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Abstract: We consider a model of medium-term commodity contracts management. Randomness takes place only in the prices on which the commodities are exchanged whilst state variable is multi-dimensional. In [9], we proposed an algorithm to deal with such problem, based on quantization of random process and a dual dynamic programming type approach. We obtained accurate estimates of the optimal value and a suboptimal strategy from this algorithm. In this paper, we analyse the sensitivity with respect to parameters driving the price model. We discuss the estimate of marginal price based on the Danskin's theorem. Finally, some numerical results applied to realistic energy market problems have been performed. Comparisons between results obtained in [9] and other classical methods are provided and evidence the accuracy of the estimate of marginal prices.

Key-words: sensitivity, stochastic programming, Danskin theorem, dual dynamic programming

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Analyse de sensibilité de problémes de gestion de contrats d'énergie par des techniques de programmation stochastique

Résumé : Nous considérons un modèle de gestion de contrats de produit de base en moyen terme. Les aléas ne portent que sur le dynamique de prix. Dans [9], nous avons proposé un algorithme pour ce type de problème, basé sur la méthode de quantification de processus aléatoire, et l'approche de programmation dynamique duale. Nous avons obtenu un estimateur précis de la valeur optimale et une stratégie sous-optimale par cet algorithme. Dans cet article, nous analysons la sensibilité par rapport aux paramètres dans le processus aléatoire (modèle de prix). Nous discutons un estimateur de coût marginal basé sur le théorème de Danskin. A la fin, des tests numériques appliqués à des problèmes réels dans le marché l'énergie ont été réalisés. Les comparaisons du résultat par la méthode dans [9] et celui par des autres méthodes, donnent des estimateurs précis sur le coût marginal.

Mots-clés : sensibilité, programmation stochastique, théorème Danskin, programmation dynamique duale

We consider a model of medium-term commodity contracts management. Randomness takes place only in the prices on which the commodities are exchanged whilst state variable is multi-dimensional. In [9], we proposed an algorithm to deal with such a problem, based on quantization of the random process and a dual dynamic programming type approach. We obtained accurate estimates of the optimal value and a suboptimal strategy from this algorithm. In this paper, we analyse the sensitivity with respect to parameters driving the price model. We discuss the estimate of marginal price based on the Danskin's theorem. Finally, some numerical results applied to realistic energy market problems have been performed. Comparisons between results obtained in [9] and other classical methods are provided and give evidence of the good accuracy of the estimate of marginal prices.

1 Motivation

We study a class of problems in the following settings. First of all, we consider a uniform discretization of time horizon $[0, T]$, $0 = t_0 < t_1 < \cdots < t_K = T$, and a discrete time Markov process $(\boldsymbol{\xi}_k) := (\boldsymbol{\xi}_{t_k})$ (in the following, we will replace in index t_k by k for sake of clarity) in the probability space $L^2(\Omega, (\mathcal{F}_k), \mathbf{P}; \mathbb{R}^d)$. The canonic filtration associated with $(\boldsymbol{\xi}_k)$ is denoted by $\mathcal{F}_k := \sigma(\boldsymbol{\xi}_s, 0 \le s \le k)$. We may write its dynamics as

$$\boldsymbol{\xi}_{k+1} = f(W_k, \boldsymbol{\xi}_k, \alpha_k) \quad 0 \le k \le K-2, \tag{1}$$

where $\boldsymbol{\xi}_0$ is deterministic, (α_k) is a sequence of deterministic parameters, and (W_k) is a sequence of independent squared integrable random variables, independent of $\mathcal{F}_k$, finally f is a measurable mapping.

The stochastic dynamic decision problem has the following expression:

$$\begin{aligned} &\inf_{(u_k)\in U} && \mathbb{E}\left[\sum_{k=0}^{K-1} c_k(\boldsymbol{\xi}_k)\cdot u_k\right] \\ &\text{subject to} && u_k \in \mathfrak{U}_k, \quad \text{almost surely}, \\ &&& x_{k+1} = x_k + A_k u_k,\ x_0 = 0, \\ &&& x_T \in \mathfrak{X}_T \quad \text{almost surely}; \end{aligned} \tag{2}$$

where $u_k \in \mathbb{R}^n$ is the control variable, $U := L^2(\Omega, (\mathcal{F}_k), \mathbf{P}; \mathbb{R}^n)$ is the functional space of the control variable u_k, $\mathfrak{U}_k$ is a compact nonempty polyhedral set in $\mathbb{R}^n$, $x_k \in \mathbb{R}^m$ represents the state variable, $\mathfrak{X}_T$ is the set of admissible final states assumed to be a nonempty polyhedral set in $\mathbb{R}^m$ taking place in the final stage constraint over state variable x_T, $A_k \in \mathbb{R}^{m\times n}$ is the dynamic matrix, $c_k(\cdot) : \mathbb{R}^d \to \mathbb{R}^n$ is the running cost per unit, assumed to be Lipschitz.

