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A new family of second-order absorbing boundary conditions for the acoustic wave equation - Part II : Mathematical and numerical studies of a simplified formulation

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Abstract: We interest ourselves on the mathematical and the numerical properties of a simplified formulation of a new family of absorbing boundary conditions (ABCs) for the acoustic wave equation introduced in the first part of this work. Considering a sound-soft scatterer, we prove the well-posedness of the corresponding boundary value problem and the exponential decay of the solution. We perform a numerical analysis. We propose to use a Discontinuous Galerkin method for the space discretization and by defining a discrete energy we prove the stability of the numerical scheme. Numerical results confirm the theoretical properties we have obtained and illustrate the performances of the new ABCs.

Key-words: Absorbing boundary conditions, acoustic wave equation, well-posedness, exponential decay, Discontinuous Galerkin method

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Une nouvelle famille de conditions aux limites absorbantes pour l'équation des ondes acoustiques -Partie II : Etudes mathématique et numérique d'une formulation simplifiée

Résumé : Nous nous intéressons aux propriétés mathématiques et numériques d'une formulation simplifiée d'une nouvelle famille de conditions aux limites absorbantes (CLAs) pour l'équation des ondes acoustiques introduite dans la première partie de ce travail. En considérant une condition de surface libre sur le bord de l'obstacle, on montre que le problème aux limites correspondant est bien posé et que la solution décroît exponentiellement. On réalise aussi une analyse numérique. On propose d'utiliser une méthode de Galerkine discontinue pour la discrétisation en espace et en définissant une énergie discrète, on montre la stabilité du schéma numérique. Des résultats numériques confirment les propriétés théoriques obtenues et illustrent les performances des nouvelles CLAs.

Mots-clés : Conditions aux limites absorbantes, équation des ondes acoustiques, problème bien posé, décroissance exponentielle, méthode de Galerkine discontinue

1 Introduction

This work follows a previous one in which we have constructed a new family of ABCs depending on a parameter. These conditions are obtained from the micro-diagonalization of the acoustic wave equation, following the principle of factorization described in [15] and formerly used by Engquist and Majda [9] for the construction of ABCs for flat surfaces. We have obtained second-order conditions which can be applied on arbitrarily shaped surfaces and in [6], we have shown that the corresponding boundary value problem is well-posed. Now, regarding the numerical properties of the conditions, we have shown that these conditions can be included variationaly only if we consider a high-order functional that is obtained after deriving the wave equation. Nevertheless, this might lead to create spurious solutions because we have shown that the corresponding system admits stationnary solutions that are non physical. An alternative consists in introducing an auxiliary unknown. By this way, the new conditions can be included variationaly. The aim of this paper is to study the boundary value problem that is obtained by introducing an auxiliary unknown. We prove that the problem is well-posed and we show that the solution is exponentially decreasing. The ABCs that we have constructed in [6] are given by

$$\partial_t \left(\partial_n u + \partial_t u\right) = \left(\frac{\kappa}{4} - \gamma\right) \partial_n u - \left(\frac{\kappa}{4} + \gamma\right) \partial_t u \text{ on } \Sigma. \tag{1.1}$$

They are second-order conditions depending on a parameter γ. When $\gamma = 0$, we recover the condition

$$\partial_t \left(\partial_n u + \partial_t u\right) = \frac{\kappa}{4}\partial_n u - \frac{\kappa}{4}\partial_t u \text{ on } \Sigma$$

and we have shown that this condition is equivalent to the curvature condition

$$\partial_n u + \partial_t u + \frac{\kappa}{2} u = 0 \text{ on } \Sigma.$$

Regarding numerical implementation, we have, at least formally

$$\int_\Omega \partial_t^2 u \varphi dx + \int_\Omega \nabla u \cdot \nabla \varphi dx = \int_\Sigma \partial_n u \varphi d\sigma.$$

Hence the new condition can be included variationaly if it reads as $\partial_n u = Bu$ on Σ where B is a boundary operator. If B is differential, the sparsity of the discretization matrices is preserved. Now, including (1.1) directly, B is pseudo-differential. Hence the condition generates a high computational cost. That is why we propose to define an auxiliary unknown to obtain an ABC easier to introduce in the formulation. The ABC (1.1) is rewritten as follows :

$$\partial_n u = -\partial_t u - \frac{\kappa}{2} \left(\partial_t - \frac{\kappa}{4} + \gamma\right)^{-1} \partial_t u \text{ on } \Sigma$$

and we define ψ as the surface field satisfying

$$\left(\partial_t - \frac{\kappa}{4} + \gamma\right) \psi = \partial_t u \text{ on } \Sigma.$$

Then the solution u satisfies

$$\partial_n u + \partial_t u + \frac{\kappa}{2} \psi = 0 \text{ on } \Sigma,$$

which can be easily included into the variational formulation.

2 Mathematical analysis

We consider the more general mixed problem: find $(u, \partial_t u, \psi)$ solution to

$$\begin{cases} \partial_t^2 u - \triangle u = 0 & \text{in } \Omega \times (0, +\infty)\,; \\ u(0,x) = u_0(x),\, \partial_t u(0,x) = u_1(x) & \text{in } \Omega; \\ \psi(0,x) = \psi_0(x) & \text{on } \Sigma; \\ u = 0 & \text{on } \Gamma \times (0, +\infty)\,; \\ \partial_n u + \partial_t u + \dfrac{\kappa}{2}\psi = 0 & \text{on } \Sigma \times (0, +\infty)\,; \\ \left(\partial_t - \dfrac{\kappa}{4} + \gamma\right)\psi = \partial_t u & \text{on } \Sigma \times (0, +\infty)\,. \end{cases} \tag{2.1}$$

The function κ is the curvature of Σ and γ is a regular parameter defined on Σ. The domain Ω is a bounded domain and its boundary $\partial\Omega = \Gamma \cup \Sigma$ is assumed to be regular, with $\Gamma \cap \Sigma = \varnothing$ (see Fig 1).
In the following, we will assume that the parameter function γ checks the following condtion

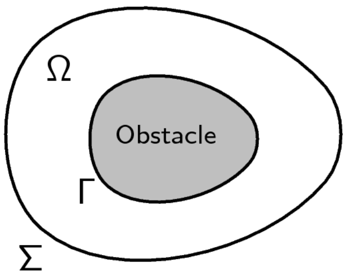

Figure 1: Studied domain

$$\gamma(x) > \frac{\kappa(x)}{4}, \forall x \in \Sigma. \tag{2.2}$$

We propose to study the problem (2.1) by using the theory of Hille-Yosida. We first transform (2.1) in a first order system in time. We introduce an auxiliary unknown v defined by $v = \partial_t u$. The vector $U = (u, v, \psi)$ is thus solution to

$$\frac{dU}{dt} = AU,\, A = \begin{pmatrix} 0 & Id & 0 \\ \triangle & 0 & 0 \\ 0 & 1 & \frac{\kappa}{4} - \gamma \end{pmatrix} \tag{2.3}$$

with the boundary conditions

$$u = 0 \text{ on } \Gamma \times (0, +\infty) \tag{2.4}$$

and

$$\partial_n u + \partial_t u + \frac{\kappa}{2}\psi = 0 \text{ on } \Sigma \times (0, +\infty)\,. \tag{2.5}$$

In the following, we will concentrate on the problem (2.3) and we will interest ourselves on its solution in suitable Hilbert spaces.
Let us first introduce H as the product space defined by

$$H = H^1_\Gamma(\Omega) \times L^2(\Omega) \times L^2(\Sigma)$$

where

$$H^1_\Gamma(\Omega) = \{h_1 \in H^1(\Omega),\, h_1 = 0 \text{ on } \Gamma\}.$$

We equip H with the Hilbertian graph norm

$$\|(h_1,h_2,h_3)\|_H = \left(\|h_1\|^2_{L^2(\Omega)} + \|\nabla h_1\|^2_{L^2(\Omega)} + \|h_2\|^2_{L^2(\Omega)} + \|h_3\|^2_{L^2(\Sigma)}\right)^{1/2}.$$

Let V be the product space defined by

$$V = \{(v_1,v_2,\varphi) \in H, A(v_1,v_2,\varphi) \in H, \partial_n v_1 + v_2 + \frac{\kappa}{2}\varphi = 0 \text{ on } \Sigma\}.$$

The space V corresponds to the domain of A. By enforcing $A(v_1,v_2,\varphi) \in H$, we improve the regularity of each component of the unknown. Indeed, we then have $v_2 \in H^1_\Gamma(\Omega)$ and $\triangle v_1 \in L^2(\Omega)$. Then $v_2 \in H^1_\Gamma(\Omega)$ implies that $v_{2|_\Sigma}$ is defined in $H^{1/2}(\Sigma)$ and $\triangle v_1 \in L^2(\Omega)$ implies that $\partial_n v_{1|_\Sigma} \in H^{-1/2}(\Sigma)$, knowing that $v_1 \in H^1(\Omega)$. Moreover, the relation $\partial_n v_1 + v_2 + \frac{\kappa}{2}\varphi = 0$ on Σ improves the regularity of $\partial_n v_{1|_\Sigma}$ since $v_2 + \frac{\kappa}{2}\varphi \in L^2(\Sigma)$. Hence, as a résumé, we have

$$\begin{aligned} V = \ & \{(v_1,v_2,\varphi) \in H,\, \triangle v_1 \in L^2(\Omega),\, v_2 \in H^1_\Gamma(\Omega),\, \partial_n v_{1|_\Sigma} \in L^2(\Sigma), \\ & \partial_n v_1 + v_2 + \frac{\kappa}{2}\varphi = 0 \text{ on } \Sigma\}. \end{aligned}$$

First of all we recall the Green formula we will use: for all $(u,v) \in H^1(\Omega)\times H^1(\Omega)$ such that $\triangle u \in L^2(\Omega)$, we have

$$\int_\Omega \triangle u v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + < \partial_n u, v >_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} . \tag{2.6}$$

Lemma 2.1. *Let κ be given in $L^\infty(\Sigma)$ and such that $\min_{x\in\Sigma}\kappa(x) = \kappa_0 > 0$. Then, for all $h \in H$, the quantity*

$$|||h||| = \left(\int_\Omega |\nabla h_1|^2 + |h_2|^2 \, dx + \int_\Sigma \frac{\kappa}{2}|h_3|^2 \, d\sigma\right)^{1/2}$$

is a norm on H equivalent to the norm $\|h\|_H$.

Proof. It is well-known that $\|\nabla \cdot\|_{L^2(\Omega)}$ defines a norm on $H^1_\Gamma(\Omega)$ which is equivalent to the standard nom in $H^1(\Omega)$, as a consequence of the Poincaré inequality. Hence, since the curvature κ is supposed to be in $L^\infty(\Sigma)$ with $\min_{x\in\Sigma}|\kappa(x)| > 0$, it is straightforward that $|||\cdot|||$ defines a norm on H equivalent to the conventional norm $\|\cdot\|_H$. □

In the following, we denote by (,) the scalar product derived from the norm $|||\ |||$.

Lemma 2.2. *Let κ and γ be given such that (2.2) is checked. Then, for all $v \in V$, we have*

$$(Av, v) \leq 0.$$

Proof. Let $v = (v_1, v_2, \varphi)$ in V. By definition of A, $Av = \left(v_2, \triangle v_1, v_2 + \left(\frac{\kappa}{4} - \gamma\right)\varphi\right)$. Then, we have

$$(Av, v) = \int_\Omega \nabla v_2 \cdot \nabla v_1 \, dx + \int_\Omega \triangle v_1 v_2 \, dx + \int_\Sigma \frac{\kappa}{2}\left(v_2 + \left(\frac{\kappa}{4} - \gamma\right)\varphi\right)\varphi \, d\sigma.$$

Using the Green formula (2.6), we get

$$\begin{aligned}(Av, v) = \quad & \int_\Omega \nabla v_2 \cdot \nabla v_1 \, dx - \int_\Omega \nabla v_2 \cdot \nabla v_1 \, dx + \int_\Sigma \partial_n v_1 v_2 \, d\sigma + \int_\Sigma \frac{\kappa}{2} v_2 \varphi \, d\sigma \\ & + < \partial_n v_1, \, v_2 >_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} + \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)|\varphi|^2 \, d\sigma.\end{aligned}$$

Moreover, in V, we have $v_{1|_\Gamma} = v_{2|_\Gamma} = 0$ and on Σ, $\partial_n v_1 = -v_2 - \frac{\kappa}{2}\varphi$. Hence,

$$(Av, v) = -\int_\Sigma |v_2|^2 \, d\sigma - \int_\Sigma \frac{\kappa}{2}\varphi v_2 \, d\sigma + \int_\Sigma \frac{\kappa}{2}\varphi v_2 \, d\sigma + \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)|\varphi|^2 \, d\sigma.$$

Now, since $\frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right) \leq 0$ on Σ, we get that for all v in V

$$(Av, v) = -\int_\Sigma |v_2|^2 \, d\sigma + \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)|\varphi|^2 \, d\sigma \leq 0,$$

which completes the proof of Lemma 2.2. □

Lemma 2.3. *The operator A, with domain V, is maximal.*

Proof. Given $f = (f_1, f_2, f_3)$ in H, we consider the following mixed problem: find $v \in V$ such that $(A - I)\, v = f$.
We thus seek $v = (v_1, v_2, \varphi) \in V$ such that

$$\begin{cases} v_2 - v_1 = f_1 \text{ in } \Omega; \\ \triangle v_1 - v_2 = f_2 \text{ in } \Omega; \\ v_2 + \left(\frac{\kappa}{4} - \gamma - 1\right)\varphi = f_3 \text{ on } \Sigma; \\ v_1 = 0 \text{ on } \Gamma; \\ \partial_n v_1 + v_2 + \frac{\kappa}{2}\varphi = 0 \text{ on } \Sigma. \end{cases} \tag{2.7}$$

