Weakly Self Affine Functions and Applications in Signal Processing

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Dedicated to Pr. Luis Santalo

Abstract

We study a class of functions, called Weakly Self Affine functions, which are a generalization of Fractal Interpolation Functions where the contraction ratios are allowed to evolve in scale. We show how to compute the multifractal spectrum of such functions, and mention an application to the multifractal segmentation of signals

1 Introduction

Fractal Interpolation Functions (FIF, [1, 3]) or Self-Similar functions ([5]) possess a strong multiplicative structure that allows to compute many related fractal and multifractal quantities of interest. FIF are constructed in a recursive way: Each scale is deduced from the preceding one by applying a finite number of contractive functions. These contractive functions are fixed once and for all. Specially for applications in Signal Processing, a strict multiplicative structure is often too restrictive a requirement to impose. We consider in this paper a generalization of FIF, called *Weakly Self Affine functions* (WSA), where the contractive functions are allowed to vary at each scale. This added flexibility permits to model and process a much larger class of signals, keeping at the same time the computation of the associated multifractal features reasonably straightforward.

This paper is organized as follows. Section 2 recalls the definition and multifractal properties of FIF. We introduce WSA functions in section 3. Section 4 is devoted to the computation of their multifractal spectrum. Finally, we explain in section 5 how WSA functions may be used to perform the segmentation of a signal into parts which are "multifractally homogeneous", i.e. have a well defined multifractal spectrum.

2 Recalls on Fractal Interpolation Functions

To fix notations, we recall the following facts about self-similar functions (see for instance [5] for proofs).

Definition 1 *A function $F : \mathbb{R}^m \to \mathbb{R}$ is called self-similar of order k iff:*

- *There exists an open set Ω and d contractive similitudes $S_1, ..., S_d$ (i.e. compositions of isometries and homotheties $x \to \mu_i x$ of ratios $\mu_i < 1$) such that:*
$$\begin{array}{c} S_i(\Omega) \subset \Omega \ \forall i \in \{1, ..., d\} \\ S_i(\Omega) \cap S_j(\Omega) = \emptyset \quad if \quad i \neq j \end{array} \tag{1}$$
- *There exist d reals $(\lambda_i)_{i=1,...,d}$ and a compactly supported function $g \in C^k$ such that:*
$$F(x) = \sum_{i=1}^{d} \lambda_i F(S_i^{-1}(x)) + g(x). \tag{2}$$
- *F is not uniformly C^k on a closed subset of Ω.*

Condition (1) is usually called the "separation condition". A formal solution of (2) is:

$$F(x) = \sum_{n=0}^{\infty} \sum_{(i_1,...,i_n)} \prod_{j=1}^{n} \lambda_{i_j} g(S_{i_n}^{-1} \circ ... \circ S_{i_1}^{-1}(x)) \tag{3}$$

where $(i_1, ..., i_n) \in \{0, ..., d-1\}^n$.
For an example of a self-similar function, consider simply the Weierstrass function:

$$F(x) = \sum_{n=0}^{\infty} 2^{-ns} sin(2\pi 2^n x)$$

where $x \in [0\,;1]$ and $s \in]0\,;1]$. Indeed, letting

$$g(x) = \begin{cases} sin(2\pi x) & \text{if } x \in [0\,;1] \\ 0 & \text{otherwise} \end{cases}$$

we get:

$$F(x) = 2^{-s}F(2x) + 2^{-s}F(2x-1) + g(x).$$

Let us now recall a few facts about multifractal analysis. Multifractal analysis is concerned with the study of the regularity structure of functions or processes, both from a local and global point of view. More precisely, one starts by measuring in some way the pointwise regularity, usually with some kind of Hölder exponents. The second step is to give a global description of this regularity. This can be done in a geometric fashion using Hausdorff dimensions, or in a statistical one through a large deviation analysis. Formally, one defines the pointwise Hölder exponent of F at x as:

$$\alpha(x) := \liminf_{\epsilon \to 0} \frac{\log(osc_F(x,\epsilon))}{\log(\epsilon)}$$

where $osc_F(x,\epsilon) = \sup_{s,s' \in B(x,\epsilon)} |F(s) - F(s')|$ and $B(x,\epsilon)$ denotes the ball centered at x with radius ϵ.