Such a formulation (2) is motivated by the fact that most medium and long-term energy contracts can be modelled as multi stage stochastic dynamic programs. Indeed, most of the contracts define the terms on how the commodity will be exchanged throughout the duration of the contract. These terms generally state that the quantity to be taken at time t_k (modelled by the control u_k) is bounded by minimal and maximal amounts. They also specify the price at which the commodity will be exchanged at time t_k; this price is generally a function of the spot price (modelled by $c_k(\boldsymbol{\xi}_k)$) observed at that period. Since the spot price is unknown at the beginning, the problem becomes stochastic. The aim of the holder of such a contract is to optimize its decisions with respect to the price information available to him at that time. Here, we choose to maximize the expectation of the revenue. In order to be compliant with the formulation of (2), we will flip the sign of the real cost-revenue functions to transform a maximizing problem into a standard minimization problem.

Most of the articles in the literature focus on the procedures on how to solve problem (2) and hence how to get the contract's price (see Carmona and Touzi [13], Carmona and Ludkovski [12], and Barrera-Esteve et al [6]). Although essential, this information needs to be coupled with sensitivity information in a trading perspective. Indeed, the values of sensitivities (called the greeks) allow traders to replicate the price variations originating from the contracts they have to manage on a regular basis. See [21, Chapter 17] for further discussions on hedging strategies using the greeks applied to vanilla options.

A lot of efforts have been made in order to assess proper sensitivities from discontinuous payoffs such as digital or barrier options. However, only a handful of papers focus on optimal exercise type options and most of them concern Bermudan options which are common in the banking industry. Recently, Cont and Fournié have developed functional Itô formulae in order to study the sensitivities of path dependent options without control [14, 15]. Our purpose in this paper is to provide an alternative method to estimate sensitivities for problem (2) based on Danskin's theorem which is well known among optimization practitioners. The advantage of Danskin's theorem is that it can be associated to any discretization type method such as partial differential equations (PDE), Monte Carlo or optimal quantization. Also, it does not require to reestimate the conditional expectations once they have been computed during the price estimation procedure.

This paper is organised as follows: section 2 presents the resolution framework of problem (2) articulated around optimal quantization and stochastic dual dynamic programming (SDDP). Section 3 introduces the model driving the prices and section 4 focuses on Danskin's theorem through the computation of sensitivities. Convergence results are also provided in section 4. Finally, numerical tests are performed in section 5.

2 A review of quantization discretization and stochastic dual dynamic programming approach

In this section, we briefly present the algorithm which was introduced in our previous article [9].

2.1 Discretization

In many real life problems, the discrete time random process $(\boldsymbol{\xi}_k)$ may possibly take values within an infinite set. In order to make the problem numerically tractable, we start by discretizing the random process by a tree. Because of the Markov property of the random process, a recombining tree is the method of choice for reducing complexity. Bally et al [3, 2] introduced the vectorial quantization tree method as a new discretization scheme. The main idea is to replace $\boldsymbol{\xi}_k$ by the approximation $\hat{\boldsymbol{\xi}}_k$ whose support is a finite set $\Gamma_k = \left\{\xi_k^i, 1 \le i \le N_k\right\}$.

The quantization $\hat{\boldsymbol{\xi}}$ is said to be optimal if it minimizes the error associated with the L^2 norm:

$$\begin{aligned}\hat{\boldsymbol{\xi}} \in \operatorname{argmin} \Bigg\{ \|\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}\|_2 := \left(\sum_{k=0}^{K-1} \|\boldsymbol{\xi}_k - \hat{\boldsymbol{\xi}}_k\|_2^2 \right)^{1/2} \\ : \hat{\boldsymbol{\xi}}_k \in \Gamma_k, \sum_{k=0}^{K-1} \#(\Gamma_k) = \sum_{k=0}^{K-1} N_k \leq N \Bigg\}\end{aligned} \tag{3}$$

where N is the number of quantized points in the tree. Then, we define a *Voronoi tessellation* associated with the quantized points $\xi_k^i \in \Gamma_k$ by

$$C_k^i := \left\{ \boldsymbol{\xi}_k : |\boldsymbol{\xi}_k - \xi_k^i| \leq \min_{j \neq i} |\boldsymbol{\xi}_k - \xi_k^j| \right\}.$$

The final step consists in building the recombining tree by computing probability transitions:

$$p_k^{ij} := \mathbf{P}\left[\hat{\boldsymbol{\xi}}_{k+1} = \xi_{k+1}^j \,|\, \hat{\boldsymbol{\xi}}_k = \xi_k^i \right] = \mathbf{P}\left[\boldsymbol{\xi}_{k+1} \in C_{k+1}^j \,|\, \boldsymbol{\xi}_k \in C_k^i \right]$$

based on the law of $(\boldsymbol{\xi}_k)$. Bally and Pagès also proposed in [3] the competitive learning vectorial quantization (CLVQ) algorithm aimed at building a vectorial quantization tree based on the stochastic gradient method.