First of all, we assume that the problem (2.7) has a solution in V. Then, by removing v_2 thanks to the equation

$$v_2 = f_1 + v_1 \text{ in } \Omega,$$

and φ thanks to the third equation

$$\varphi = \frac{f_3 - f_1 - v_1}{\frac{\kappa}{4} - \gamma - 1}. \tag{2.8}$$

we obtain that v_1 is solution to the boundary-value problem

$$\left\{\begin{array}{l} -\triangle v_1 + v_1 = \tilde{f} \text{ in } \Omega; \\ \\ v_1 = 0 \text{ on } \Gamma; \\ \\ \partial_n v_1 + \alpha v_1 = \tilde{g} \text{ on } \Sigma. \end{array}\right. \tag{2.9}$$

with

$$\tilde{f} := -\left(f_2 + f_1\right) \text{ in } L^2(\Omega),$$

$$\tilde{g} := -\left(1 + \frac{\kappa}{2\left(1+\gamma-\frac{\kappa}{4}\right)}\right) f_1 + \frac{\kappa}{2\left(1+\gamma-\frac{\kappa}{4}\right)} f_3 \text{ in } L^2(\Sigma),$$

and

$$\alpha = 1 + \frac{\kappa}{2\left(1+\gamma-\frac{\kappa}{4}\right)} > 0.$$

It is obvious that, since $\kappa \in L^\infty(\Sigma)$ and $\alpha > 0$ by hypothesis, the functional

$$|v|_{1,\alpha} = \left(\|v\|^2_{H^1(\Omega)} + \int_\Sigma \alpha |v|^2 d\sigma\right)^{1/2}$$

defines a norm in $H^1(\Omega)$ which is equivalent to the standard norm $\| \, \|_{H^1(\Omega)}$. Let $\mathcal{T}\left(\overline{\Omega}\right)$ be the space of test functions defined by

$$\mathcal{T}\left(\overline{\Omega}\right) = \{\phi \in \mathcal{D}\left(\overline{\Omega}\right),\, \phi_{|_\Gamma} = 0\}$$

It is dense in $H^1_\Gamma(\Omega)$ and if we assume that the problem (2.9) has a solution, we have

$$\forall \phi \in \mathcal{T}\left(\overline{\Omega}\right),\, -\int_\Omega \triangle v_1 \phi \, dx + \int_\Omega v_1 \phi \, dx = \int_\Omega \tilde{f} \phi \, dx.$$

By using the Green formula (2.6), we get

$$\forall \phi \in \mathcal{T}\left(\overline{\Omega}\right),\, \int_\Omega \nabla v_1 \nabla \phi \, dx - < \partial_n v_1,\, \phi >_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} + \int_\Omega v_1 \phi \, dx = \int_\Omega \tilde{f} \phi \, dx.$$

Moreover, $\phi_{|_\Gamma} = 0$. Therefore,

$$\forall \phi \in \mathcal{T}\left(\overline{\Omega}\right),\, \int_\Omega \nabla v_1 \nabla \phi \, dx - \int_\Sigma \partial_n v_1 \phi \, d\sigma + \int_\Omega v_1 \phi \, dx = \int_\Omega \tilde{f} \phi \, dx.$$

Then by using that $\partial_n v_1 = \tilde{g} - \alpha v_1$ on Σ, we obtain

$$\forall \phi \in \mathcal{T}\left(\overline{\Omega}\right),\, \int_\Omega \nabla v_1 \nabla \phi \, dx + \int_\Sigma \alpha v_1 \phi \, d\sigma + \int_\Omega v_1 \phi \, dx = \int_\Omega \tilde{f} \phi \, dx + \int_\Sigma \tilde{g} \phi \, d\sigma. \tag{2.10}$$

Let $a(\cdot,\cdot)$ the bilinear form defined by

$$a(v_1, \phi) = \int_\Omega \nabla v_1 \nabla \phi \, dx + \int_\Sigma \alpha v_1 \phi \, d\sigma + \int_\Omega v_1 \phi \, dx.$$

It is obvious that $a(\cdot,\cdot)$ is continuous on $H^1_\Gamma(\Omega) \times H^1_\Gamma(\Omega)$ and $H^1_\Gamma(\Omega)$-coercive, since $a(v_1, \phi)$ corresponds exactly to the scalar product defined from the norm $|\cdot|_{1,\alpha}$. Let $l(\cdot)$ be the linear form defined by

$$l(\phi) = \int_\Omega \tilde{f} \phi \, dx + \int_\Sigma \tilde{g} \phi \, d\sigma.$$

Since the pair $(\tilde{f},\tilde{g})$ belongs to $L^2(\Omega)\times L^2(\Sigma)$, $l(\cdot)$ is continuous in $H^1_\Gamma(\Omega)$. Then, according to the fact that $\mathcal{T}\left(\overline{\Omega}\right)$ is dense in $H^1_\Gamma(\Omega)$, the formulation (2.10) can be extended to $H^1_\Gamma(\Omega)$:

$$\forall\phi\in H^1_\Gamma(\Omega),\ a(v_1,\phi)=l(\phi)$$

and, according to Lax-Milgram theorem, the problem

$$\forall\phi\in H^1_\Gamma(\Omega), a(v_1,\phi)=l(\phi)$$

has a unique solution v_1 in $H^1_\Gamma(\Omega)$. In particular,

$$\forall\phi\in\mathcal{D}(\Omega)\subset H^1_\Gamma(\Omega),\ \int_\Omega\nabla v_1\nabla\phi\,dx+\int_\Omega v_1\phi\,dx=\int_\Omega\tilde{f}\phi\,dx.$$

We then deduce that

$$\forall\phi\in\mathcal{D}(\Omega),\ <v_1-\triangle v_1-\tilde{f},\phi>=0,$$

which means that

$$v_1-\triangle v_1=\tilde{f}\text{ in }\mathcal{D}'(\Omega).$$

This identity allows us to give a sense to $\triangle v_1$ in $L^2(\Omega)$. Therefore, $\partial_n v_{1|\partial\Omega}\in H^{-1/2}(\partial\Omega)$ and we also have

$$\forall\phi\in\mathcal{T}\left(\overline{\Omega}\right),\ \int_\Omega\nabla v_1\nabla\phi\,dx+\int_\Sigma\alpha v_1\phi\,d\sigma+\int_\Omega v_1\phi\,dx=\int_\Omega\tilde{f}\phi\,dx+\int_\Sigma\tilde{g}\phi\,d\sigma.$$

Using the Green formula (2.6), we get $\forall\phi\in\mathcal{T}\left(\overline{\Omega}\right)$,

$$\int_\Omega\triangle v_1\phi\,dx+<\partial_n v_1,\phi>_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)}+\int_\Sigma\alpha v_1\phi\,d\sigma+\int_\Omega v_1\phi\,dx=$$
$$\int_\Omega\tilde{f}\phi\,dx+\int_\Sigma\tilde{g}\phi\,d\sigma,$$

i.e.

$$\forall\phi\in\mathcal{T}\left(\overline{\Omega}\right),\ +<\partial_n v_1,\phi>_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)}+\int_\Sigma\alpha v_1\phi\,d\sigma=\int_\Sigma\tilde{g}\phi\,d\sigma.$$

Now, we have $\phi_{|\Gamma}=0$ which implies that

$$<\partial_n v_1+\left(1+\frac{\kappa}{2\left(1+\gamma-\frac{\kappa}{4}\right)}\right)v_1-\tilde{g},\phi>_{H^{-1/2}(\Sigma),H^{1/2}(\Sigma)}=0,$$

and we then have

$$\partial_n v_1+\left(1+\frac{\kappa}{2\left(1+\gamma-\frac{\kappa}{4}\right)}\right)v_1=\tilde{g}\text{ on }\Sigma.$$

The existence of v_1 solution to (2.9) is thus proved.
Since v_1 and f_1 are in $H^1(\Omega)$, we deduce the existence of $v_2=f_1+v_1$ in $H^1(\Omega)$. Moreover since $v_{2|\Sigma}$ and f_3 are in $L^2(\Sigma)$, we deduce the existence of

$\varphi = \dfrac{f_3 - f_1 - v_1}{\frac{\kappa}{4} - \gamma - 1}$ in $L^2(\Sigma)$.
To complete the proof, we have to check that actually $(v_1, v_2, \varphi) \in V$, which is obvious from

$$v_2 = f_1 + v_1 \text{ in } \Omega$$

$$\alpha\varphi = f_3 - v_2 \text{ on } \Sigma$$

and $f_1 \in H^1_\Gamma(\Omega)$, $f_3 \in L^2(\Sigma)$. □

As a conclusion,

Theorem 2.4. *Let (u_0, u_1, φ_0) in V. The problem (2.1) admits a unique solution u such that*

$$(u,\, \partial_t u,\, \varphi) \in C^1([0, +\infty[; V) \cap C^0([0, +\infty[; H). \tag{2.11}$$

Proof. The two previous lemmas show that the operator A is a maximal dissipative operator in its domain V. According to the Hille-Yosida theorem [12] the problem (2.1) has one and only one solution $U = (u,\, v,\, \psi)$ such that

$$(u,\, v,\, \psi) \in C^1([0, +\infty[; V) \cap C^0([0, +\infty[; H). \tag{2.12}$$

A is thus the infinitesimal generator of a semi-group of contraction $Z(t)$ and we can define the finite energy solution of (2.1) with initial data (u_0, u_1, ψ_0) in V in such a way that

$$(u,\, v,\, \psi) = Z(t)(u_0, u_1, \psi_0) \in C^1([0, +\infty[; V) \cap C^0([0, +\infty[; H)\,;$$

which ends the proof of the theorem. □

3 Long time behavior

The results of Section 2 can be enriched by introducing the functional defined on H by

$$\mathcal{E}(h_1, h_2, h_3) = \frac{1}{2}\int_\Omega \left(|\nabla h_1|^2 + |h_2|^2\right) dx + \frac{1}{2}\int_\Sigma \frac{\kappa}{2}|h_3|^2 d\sigma.$$

If $\kappa(x) > 0$ for all $x \in \Sigma$, $\mathcal{E}$ defines an energy on H and $\mathcal{E}^{1/2}$ is obviously a norm in H equivalent to the norm $\|\,\|$, according to Lemma 2.1. Then, $\mathcal{E}(u, \partial_t u, \psi)$ defines an energy on H.
Moreover,

Lemma 3.1. *For all $(u_0, u_1, \psi_0) \in V$, $t \longmapsto \mathcal{E}(u, \partial_t u, \psi)$ is differentiable and is decreasing under the condition (2.2):*

$$\gamma(x) > \frac{\kappa(x)}{4}, \forall x \in \Sigma.$$

Proof. In the previous section, we have seen that if the initial conditions (u_0, u_1, ψ_0) are in V, $\mathcal{E}(u, \partial_t u, \psi) \in C^1([0, +\infty[)$. Moreover,

$$\frac{d}{dt}\mathcal{E}(u, \partial_t u, \psi) = \int_\Omega \nabla(\partial_t u) \cdot \nabla u \, dx + \int_\Omega \partial_t u \partial_t^2 u \, dx + \int_\Sigma \frac{\kappa}{2} \psi \partial_t \psi \, d\sigma. \tag{3.1}$$

Using the Green formula (2.6) and the relation $\partial_t^2 u = \triangle u$ in Ω, we obtain

$$\frac{d}{dt}\mathcal{E}(u, \partial_t u, \psi) = -\int_\Omega \partial_t u \triangle u \, dx + \int_{\partial\Omega} \partial_n u \partial_t u \, d\sigma + \int_\Omega \partial_t u \triangle u \, dx + \int_\Sigma \frac{\kappa}{2} \psi \partial_t \psi \, d\sigma.$$

Since $\partial_t u_{|\Gamma} = 0$ and $\partial_n u = -\partial_t u - \frac{\kappa}{2}\psi$ on Σ,

$$\frac{d}{dt}\mathcal{E}(u, \partial_t u, \psi) = -\int_\Sigma |\partial_t u|^2 d\sigma - \int_\Sigma \frac{\kappa}{2} \psi \partial_t u d\sigma.$$

At last, since $\partial_t \psi = \partial_t u - \left(\gamma - \frac{\kappa}{4}\right)\psi$, we get

$$\frac{d}{dt}\mathcal{E}(u, \partial_t u, \psi) = -\left[\int_\Sigma \frac{\kappa}{2}\left(\gamma - \frac{\kappa}{4}\right)|\psi|^2 \, d\sigma + \int_\Sigma |\partial_t u|^2 \, d\sigma\right].$$

which implies that

$$\frac{d}{dt}\mathcal{E}(u, \partial_t u, \psi) \leq 0,$$

and completes the proof of Lemma 3.1. □

In the following, we will assume that $\psi_0 = 0$ on Σ. This is a necessary condition for (2.1) to be equivalent to the initial problem

$$\begin{cases} \partial_t^2 u - \triangle u = 0 \text{ in } \Omega \times (0, +\infty)\,; \\ u(0, x) = u_0(x),\ \partial_t u(0, x) = u_1(x) \text{ in } \Omega; \\ u = 0 \text{ or } \partial_n u = 0 \text{ on } \Gamma \times (0, +\infty)\,; \\ \partial_t\left(\partial_n u + \partial_t u\right) = \left(\frac{\kappa}{4} - \gamma\right)\partial_n u - \left(\frac{\kappa}{4} + \gamma\right)\partial_t u \text{ on } \Sigma \times (0, +\infty)\,. \end{cases}$$

Theorem 3.2. *Under (2.2), for all $(u_0, u_1, 0) \in V$,*

$$\lim_{t \to +\infty} \mathcal{E}(u, \partial_t u, \psi) = 0.$$

Proof. We have already seen that A is the generator of a continuous contraction semi-group $Z(t)$. As it is sufficient to prove the theorem on a dense subspace of $V = D(A)$, we consider the initial data $(u_0, u_1, 0)$ in $D(A^2)$, where

$$D(A^2) = \{(v_1, v_2, \psi) \in V, A(v_1, v_2, \psi) \in V\}$$

is equipped with the norm graph

$$\|(v_1, v_2, \psi)\|_{D(A^2)} = \|(v_1, v_2, \psi)\|_V + \|A(v_1, v_2, \psi)\|_V + \|A^2(v_1, v_2, \psi)\|_V.$$

For any solution to (2.1), we have

$$\begin{aligned}\|(u,\partial_t u,\psi)\|_{D(A^2)} &= \|Z(t)(u_0,u_1,0)\|_{D(A^2)}\\ &= \|Z(t)(u_0,u_1,0)\|_V + \|A(Z(t)(u_0,u_1,0))\|_V\\ &+\|A^2(Z(t)(u_0,u_1,0))\|_V.\end{aligned}$$

As A, A^2 and $Z(t)$ are commuting on $D(A^2)$,

$$\|(u,\partial_t u,\psi)\|_{D(A^2)} = \|Z(t)(u_0,u_1,0)\|_V + \|Z(t)A(u_0,u_1,0)\|_V + \|Z(t)A^2(u_0,u_1,0)\|_V.$$