The Hausdorff multifractal spectrum describes the structure of the function $x \to \alpha(x)$ by evaluating the size of its level sets. More precisely, let:

$$E_\alpha = \{x : \alpha(x) = \alpha\}$$

The Hausdorff multifractal spectrum is the function:

$$d(\alpha) = dim_H(E_\alpha)$$

where $dim_H(E)$ denotes the Hausdorff dimension of the set E.

Other multifractal spectra are also defined, for instance the so-called large deviation and Legendre multifractal spectra. We shall not consider them here.

The multifractal spectrum of self-similar functions assumes a particularly simple form. Define:

$$\alpha_{min} = \inf_{i=1,\dots,d} \frac{\log|\lambda_i|}{\log|\mu_i|} \qquad \alpha_{max} = \sup_{i=1,\dots,d} \frac{\log|\lambda_i|}{\log|\mu_i|}$$

Let τ the function defined implicitly by:

$$\sum_{i=1}^{d} \lambda_i^q \mu_i^{-\tau(q)} = 1.$$

Then we have:

Theorem 1 *Assume that k is larger than α_{max}. Suppose in addition that, for all i, $|\lambda_i| < 1$. If $\alpha_{min} > 0$, then, for all α:*

$$d(\alpha) = \begin{cases} -\infty & \text{if } \alpha \notin [\alpha_{min}\,;\alpha_{max}] \\ \\ \inf_{q \in \mathbb{R}} (q\alpha - \tau(q)) & \text{otherwise} \end{cases}$$

This theorem is a version of the so-called "multifractal formalism", which, when applicable, allows to obtain $d(\alpha)$ as the Legendre transform of a function which is easy to compute.

3 Weakly Self-Affine functions

As said in the introduction, WSA functions are defined as a generalization of self-similar functions where the similarity ratios are allowed to vary at each scale. Formally:

Definition 2 *A function $F : [0,1] \to \mathbb{R}$ is called WSA iff:*
i) There exists an open set $\Omega \subset [0,1]$ and contractive similitudes $S_0, ..., S_{d-1}$ such that:

- $S_i(\Omega) \subset \Omega \;\; \forall i \in \{0, ..., d-1\}$
- $S_i(\Omega) \cap S_j(\Omega) = \emptyset \;\; \text{if} \;\; i \neq j$

ii) There exists d positive sequences $(\lambda_0^j)_{j \in \mathbb{N}^*}, ..., (\lambda_{d-1}^j)_{j \in \mathbb{N}^*}$ satisfying $0 < \lambda_i^j < 1$ for every $i \in \{0, ..., d-1\}$ and $j \in \mathbb{N}^*$, and there exists a compactly supported continuous function g such that F verifies:

$$F(x) = g(x) + \sum_{n=1}^{\infty} \sum_{(i_1,...,i_n) \in \{0,...,d-1\}^n} \left(\prod_{j=1}^{n} \epsilon^j_{\sum_{p=1}^{j} i_p 2^{j-p}} \lambda^j_{i_j} \right) g(S_{i_n}^{-1} \circ ... \circ S_{i_1}^{-1}(x)) \tag{4}$$

where, for each $j \geq 1$ and $k \in \{0, ..., d^j - 1\}$, we have : $\epsilon_k^j = \pm 1$.