2.2 Stochastic dual dynamic programming algorithm

After having discretized the random process $(\boldsymbol{\xi}_k)$ with the help of vectorial quantization, we now focus on the resolution of problem (2). This problem can be naturally decomposed in time by the dynamic programming principle. We denote by $Q(k, x_k, \boldsymbol{\xi}_k)$ the Bellman value:

$$\begin{aligned} Q(k, x_k, \boldsymbol{\xi}_k) = \inf_{(u_s) \in U} \quad & \mathbb{E}\left[\sum_{s=k}^{K-1} c_s(\boldsymbol{\xi}_s) \cdot u_s \right] \\ \text{subject to} \quad & u_s \in \mathfrak{U}_s \quad \text{almost surely}, \\ & x_{s+1} = x_s + A_s u_s, \; x_0 = 0, \\ & x_T \in \mathfrak{X}_T \quad \text{almost surely.} \end{aligned} \tag{4}$$

Then, the dynamic programming principle can be written as

$$Q(k, x_k, \boldsymbol{\xi}_k) = \inf_{u_k \in \mathfrak{U}_k} c_k(\boldsymbol{\xi}_k) \cdot u_k + \mathcal{Q}(k+1, x_{k+1}, \boldsymbol{\xi}_k) \tag{5}$$

where

$$\mathcal{Q}(k+1, x_{k+1}, \boldsymbol{\xi}_k) = \mathbb{E}[Q(k+1, x_{k+1}, \boldsymbol{\xi}_{k+1}) \,|\, \mathcal{F}_k]$$

and final time condition:

$$\mathcal{Q}(T, x_T, \boldsymbol{\xi}_{T-1}) = \chi_{\mathfrak{X}_T}(x_T) = \begin{cases} 0 & \text{if } x_T \in \mathfrak{X}_T \\ \infty & \text{otherwise} \end{cases}$$

In that perspective, we have opted for the stochastic dual dynamic programming (SDDP) algorithm which is well suited to our problem. The stochastic dual dynamic programming method was first introduced by Pereira and Pinto [22]. Recently, Philpott and Guan [23] and Shapiro [25] have analysed the convergence property of this method. The main idea of SDDP algorithm is to approximate the Bellman value function $\mathcal{Q}(k, x_k, \boldsymbol{\xi}_{k-1})$ by the supremum of a family of affine function of x_k, depending on t_k and $\boldsymbol{\xi}_k$, denoted by $\vartheta(k+1, x_{k+1}, \hat{\boldsymbol{\xi}}_k, \mathcal{O}_{k+1}^{\hat{\boldsymbol{\xi}}_k})$. The method is based on successive iterations of forward and backward passes.

The forward pass generates a set of random trajectories following the dynamic of $(\boldsymbol{\xi}_k)$ and runs the optimization on the fly over these trajectories using the following approximation of dynamic programming:

$$
\begin{aligned}
\inf_{u_k \in \mathfrak{U}_k} \quad & c_k(\boldsymbol{\xi}_k) \cdot u_k + \vartheta(k+1, x_{k+1}, \hat{\boldsymbol{\xi}}_k, \mathcal{O}_{k+1}^{\hat{\boldsymbol{\xi}}_k}) \\
\text{subject to} \quad & x_{k+1} = x_k + A_k u_k, \\
& \vartheta(k+1, x_{k+1}, \xi_k^i, \mathcal{O}_{k+1}^{\hat{\boldsymbol{\xi}}_k}) \geq x^* \cdot x_{k+1} - e; \;\; \forall (x^*, e) \in \mathcal{O}_{k+1}^{\hat{\boldsymbol{\xi}}_k},
\end{aligned}
\tag{6}
$$

where $\mathcal{O}_{k+1}^{\hat{\boldsymbol{\xi}}_k}$ is the set of optimality cuts cumulated at vertex $\hat{\boldsymbol{\xi}}_k \in \Gamma_k$. The statistical average performed over the value associated to each forward pass provides an upper bound of the optimal value of (2).

The backward pass updates the optimality cuts $\mathcal{O}_{k+1}^{\hat{\boldsymbol{\xi}}_k}$ on each vertex in the tree using:

$$
\begin{aligned}
\inf_{u_k \in \mathfrak{U}_k} \quad & c_k(\hat{\boldsymbol{\xi}}_k) \cdot u_k + \vartheta(k+1, x_{k+1}, \hat{\boldsymbol{\xi}}_k, \mathcal{O}_{k+1}^{\hat{\boldsymbol{\xi}}_k}) \\
\text{subject to} \quad & x_{k+1} = x_k + A_k u_k, \\
& \vartheta(k+1, x_{k+1}, \xi_k^i, \mathcal{O}_{k+1}^{\hat{\boldsymbol{\xi}}_k}) \geq x^* \cdot x_{k+1} - e; \;\; \forall (x^*, e) \in \mathcal{O}_{k+1}^{\hat{\boldsymbol{\xi}}_k}.
\end{aligned}
\tag{7}
$$

Therefore the backward pass generates a lower bound of the optimal value at the first stage.

The stopping criteria has been discussed in [25]. The criteria taken in this article is that

$$|\overline{v} - \underline{v}| \leq \rho\sigma(\overline{v}) \tag{8}$$

where $\overline{v}$ is the forward pass value $\underline{v}$ is the backward pass value $\sigma(\overline{v})$ is the standard deviation of the forward pass value and ρ is a parameter.

Remark 2.1. Note that we are using the original continuous random process $(\boldsymbol{\xi}_k)$ in the forward pass, whereas the discretized random process $(\hat{\boldsymbol{\xi}}_k)$ is used in the backward pass. The convergence property follows 3 steps.

- The convergence result of L-Shape algorithm on finite probability distribution is given in Birge and Louveaux [7, chapter 7, theorem 1], where the main argument is the finiteness of optimality cuts and feasibility cuts. The convergence of classical SDDP method is studied in Philpott and Guan [23] in finite probability framework, and Shapiro [25] in general probability framework.
- During the numerical test, we can first simulate the forward pass following the discretized distribution $(\hat{\boldsymbol{\xi}}_k)$ and let the forward value and the backward value converge. This convergence follows that the forward value of Monte Carlo method is itself a random variable whose mean is the upper bound of L-Shape method which converges to lower bound by finite convergence of L-Shape method.