Since $Z(t)$ is continuous in V, we deduce easily that there exists a positive constant C such that

$$\|(u,\partial_t u,\psi)\|_{D(A^2)} \leq C\|(u_0,u_1,0)\|_{D(A^2)}.$$

We thus have a bounded sequence of solutions in $D(A^2)$ which implies that we can extract a subsequence denoted by $Z(t_k)(u_0,u_1,0)$ which weakly converges to $(u_\infty,v_\infty,\psi_\infty)$ in $D(A^2)$. Now, let us denote by $(u(t_k),v(t_k))$ the sequence that is converging to (u_∞,v_∞). By definition of $D(A^2)$, $(u(t_k),v(t_k))$ is bounded in $H^{3/2}(\Omega)\times H^{3/2}(\Omega)$ and $\triangle u(t_k)$ is bounded in $H^1(\Omega)$. Indeed, any (u,v) in $D(A^2)$ satisfies

$$\left\{\begin{array}{l} u\in H^1(\Omega),\ \triangle u\in H^1(\Omega),\ u=0 \text{ on } \Gamma,\ \partial_n u_{|_\Sigma}\in L^2(\Sigma)\\ v\in H^1(\Omega),\ \triangle v\in H^1(\Omega),\ v=0 \text{ on } \Gamma,\ \partial_n v_{|_\Sigma}\in L^2(\Sigma).\end{array}\right.$$

We can thus deduce that $(u(t_k),v(t_k))$ strongly converges to (u_∞,v_∞) in $H^1(\Omega)\times H^1(\Omega)$. Moreover, since $\triangle u(t_k)$ strongly converges in $L^2(\Omega)$ and $\triangle u(t_k)$ converges to $\triangle u_\infty$ in $\mathcal{D}'(\Omega)$, $\triangle u(t_k)$ strongly converges to $\triangle u_\infty$ in $L^2(\Omega)$ since the limit is unique.
We then have that $\partial_n u(t_k)_{|_\Sigma}$ strongly converges to $\partial_n u_{\infty|_\Sigma}$ in $H^{-1/2}(\Sigma)$ and that $v(t_k)_{|_\Sigma}$ strongly converges to $v_{\infty|_\Sigma}$ in $H^{1/2}(\Sigma)$. This implies that $\phi(t_k)$ strongly converges to ϕ_∞ in $H^{-1/2}(\Sigma)$. Nevertheless, this result of convergence is not sufficient to have that $(u(t_k),v(t_k),\phi(t_k))$ strongly converges to $(u_\infty,v_\infty,\psi_\infty)$ in V. That is why we consider the equation that defines $\phi(t_k)$ to show that in fact, $\phi(t_k)$ strongly converges in $L^2(\Sigma)$. By construction, ϕ is solution to:

$$\left\{\begin{array}{ll} \partial_t\phi+\left(\gamma-\dfrac{\kappa}{4}\right)\phi=\partial_t u & \text{on } \Sigma\times(0,+\infty)\\ \phi(0,x)=0 & \text{on } \Sigma\end{array}\right.$$

and according to the Duhamel formula, we have

$$\phi(t,x)=\int_0^t e^{s-t}v(s,x)ds.$$

We know that v is bounded in $H^{1/2}(\Sigma)$. Hence we have: for any $\xi\in H^{-1/2}(\Sigma)$,

$$\begin{aligned}<\phi,\xi>_{H^{1/2},H^{-1/2}} &=<\int_0^t e^{s-t}v ds,\xi>_{H^{1/2},H^{-1/2}}\\ &=\int_0^t e^{s-t}<v,\xi>_{H^{1/2},H^{-1/2}} ds\end{aligned}$$

according to the fact that $v(s)$ is uniformly bounded with respect to s. Thus we obtain

$$|<\phi,\xi>_{H^{1/2},H^{-1/2}}| \leq \|v\|_{H^{1/2}(\Sigma)}\|\xi\|_{H^{-1/2}(\Sigma)}(1-e^{-t})$$

which implies

$$\|\phi\|_{H^{1/2}(\Sigma)} \leq (1-e^{-t})\|v\|_{H^{1/2}(\Sigma)}.$$

We thus have proved that $\phi(t)$ is uniformly bounded in $H^{1/2}(\Sigma)$ and we can then deduce that $\phi(t_k)$ strongly converges to ϕ_∞ in $L^2(\Sigma)$, as a consequence of the compact injection from $H^{1/2}(\Sigma)$ into $L^2(\Sigma)$.
As a conclusion, $(u(t_k), v(t_k), \phi(t_k))$ strongly converges to $(u_\infty, v_\infty, \psi_\infty)$ in V. Since $t \longmapsto \mathcal{E}(u,\partial_t u)$ is continuous, we thus have

$$\begin{aligned}\lim_{t\to+\infty}\mathcal{E}(u,\partial_t u,\psi) &= \lim_{t\to+\infty}\mathcal{E}(Z(t)(u_0,u_1,0))\\ &= \lim_{t_k\to+\infty}\mathcal{E}(Z(t_k)(u_0,u_1,0))\\ &= \mathcal{E}(u_\infty,v_\infty,\psi_\infty).\end{aligned}$$

We also have, for all s positive,

$$\lim_{t\to+\infty}\mathcal{E}(Z(t+s)(u_0,u_1,0)) = \lim_{t_k\to+\infty}\mathcal{E}(Z(s)Z(t_k)(u_0,u_1,0)) = \mathcal{E}(Z(s)(u_\infty,v_\infty,\psi_\infty)).$$

Then if $(w,\partial_t w,\varphi) = Z(t)(u_\infty,v_\infty,\psi_\infty)$ denotes the solution to problem (2.1), with initial data $(u_\infty,v_\infty,\psi_\infty)$ in $D(A)$, we have

$$\mathcal{E}(w,\partial_t w),\varphi = \mathcal{E}(u_\infty,v_\infty,\psi_\infty) \text{ for all } t \text{ positive.}$$

Hence, according to the proof of Lemma 3.1, since

$$\frac{d}{dt}\mathcal{E}(w,\partial_t w,\varphi) = -\left[\int_\Sigma \frac{\kappa}{2}\left(\gamma-\frac{\kappa}{4}\right)|\varphi|^2\,d\sigma + \int_\Sigma |\partial_t w|^2\,d\sigma\right],$$

we necessarily have $\partial_t w = 0$ on Σ and $\varphi = 0$ on Σ. We then deduce that w is solution to the following problem

$$\left\{\begin{array}{ll}\partial_t^2 w - \triangle w = 0 & \text{in } \Omega\times[0,+\infty[\\ w(x,0)=u_\infty,\ \partial_t w(x,0)=v_\infty & \text{in } \Omega\\ w=0 & \text{on } \Gamma\times[0,+\infty[\\ \partial_n w = 0 & \text{on } \Sigma\times(0,+\infty)\end{array}\right.$$

and $z := \partial_t w$ is solution to

$$\left\{\begin{array}{ll}\partial_t^2 z - \triangle z = 0 & \text{in } \Omega\times[0,+\infty[\\ z(x,0)=v_\infty,\ \partial_t z(x,0)=\triangle u_\infty & \text{in } \Omega\\ z=0 & \text{on } \Gamma\times[0,+\infty[\\ \partial_n z = z = 0 & \text{on } \Sigma\times(0,+\infty)\,.\end{array}\right.$$

Since $\partial_n z = z = 0$ on Σ, we deduce that $z = 0$ in $\Omega \times [0, +\infty[$, as a consequence of the Holmgren theorem (see Lions [14]). Therefore, w is solution to

$$\begin{cases} \triangle w = 0 & \text{in } \Omega \times [0, +\infty[\\ w(x,0) = u_\infty & \text{in } \Omega \\ w = 0 & \text{on } \Gamma \times [0, +\infty[\\ \partial_n w = 0 & \text{on } \Sigma \times (0, +\infty) \end{cases}$$

which implies that $w = 0$ in $\Omega \times [0, +\infty[$ since Ω is connected.The pair (u_∞, v_∞) is thus equal to zero. Consequently, we also get that ψ_∞ is also equal to zero and *a fortiori*, we have $\mathcal{E}(u_\infty, v_\infty, \psi_\infty) = 0$. □

We are now willing to propose

Theorem 3.3. *Let $(u_0, u_1, 0)$ in V. Then the solution u to (2.1) satisfies*

$$\lim_{t \to +\infty} (u, \partial_t u, \psi) = (0, 0, 0) \textit{ in } H.$$

Proof. If the pair $(u_0, u_1, 0)$ is in V, we know that (cf. Theorem 3.2)

$$\lim_{t \to +\infty} \mathcal{E}(u, \partial_t u, \psi) = 0$$

and that $\mathcal{E}$ is a norm on H, equivalent to $\| \, \|$.
Therefore,

$$\lim_{t \to +\infty} (u, \partial_t u, \psi) = (0, 0, 0) \text{ in } H.$$

□

4 Exponential Decay

The aim of this section is to study the exponential energy decay of the solution of the problem (2.1) when the obstacle is a sound-soft scatterer ($u = 0$ on Γ). We want to prove that there exists two positive constant C and β such that for all initial data,

$$\mathcal{E}(u, \partial_t u, \psi) \le C e^{-\beta t} \mathcal{E}(u, \partial_t u, \psi)_{|t=0}$$

where

$$\mathcal{E}(u, \partial_t u, \psi) = \frac{1}{2} \int_\Omega \left(|\partial_t u|^2 + |\nabla u|^2 \right) dx + \frac{1}{2} \int_\Sigma \frac{\kappa}{2} |\psi|^2 d\sigma. \tag{4.1}$$

To obtain such an inequality, we will use the Gronwall's lemma and it is thus sufficient to prove that there exists a positive constant C such that

$$\int_S^T \mathcal{E}(u, \partial_t u, \psi) dt \le C \mathcal{E}(u, \partial_t u, \psi)_{|t=S} \tag{4.2}$$

with $0 \le S < T < +\infty$.

4.1 Preliminary results

Lemma 4.1. *Let* $(u_0, u_1, \psi_0) \in V$. *Then,*

$$\mathcal{E}(T) - \mathcal{E}(S) = -\int_S^T \int_\Sigma |\partial_t u|^2 d\sigma dt + \int_S^T \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)|\psi|^2 d\sigma dt, \tag{4.3}$$

with

$$\mathcal{E}(t) = \mathcal{E}(u, \partial_t u, \psi)(t).$$

Proof. We have proved that $\mathcal{E}$ is differentiable and

$$\frac{d\mathcal{E}}{dt}(u, \partial_t u, \psi) = -\int_\Sigma |\partial_t u|^2 d\sigma + \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)|\psi|^2 d\sigma.$$

Then after integrating on $[S, T]$, we get

$$\mathcal{E}(T) - \mathcal{E}(S) = -\int_S^T \int_\Sigma |\partial_t u|^2 d\sigma dt + \int_S^T \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)|\psi|^2 d\sigma dt.$$

□

In the following, we assume that

$$\gamma(x) > \frac{\kappa(x)}{4},\ \forall x \in \Sigma \tag{4.4}$$

and that Σ is strictly convex so that

$$\kappa(x) > 0,\ \forall x \in \Sigma. \tag{4.5}$$

Lemma 4.2. *We have*

$$\left\{\begin{array}{l} \displaystyle\int_S^T \int_\Sigma |\partial_t u|^2 \, d\sigma \, dt \leq \mathcal{E}(S) \\ \displaystyle\int_S^T \int_\Sigma \frac{\kappa}{2}\left(\gamma - \frac{\kappa}{4}\right)|\psi|^2 \, d\sigma \, dt \leq \mathcal{E}(S) \end{array}\right. \tag{4.6}$$

and if $\alpha_{min} = \min\limits_{x\in\Sigma}\left(\gamma - \frac{\kappa}{4}\right)$, *we have*

$$\int_S^T \int_\Sigma \frac{\kappa}{2}|\psi|^2 \, d\sigma \, dt \leq \frac{1}{\alpha_{min}}\mathcal{E}(S) \tag{4.7}$$

Proof. We know that

$$\mathcal{E}(T) - \mathcal{E}(S) = -\int_S^T \int_\Sigma |\partial_t u|^2 d\sigma dt + \int_S^T \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)|\psi|^2 d\sigma dt$$

which is equivalent to

$$\mathcal{E}(S) = \int_S^T \int_\Sigma |\partial_t u|^2 d\sigma dt + \int_S^T \int_\Sigma \frac{\kappa}{2}\left(\gamma - \frac{\kappa}{4}\right)|\psi|^2 d\sigma dt + \mathcal{E}(T).$$

Since each term is positive we get

$$\left\{ \begin{array}{l} \displaystyle\int_S^T \int_\Sigma |\partial_t u|^2 \, d\sigma \, dt \leq \mathcal{E}(S); \\ \displaystyle\int_S^T \int_\Sigma \frac{\kappa}{2} \left(\gamma - \frac{\kappa}{4}\right) |\psi|^2 \, d\sigma \, dt \leq \mathcal{E}(S). \end{array} \right.$$

From $\displaystyle\int_S^T \int_\Sigma \frac{\kappa}{2} \left(\gamma - \frac{\kappa}{4}\right) |\psi|^2 \, d\sigma \, dt \leq \mathcal{E}(S)$, we obviously get

$$\int_S^T \int_\Sigma \frac{\kappa}{2} |\psi|^2 \, d\sigma \, dt \leq \frac{1}{\alpha_{\min}} \mathcal{E}(S)$$

which ends the proof of Lemma 4.6. □

Lemma 4.3. *There exists a constant $C > 0$ such that, for any $t \leq S \leq 0$,*

$$\int_\Sigma |u(t,x)|^2 d\sigma \leq C\mathcal{E}(S). \tag{4.8}$$

Proof. The trace map from $H^1(\Omega)$ into $H^{1/2}(\Sigma)$ is continuous and we have

$$\|u\|^2_{L^2(\Sigma)} \leq C\|u\|^2_{H^{1/2}(\Sigma)} \leq C\|u\|^2_{H^1(\Omega)}.$$

Moreover, u satisfies the Poincaré inequality which is
There exists a positive constant C such that

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}. \tag{4.9}$$

This implies that

$$\|u\|^2_{L^2(\Sigma)} \leq C\mathcal{E}(t).$$

We conclude easily since $t \longmapsto \mathcal{E}(t)$ is decreasing. □

Let $m(x)$ be a function in $\mathcal{C}^1(\bar{\Omega})^3$.