If there exist d reals $\lambda_0, ..., \lambda_{d-1}$ such that

$$\epsilon_k^j \lambda_i^j = \lambda_i, \quad \forall i \in \{0, ..., d-1\}, \forall j \geq 1 \text{ and } \; \forall k \in \{0, ..., d^j - 1\},$$

then one recovers the classical self-similar functions. The weak self-affinity of F is apparent when one realizes that Definition 2 implies that F can be obtained as the limit of the sequence $(F_j)_{j\in\mathbb{N}}$, where $F_0(x) = g(x)$ and, for $j \geq 1$, F_j is recursively computed as follows:

$$F_j(x) = \sum_{i=0}^{d-1} \epsilon_i^j \lambda_i^j F_{j-1}(S_i^{-1}(x)) + g(x).$$

We establish now a condition that ensures the continuity of WSA functions. Let:

$$I_d^n = \{\iota_n = (i_1, i_2, \ldots, i_n) \; : i_j \in \{0, \ldots, d-1\}, \; j \in \{1, ..., n\}\}.$$

$$I_d^\infty = \{\iota = (i_1, i_2, \ldots) \; : i_j \in \{0, \ldots, d-1\}, \; j \in \mathbb{N}^*\}.$$

Proposition 1
Assume that:

$$\lim_{N\to\infty} \sup_{(i_1,i_2,\ldots)\in I_d^\infty} \left\{ \sum_{n=N}^{\infty} \prod_{j=1}^{n} |\lambda_{i_j}^j| \right\} = 0. \tag{5}$$

Then F is continuous.

Proof:
Write

$$F(x) = \lim_{N\to\infty} F_N(x)$$

where

$$F_N(x) = \sum_{n=0}^{N} \sum_{(i_1,\ldots,i_n)} \prod_{j=1}^{n} \lambda_{i_j}^j g(S_{i_n}^{-1} \circ \ldots \circ S_{i_1}^{-1}(x)).$$

Clearly, F_N is continuous for all $N \in \mathbb{N}$. We shall show that the sequence (F_N) converges uniformly to F. We have:

$$\begin{aligned} |F(x) - F_N(x)| &= \left| \sum_{n=N+1}^{\infty} \sum_{(i_1,\ldots,i_n)} \prod_{j=1}^{n} \lambda_{i_j}^j g(S_{i_n}^{-1} \circ \ldots \circ S_{i_1}^{-1}(x)) \right| \\ &\leq \sum_{n=N+1}^{\infty} \sum_{(i_1,\ldots,i_n)} \prod_{j=1}^{n} \left|\lambda_{i_j}^j\right| \left|g(S_{i_n}^{-1} \circ \ldots \circ S_{i_1}^{-1}(x))\right| \end{aligned}$$

Since g is compactly supported, there exists a constant $C \geq 1$, such that, for all $n \in \mathbb{N}^*$ and for all $x \in [0\,;1]$, we have:

$$card \; \left\{\iota_n = (i_1, ..., i_n) \in I_d^n \; : S_{i_n}^{-1} \circ \ldots \circ S_{i_1}^{-1}(x) \in supp(g)\right\} \leq C.$$

Thus, for all x:

$$|F(x) - F_N(x)| \leq C \sup_{(i_1,i_2,...)\in I_d^\infty} \left\{ \sum_{n=N}^{\infty} \prod_{j=1}^{n} |\lambda_{i_j}^j| \right\}.$$

Using (5), this implies that (F_N) converges uniformly to F. ■

Remark 1: Condition (5) is analogous to the one ensuring the continuity of GIFS. GIFS are yet another generalization of FIF where not only the contractive functions are allowed to change at each scale, but the number of functions S_i may also vary, and, in particular, tend to infinity with n (see [3].

Remark 2: Condition (5) is obviously satisfied if there exist a and b such that, for all i and j, $0 < a < |\lambda_i^j| < b < 1$.

Remark 3: The condition that g is compactly supported is unnecessary for both the definition of WSA functions and the continuity criterion. It just allows to simplify the analysis. A well localized function g would lead to the same results.

4 Multifractal formalism for WSA functions

We compute in this section the multifractal spectrum $d(\alpha)$ of WSA functions. It is a remarkable fact that, as is the case for self-similar functions, a multifractal formalism holds for WSA functions. Thus, WSA modeling, while allowing much greater generality than strict self-similarity, also leads to a quite simple multifractal analysis.

To avoid technicalities, we shall restrict our attention from now on to the case where

$$\mu_0 = ... = \mu_{d-1} = \mu = \frac{1}{d}$$

and, for all $i \in \{0, ..., d-1\}$:

$$S_i(x) = \frac{x+i}{d}$$

More general forms can be treated in the same way at the expense of various complications.