- Finally, we can change the forward simulation to following continuous distribution. Using continuous distribution in forward sampling gives an indication on the error induced by the discretization of the continuous distribution.

Remark 2.2. The method presented here deviates from standard stochastic dual dynamic programming in [22] because the random variable intervenes only in the objective function and is not present in the right hand side of state dynamics. This causes the partial convexity of $Q(t, x_t, \boldsymbol{\xi}_t)$, which is non convex with respect to $\boldsymbol{\xi}_t$, forcing us to proceed to discretization of $\boldsymbol{\xi}_t$. As a result, the optimality cuts cannot be shared among discretized points $\xi_t^i \in \Gamma_t$. Therefore, this method is clearly outperformed by others [6, 12] in terms of CPU time compared when state variable is in low dimension, as it will be shown in section 5.3. However, the main advantage resides in its ability to deal with high dimension (state variable) problems.

3 Price model

In this paper, we assume that the random process, taking part in the modeling of futures commodity price dynamics, follows the celebrated Black model [8]. Although simplistic, this model is equivalent to the one factor Gaussian HJM model (see [16]) which is well adapted for forward curve modeling. This model is discussed in the practice-based literature, Eydeland and Wolyniec[18, Chapter 5], and Lai et al [19]. Since we are using the vectorial quantization method to discretize the underlying random process, we are not limited to that model. Other financial price models can also be applied.

The price model takes as parameters the market volatilities σ, correlations ρ as well as the forward curves F_0. The Black model states that future prices of commodities are martingales under the risk neutral probability $\mathbf{P}$ whose expectations correspond to the original forward curve level F_0 due to lack of arbitrage opportunities. The forward contract price F_s^t at time s, with maturity (also called tenor) t, follows the dynamic:

$$\ln \frac{(F_{k+1}^t)^i}{(F_k^t)^i} = \sigma^i W_k^i - \frac{1}{2}(\sigma^i)^2 \quad i = 1, \dots, d; \tag{9}$$

where W_k^i is a standard normal distribution $\mathcal{N}(0, 1)$, with correlation

$$\mathrm{corr}(W_s^i, W_s^j) = \rho_{ij}.$$

Then, the spot prices at time t_k are simply obtained by taking the t_k-expiring forward contracts at time t_k: $\boldsymbol{\xi}_k = F_k^k$,

$$\boldsymbol{\xi}_k^i := (F_k^k)^i = (F_0^k)^i \exp\left(\sum_{s=0}^{k-1} \sigma^i W_s^i - \frac{1}{2}(\sigma^i)^2 k\right) \quad i = 1, \dots, d; \tag{10}$$

where (F_0^k) is the original forward curve observed. Setting

$$S_k^i := \exp\left(\sum_{s=0}^{k-1} \sigma^i W_s^i - (\sigma^i)^2 k/2\right),$$

we then have $\boldsymbol{\xi}_k^i = (F_0^k)^i S_k^i$.

Finally, a simple computation on the price model (10) shows that:

$$\frac{\partial \boldsymbol{\xi}_k^i}{\partial (F_0^k)^i} = S_k^i, \tag{11}$$

$$\frac{\partial \boldsymbol{\xi}_k^i}{\partial \sigma^i} = \boldsymbol{\xi}_k^i \left(\sum_{s=0}^{k-1} W_s^i - \sigma^i k \right). \tag{12}$$

Remark 3.1. In the context of our problem, it is reasonable to consider that $(F, \sigma) \in \mathbb{R}_+^{d \times K} \times \mathbb{R}_+^d$, where $\mathbb{R}_+ = \{x > 0\}$ is positive value. Under this condition, both $\boldsymbol{\xi}_t$, $D_{F,\sigma}\boldsymbol{\xi}_t$ have probability measure absolutely continuous with respect to Lebesgue measure, i.e. the value set does not reduce to singleton. Furthermore, the density function of log normal is bounded. This point is essential to continue analysis below since this property provides continuity even if the integrand is not continuous.

4 Sensitivity analysis

The principal contribution of this paper lies in the application of Danskin's theorem for computing certain sensitivities of problem (2). In this section, we begin by stating this theorem and apply it to our sensitivity analysis. Then, we provide a result on the convergence properties of the sensitivity estimate of our approximation algorithm in section 2 to the actual sensitivities of the original problem (2).

In the subsequent analysis, given two Banach spaces X and Y, we say that $\phi : X \to Y$ is directionally differentiable at a point $x \in X$ in the direction $h \in X$ if the following limit exists:

$$\phi'(x; h) := \lim_{t \downarrow 0} \frac{\phi(x + th) - \phi(x)}{t}. \tag{13}$$

4.1 Danskin's theorem and its applications

Let us denote by $v(F, \sigma)$ (resp. $U^*(F, \sigma)$) the optimal value function (resp. the optimal solution set) of the original problem (2). In this section, we show that $v(F, \sigma)$ is directionally differentiable at every $F \in \mathbb{R}_+^{d \times K}$ and $\sigma \in \mathbb{R}_+^d$, and Fréchet differentiable almost everywhere.