Lemma 4.4. *We have*

$$\left[\int_\Omega \partial_t u \, (m \cdot \nabla u) \, dx\right]_S^T - \frac{1}{2} \int_S^T \int_{\partial\Omega} (m \cdot n) \, |\partial_t u|^2 d\sigma \, dt + \frac{1}{2} \int_S^T \int_\Omega div \; m |\partial_t u|^2 dx \, dt +$$
$$\int_S^T \int_\Omega \nabla u \cdot \nabla \left(m \cdot \nabla u\right) dx \, dt - \int_S^T \int_{\partial\Omega} \partial_n u \, (m \cdot \nabla u) \, d\sigma \, dt = 0. \tag{4.10}$$

Proof. This identity has been used in several papers. We refer for instance to [13]. Nevertheless, we recall its construction.
Since u is the solution of the wave equation, we know that

$$\int_S^T \int_\Omega \left(\partial_t^2 u - \triangle u\right) (m \cdot \nabla u) \, dx \, dt = 0$$

that is to say that

$$\left[\int_{\Omega} \partial_t u\,(m\cdot\nabla u)\,dx\right]_S^T - \int_S^T \int_{\Omega} \partial_t u\,(m\cdot\nabla\partial_t u)\;dx\,dt + \int_S^T \int_{\Omega} \nabla u\cdot\nabla\,(m\cdot\nabla u)\;dx\,dt$$
$$-\int_S^T \int_{\partial\Omega} \partial_n u\,(m\cdot\nabla u)\;d\sigma\,dt = 0$$

Moreover, we can check that in Ω

$$\partial_t u\,(m\cdot\nabla\partial_t u) = \frac{1}{2} m\cdot\nabla|\partial_t u|^2$$

so that

$$\begin{aligned}\int_{\Omega} \partial_t u\,(m\cdot\nabla\partial_t u) &= \frac{1}{2}\int_{\Omega} m\cdot\nabla|\partial_t u|^2\,dx\\ &= \frac{1}{2}\int_{\partial\Omega} m\cdot n|\partial_t u|^2\,d\sigma - \frac{1}{2}\int_{\Omega} \text{div } m|\partial_t u|^2\,dx\end{aligned}$$

Therefore, we get

$$\left[\int_{\Omega} \partial_t u\,(m\cdot\nabla u)\,dx\right]_S^T - \frac{1}{2}\int_S^T \int_{\partial\Omega} (m\cdot n)\,|\partial_t u|^2 d\sigma\,dt + \frac{1}{2}\int_S^T \int_{\Omega} \text{div } m|\partial_t u|^2 dx\,dt +$$
$$\int_S^T \int_{\Omega} \nabla u\cdot\nabla\,(m\cdot\nabla u)\,dx\,dt - \int_S^T \int_{\partial\Omega} \partial_n u\,(m\cdot\nabla u)\,d\sigma\,dt = 0,$$

which ends the proof of the lemma. □

4.2 Proof of the exponential energy decay

In this section, we set $m(x) = x - x_0$ where $x_0 \in \mathbb{R}^N$ ($N = 2, 3$ denotes the space dimension). We suppose that x_0 is chosen such that

$$\Gamma = \{x \in \partial\Omega, m.n \le 0\} \tag{4.11}$$

and

$$\Sigma = \{x \in \partial\Omega, m.n \ge 0\}. \tag{4.12}$$

This hypothesis is satisfied if Ω is star-shaped with respect to x_0.
In that case, we know that div $m = N$. For the sake of simplicity, we suppose that $N = 3$ but there is no difficulty to obtain the same result for $N = 2$.

To prove that there exists a positive constant C that satisfies (4.2), we only have to find an upper bound of

$$\frac{1}{2}\int_S^T \int_{\Omega} \left(|\partial_t u|^2 + |\nabla u|^2\right)\,dx\,dt$$

since we already know from Lemma 4.2 that

$$\int_S^T \int_{\Sigma} \frac{\kappa}{2}|\psi|^2\,d\sigma\,dt \le \frac{1}{\alpha_{\min}}\mathcal{E}(S).$$

Lemma 4.5. *Let $(u_0, u_1, \psi_0) \in V$. Then, we have*

$$\begin{aligned}
&\frac{1}{2}\int_S^T \int_\Omega \left(|\partial_t u|^2 + |\nabla u|^2\right)\, dx\, dt = -\left[\int_\Omega \partial_t u \left((m\cdot\nabla u) + u\right) dx\right]_S^T \\
&+\frac{1}{2}\int_S^T \int_{\partial\Omega} (m\cdot n)\left[|\partial_t u|^2 - |\nabla u|^2\right]\, d\sigma\, dt + \int_S^T \int_{\partial\Omega} \partial_n u u\, d\sigma\, dt \qquad (4.13)\\
&+\int_S^T \int_{\partial\Omega} \partial_n u \left(m\cdot\nabla u\right)\, d\sigma\, dt.
\end{aligned}$$

Proof. From Lemma 4.4, we know that

$$\left[\int_\Omega \partial_t u\,(m\cdot\nabla u)\, dx\right]_S^T - \frac{1}{2}\int_S^T \int_{\partial\Omega} (m\cdot n)\,|\partial_t u|^2 d\sigma\, dt + \frac{1}{2}\int_S^T \int_\Omega \operatorname{div}\, m|\partial_t u|^2 dx\, dt +$$
$$\int_S^T \int_\Omega \nabla u\cdot\nabla\,(m\cdot\nabla u)\, dx\, dt - \int_S^T \int_{\partial\Omega} \partial_n u\,(m\cdot\nabla u)\, d\sigma\, dt = 0.$$

Moreover, we can check that

$$\partial_j\,(m\cdot\nabla u) = (m\cdot\nabla)\,\partial_j u + \nabla u\cdot\partial_j m \text{ for } j = 1, 2, 3;$$

and since $m(x) = x - x_0$, we get

$$\nabla\,(m\cdot\nabla u) = (m\cdot\nabla)\,\nabla u + \nabla u.$$

Consequently, we have

$$\begin{aligned}
\int_S^T \int_\Omega \nabla u\cdot\nabla\,(m\cdot\nabla u)\, dx\, dt &= \int_S^T \int_\Omega |\nabla u|^2\, dx\, dt + \int_S^T \int_\Omega ((m\cdot\nabla)\,\nabla u)\cdot\nabla u\, dx\, dt \\
&= \int_S^T \int_\Omega |\nabla u|^2\, dx\, dt + \frac{1}{2}\int_S^T \int_\Omega m\cdot\nabla|\nabla u|^2\, dx\, dt \\
&= \int_S^T \int_\Omega |\nabla u|^2\, dx\, dt + \frac{1}{2}\int_S^T \int_{\partial\Omega} (m\cdot n)\,|\nabla u|^2\, d\sigma\, dt \\
&-\frac{1}{2}\int_S^T \int_\Omega \operatorname{div}\, m|\nabla u|^2\, dx\, dt
\end{aligned}$$

Therefore, we obtain using that div $m = 3$,

$$\begin{aligned}
&\left[\int_\Omega \partial_t u\,(m\cdot\nabla u)\, dx\right]_S^T - \frac{1}{2}\int_S^T \int_{\partial\Omega} (m\cdot n)\,|\partial_t u|^2 d\sigma\, dt + \frac{3}{2}\int_S^T \int_\Omega |\partial_t u|^2 dx\, dt \\
&-\frac{1}{2}\int_S^T \int_\Omega |\nabla u|^2 dx\, dt + \frac{1}{2}\int_S^T \int_{\partial\Omega} (m\cdot n)\,|\nabla u|^2 d\sigma\, dt \\
&-\int_S^T \int_{\partial\Omega} \partial_n u\,(m\cdot\nabla u)\, d\sigma\, dt = 0
\end{aligned}$$

that is to say

$$\begin{aligned}
&\frac{3}{2}\int_S^T\int_\Omega |\partial_t u|^2 dx\, dt - \frac{1}{2}\int_S^T\int_\Omega |\nabla u|^2 dx\, dt = \\
&-\left[\int_\Omega \partial_t u\,(m\cdot\nabla u)\, dx\right]_S^T + \frac{1}{2}\int_S^T\int_{\partial\Omega}(m\cdot n)\,|\partial_t u|^2 d\sigma\, dt \\
&-\frac{1}{2}\int_S^T\int_{\partial\Omega}(m\cdot n)\,|\nabla u|^2 d\sigma\, dt + \int_S^T\int_{\partial\Omega}\partial_n u\,(m\cdot\nabla u)\, d\sigma\, dt.
\end{aligned} \tag{4.14}$$

Moreover we know that

$$\int_S^T\int_\Omega \left(\partial_t^2 u - \triangle u\right) u = 0$$

which is equivalent to

$$\left[\int_\Omega \partial_t u u\, dx\right]_S^T - \int_S^T\int_\Omega |\partial_t u|^2\, dx\, dt + \int_S^T\int_\Omega |\nabla u|^2\, dx\, dt - \int_S^T\int_{\partial\Omega}\partial_n u u\, d\sigma\, dt = 0. \tag{4.15}$$

Then, adding (4.14) and (4.15), we obtain

$$\begin{aligned}
&\left[\int_\Omega \partial_t u u\, dx\right]_S^T + \frac{1}{2}\int_S^T\int_\Omega |\partial_t u|^2\, dx\, dt + \frac{1}{2}\int_S^T\int_\Omega |\nabla u|^2\, dx\, dt - \int_S^T\int_{\partial\Omega}\partial_n u u\, d\sigma\, dt = \\
&-\left[\int_\Omega \partial_t u\,(m\cdot\nabla u)\, dx\right]_S^T + \frac{1}{2}\int_S^T\int_{\partial\Omega}(m\cdot n)\,|\partial_t u|^2 d\sigma\, dt \\
&-\frac{1}{2}\int_S^T\int_{\partial\Omega}(m\cdot n)\,|\nabla u|^2 d\sigma\, dt + \int_S^T\int_{\partial\Omega}\partial_n u\,(m\cdot\nabla u)\, d\sigma\, dt.
\end{aligned}$$

Finally, we get

$$\begin{aligned}
&\frac{1}{2}\int_S^T\int_\Omega \left(|\partial_t u|^2 + |\nabla u|^2\right)\, dx\, dt = -\left[\int_\Omega \partial_t u\,((m\cdot\nabla u) + u)\, dx\right]_S^T \\
&+\frac{1}{2}\int_S^T\int_{\partial\Omega}(m\cdot n)\left[|\partial_t u|^2 - |\nabla u|^2\right]\, d\sigma\, dt + \int_S^T\int_{\partial\Omega}\partial_n u u\, d\sigma\, dt \\
&+\int_S^T\int_{\partial\Omega}\partial_n u\,(m\cdot\nabla u)\, d\sigma\, dt,
\end{aligned}$$

which completes the proof of Lemma 4.5. □

In the following we will denote equally by C all the constants.

Lemma 4.6. *We have the following inequality*

$$\frac{1}{2}\int_S^T\int_\Gamma (m\cdot n)\left[|\partial_t u|^2 - |\nabla u|^2\right]\, d\sigma\, dt + \int_S^T\int_\Gamma \partial_n u u\, d\sigma\, dt + \int_S^T\int_\Gamma \partial_n u\,(m\cdot\nabla u)\, d\sigma\, dt \le 0.$$

Proof. We know that $u = 0$ on Γ so that $\nabla_\Gamma u = 0$ and $\partial_t u = 0$ on Γ. Moreover, since $\nabla u = \nabla_\Gamma u + (\nabla u \cdot n)\, n$, $\nabla u = (\nabla u \cdot n)\, n$ on Γ. Then we get

$$\begin{aligned}
&\frac{1}{2}\int_S^T \int_\Gamma (m\cdot n)\left[|\partial_t u|^2 - |\nabla u|^2\right]\, d\sigma\, dt + \int_S^T \int_\Gamma \partial_n u u \, d\sigma\, dt + \int_S^T \int_\Gamma \partial_n u \,(m\cdot\nabla u)\; d\sigma\, dt \\
&= -\frac{1}{2}\int_S^T \int_\Gamma (m\cdot n)\,|\partial_n u|^2 \, d\sigma\, dt + \int_S^T \int_\Gamma (m\cdot n)\,|\partial_n u|^2 \, d\sigma\, dt \\
&= \frac{1}{2}\int_S^T \int_\Gamma (m\cdot n)\,|\partial_n u|^2 \, d\sigma\, dt.
\end{aligned}$$

By hypothesis, we have that $m \cdot n$ is negative on Γ. Therefore, we obtain

$$\frac{1}{2}\int_S^T \int_\Gamma (m\cdot n)\left[|\partial_t u|^2 - |\nabla u|^2\right]\, d\sigma\, dt + \int_S^T \int_\Gamma \partial_n u u \, d\sigma\, dt + \int_S^T \int_\Gamma \partial_n u \,(m\cdot\nabla u)\; d\sigma\, dt \le 0.$$

□

Lemma 4.7. *We have*

$$\int_S^T \int_\Sigma \partial_n u \,(m\cdot\nabla u)\; d\sigma\, dt - \frac{1}{2}\int_S^T \int_\Sigma (m\cdot n)\,|\nabla u|^2 \, d\sigma\, dt \le C\mathcal{E}(S).$$

Proof. Since $m \cdot n > 0$ on Σ by hypothesis, we have

$$\partial_n u \,(m\cdot\nabla u) = \sqrt{m\cdot n}\,(m\cdot\nabla u)\,\frac{\partial_n u}{\sqrt{m\cdot n}}.$$

Therefore, if we denote by $R = \max_{x\in\Sigma}|m(x)|$, we get

$$\begin{aligned}
|\partial_n u \,(m\cdot\nabla u)\,| &\le 2R\frac{|\partial_n u|}{\sqrt{m\cdot n}}\frac{\sqrt{m\cdot n}}{2}|\nabla u| \\
&\le \frac{R^2}{m\cdot n}|\partial_n u|^2 + \frac{m\cdot n}{4}|\nabla u|^2 \\
&\le C|\partial_n u|^2 + \frac{m\cdot n}{4}|\nabla u|^2
\end{aligned}$$

From this inequality, we deduce that

$$\begin{aligned}
\int_S^T \int_\Sigma \partial_n u \,(m\cdot\nabla u)\; d\sigma\, dt - \frac{1}{2}\int_S^T \int_\Sigma (m\cdot n)\,|\nabla u|^2 \, d\sigma\, dt &\le -\frac{1}{4}\int_S^T \int_\Sigma (m\cdot n)\,|\nabla u|^2 \, d\sigma\, dt \\
&\quad + C\int_S^T \int_\Sigma |\partial_n u|^2 \, d\sigma\, dt \\
&\le C\int_S^T \int_\Sigma |\partial_n u|^2 \, d\sigma\, dt.
\end{aligned}$$