Our first task is to compute the pointwise Hölder exponents. The following proposition describes the pointwise regularity of WSA functions:

Proposition 2
For all x, let $I_n(x)$ be the d-adic interval of size d^{-n} containing x. Denote by $I_n^-(x)$ and $I_n^+(x)$ the two d-adic intervals of size d^{-n} neighbouring $I_n(x)$. Let $(i_1,...,i_n)$ be the coefficients of the d-adic expansion of x up to rank n, and $(i_1^-,...,i_n^-)$ (resp. $(i_1^+,...,i_n^+)$) be the the coefficients of the d-adic expansion up to rank n of any t in $I_n^-(x)$ (resp.$I_n^+(x)$). Then:

$$\alpha_f(x) = \liminf_{n\to\infty} \min\left(-\frac{\sum_{m=1}^{n} \log_d |\lambda_{i_m}^m|}{n}, -\frac{\sum_{m=1}^{n} \log_d |\lambda_{i_m^-}^m|}{n}, -\frac{\sum_{m=1}^{n} \log_d |\lambda_{i_m^+}^m|}{n}\right) \tag{6}$$

We shall denote in the sequel $B_n(x)$ the set $\{(i_1,...,i_n),(i_1^-,...,i_n^-),(i_1^+,...,i_n^+)\}$. When g is a piecewise linear function that interpolates $d+1$ equidistant points that do not lie on a straight line, F is an SGIFS as defined in [3], and the proposition above is a simple consequence of proposition 10 in [3]. In the general case, the proof follows closely the one in [5] for self-similar functions.

Let us now move to the multifractal spectrum of F. Define, for every integer $j\geq 1$, the d-tuple $(u_0^j,...,u_{d-1}^j)$ by:

$$(u_0^j,...,u_{d-1}^j) = (\lambda_{i_0}^j,...,\lambda_{i_{d-1}}^j),$$

where $(i_0,...,i_{d-1})$ is a permutation of $(0,...,d-1)$ which yields:

$$\lambda_{i_0}^j \leq ... \leq \lambda_{i_{d-1}}^j.$$

In other words, for each j, $(u_0^j,...,u_{d-1}^j)$ is the d-tuple $(\lambda_0^j,...,\lambda_{d-1}^j)$ rearranged in increasing order.

Theorem 2 *Suppose that there exists two reals $a>0$ and $b>0$ such that, for every $i\in\{0,...,d-1\}$ and $j\geq 1$ we have:*

$$0 < a \leq u_i^j \leq b < 1$$

Suppose also that:

$$p(x_0,...,x_{d-1}) = \lim_n \frac{card\left\{j\in\{1,...,n\}\ : u_i^j \leq x_i \forall i=0,...,d-1\right\}}{n} \tag{7}$$

exists for every $(x_0,...,x_{d-1})\in[a\,;b]^d$. Suppose finally that g is uniformly more regular than F. Then the Hausdorff multifractal spectrum of F is:

- $d(\alpha) = -\infty$ *if* $\alpha \notin [\alpha_{min}\,; \alpha_{max}]$ *where*

$$\left\{ \begin{array}{l} \alpha_{min} = \lim_n -\frac{\log_d(u^1_{d-1})+...+\log_d(u^n_{d-1})}{n} \\ \\ \alpha_{max} = \lim_n -\frac{\log_d(u^1_0)+...+\log_d(u^n_0)}{n} \end{array} \right.$$

- *if* $\alpha \in [\alpha_{min}\,; \alpha_{max}]$, *then*

$$d(\alpha) = \inf_{q \in \mathbb{R}} (q\alpha - \tau(q))$$

where

$$\tau(q) = \liminf_{n\to\infty} -\frac{\sum\limits_{j=1}^{n} \log_d \left((\lambda^j_0)^q + ... + (\lambda^j_{d-1})^q \right)}{n} =: \liminf_{n\to\infty} \tau_n(q)$$