We first study the objective function of problem (2) which can be written as:

$$f(F, \sigma, u) = \mathbb{E} \left[\sum_{k=0}^{K-1} c_k(\boldsymbol{\xi}_k(F_0^k, \sigma)) \cdot u_k \right]. \tag{14}$$

$f(F, \sigma, u)$ is a continuous function on $\mathbb{R}_+^{d \times K} \times \mathbb{R}_+^d \times U$ according to the continuity of c_k and the continuity of $(\boldsymbol{\xi}_k)$ in the price model (10). Then, we show the Fréchet differentiability by the following lemma.

Lemma 4.1. *The objective function $f(F, \sigma, u)$ is Fréchet differentiable with respect to $F \in \mathbb{R}_+^{d \times K}$ and $\sigma \in \mathbb{R}^d$, and its derivative is*

$$D_{F,\sigma} f(F, \sigma, u) = \mathbb{E} \left[\sum_{k=0}^{K-1} D_{F,\sigma} c_k \cdot u_k \right]. \tag{15}$$

Proof. In view of the Lipschitz continuity of c_k, Rademacher's theorem [24, Theorem 9.60] states that it is almost everywhere Fréchet differentiable, and its derivative is bounded. Note that in the derivative of $f(F,\sigma,u)$ will appear the derivative of c_k as an integrand against measure of $\boldsymbol{\xi}_k$. Furthermore, at time t_k, the random variable $\boldsymbol{\xi}_k$ follows a log-normal distribution according to our price model, whose probability measure is absolutely continuous with respect to the Lebesgue measure. Thus, using chain rule of derivative on the composite mapping as well as the dominated convergence theorem ends the proof. $\square$

Following this lemma, we immediately deduce another property of $f(F,\sigma,u)$.

Proposition 4.2. *The derivative $D_{F,\sigma}f$ is continuous with respect to $(F,\sigma) \in \mathbb{R}_+^{d\times K} \times \mathbb{R}_+^d$, and it is weakly continuous with respect to u.*

Proof. In view of (15), $D_{F,\sigma}f(F,\sigma,u)$ is weakly continuous with respect to u.

Now, let us consider a sequence $(F^n,\sigma^n) \to (F^*,\sigma^*)$ in $\mathbb{R}_+^{d\times K} \times \mathbb{R}_+^d$, and denote by $\boldsymbol{\xi}_k^n = \boldsymbol{\xi}_k(F^n,\sigma^n)$ and $\boldsymbol{\xi}_k^* = \boldsymbol{\xi}_k(F^*,\sigma^*)$. We have to show that $D_{F,\sigma}c_k(\boldsymbol{\xi}_k^n) \xrightarrow{L^1} D_{F,\sigma}c_k(\boldsymbol{\xi}_k^*)$. By chain rule of derivative, at the points where c_k is differentiable, we have

$$D_{F,\sigma}c_k(\boldsymbol{\xi}_k) = D_{\boldsymbol{\xi}_k}c_k(\boldsymbol{\xi}_k)D_{F,\sigma}\boldsymbol{\xi}_k(F,\sigma). \tag{16}$$

The term $D_{F,\sigma}\boldsymbol{\xi}_k(F,\sigma)$ is computed in (11) and (12). As mentioned in remark 3.1, $\boldsymbol{\xi}_k^*$ and its derivative $D_{F,\sigma}\boldsymbol{\xi}_k^*$ (resp. $\boldsymbol{\xi}_k^n$ and $D_{F,\sigma}\boldsymbol{\xi}_k^n$) both have bounded density function pdf_k^* (resp. pdf_k^n) with respect to Lebesgue measure. Since $c_k : \mathbb{R}^d \to \mathbb{R}^n$ is Lipschitz, $Dc_k \in L^\infty(\Omega,\mathcal{F},\mathbf{P}) \subset L^q(\Omega,\mathcal{F},\mathbf{P})$, $1 \le q < \infty$. Since the space of continuous functions with compact support $C_c(\Omega)$ is dense in $L^q(\Omega,\mathcal{F},\mathbf{P})$, we have $\forall\epsilon > 0$, $\exists g$, such that $\|g - Dc_k\|_q \le \epsilon$.

$$\begin{aligned}
\|g(\boldsymbol{\xi}_k^n) - Dc_k(\boldsymbol{\xi}_k^n)\|_q &= \left(\int_{\boldsymbol{\xi}} |g(\boldsymbol{\xi}) - Dc_k(\boldsymbol{\xi})|^q \mathrm{pdf}_k^n(\boldsymbol{\xi})d\boldsymbol{\xi}\right)^{1/q} \\
&\le \|\mathrm{pdf}_k^n\|_\infty^{1/q}\|g - Dc_k\|_q \\
&\le \|\mathrm{pdf}_k^n\|_\infty^{1/q}\epsilon
\end{aligned}$$

We have same result for (F^*,σ^*)

$$\|g(\boldsymbol{\xi}_k^*) - Dc_k(\boldsymbol{\xi}_k^*)\|_q \le \|\mathrm{pdf}_k^*\|_\infty^{1/q}\epsilon$$

