sigma(s) = s^2 - c + \frac{1}{2}(1-s)$, and since the derivative $\sigma'(s) = 2s - \frac{1}{2}$ vanishes at the point $s = 1/4$, this point is the minimum of σ on $(-\infty, 1]$. Since

$$\sigma(1/4) = 7/16 - c < \sigma(1),$$

the point $s = 1/4$ is the absolute minimum of the function $\sigma(s)$ on $\mathbb{R}$, and the minimal value of σ on $\mathbb{R}$ is equal to $7/16 - c$.

Therefore the inequality $7/16 - c > 0$ is equivalent to the condition of the local quadratic growth of the Hamiltonian at the point $(\bar{u}, \bar{y})$. Thus, the local quadratic growth does not hold iff $c \geq 7/16$ (earlier we showed this for $c \geq 1/2$).

Now, it is easy to find a control function which show that a local quadratic growth does not hold for $c \geq 7/16$. Namely, take $\bar{\bar{u}}_1(t) = \bar{\bar{u}}_2(t) = (4/3)t$. This control is feasible, and

$$f(t, \bar{\bar{u}}(t), \bar{\bar{y}}(t)) = \left(\frac{t}{3}\right)^2 + \tfrac{1}{2}t \cdot \frac{4}{3}t - c \cdot \frac{16}{9} \cdot t^2 = \frac{1}{9}(7 - 16c)t^2.$$

Again, we see that for $c \geq 7/16$ the condition of the local quadratic growth of the Hamiltonian at the point $(\bar{u}, \bar{y})$ does not hold.

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