To end the proof of the lemma, we have to check that

$$\int_S^T \int_\Sigma |\partial_n u|^2 \, d\sigma\, dt \le C\mathcal{E}(S).$$

We know that
$$\partial_n u = -\partial_t u - \frac{\kappa}{2}\psi \text{ on } \Sigma.$$
Since $\kappa > 0$ on Σ according to (4.5), we get
$$|\partial_n u|^2 \leq \kappa \left(\frac{\kappa}{2}|\psi|^2 + \frac{2}{\kappa}|\partial_t u|^2 \right).$$
Then, if we denote by $\kappa_{\max}$ the maximum of κ on Σ, we get
$$\int_S^T \int_\Sigma |\partial_n u|^2 \, d\sigma \, dt \leq \frac{\kappa_{\max}}{\alpha_{\min}} \mathcal{E}(S) + 2 \int_S^T \int_\Sigma |\partial_t u|^2 \, d\sigma \, dt.$$
From Lemma 4.2, we know that
$$\int_S^T \int_\Sigma |\partial_t u|^2 \, d\sigma \, dt \leq C\mathcal{E}(S)$$
which proves that
$$\int_S^T \int_\Sigma |\partial_n u|^2 \, d\sigma \, dt \leq C\mathcal{E}(S)$$
and therefore that
$$\int_S^T \int_\Sigma \partial_n u \, (m \cdot \nabla u) \, d\sigma \, dt - \frac{1}{2} \int_S^T \int_\Sigma (m \cdot n) \, |\nabla u|^2 \, d\sigma \, dt \leq C\mathcal{E}(S).$$
□

Lemma 4.8. *We have*
$$\frac{1}{2} \int_S^T \int_\Sigma (m \cdot n) \, |\partial_t u|^2 \, d\sigma \, dt \leq C\mathcal{E}(S).$$

Proof. By definition of R, we have $|m \cdot n| \leq R$. Therefore, we obtain
$$\frac{1}{2} \int_S^T \int_\Sigma (m \cdot n) \, |\partial_t u|^2 \, d\sigma \, dt \leq \frac{R}{2} \int_S^T \int_\Sigma |\partial_t u|^2 \, d\sigma \, dt$$
and (4.6) implies that
$$\frac{1}{2} \int_S^T \int_\Sigma (m \cdot n) \, |\partial_t u|^2 \, d\sigma \, dt \leq C\mathcal{E}(S)$$
□

Lemma 4.9. *We have*
$$\int_S^T \int_\Sigma \partial_n u u \, d\sigma \, dt \leq C\mathcal{E}(S).$$

Proof. We recall that on Σ, $\partial_n u = -\partial_t u - \frac{\kappa}{2}\psi$. Therefore,

$$\begin{aligned}
\int_S^T \int_\Sigma \partial_n u u \, d\sigma \, dt &= \int_S^T \int_\Sigma \left(-\partial_t u - \frac{\kappa}{2}\psi\right) u \, d\sigma \, dt; \\
&= -\frac{1}{2}\left[\int_\Sigma |u|^2 \, d\sigma\right]_S^T - \int_S^T \int_\Sigma \frac{\kappa}{2}\psi u \, d\sigma \, dt; \\
&= -\frac{1}{2}\int_\Sigma |u(T)|^2 \, d\sigma + \frac{1}{2}\int_\Sigma |u(S)|^2 \, d\sigma - \int_S^T \int_\Sigma \frac{\kappa}{2}\psi u \, d\sigma \, dt; \\
&\leq \frac{1}{2}\int_\Sigma |u(S)|^2 \, d\sigma - \int_S^T \int_\Sigma \frac{\kappa}{2}\psi u \, d\sigma \, dt.
\end{aligned}$$

Since the trace operator is continuous, we know that

$$\int_\Sigma |u(S)|^2 \, d\sigma \leq C\|u(S)\|_{H^1}^2.$$

From the Poincaré inequality (4.9), we get that there exists a positive constant denoted by C such that

$$\|u(S)\|_{H^1}^2 \leq C \int_\Omega |\nabla u(S)|^2 \, dx.$$

Finally we obtain

$$\int_S^T \int_\Sigma \partial_n u u \, d\sigma \, dt \leq C\mathcal{E}(S) - \int_S^T \int_\Sigma \frac{\kappa}{2}\psi u \, d\sigma \, dt.$$

Now, we interest ourselves in the control of $-\displaystyle\int_S^T \int_\Sigma \frac{\kappa}{2}\psi u \, d\sigma \, dt$. We know that

$$\partial_t \psi = \partial_t u + \left(\frac{\kappa}{4} - \gamma\right)\psi \text{ on } \Sigma,$$

which is equivalent to

$$\psi = \left(\frac{\kappa}{4} - \gamma\right)^{-1}(\partial_t \psi - \partial_t u).$$

Then we get

$$\begin{aligned}
-\int_S^T \int_\Sigma \frac{\kappa}{2}\psi u \, d\sigma \, dt &= -\int_S^T \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1}(\partial_t \psi - \partial_t u) u \, d\sigma \, dt \\
&= -\int_S^T \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1}\partial_t \psi u \, d\sigma \, dt + \int_S^T \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1}\partial_t u u \, d\sigma \, dt \\
&= -\int_S^T \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1}\partial_t \psi u \, d\sigma \, dt + \frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1}\left[|u|^2\right]_S^T \, d\sigma \\
&= \frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1}\left[|u|^2\right]_S^T \, d\sigma + \int_S^T \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1}\psi \partial_t u \, d\sigma \, dt \\
&\quad - \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4} - \gamma\right)^{-1}\left[\psi u\right]_S^T \, d\sigma.
\end{aligned}$$

First, we know that

$$\frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4}-\gamma\right)^{-1}\left[|u|^2\right]_S^T d\sigma \quad = \frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4}-\gamma\right)^{-1}|u(T)|^2\,d\sigma - \frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4}-\gamma\right)^{-1}|u(S)|^2\,d\sigma$$

and since $\gamma \geq \frac{\kappa}{4}$

$$\frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4}-\gamma\right)^{-1}|u(T)|^2\,d\sigma \leq 0.$$

Therefore,

$$\begin{aligned}\frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4}-\gamma\right)^{-1}\left[|u|^2\right]_S^T d\sigma \quad &= -\frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\gamma-\frac{\kappa}{4}\right)^{-1}\left[|u|^2\right]_S^T d\sigma\\ &= -\frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\gamma-\frac{\kappa}{4}\right)^{-1}|u(T)|^2\,d\sigma + \frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\gamma-\frac{\kappa}{4}\right)^{-1}|u(S)|^2\,d\sigma.\end{aligned}$$

Since $\frac{\kappa}{2}\left(\gamma-\frac{\kappa}{4}\right)^{-1} \leq 0$,

$$\begin{aligned}\frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4}-\gamma\right)^{-1}\left[|u|^2\right]_S^T d\sigma \quad &\leq \frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\gamma-\frac{\kappa}{4}\right)^{-1}|u(S)|^2\,d\sigma\\ &\leq \frac{\kappa_{\max}}{4\alpha_{\min}}\int_\Sigma |u(S)|^2\,d\sigma.\end{aligned}$$

According to Lemma 4.3, we get

$$\frac{1}{2}\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4}-\gamma\right)^{-1}\left[|u|^2\right]_S^T d\sigma \leq \frac{\kappa_{\max}}{4\alpha_{\min}} C\mathcal{E}(S).$$

Moreover, we know that

$$\int_S^T \int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4}-\gamma\right)^{-1}\psi\partial_t u\,d\sigma\,dt \quad \leq C\mathcal{E}(S)$$

using Lemma 4.2.
Finally,

$$\begin{aligned}-\int_\Sigma \frac{\kappa}{2}\left(\frac{\kappa}{4}-\gamma\right)^{-1}\left[\psi u\right]_S^T d\sigma \quad &= \int_\Sigma \frac{\kappa}{2}\left(\gamma-\frac{\kappa}{4}\right)^{-1}\psi(T)u(T)\,d\sigma - \int_\Sigma \frac{\kappa}{2}\left(\gamma-\frac{\kappa}{4}\right)^{-1}\psi(S)u(S)\,d\sigma\\ &\leq \frac{1}{\alpha_{\min}}\left(\int_\Sigma \frac{\kappa}{2}\psi(T)u(T)\,d\sigma - \int_\Sigma \frac{\kappa}{2}\psi(S)u(S)\,d\sigma\right)\end{aligned}$$

We have

$$\begin{aligned}\frac{\kappa}{2}\psi(T)u(T) \quad &= \left(\frac{\sqrt{\kappa}}{\sqrt{2}}\psi(T)\right)\left(\frac{\sqrt{\kappa}}{\sqrt{2}}u(T)\right)\\ &\leq \frac{1}{2}\left(\frac{\kappa}{2}|\psi(T)|^2 + \frac{\kappa}{2}|u(T)|^2\right)\end{aligned}$$

and

$$\begin{aligned}-\frac{\kappa}{2}\psi(S)u(S) \quad &= -\left(\frac{\sqrt{\kappa}}{\sqrt{2}}\psi(S)\right)\left(\frac{\sqrt{\kappa}}{\sqrt{2}}u(S)\right)\\ &\leq \frac{1}{2}\left(\frac{\kappa}{2}|\psi(T)|^2 + \frac{\kappa}{2}|u(S)|^2\right)\end{aligned}$$

Therefore,

$$-\int_{\Sigma}\frac{\kappa}{2}\left(\frac{\kappa}{4}-\gamma\right)^{-1}[\psi u]_S^T\, d\sigma \leq \frac{1}{2\alpha_{\min}}\int_{\Sigma}\left(\frac{\kappa}{2}|\psi(T)|^2+\frac{\kappa}{2}|u(T)|^2+\frac{\kappa}{2}|\psi(S)|^2+\frac{\kappa}{2}|u(S)|^2\right)\, d\sigma.$$

We already know that

$$\int_{\Sigma}\left(\frac{\kappa}{2}|\psi(T)|^2+\frac{\kappa}{2}|\psi(S)|^2\right)\, d\sigma \leq C\left(\mathcal{E}(T)+\mathcal{E}(S)\right)$$

and according to Lemma 4.3,

$$\int_{\Sigma}\left(\frac{\kappa}{2}|u(T)|^2+\frac{\kappa}{2}|u(S)|^2\right)\, d\sigma \leq C\left(\mathcal{E}(T)+\mathcal{E}(S)\right).$$

Thus, since $t \longmapsto \mathcal{E}(t)$ is decreasing, we get

$$-\int_{\Sigma}\frac{\kappa}{2}\left(\frac{\kappa}{4}-\gamma\right)^{-1}[\psi u]_S^T\, d\sigma \leq C\mathcal{E}(S)$$

Finally, we have shown that

$$-\int_S^T\int_{\Sigma}\frac{\kappa}{2}\psi u\, d\sigma\, dt \leq C\mathcal{E}(S)$$

which proves that

$$\int_S^T\int_{\Sigma}\partial_n u u\, d\sigma\, dt \leq C\mathcal{E}(S)$$

□

Lemma 4.10. *We have*

$$-\left[\int_{\Omega}\partial_t u\left((m\cdot\nabla u)+u\right)dx\right]_S^T \leq C\mathcal{E}(S).$$

Proof. We know that

$$-\left[\int_{\Omega}\partial_t u\left((m\cdot\nabla u)+u\right)dx\right]_S^T = -\int_{\Omega}\partial_t u\left(m\cdot\nabla u\right)_{|t=T}\, dx-\int_{\Omega}\partial_t u u_{|t=T}\, dx$$
$$+\int_{\Omega}\partial_t u\left(m\cdot\nabla u\right)_{|t=S}\, dx+\int_{\Omega}\partial_t u u_{|t=S}\, dx.$$

Moreover, we have

$$\left\{\begin{array}{l} -\int_{\Omega}\partial_t u\left(m\cdot\nabla u\right)_{|t=T}\, dx \leq C\left(\int_{\Omega}|\partial_t u_{|t=T}|^2\, dx+\int_{\Omega}|\nabla u_{|t=T}|^2\, dx\right)\\ \int_{\Omega}\partial_t u\left(m\cdot\nabla u\right)_{|t=S}\, dx \leq C\left(\int_{\Omega}|\partial_t u_{|t=S}|^2\, dx+\int_{\Omega}|\nabla u_{|t=S}|^2\, dx\right)\end{array}\right.$$

and

$$\left\{\begin{array}{l} -\int_{\Omega}\partial_t u u_{|t=T}\, dx \leq C\left(\int_{\Omega}|\partial_t u_{|t=T}|^2\, dx+\int_{\Omega}|u_{|t=T}|^2\, dx\right)\\ \int_{\Omega}\partial_t u u_{|t=S}\, dx \leq C\left(\int_{\Omega}|\partial_t u_{|t=S}|^2\, dx+\int_{\Omega}|u_{|t=S}|^2\, dx\right)\end{array}\right. .$$

Using the Poincaré inequality (4.9) and that $\mathcal{E}$ decreases, we obviously get

$$-\left[\int_\Omega \partial_t u\left((m\cdot\nabla u)+u\right)dx\right]_S^T \leq C\mathcal{E}(S),$$

which ends the proof. □

Theorem 4.11. *There exists a positive constant C such that for all $0 \leq S < T < +\infty$,*

$$\int_S^T \mathcal{E}(u,\partial_t u,\psi)dt \leq C\mathcal{E}(S). \tag{4.16}$$

Proof. From Lemma 4.5 to Lemma 4.10, we get

$$\frac{1}{2}\int_S^T\int_\Omega \left(|\partial_t u|^2+|\nabla u|^2\right)\,dx\,dt \leq C\mathcal{E}(S). \tag{4.17}$$

Combining (4.17) with the result of Lemma 4.2,

$$\int_S^T\int_\Sigma \frac{\kappa}{2}|\psi|^2\,d\sigma\,dt \leq C\mathcal{E}(S). \tag{4.18}$$

We then obtain

$$\frac{1}{2}\int_S^T\int_\Omega \left(|\partial_t u|^2+|\nabla u|^2\right)\,dx\,dt + \frac{1}{2}\int_S^T\int_\Sigma \frac{\kappa}{2}|\psi|^2\,d\sigma\,dt \leq C\mathcal{E}(S) \tag{4.19}$$

and the proof of Theorem 4.11 is completed. □

Theorem 4.12. *There exists a positive constant C such that for all initial data in V,*

$$\mathcal{E}(u,\partial_t u,\psi) \leq e^{-(t-C)/C}\mathcal{E}(u,\partial_t u,\psi)_{|t=0}. \tag{4.20}$$

Proof. In Theorem 4.11 we have shown that there exists a positive constant C such that for all $0 \leq S < T < +\infty$,

$$\int_S^T \mathcal{E}(u,\partial_t u,\psi)dt \leq C\mathcal{E}(u,\partial_t u,\psi)_{|t=S}.$$