Moreover, $\boldsymbol{\xi}_k^n \xrightarrow{a.s.} \boldsymbol{\xi}_k^*$ and $D_{F,\sigma}\boldsymbol{\xi}_k^n \xrightarrow{L^p} D_{F,\sigma}\boldsymbol{\xi}_k^*, 1 \le p < \infty$. By continuity of g and the dominated convergence theorem, $\exists N_p$, we have that $\forall n \ge N_p$

$$\|g(\boldsymbol{\xi}_k^n) - g(\boldsymbol{\xi}_k^*)\|_q \le \epsilon.$$

Thus, for $n \ge N_q$

$$\begin{aligned}
&\|Dc_k(\boldsymbol{\xi}_k^n) - Dc_k(\boldsymbol{\xi}_k^*)\|_q \\
&\le \|Dc_k(\boldsymbol{\xi}_k^n) - g(\boldsymbol{\xi}_k^n)\|_q + \|g(\boldsymbol{\xi}_k^n) - g(\boldsymbol{\xi}_k^*)\|_q + \|g(\boldsymbol{\xi}_k^*) - Dc_k(\boldsymbol{\xi}_k^*)\|_q \\
&\le (\|\mathrm{pdf}_k^n\|_\infty^{1/q} + \|\mathrm{pdf}_k^*\|_\infty^{1/q} + 1)\epsilon
\end{aligned} \tag{17}$$

Let $\epsilon \to 0$, we obtain $Dc_k(\boldsymbol{\xi}_k^n) \xrightarrow{L^q} Dc_k(\boldsymbol{\xi}_k^*)$.

Finally, taking $p^{-1} + q^{-1} = 1$, $D_{F,\sigma}c_k(\boldsymbol{\xi}_k^n) \xrightarrow{L^1} D_{F,\sigma}c_k(\boldsymbol{\xi}_k^*)$ follows by combination of convergence of $D_{\boldsymbol{\xi}_k}c_k(\boldsymbol{\xi}_k)$ in L^p and convergence of $D_{F,\sigma}\boldsymbol{\xi}_k$ in L^q. $\square$

Next, we recall Danskin's theorem:

Theorem 4.3. *[10, Theorem 4.13] Let V be a nonempty, compact topological space and $\psi : \mathbb{R}^n \times V \to \mathbb{R}$. Let $\phi(x)$ be the optimal value function and $V^*(x)$ be the optimal solution set:*

$$\phi(x) := \min_{v \in V} \psi(x,v) \qquad V^*(x) := \operatorname*{argmin}_{v \in V} \psi(x,v)$$

If $\psi(\cdot,v)$ is differentiable for every $v \in V$ and $D_x\psi(x,v)$ is continuous on $\mathbb{R}^n \times V$, then $\phi(x)$ is locally Lipschitz continuous, directionally differentiable, and

$$\phi'(x;h) = \inf_{v \in V^*(x)} D_x\psi(x,v)h. \tag{18}$$

In particular, if for some $x \in \mathbb{R}^n$, the set $V^*(x) = \{v^*\}$ is a singleton, then the min-function is differentiable at x and

$$\phi'(x) = D_x\psi(x,v^*). \tag{19}$$

In order to show the directional differentiability of $v(F,\sigma)$, we only need to verify the compactness assumption of the feasible solution space in Danskin's theorem 4.3. In our framework, the feasible solution set U taking value in $(\mathfrak{U}_k)$ is a bounded closed subset of $L^2 \cap L^\infty$. From the Banach Alaoglu Bourbaki theorem [11, theorem 3.16], it is a weak compact subset in L^2.

Corollary 4.4. *The optimal function $v(F,\sigma)$ is Fréchet differentiable at almost every $F \in \mathbb{R}_+^{d\times K}$ and $\sigma \in \mathbb{R}_+^d$, and at the point where $v(F,\sigma)$ is differentiable, we have*

$$Dv(F,\sigma) = D_{F,\sigma} f(F,\sigma,u), \quad \text{for all } u \in U^*. \tag{20}$$

Proof. The almost every Fréchet differentiability is a direct consequence of Rademacher's theorem [24, Theorem 9.60] and the fact that $v(F,\sigma)$ is locally Lipschitz by Danskin's theorem. And if u_1 and u_2 are two optimal solutions such that $D_{F,\sigma} f(F,\sigma,u_1) \neq D_{F,\sigma} f(F,\sigma,u_2)$, then (18) states that f is not Fréchet differentiable at (F,σ). Then, we have a contradiction. □

The direct application of Danskin's theorem leads to one method to compute the value of sensitivities. We will focus on the sensitivities $\delta_F(dF)$ and $\delta_\sigma(d\sigma)$ and give an expression the directional derivative of the optimal value $v(F,\sigma)$ with respect to the forward curve F and volatility vector σ in the directions dF and $d\sigma$:

$$\delta_F(dF) := v'(F,\sigma;dF,0) = \inf_{(u_k^*) \in U^*} \mathbb{E}\left[\sum_{k=0}^{K-1} D_F c_k dF_0^k \cdot u_k^*\right] \tag{21}$$

$$\delta_\sigma(d\sigma) := v'(F,\sigma;0,d\sigma) = \inf_{(u_k^*) \in U^*} \mathbb{E}\left[\sum_{k=0}^{K-1} D_\sigma c_k d\sigma \cdot u_k^*\right] \tag{22}$$