To prove the theorem, we shall first make use of the following well-known property (see for instance proposition 4.9 in [7]):

Proposition 3
Let $\mathcal{H}^s$ be the s-dimensional Hausdorff measure. Let ν be a probability measure on $\mathbb{R}$. Let $E \subset \mathbb{R}$ and C be a positive constant. Then:

- *if, for all $x \in F$, $\limsup_{r\to 0} \frac{\nu(B(x,r))}{r^s} < C$, then $\mathcal{H}^s(F) \geq \frac{\nu(F)}{C}$.*

- *if, for all $x \in F$, $\limsup_{r\to 0} \frac{\nu(B(x,r))}{r^s} > C$, then $\mathcal{H}^s(F) \leq \frac{2^s}{C}$.*

<u>Proof of theorem 2 :</u>
Let:

$$F_u(x) = \sum_{n=0}^{\infty} \sum_{(i_1,...,i_n)} \prod_{j=1}^{n} u^j_{i_j}\, g(S^{-1}_{i_n} \circ ... \circ S^{-1}_{i_1}(x)). \tag{8}$$

It is easy to see that the multifractal spectra of F and F_u coincide. ¿From proposition 2, we get:

$$\alpha_{f_u}(x) = \liminf_{n\to\infty} \inf_{(i_1,...,i_n)\in B_n(x)} -\frac{\log_d|u^1_{i_1}| + ... + \log_d|u^n_{i_n}|}{n}. \tag{9}$$

Let now $s > 0$ and $q \in \mathbb{R}$. For all $j \geq 1$, let:

$$t_j = \log_d \left(|u^j_0|^q + ... + |u^j_{d-1}|^q \right)$$

and for all $i \in \{0, ..., d-1\}$,

$$P_i^j = (u_i^j)^q \mu^{t_j}.$$

Recall that $\mu = \frac{1}{d}$. As a consequence: $\sum\limits_{i=0}^{d-1} P_i^j = 1$, for all integer $j \geq 1$.
Consider the probability measure ν defined on $K = [0\,;1]$ by:

$$\forall (i_1, ..., i_n) \in \{0, ..., d-1\}^n \ , \ \nu(S_{i_1} ... S_{i_n}(K)) = P_{i_1}^1 ... P_{i_n}^n. \qquad (10)$$

(the existence of ν follows from Kolmogorov consistency theorem). Let $x \in K$ and $r > 0$ be such that:

$$d^{-n} \leq r < d^{-(n-1)}.$$

Then

$$\begin{aligned} \frac{\nu(B(x,r))}{r^s} &\sim \sum_{(i_1,...,i_n) \in B_n(x)} \frac{P_{i_1}^1 ... P_{i_n}^n}{r^s} \\ &\sim \sum_{(i_1,...,i_n) \in B_n(x)} \frac{(u_{i_1}^1 ... u_{i_n}^n)^q \mu^{t_1 + ... + t_n}}{\mu^{ns}} \\ &\sim \sup_{(i_1,...,i_n) \in B_n(x)} \left(u_{i_1}^1 ... u_{i_n}^n \mu^{n \frac{-\tau_n(q) - s}{q}} \right)^q \end{aligned}$$

Let $E_\alpha = \{x : \alpha_f(x) = \alpha\}$ and assume that $s > q\alpha - \tau(q)$.
Case $q > 0$:
There exists $a > 0$ which depends only on s et q such that $s > q\alpha - \tau(q) + qa$.
As a consequence, there exists $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, we have:

$$\frac{\tau_n(q) + s}{q} - a > \alpha.$$

Let $\delta_n = \frac{\tau_n(q)+s}{q}$ and let γ be a real number such that $\alpha < \gamma < \delta_n - a$ for all $n > n_0$.
Then there exists a finite set $\mathbb{P} \subset \mathbb{N}$ such that, for all $n \in \mathbb{P}$ and $n > n_0$ we have:

$$\inf_{(i_1,...,i_n) \in B_n(x)} - \frac{\log_d u_{i_1}^1 + ... + \log_d u_{i_n}^n}{n} < \gamma$$