When T goes to $+\infty$, we get

$$\int_S^{+\infty} \mathcal{E}(u,\partial_t u,\psi)dt \leq C\mathcal{E}(u,\partial_t u,\psi)_{|t=S}, \tag{4.21}$$

which implies that

$$\frac{d}{ds}\left(e^{S/C}\int_S^{+\infty}\mathcal{E}(u,\partial_t u,\psi)dt\right) \leq 0.$$

The map $S \longmapsto e^{S/C}\displaystyle\int_S^{+\infty}\mathcal{E}(u,\partial_t u,\psi)dt$ is thus decreasing and, using the Gronwall's lemma, we get

$$e^{S/C}\int_S^{+\infty}\mathcal{E}(u,\partial_t u,\psi)dt \leq \int_0^{+\infty}\mathcal{E}(u,\partial_t u,\psi)dt.$$

Besides, when we apply (4.21) for $S = 0$, we get

$$\int_0^{+\infty} \mathcal{E}(u, \partial_t u, \psi) dt \le C\mathcal{E}(u, \partial_t u, \psi)_{|t=0}.$$

Therefore,

$$e^{S/C} \int_S^{+\infty} \mathcal{E}(u, \partial_t u, \psi) dt \le C\mathcal{E}(u, \partial_t u, \psi)_{|t=0}. \tag{4.22}$$

Moreover, since $\mathcal{E}$ is positive

$$\int_S^{+\infty} \mathcal{E}(u, \partial_t u, \psi) dt \ge \int_S^{S+C} \mathcal{E}(u, \partial_t u, \psi) dt,$$

and since $\mathcal{E}$ decreases

$$\int_S^{S+C} \mathcal{E}(u, \partial_t u, \psi) dt \ge \int_S^{S+C} \mathcal{E}(u, \partial_t u, \psi)_{|t=S+C} = C\mathcal{E}(S+C). \tag{4.23}$$

Consequently, by plugging (4.22) into (4.23), we obtain

$$e^{S/C}\mathcal{E}(u, \partial_t u, \psi)_{|t=S+C} \le \mathcal{E}(u, \partial_t u, \psi)_{|t=0},$$

which implies that for all $t > 0$

$$\mathcal{E}(u, \partial_t u, \psi) \le e^{-(t-C)/C}\mathcal{E}(u, \partial_t u, \psi)_{|t=0}.$$

□

5 Numerical analysis

In this section, we recall the IPDG method that we use for our numerical simulations to test the performance of the ABCs we are considering in this work. Next, we introduce a discrete energy that is decreasing and corresponds to the functional $\mathcal{E}$. We only present the case of a Dirichlet boundary condition on Γ.

5.1 General setting of the IPDG method

The IPDG method has been introduced in [4] for general elliptic problems. It is a discontinuous Galerkin approximation method in which a penalization term is introduced to impose the weak continuity of the solution through each element of the mesh. As a discontinuous Galerkin method, the IPDG method is a finite element method which allows to handle triangular meshes in 2D and tetrahedral meshes in 3D. This property is very important in our case because we deal with arbitrarily-shaped domains that is to say that the domains we have to mesh have complex geometries. Moreover, when using discontinuous Galerkin methods, we get a quasi-explicit representation of the solution because the mass matrix we have to invert is block-diagonal by construction. Furthermore, since the basis functions are discontinuous, we can easily consider heterogeneous media and the hp-adaptivity is straightforward.
We consider a partition $\mathcal{T}_h$ of Ω composed of triangles K, we denote by Ω_h the set of triangles, by Σ_{abs} the set of the edges on the absorbing boundary Σ, by

Σ_{D} the set of the edges on Γ and by Σ_{i} the set of the edges in the domain such that $\Sigma_{\mathrm{i}} \cap (\Sigma_{\mathrm{D}} \cup \Sigma_{\mathrm{abs}}) = \emptyset$. For each $\sigma \in \Sigma_{\mathrm{i}}$, we have to distinguish the two triangles that share σ: we note them arbitrarily K^+ and K^- . We introduce notations to define the jump and the average over each edge:

$$[\![v]\!] := v^+\boldsymbol{\nu}^+ + v^-\boldsymbol{\nu}^- \qquad \text{and} \qquad \{\!\{v\}\!\} := \frac{v^+ + v^-}{2},$$

where v^+ and v^- respectively refers to the restriction of v in K^+ and K^- and $\boldsymbol{\nu}^\pm$ stands for the unit outward normal vector to $K^\pm$.
The IPDG formulation of the wave equation with standard boundary conditions has been introduced by [1, 10].
First, we recall that the ABC (1.1) we have constructed reads as

$$\partial_n u = -\partial_t u - \frac{\kappa}{2}\psi \text{ on } \Sigma,$$

with

$$\left(\partial_t - \frac{\kappa}{4} + \gamma\right)\psi = \partial_t u \text{ on } \Sigma.$$

We seek an approximation of the solution u in the finite element space V_h^k defined as follows

$$V_h^k = \left\{v \in L^2(\Omega); v_{|K} \in P_V^k, \forall K \in \mathcal{T}_h\right\}, k \in \mathbb{N}$$

and of ψ in the finite element space W_h^k defined by

$$W_h^k = \left\{W \in L^2(\Sigma); w_{|\sigma} \in P_W^k, \forall \sigma \in \mathcal{T}_h\right\}, k \in \mathbb{N}$$

where P_V^k (respectively P_W^k) is the set of polynomials of degree at most k on K (respectively on σ).
The discrete problem is given by

$$\begin{cases} \text{Find } u_h \in V_h^k \times (0, +\infty) \text{ and } \psi \in W_h^k \times (0, +\infty) \text{ such that,} \\ \displaystyle\sum_{K \in \mathcal{T}_h} \int_K \partial_t^2 u_h v_h + a(u_h, v_h) + \sum_{\sigma \in \Sigma_{\mathrm{abs}}} \int_\sigma \left(\partial_t u_h + \frac{\kappa}{2}\psi_h\right) v_h = \sum_{K \in \mathcal{T}_h} \int_K f v_h, \ \forall v_h \in V_h^k \\ \displaystyle\sum_{\sigma \in \Sigma_{\mathrm{abs}}} \int_\sigma \left(\partial_t - \frac{\kappa}{4} + \gamma\right) \psi_h \varphi_h = \sum_{\sigma \in \Sigma_{\mathrm{abs}}} \int_\sigma \partial_t u_h w_h, \ \forall w_h \in W_h^k \end{cases} \tag{5.1}$$

with

$$a(u,v) = \sum_K \int_K \nabla u \nabla v - \sum_{\sigma \in \Sigma_{\mathrm{i}}} \int_\sigma \Big(\{\!\{\nabla u\}\!\} [\![v]\!] + \{\!\{\nabla v\}\!\} [\![u]\!] - \alpha [\![u]\!] [\![v]\!] \Big)$$

and α a penalisation coefficient [1].
We already know that, to determine u_h (respectively ψ_h) on a given triangle (respectively on an edge) we need C_{k+2}^k degrees of freedom (respectively $1 + k$) where $C_{k+2}^k = \frac{(k+2)!}{k!2!}$. Since we consider a DG method we will have

$$N := \mathit{number\,of\,triangles} \times C_{k+2}^k$$

degrees of freedom in the whole domain to interpolate u_h and

$$M = \mathit{number\,of\,edges\,on}\,\Sigma \times (1 + k)$$

degrees of freedom on Σ to interpolate ψ_h.
Let us consider $\{v_i\,,\,1\leq i\leq N\}$ a basis of V_h^k and $\{w_i\,,\,1\leq i\leq M\}$ a basis of W_h^k . We can rewrite u_h as

$$u_h\left(x,t\right)=\sum_{i=1}^{N}U_i\left(t\right)v_i\left(x\right)$$

and ψ_h as

$$\psi_h\left(x,t\right)=\sum_{i=1}^{M}\Psi_i\left(t\right)w_i\left(x\right).$$

The algebraic form of this problem is given by

$$\left\{\begin{array}{l} M\dfrac{d^2\mathbf{U}}{dt^2}+B\dfrac{d\mathbf{U}}{dt}+B_\kappa\mathbf{\Psi}+\mathbf{K}\mathbf{U}=\mathbf{F}, \\ C\dfrac{d\mathbf{\Psi}}{dt}+C_{\kappa,\,\gamma}\mathbf{\Psi}=D\dfrac{d\mathbf{U}}{dt} \end{array}\right. \tag{5.2}$$

where $\mathbf{U}$ and $\mathbf{\Psi}$ are the vectors of unknowns and

$$\begin{array}{l} M=\left(\displaystyle\sum_{K\in\mathcal{T}_h}\int_K v_iv_j\right)_{1\leq i,j\leq N}\,,\; K=\left(a(v_i,v_j)\right)_{1\leq i,j\leq N}\,,\;\mathbf{F}=\left(\displaystyle\sum_{K\in\mathcal{T}_h}\int_K f\,v_i\right)_{1\leq i\leq N}, \\ B=\left(\displaystyle\sum_{\sigma\in\Sigma_{\mathrm{abs}}}\int_\sigma v_iv_j\right)_{1\leq i,j\leq N}\,,\; B_\kappa=\left(\displaystyle\sum_{\sigma\in\Sigma_{\mathrm{abs}}}\int_\sigma \frac{\kappa}{2}w_iv_j\right)_{\substack{1\leq i\leq M\\ 1\leq j\leq N}} \\ C=\left(\displaystyle\sum_{\sigma\in\Sigma_{\mathrm{abs}}}\int_\sigma w_iw_j\right)_{1\leq i,j\leq M}\,,\; C_{\kappa,\,\gamma}=\left(\displaystyle\sum_{\sigma\in\Sigma_{\mathrm{abs}}}\int_\sigma \left(\gamma-\frac{\kappa}{4}\right)w_iw_j\right)_{1\leq i,j\leq M}, \\ D=\left(\displaystyle\sum_{\sigma\in\Sigma_{\mathrm{abs}}}\int_\sigma v_iw_j\right)_{\substack{1\leq i\leq N\\ 1\leq j\leq M}}. \end{array} \tag{5.3}$$

As for the time discretization, we use a finite difference scheme of order two with a time step Δt and we obtain

$$\mathbf{M}\mathbf{X}^{n+1}=\begin{pmatrix}\Delta t^2\left(\mathbf{F}^n-K\mathbf{U^n}\right)+2M\mathbf{U^n}\\ 0\end{pmatrix}\mathbf{X}^n+\begin{pmatrix}-M+\dfrac{\Delta t}{2}B & -\dfrac{\Delta t^2}{2}B_\kappa\\ -D & C-\Delta tC_{\kappa,\,\gamma}\end{pmatrix}\mathbf{X}^{n-1}, \tag{5.4}$$

where

$$\mathbf{M}=\begin{pmatrix}M+\dfrac{\Delta t}{2}B & \dfrac{\Delta t^2}{2}B_\kappa\\ -D & C+\Delta tC_{\kappa,\,\gamma}\end{pmatrix}$$

and

$$\mathbf{X}^i=\begin{pmatrix}\mathbf{U}^i\\ \mathbf{\Psi}^i\end{pmatrix}.$$

Remark (a) We can remark that $\mathbf{M}$ is easily invertible because it is composed of four blocks of block-diagonal matrices.
(b) In practice, when we consider interior triangles (that is to say triangles which have no edges on Σ), we solve

$$M\mathbf{U}^{n+1} = \Delta t^2 \left(\mathbf{F}^n - K\mathbf{U}^n\right) + 2M\mathbf{U}^n - M\mathbf{U}^{n-1}.$$

We solve the system (5.4) only when we consider exterior triangles (that is to say triangles that have an edge on Σ).

5.2 A discrete energy

In this section, we study the stability of the fully discretized scheme which reads as

$$\left\{\begin{array}{l} M\dfrac{\mathbf{U}^{n+1}-2\mathbf{U}^n+\mathbf{U}^{n-1}}{\Delta t^2} + B\dfrac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t} + B_\kappa\dfrac{\mathbf{\Psi}^{n+1}+\mathbf{\Psi}^{n-1}}{2} + K\mathbf{U}^n = 0, \\ C\dfrac{\mathbf{\Psi}^{n+1}+\mathbf{\Psi}^{n-1}}{2\Delta t} + C_{\kappa,\gamma}\dfrac{\mathbf{\Psi}^{n+1}+\mathbf{\Psi}^{n-1}}{2} - D\dfrac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t} = 0 \end{array}\right. \tag{5.5}$$

For $n \in \mathbb{N}$, we set

$$\begin{aligned} E^{n+1/2} &= \left(M\frac{\mathbf{U}^{n+1}-\mathbf{U}^n}{\Delta t}, \frac{\mathbf{U}^{n+1}-\mathbf{U}^n}{\Delta t}\right) + \left(K\mathbf{U}^{n+1}, \mathbf{U}^n\right) \\ &\quad + \frac{1}{2}\left[\left(C\mathbf{\Psi}^{n+1}, I_\kappa\mathbf{\Psi}^{n+1}\right) + \left(C\mathbf{\Psi}^n, I_\kappa\mathbf{\Psi}^n\right)\right] \end{aligned} \tag{5.6}$$

with

$$I_\kappa = \left(\delta_{ij}\kappa_i\right)_{1\leq i,j\leq nbr\_ar}$$

where nbr_ar denotes the number of edges on the absorbing boundary, δ_{ij} the Kronecker symbol and κ_i the value of the curvature on the edge i supposed to be constant.
Let λ_{max} be the maximum of the eigenvalues of the matrix $M^{-1}K$.