Danskin's theorem requires all the optimal solutions which is generally difficult to obtain in stochastic programs which are not strongly convex. In most cases, we can only get *one* optimal policy (u_k^*), and the whole optimal policy set is hard to compute. Therefore, we can simply obtain an upper bound of the sensitivity value:

$$\delta_F(dF) \leq \bar{\delta}_F(dF, u^*) := \mathbb{E}\left[\sum_{k=0}^{K-1} D_F c_k dF_0^k \cdot u_k^*\right] \tag{23}$$

$$\delta_\sigma(d\sigma) \leq \bar{\delta}_\sigma(d\sigma, u^*) := \mathbb{E}\left[\sum_{k=0}^{K-1} D_\sigma c_k d\sigma \cdot u_k^*\right] \tag{24}$$

However, by corollary 4.4, $v(F, \sigma)$ is Fréchet differentiable almost everywhere. Thus, the inequalities in (23) and (24) become equality at almost every F and σ. So, knowing one optimal policy $u^* \in U^*$, allows us to get the derivative value almost everywhere.

Furthermore, we can study the derivatives associated to the standard base of F and σ. At almost every F and σ, we have:

$$(\delta_F)_k^i := \delta_F(e_k^i) = \mathbb{E}\left[\frac{\partial c_k(\boldsymbol{\xi}_k)}{\partial (F_0^k)^i} \cdot u_k^*\right] \quad i = 1, \dots, d \tag{25}$$

$$(\delta_\sigma)^i := \delta_\sigma(e^i) = \mathbb{E}\left[\sum_{k=0}^{K-1} \frac{\partial c(\boldsymbol{\xi}_k)}{\partial \sigma^i} \cdot u_k^*\right] \quad i = 1, \dots, d \tag{26}$$

Then, at almost every F and σ:

$$\delta_F(dF) = \sum_{k=0}^{K-1} \sum_{i=1}^{d} (dF)_k^i (\delta_F)_k^i \qquad \delta_\sigma(d\sigma) = \sum_{i=1}^{d} (d\sigma)^i (\delta_\sigma)^i. \tag{27}$$

4.2 Convergence of sensitivity estimate

In this section, we start by presenting two discrete approximations of problem (2) helping us describe the forward and backward schemes made in our algorithm in section 2. Then we show some properties of the solutions of the discrete approximation problems and present their convergence to the solution of problem (2). Eventually, we deduce the convergence of the sensitivity estimates of the discrete approximation problems to the sensitivities of continuous problem.

We consider a sequence of optimal vectorial quantization trees $\hat{\boldsymbol{\xi}}^m \in \Gamma^m$, such that the total number of quantized points is N^m and $\lim_{m\to\infty} N^m = \infty$. Denote by $(\hat{\mathcal{F}}_k^m)$ the canonic filtration associated to the discretized random process $(\hat{\boldsymbol{\xi}}_k^m)$: $\hat{\mathcal{F}}_k^m := \sigma(\hat{\boldsymbol{\xi}}_s^m, 0 \leq s \leq k)$.

Let us consider two discrete approximations of problem (2). The first one enforces a constant strategy in each Voronoi cell. We call it as the *quantization approximation* of problem (2):

$$\begin{aligned} v_Q^m(F, \sigma) = \inf_{(u_k^m)\in U} \quad & \mathbb{E}\left[\sum_{k=0}^{K-1} c_k(\boldsymbol{\xi}_k) \cdot u_k^m\right] \\ \text{subject to} \quad & u_k^m \in \mathfrak{U}_k, \quad \text{almost surely} \\ & u_k^m(\boldsymbol{\xi}_k, x_k^m) = \sum_{\xi_k^{i,m} \in \Gamma_k^m} u_k^{i,m}(x_k^m) \mathbb{1}_{C_k^{i,m}}(\boldsymbol{\xi}_k) \\ & x_{k+1}^m = x_k^m + A_k u_k^m, \quad x_0^m = 0 \\ & x_T^m \in \mathfrak{X}_T^m \quad \text{almost surely} \end{aligned} \tag{28}$$

By forward induction, we can deduce that u_k is $\hat{\mathcal{F}}_k^m$ adapted and that x_k is $\hat{\mathcal{F}}_{k-1}^m$ measurable. This quantization approximation describes the formulation made in the forward pass of our algorithm.

Since u_k is $\hat{\mathcal{F}}_k^m$ adapted, we may write the cost function as a function of (28) as

$$\begin{aligned} \mathbb{E}\left[\sum_{k=0}^{K-1} c_k(\boldsymbol{\xi}_k)\cdot u_k^m\right] &= \mathbb{E}\left[\sum_{k=0}^{K-1} \mathbb{E}\left[c_k(\boldsymbol{\xi}_k)\cdot u_k^m|\hat{\mathcal{F}}_k^m\right]\right] \\ &= \mathbb{E}\left[\sum_{k=0}^{K-1} \mathbb{E}\left[c_k(\boldsymbol{\xi}_k)|\hat{\mathcal{F}}_k^m\right]\cdot u_k^m\right]. \end{aligned} \tag{29}$$