Thus:

$$\sup_{(i_1,...,i_n) \in B_n(x)} \frac{\log_d u_{i_1}^1 + ... + \log_d u_{i_n}^n}{n} > -\gamma.$$

This implies that, for $n > n_0$ in $\mathbb{P}$,

$$\sup_{(i_1,\ldots,i_n)\in B_n(x)} \frac{\log_d u_{i_1}^1 + \ldots + \log_d u_{i_n}^n}{n} + \delta_n > a$$

or

$$\sup_{(i_1,\ldots,i_n)\in B_n(x)} \left(u_{i_1}^1 \ldots u_{i_n}^n \mu^{n \frac{-\tau_n(q)-s}{q}} \right)^q > d^{aqn}$$

This entails that

$$\limsup_r \frac{\nu(B(x,r))}{r^s} = +\infty$$

<u>Case $q < 0$</u>:
There exists $b > 0$ depending only on s et q such that $s > q\alpha - \tau(q) - qb$.
As a consequence, there exists $n_0 \in \mathbb{N}$ such that, for all $n > n_0$ we have:

$$\frac{\tau_n(q) + s}{q} + b < \alpha.$$

Let $\delta_n = \frac{\tau_n(q)+s}{q}$ and let γ be a real such that $\alpha > \gamma > \delta_n + b$ for all $n > n_0$.
Then there exist a set $\mathbb{P}$ and $n_1 \in \mathbb{N}$ such that, for all $n \in \mathbb{P}$ and $n > n_1$,

$$\inf_{(i_1,\ldots,i_n)\in B_n(x)} - \frac{\log_d u_{i_1}^1 + \ldots + \log_d u_{i_n}^n}{n} > \gamma$$

Thus

$$\sup_{(i_1,\ldots,i_n)\in B_n(x)} \frac{\log_d u_{i_1}^1 + \ldots + \log_d u_{i_n}^n}{n} < -\gamma.$$

This implies that, for $n > \max(n_0, n_1)$:

$$\sup_{(i_1,\ldots,i_n)\in B_n(x)} \frac{\log_d u_{i_1}^1 + \ldots + \log_d u_{i_n}^n}{n} + \delta_n < -b$$

Or:

$$\sup_{(i_1,\ldots,i_n)\in B_n(x)} \left(u_{i_1}^1 \ldots u_{i_n}^n \mu^{n \frac{-\tau_n(q)-s}{q}} \right)^q > d^{-bqn}$$

Thus

$$\limsup_r \frac{\nu(B(x,r))}{r^s} = +\infty.$$

This entails that, for any given $\alpha \in [\alpha_{min}\,; \alpha_{max}]$, for all $q \in \mathbb{R}$ and all $s > q\alpha - \tau(q)$, we have:

$$\mathcal{H}^s(E_\alpha) = 0.$$

As a consequence,

$$d(\alpha) \geq \inf_{q \in \mathbb{R}} (q\alpha - \tau(q)).$$

The proof that $s < q\alpha - \tau(q)$ implies

$$\limsup_r \frac{\nu(B(x,r))}{r^s} = 0$$

follows the same lines. Thus, to show that $d(\alpha) \leq \inf_{q \in \mathbb{R}} (q\alpha - \tau(q))$, it suffices to find q and t such that $\nu(E_\alpha) > 0$. q and t are solutions of the following system:

$$\left\{ \begin{array}{l} \sum_{i=0}^{d-1} P_i^j = 1 \ \forall j \geq 1 \\ \\ \lim_n -\frac{\sum_{j=1}^{n} \sum_{i=0}^{d-1} P_i^j \log_d u_i^j}{n} = \alpha \end{array} \right. \tag{11}$$

Lemma 1
The system (11) has a solution iff:

$$\alpha_{min} < \alpha < \alpha_{max}$$

This lemma is proved below.