Proposition 5.1. *Under the Courant-Friedrichs-Levy condition (CFL)*

$$\Delta t < \frac{2}{\sqrt{\lambda_{max}}}, \tag{5.7}$$

$E^{n+1/2}$ *defines a discrete energy.*

Proof. To show that $E^{n+1/2}$ defines a discrete energy, we only have to prove that $E^{n+1/2}$ is positive.
We have

$$\begin{aligned} E^{n+1/2} &= \left(\left(M - \frac{\Delta t^2}{4}K\right)\frac{\mathbf{U}^{n+1}-\mathbf{U}^n}{\Delta t}, \frac{\mathbf{U}^{n+1}-\mathbf{U}^n}{\Delta t}\right) + \left(K\frac{\mathbf{U}^{n+1}+\mathbf{U}^n}{2}, \frac{\mathbf{U}^{n+1}+\mathbf{U}^n}{2}\right) \\ &\quad + \frac{1}{2}\left[\left(CI_\kappa\mathbf{\Psi}^{n+1}, \mathbf{\Psi}^{n+1}\right) + \left(CI_\kappa\mathbf{\Psi}^n, \mathbf{\Psi}^n\right)\right] \end{aligned}$$

It is obvious that $E^{n+1/2}$ is positive if $M - \dfrac{\Delta t^2}{2}K$, K and CI_κ are positive. We know that K and CI_κ are positive definite matrices by construction. Moreover since M is a symmetric positive definite matrix, the positivity of the first matrix is equivalent to the positivity of $I - \dfrac{\Delta t^2}{4} M^{-1}K$. Hence, if λ_{max} denotes the largest eigenvalue of $M^{-1}K$, $I - \dfrac{\Delta t^2}{4} M^{-1}K$ is positive if

$$\lambda_{max} \leq \frac{4}{\Delta t^2}.$$

□

Proposition 5.2. *When the space step h is small enough, the CFL condition (5.7) is equivalent to*

$$\Delta t < \frac{2}{\sqrt{\alpha_{LF}}} h, \tag{5.8}$$

where α_{LF} is a constant depending on the mesh and on the space discretisation method.

Proof. The eigenvalue λ_{max} depends on the space discretisation and satisfies $\lambda_{max} \simeq \frac{\alpha_{LF}}{h^2}$, with α_{LF} a constant. Therefore the CFL condition (5.7) can be written as

$$\Delta t < \frac{2}{\sqrt{\alpha_{LF}}} h.$$

□

Remark The CFL condition only depends on the matrices M and K and not on the boundary matrices. This proves that the ABCs do not penalize the CFL.

Proposition 5.3. *Under the CFL condition (5.8), the energy $E^{n+1/2}$ is decreasing.*

Proof. To prove this result, we multiply the first equation of (5.5) by $\dfrac{\mathbf{U}^{n+1} - \mathbf{U}^{n-1}}{2\Delta t}$, the second equation of (5.5) by $I_\kappa \dfrac{\mathbf{\Psi}^{n+1} + \mathbf{\Psi}^{n-1}}{2}$ and we sum them.
We get

$$\begin{aligned}
&\left(M\frac{\mathbf{U}^{n+1} - 2\mathbf{U}^n + \mathbf{U}^{n-1}}{\Delta t^2}, \frac{\mathbf{U}^{n+1} - \mathbf{U}^{n-1}}{2\Delta t}\right) + \left(B\frac{\mathbf{U}^{n+1} - \mathbf{U}^{n-1}}{2\Delta t}, \frac{\mathbf{U}^{n+1} - \mathbf{U}^{n-1}}{2\Delta t}\right)\\
&+ \left(B_\kappa\frac{\mathbf{\Psi}^{n+1} + \mathbf{\Psi}^{n-1}}{2}, \frac{\mathbf{U}^{n+1} - \mathbf{U}^{n-1}}{2\Delta t}\right) + \left(K\mathbf{U}^n, \frac{\mathbf{U}^{n+1} - \mathbf{U}^{n-1}}{2\Delta t}\right)\\
&+ \left(C\frac{\mathbf{\Psi}^{n+1} + \mathbf{\Psi}^{n-1}}{2\Delta t}, I_\kappa\frac{\mathbf{\Psi}^{n+1} + \mathbf{\Psi}^{n-1}}{2}\right) + \left(C_{\kappa,\gamma}\frac{\mathbf{\Psi}^{n+1} + \mathbf{\Psi}^{n-1}}{2}, I_\kappa\frac{\mathbf{\Psi}^{n+1} + \mathbf{\Psi}^{n-1}}{2}\right)\\
&- \left(D\frac{\mathbf{U}^{n+1} - \mathbf{U}^{n-1}}{2\Delta t}, I_\kappa\frac{\mathbf{\Psi}^{n+1} + \mathbf{\Psi}^{n-1}}{2}\right) = 0
\end{aligned}$$

But

$$\begin{aligned}
&\left(M\frac{\mathbf{U}^{n+1} - 2\mathbf{U}^n + \mathbf{U}^{n-1}}{\Delta t^2}, \frac{\mathbf{U}^{n+1} - \mathbf{U}^{n-1}}{2\Delta t}\right) = \left(M\frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\Delta t^2}, \frac{\left(\mathbf{U}^{n+1} - \mathbf{U}^n\right) + \left(\mathbf{U}^n - \mathbf{U}^{n-1}\right)}{2\Delta t}\right)\\
&- \left(M\frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{\Delta t^2}, \frac{\left(\mathbf{U}^{n+1} - \mathbf{U}^n\right) + \left(\mathbf{U}^n - \mathbf{U}^{n-1}\right)}{2\Delta t}\right).
\end{aligned}$$

Therefore, since M is symmetric, we have

$$\left(M\frac{\mathbf{U}^{n+1}-2\mathbf{U}^{n}+\mathbf{U}^{n-1}}{\Delta t^2},\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t}\right)=\frac{1}{2\Delta t}\left[\left(M\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n}}{\Delta t},\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n}}{\Delta t}\right)\right.$$
$$\left.-\left(M\frac{\mathbf{U}^{n}-\mathbf{U}^{n-1}}{\Delta t},\frac{\mathbf{U}^{n}-\mathbf{U}^{n-1}}{\Delta t}\right)\right].$$

In the same way, since K and C are both symmetric matrices, we obtain

$$\left(K\mathbf{U}^{n},\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t}\right)=\frac{1}{2\Delta t}\left[\left(K\mathbf{U}^{n+1},\mathbf{U}^{n}\right)-\left(K\mathbf{U}^{n},\mathbf{U}^{n-1}\right)\right]$$

and

$$\left(C\frac{\mathbf{\Psi}^{n+1}+\mathbf{\Psi}^{n-1}}{2\Delta t},I_\kappa\frac{\mathbf{\Psi}^{n+1}+\mathbf{\Psi}^{n-1}}{2}\right)=\frac{1}{4\Delta t}\left[\left(C\mathbf{\Psi}^{n+1},I_\kappa\mathbf{\Psi}^{n+1}\right)-\left(C\mathbf{\Psi}^{n-1},I_\kappa\mathbf{\Psi}^{n-1}\right)\right].$$

Moreover, by definition, we have $(DI_\kappa)^T=B_\kappa$ which implies

$$\left(B_\kappa\frac{\mathbf{\Psi}^{n+1}+\mathbf{\Psi}^{n-1}}{2},\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t}\right)-\left(D\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t},I_\kappa\frac{\mathbf{\Psi}^{n+1}+\mathbf{\Psi}^{n-1}}{2}\right)=0.$$

Consequently, (5.2) is equivalent to

$$\frac{1}{2\Delta t}\left[\left(M\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n}}{\Delta t},\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n}}{\Delta t}\right)-\left(M\frac{\mathbf{U}^{n}-\mathbf{U}^{n-1}}{\Delta t},\frac{\mathbf{U}^{n}-\mathbf{U}^{n-1}}{\Delta t}\right)\right]+$$
$$\left(B\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t},\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t}\right)+\frac{1}{2\Delta t}\left[\left(K\mathbf{U}^{n+1},\mathbf{U}^{n}\right)-\left(K\mathbf{U}^{n},\mathbf{U}^{n-1}\right)\right]$$
$$+\frac{1}{4\Delta t}\left[\left(C\mathbf{\Psi}^{n+1},I_\kappa\mathbf{\Psi}^{n+1}\right)-\left(C\mathbf{\Psi}^{n-1},I_\kappa\mathbf{\Psi}^{n-1}\right)\right]+\left(C_{\kappa,\gamma}\frac{\mathbf{\Psi}^{n+1}+\mathbf{\Psi}^{n-1}}{2},I_\kappa\frac{\mathbf{\Psi}^{n+1}+\mathbf{\Psi}^{n-1}}{2}\right)=0.$$

To get the expresion of $E^{n+1/2}$ in the previous equation, we artificially add

$$\frac{1}{4\Delta t}\left[\left(C\mathbf{\Psi}^{n},I_\kappa\mathbf{\Psi}^{n}\right)-\left(C\mathbf{\Psi}^{n},I_\kappa\mathbf{\Psi}^{n}\right)\right].$$

We have shown that

$$\frac{1}{2\Delta t}\left(E^{n+1/2}-E^{n-1/2}\right)=-\left(B\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t},\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t}\right)$$
$$-\left(C_{\kappa,\gamma}\frac{\mathbf{\Psi}^{n+1}+\mathbf{\Psi}^{n-1}}{2},I_\kappa\frac{\mathbf{\Psi}^{n+1}+\mathbf{\Psi}^{n-1}}{2}\right).$$

Since B and $C_{\kappa,\gamma}$ are positive definite matrices and I_κ is a diagonal matrix with a positive diagonal, we get

$$\frac{1}{2\Delta t}\left(E^{n+1/2}-E^{n-1/2}\right)<0,$$

which ends the proof of the proposition. □

6 Numerical results

In this section, we first study numerically the behavior of the discrete energy to emphasize the validity of Theorem (4.12) before investigating the performances of the ABC (1.1).

6.1 Behavior of the discrete energy

In this part, we look at the behavior of the numerical energy (5.6) to check numerically the theoretical results of Section 4.
We consider the following configuration denoted by Configuration 1: the two-dimensional domain Ω_1 (see Fig. 2) is delimited by an exterior boundary Σ_1

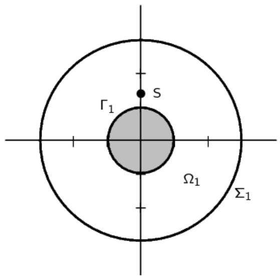

Figure 2: Computational domain - Configuration 1

and by an interior boundary Γ_1. Σ_1 is a circle of radius $R_{\text{ext}} = 3$m centered in the origin and Γ_1 is the boundary of a circular obstacle of radius 1m centered in the origin. The point source is set at (0m, 1.3m) and is defined as a second-derivative of a Gaussian with a dominant frequency of 1Hz,

$$f = \delta_{x_0} 2\lambda \left(\lambda \left(t - t_0\right)^2 - 1 \right) e^{-\lambda (t-t_0)^2},$$

with $x_0 = (0\text{m}, 1.3\text{m})$, $\lambda = \pi^2 f_0^2$, $f_0 = 1$ and $t_0 = 1/f_0$.
We first want to check that the discrete energy is actually decreasing when the source is switched off. In a second part, we illustrate the fact that the condition $\gamma \geq \frac{\kappa}{4}$ is an optimal condition to get a well-posed problem. In the last part, we check that we can control the discrete energy thanks to an exponential decreasing function.

6.1.1 Energy decay

We first want to check that the discrete energy defined in (5.6) is decreasing in time. In Fig. 3, we represent the evolution of the discrete energy along the time when $\gamma = \kappa$. We see that the energy first increases, which is due to the fact that Theorem 4.12 is only valid when the source is switched off. Once $f(t) = 0$, we remark that the energy is constant as long as the wave has not reached the boundary of the domain. Finally, the energy starts decreasing as predicted by Theorem 4.12.

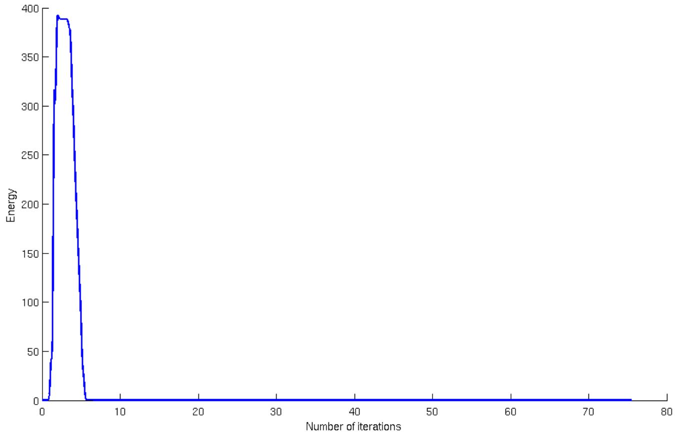

Figure 3: Energy vs time for $\gamma = \kappa$

6.1.2 Stability of the ABC

We have seen in Section 3 that γ has to be greater than $\frac{\kappa}{4}$ to get a well-posed problem. To illustrate this property, we compare the evolution of the discrete energy for $\gamma = 0.249 * \kappa$ and $\gamma = 0.25 * \kappa$. In Fig. 4 (resp. in Fig. 5), we represent the evolution of the discrete energy during 1 000 000 iterations when $\gamma = 0.249\kappa$ (resp. when $\gamma = 0.25\kappa$). If we just look at these figures, it seems that both scheme are stable even when $\gamma < \frac{\kappa}{4}$. But if we look at Fig. 6 (resp. Fig. 7) where we have magnified the y-scale by a factor 10^4, for $\gamma = 0.249\kappa$ (resp. when $\gamma = 0.25\kappa$), we see that the scheme is not stable when $\gamma < \frac{\kappa}{4}$. The numerical tests confirm that $\frac{\kappa}{4}$ is a critical value for γ.