For $j \geq 1$, denote (X_j) a sequence of iid random variables that take the value $-\log(u_i^j)$ with probability P_i^j. Let:

$$S_n = \sum_{j=1}^{n} X_j$$

With $\mathcal{E}(S_n)$ denoting the expectation of S_n, the strong law of large numbers entails that:

$$\lim_n \frac{S_n - \mathcal{E}(S_n)}{n} = 0 \quad \nu\text{-almost surely}$$

This implies that:

$$\lim_n \frac{\log_d u_{i_1}^1 + ... + \log_d u_{i_n}^n}{n} - \frac{\sum_{j=1}^{n} \sum_{i=0}^{d-1} P_i^j \log_d u_i^j}{n} = 0$$

for ν-almost all $\iota = (i_1, ..., i_n, ...) \in I_d^\infty$.
Using (11), we get

$$\lim_n - \frac{\log_d u^1_{i_1} + ... + \log_d u^n_{i_n}}{n} = \alpha$$

for ν-almost all $\iota = (i_1, ..., i_n, ...) \in I_d^\infty$.
Proposition 2 allows to conclude that

$$\alpha_{f_u}(x) = \alpha$$

for ν-almost all $x \in E_\alpha$ and thus

$$\nu(E_\alpha) = 1$$

■

Proof of lemma 1 : To prove the lemma, we shall need to prove the following slight generalization of a theorem of Hardy.

Proposition 4
Let $(u^n)_{n\geq 1}$ be sequence in $[a\,;b]^d \subset \mathbb{R}^d$ such that (7) is verified. Then:

$$\lim_n \frac{f(u^1) + ... + f(u^n)}{n} = \int_{[a,b]^d} f(x)dg(x) \tag{12}$$

for all continuous functions $f : [a\,;b]^d \to \mathbb{R}$.

Note that, since f is continuous and g is of bounded variations, the Stieltjes integral in (12) does exist.

Proof :
Let $I_{0,...,d-1} = I_0 \times ... \times I_{d-1} \subset [a,b]^d$ where $I_j = [a_i\,;b_i]$ for all j. Let $1_{I_{0,...,d-1}}$ be the characteristic function of $I_{0,...,d-1}$. Then:

$$\begin{aligned}\lim_n \frac{1_{I_{0,...,d-1}}(u^1_0, ..., u^1_{d-1}) + ... + 1_{I_{0,...,d-1}}(u^n_0, ..., u^n_{d-1})}{n} &= g(b_0, ..., b_{d-1}) - g(a_0, ..., a_{d-1}) \\ &= \int_{[a,b]^d} 1_{I_{0,...,d-1}} dg(x)\end{aligned}$$

Thus (12) is true for any characteristic functions on $[a,b]^d$. By linearity, (12) is true for any step function, and, by continuity, it is true for continuous functions on $[a,b]^d$. ■

Recall now that proving lemma 1 amounts to proving that the function

$$T(q) = \lim_{n} -\frac{\sum_{j=1}^{n} \frac{\sum_{i=0}^{d-1} (u_i^j)^q \log_d u_i^j}{\sum_{i=0}^{d-1} (u_i^j)^q}}{n}$$

exists and is continuous for all $q \in \mathbb{R}$.
Consider, for all $q \in \mathbb{R}$, the function f_q defined on $[a\,;b]^d$ by:

$$f_q(x_0, ..., x_{d-1}) = \frac{\sum_{i=0}^{d-1} (x_i)^q \log_d x_i}{\sum_{i=0}^{d-1} (x_i)^q}$$

Applying proposition 4 to each f_q allows to deduce that $T(q)$ is well defined over $\mathbb{R}$. The continuity of $T(q)$ then stems from the fact that $f_q(x)$ is continuous in q uniformly in x.

Finally, it is trivial to check that $\lim_{q \to +\infty} T(q) = \alpha_{min}$ and $\lim_{q \to -\infty} T(q) = \alpha_{max}$.
■

5 Application in Signal Processing

It is well known that certain natural signals display some kind of self-similar behaviour (see [2] for examples). However, in most applications, even an approximate self-similarity does not hold. The scope of FIF modeling is thus quite restricted. Obviously, a much larger class of signals may be represented with WSA functions, because this modeling imposes far less constraints on the data. It allows in particular the small scale features to be different from the large scale ones. The interest of developping a method that finds, for a given signal, a WSA function that represents it is twofold. First, it permits to give a compact description of the signal, even in the case where it does not have definite fractal properties. Second, thanks to theorem 2, the WSA representation allows to compute the multifractal spectrum of the signal.

In practice, and specially when one deals with strongly non stationnary signals, a modeling with a single WSA function will still not be flexible enough. A natural extension is to represent the data with a lumping of WSA functions, thus taking into account the fact that several weak self-affine mechanisms may come into play at different periods of time. Formally, the problem may then be stated as follows: given a L^2 function F supported on the interval $[a, b]$, find a partition of $[a, b]$ into p subintervals $(I_j)_{j=1,\dots,p}$ and an associated set of p WSA functions $(F_j)_{j=1,\dots,p}$, each F_j being supported on I_j, such that the lumping of the F_j is the best L^2 approximation of F. This representation possesses the additional feature that it allows to segment F into parts which are multifractally homogeneous: This means that, for all j, the restriction of F to any subinterval of positive measure of (I_j) has the same multifractal spectrum as F_j. This new kind of stationarity may prove important in certain applications such as TCP traffic analysis (see [6]).

In all generality, the problem above seems hard to solve. However, it is possible to design a greedy algorithm that finds an acceptable sub-optimal solution for many real-world signals. We cannot develop this method here, and refer instead the interested reader to [4] for a complete description. We just show an example of application of this technique to the segmentation of a voice signal. The original signal is the word "welcome" uttered by a male speaker, containing 2^{15} samples. The WSA modeling yields a representation with seven functions F_j. As can be seen on figure 1, the original signal and the model are visually almost indistinguishable. More importantly, they sound practically the same, as the interested reader may check by pointing to http://www-rocq.inria.fr/fractales. In addition, the segmentation (see the red crosses) is phonetically relevant, since the marks almost perfectly coincide with the following sounds : silence, /w/, /ϵl/, silence, /k/, /ɔm/, silence. The slight discrepancy between the position of the segmentation marks and the exact location of the phonetic units is due to the fact that, in the current implementation of the method, the marks are restricted to be on dyadic points.

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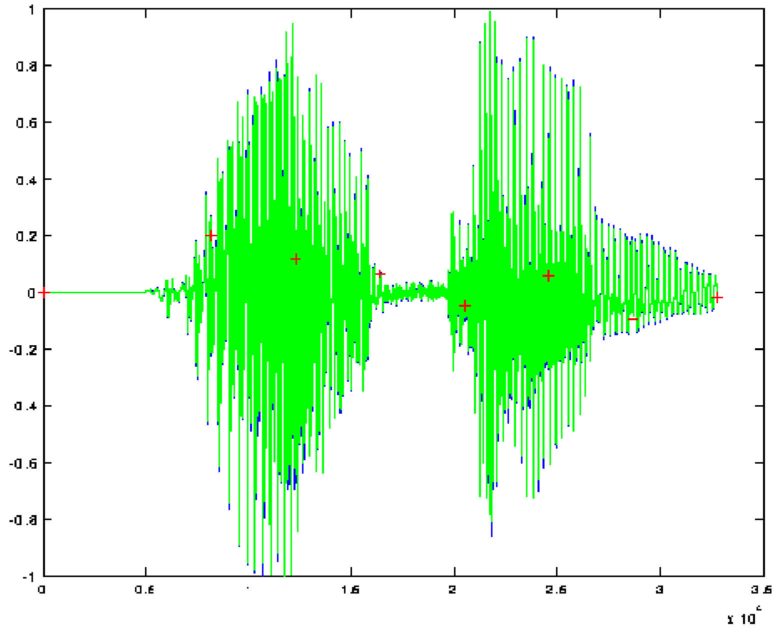

Figure 1: The word "welcome" uttered by a male speaker (in blue) along with its approximation (superimposed in green) and the segmentation marks (red crosses).