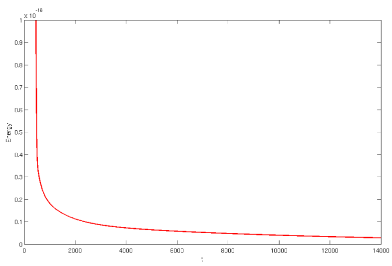

Figure 4: Energy for $\gamma = 0.249\kappa$

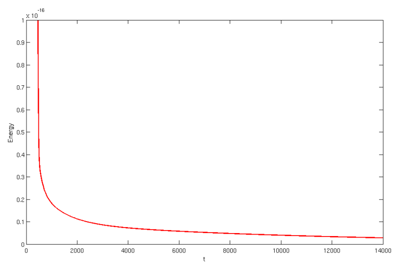

Figure 5: Energy for $\gamma = 0.25\kappa$

6.1.3 Exponential decay of the discrete energy

From Section 4, we know that the continuous energy can be controlled by an exponential decreasing function. In Fig. 8, we compare the evolution of the discrete energy obtain with $\gamma = \kappa$ (black curve) with the following exponential

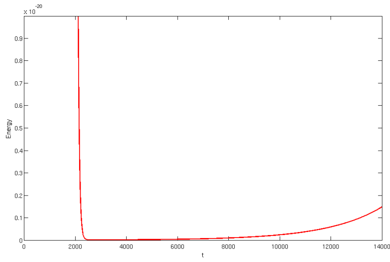

Figure 6: Energy for $\gamma = 0.249\kappa$ - zoom

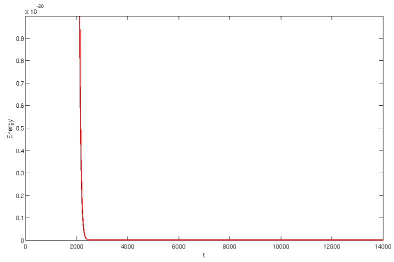

Figure 7: Energy for $\gamma = 0.25\kappa$ - zoom

function (blue dashed curve)

$$g(x) = 10^{-2} exp(-0.1 * (x - 11)), \tag{6.1}$$

the y scale is magnified by 100 in the second picture and by 10^8 in the third one. It is clear that g is always greater than the discrete energy which illustrates

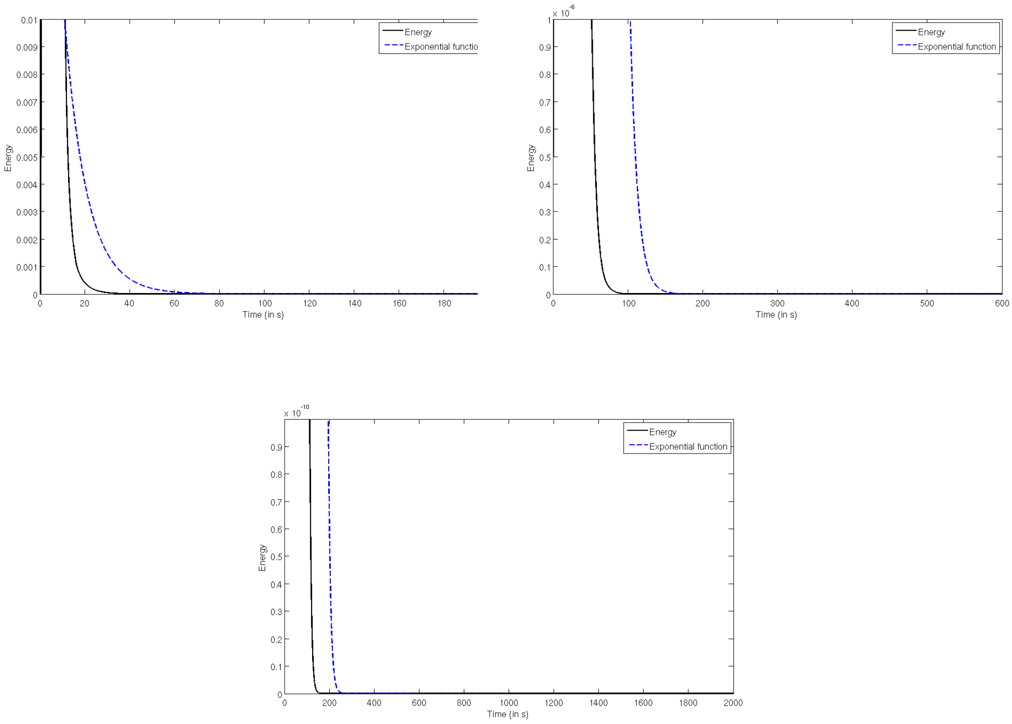

Figure 8: Exponential decay of the energy

the exponential decay of the energy.

6.2 Accuracy of the ABC

Now, we want to analyse the performances of the ABC (1.1) compared to the C-ABC for different values of γ. To this aim, we will consider once again Configuration 1 and a second configuration denoted Configuration 2 (see Fig. 9): Ω_2 is a two-dimensional domain delimited by an exterior boundary Σ_2 and by an interior boundary Γ_2. Σ_2 is an ellipse of semi-major axis $a_{\text{ext}} = 6$m and semi-minor axis $b_{\text{ext}} = 3$m centered in the origin. Γ_2 is the boundary of an ellipsical obstacle of semi-major axis a =2m and semi-minor axis b =1m centered

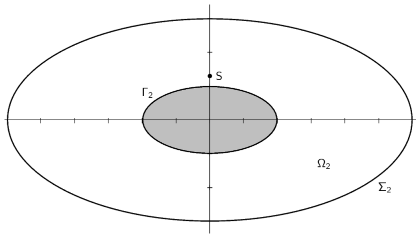

Figure 9: Computational domain -Configuration 2

in the origin.
To compare the efficiency of the ABC (1.1) for different values of γ, we first compute the relative L^2-error in time at a given receiver set near the exterior boundary. To evaluate this relative error, we have to compute the exact solution at each receiver. We did not actually compute the exact solution, but we computed an approximated solution in much larger domain than Ω_1 and Ω_2. Indeed, we multiplied the dimensions of Σ_1 and Σ_2 by 3. The relative $L^2_{(x,y)}([0,T])$ error at point (x, y) is defined by

$$\frac{\left(\int_0^T (u_{app}(t,(x,y)) - u_{ex}(t,(x,y)))^2 dt\right)^{1/2}}{\left(\int_0^T (u_{ex}(t,(x,y)))^2 dt\right)^{1/2}}, \tag{6.2}$$

where u_{app} is the numerical solution and u_{ex} is the exact solution. The error is given after 6000 iterations (with a time step equal to 3.6^{-3}s).
We consider three receivers set near the boundary with coordinates (0, 2.85m), (0.7m, 2.75m) and (1.4m, 2.45m) for Configuration 1 and four receivers whose coordinates are (6m, 2.05m), (5m, 2.3m), (6m, −2.05m) and (5m, −2.3m) for Configuration 2. The results are given in Tab. 1 for Configuration 1 and Tab. 2 for Configuration 2. From this results, we deduce that the relative L^2-error is quite the same for all the values of γ and is similar to the errors obtained with the C-ABC. We also see that the error increases when γ is greater than κ. Actually, for high values of γ the error is increasing at all the receivers.
In the following, we will arbitrarily consider $\gamma = \kappa$.
Now, we interest ourselves on the evolution of those errors when we move the absorbing boundary, to find out where the boundary should be to obtain suitable results without high computational burdens. We first consider Configuration 1. We have tested six different radius R_{ext}: 1.5m, 2m, 3m, 4m, 5m and 6m in order to find the critical value of the radius of the exterior circle we should take to

	(0, 2.85m)	(0.7m, 2.75m)	(1.4m, 2.45m)
curvature	4.92	13.11	13.3
$\gamma = \frac{\kappa}{4}$	4.92	13.11	13.30
$\gamma = \frac{\kappa}{3}$	4.92	13.10	13.31
$\gamma = \kappa$	4.89	13.19	13.37
$\gamma = 3\kappa$	4.92	13.42	13.83
$\gamma = 10\kappa$	5.33	14.04	14.22

Table 1: Relative L^2 error (in %) - Source in (1.5,1) - Configuration 1

	(6m, 2.05m)	(5m, 2.3m)	(6m, −2.05m)	(5m, −2.3m)
curvature	2.49	3.65	6.77	11.53
$\gamma = \frac{\kappa}{4}$	2.49	3.65	6.77	11.53
$\gamma = \frac{\kappa}{3}$	2.49	3.65	6.79	11.56
$\gamma = \kappa$	2.50	3.66	6.93	11.75
$\gamma = 3\kappa$	2.52	3.67	7.35	12.31
$\gamma = 10\kappa$	2.60	3.74	7.89	13.10

Table 2: Relative L^2 error (in %) - Source in (1.5,1) - Configuration 2

obtain accurate results.
First, we set three receivers of coordinates (1.025m, 1.025m), (1.256m, 0.725m) and (1.4m, 0.375m) near the obstacle and we evaluate the relative L^2-error in time defined in (6.2). The results we have obtained are presented in Tab. 3. We

	(1.025m, 1.025m)	(1.256m, 0.725m)	(1.4m, 0.375m)
$R_{\text{ext}} = 1.5$m	29.87	35.08	41.75
$R_{\text{ext}} = 2$m	6.49	10.84	18.17
$R_{\text{ext}} = 3$m	1.46	2.46	4.52
$R_{\text{ext}} = 4$m	0.96	1.42	2.03
$R_{\text{ext}} = 5$m	0.94	1.33	1.11
$R_{\text{ext}} = 6$m	0.93	0.87	1.07

Table 3: Relative L^2 error (in %) - Source in (1.3,0) - Configuration 1

see that when the artificial boundary is set near the boundary of the obstacle, the relative errors are very important because there are many reflections coming from the exterior boundary. When the radius of the artificial boundary is greater than 3m, we obtain small relative errors. However, when we take the exterior radius equal to 5m or 6m, the relative errors obtained are quite the same as for an exterior radius of 3m or 4m but the computation is much more expensive. Therefore when the obstacle is a circle we recommend to take for the artificial boundary a circle whose radius is equal to three times the radius of the obstacle.
Now, we perform the same kind of analysis but looking at the global error in

space and in time, i.e. we evaluate the relative $L^2([0,t]\times\Omega)$-error defined by

$$\left(\frac{\displaystyle\int_0^t\int_\Omega |u_{app}(s,x)-u_{ex}(s,x)|^2 dxds}{\displaystyle\int_0^t\int_\Omega |u_{ex}(s,x)|^2 dxds}\right)^{1/2}.$$

The results we have obtained are presented in Fig. 10. For external radius smaller than 3m there are a lot of reflections coming from the external boundary and so the computed relative errors are between 3 and 25%. When the external radius is greater than 3m, we obtain small relative errors and the differences between the relative errors are small. Hence, choosing an external radius of 3m when the interior radius is equal to 1m seems to be the best choice to obtain good approximations of the solution of the wave equation without high computation burdens.

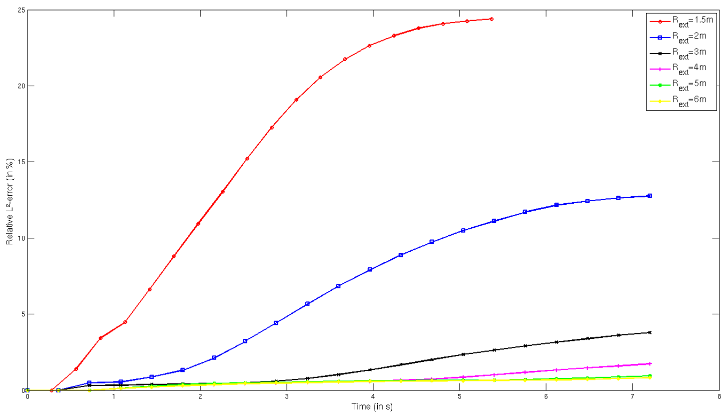

Figure 10: Relative error for different distances to the obstacle - Configuration 1

Now, we want to check if this conclusion is the same when dealing with elliptic obstacle. Here again, we consider six different domains. In each configuration, the interior boundary is the same as in Configuration 2 that is to say that the obstacle is an ellipse centered in the origin whose semi-major axis a is equal to 2m and whose semi-minor axis b is equal to 1m. The exterior boundary is an ellipse centered in the origin of semi-major axis a_{ext} and semi-minor axis b_{ext}. a_{ext} and b_{ext} are respectively obtained by multiplying a and b by 1.5, 2, 3, 4, 5 and 6. We have set three receivers near the obstacle of coordinates (0.75m, 1.4m), (2.51m, 0.725m) and (2.8m, 0.375m) and we evaluate the relative L^2-error in time defined in (6.2). The results we have obtained are presented in Tab. 4. We can see that in this case, we obtain really good results when $a_{\text{ext}} \geq 4a$ and $b_{\text{ext}} \geq 4b$. If we take $a_{\text{ext}} = 6$m and $b_{\text{ext}} = 3$m we obtain suitable results but the difference with the case when $a_{\text{ext}} = 8$m and $b_{\text{ext}} = 4$m is important. Now, we want to check if we obtain similar results when we compute the relative $L^2([0,t]\times\Omega)$-error. The results are depicted in Fig. 11. We

	(0.75m, 1.4m)	(2.51m, 0.725m)	(2.8m, 0.375m)
$a_{\text{ext}} = 3\text{m}$, $b_{\text{ext}} = 1.5\text{m}$	35.03	61.61	69.34
$a_{\text{ext}} = 4\text{m}$, $b_{\text{ext}} = 2\text{m}$	4.27	22.69	30.77
$a_{\text{ext}} = 6\text{m}$, $b_{\text{ext}} = 3\text{m}$	4.1	8.75	10.52
$a_{\text{ext}} = 8\text{m}$, $b_{\text{ext}} = 4\text{m}$	3.72	2.48	3.27
$a_{\text{ext}} = 10\text{m}$, $b_{\text{ext}} = 5\text{m}$	0.59	1.15	1.77
$a_{\text{ext}} = 12\text{m}$, $b_{\text{ext}} = 6\text{m}$	0.44	0.55	1.38

Table 4: Relative L^2 error (in %) - Source in (1.3,0) - Configuration 2

deduce that in such a configuration, we should take $a_{\text{ext}} = 3a$ and $b_{\text{ext}} = 3b$ to obtain small relative errors as in the case of a circular scatterer.

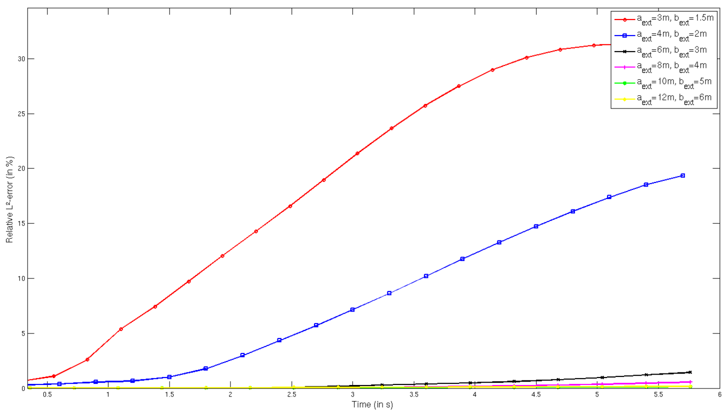

Figure 11: Relative error for different distances to the obstacle - Configuration 2

7 Conclusion

In this work, we have studied a simplified formulation of the ABCs constructed in [6]. We proposed to introduce an auxiliary unknown so that the new condition can be easily included into a variational formulation. We have proved that the corresponding boundary value problem is well-posed and that the solution is exponentially decreasing. We have also shown that the new ABC can be included into an IPDG formulation without hampering the CFL condition of the scheme. Finally, we have performed numerical experiments to emphasize the validity of the theoretical results and to investigate the performances of the ABC. We have observed numerically that the parameter γ, on which the ABCs depend, does not affect the accuracy of the solution, provided γ is small enough. Finally, we have carried out a numerical study to determine the optimal distance between the boundary and the obstacle.

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