In this formulation the decision $u_k^{i,m}$ to be taken when the state belongs to the cell C_k^i at step t_k actually depends on the previous cells along the realization of ξ. We may give another approximation formulation by replacing in each Voronoi cell $\mathbb{E}\left[c_k(\boldsymbol{\xi}_k)|\hat{\mathcal{F}}_k^m\right]$ by $c_k(\mathbb{E}[\boldsymbol{\xi}_k \mid \boldsymbol{\xi}_k \in C_k^i])$ in the previous formulation. It can be shown, see [26, Chapter 3], that it is equivalent to one large size formulation on scenarios. We call it be the *scenario approximation* of problem (2):

$$\begin{aligned} v_S^m(F,\sigma) = \inf_{(u_k^m)\in U} \quad & \sum_{(\xi_k^{i,m})_k} p^{i,m}\left(\sum_{k=0}^{K-1} c_k(\boldsymbol{\xi}_k^{i,m})\cdot u_k^{i,m}\right) \\ \text{subject to} \quad & u_k^{i,m} \in \mathfrak{U}_k \\ \text{(non-anticipativity)} \quad & u_k^{i,m} = u_k^{j,m}, \quad \text{if } \xi_{[k]}^{i,m} = \xi_{[k]}^{j,m} \\ & x_{k+1}^{i,m} = x_k^{i,m} + A_k u_k^{i,m}, \quad x_0^{i,m} = 0 \\ & x_K^{i,m} \in \mathfrak{X}_K^{i,m} \end{aligned} \tag{30}$$

where $\xi_{[k]}^{i,m} := (\xi_0^{i_0,m},\dots,\xi_k^{i_k,m}) \in (\Gamma_0^m,\dots,\Gamma_k^m)$ is a sub-trajectory in the quantization tree until time t_k, $p^{i,m} := \prod_{k=0}^{K-2} p_k^{i_k i_{k+1},m}$ is the probability associated to one trajectory $\xi_{[K-1]}^{i,m}$. This scenario approximation describes the formulation made in the backward pass in our algorithm.

Denote by $v_Q^m(F,\sigma)$ and $u_Q^{*,m}(F,\sigma)$ (resp. $v_S^m(F,\sigma)$ and $u_S^{*,m}(F,\sigma)$) the optimal value function and one optimal solution of the quantization approximation problem (28) (resp. of the scenario approximation problem (30)).

In the same way as (25) and (26), we obtain the sensitivity estimates corresponding to the above discretized approximation problems by Danskin's theorem:

$$(\delta^m_{Q,F})^i_k := (v^m_Q)'(F,\sigma;e^i_k,0) \le (\bar\delta^m_{Q,F})^i_k := \mathbb{E}\left[\frac{\partial c_k(\boldsymbol{\xi}_k)}{\partial (F^k_0)^i}\cdot (u^{*,m}_Q)_k\right] \quad i=1,\dots,d \tag{31}$$

$$(\delta^m_{Q,\sigma})^i := (v^m_Q)'(F,\sigma;0,e^i) \le (\bar\delta^m_{Q,\sigma})^i := \mathbb{E}\left[\sum_{k=0}^{K-1}\frac{\partial c(\boldsymbol{\xi}_k)}{\partial \sigma^i}\cdot (u^{*,m}_Q)_k\right] \quad i=1,\dots,d \tag{32}$$

$$(\delta^m_{S,F})^i_k := (v^m_S)'(F,\sigma;e^i_k,0) \le (\bar\delta^m_{S,F})^i_k := \mathbb{E}\left[\frac{\partial c_k(\hat{\boldsymbol{\xi}}_k)}{\partial (F^k_0)^i}\cdot (u^{*,m}_S)_k\right] \quad i=1,\dots,d \tag{33}$$

$$(\delta^m_{S,F})^i := (v^m_S)'(F,\sigma;e^i,0) \le (\bar\delta^m_{S,\sigma})^i := \mathbb{E}\left[\sum_{k=0}^{K-1}\frac{\partial c(\hat{\boldsymbol{\xi}}_k)}{\partial \sigma^i}\cdot (u^{*,m}_S)_k\right] \quad i=1,\dots,d \tag{34}$$

Lemma 4.5. *If the distribution of $\boldsymbol{\xi}_k$ is absolutely continuous with respect to the Lebesgue measure, then the optimal value functions of the two discrete discrete approximation problems $v^m_S(F,\sigma)$ and $v^m_Q(F,\sigma)$ are Fréchet differentiable at almost every $F \in \mathbb{R}^{d\times K}$ and $\sigma \in \mathbb{R}^d_+$.*

Proof. The scenario approximation problem (30) can be viewed as a large linear program. According to the linear programming theory, v^m_S is piecewise linear, concave with respect to the coefficient of the objective function defined in whole space, so v^m_S is locally Lipschitz and then almost everywhere Fréchet differentiable by Rademacher's theorem [24, Theorem 9.60]. Under the assumption on the distribution of $\boldsymbol{\xi}_k$, and the fact that $(F,\sigma)\to\hat{\boldsymbol{\xi}}^m(F,\sigma)$ is injective and Fréchet differentiable, the chain rule of Fréchet derivative of composite mapping holds almost everywhere and we deduce the lemma for scenario approximation problem (30).

The quantization approximation problem (28) only differs from (30) by replacing the coefficient in objective function $\mathbb{E}[c(\boldsymbol{\xi}_t)\,|\,\boldsymbol{\xi}_t\in C^i_t]$ by $c(\mathbb{E}[\boldsymbol{\xi}_t\,|\,\boldsymbol{\xi}_t\in C^i_t])$. Thus, concavity of value function v^m_Q with respect to the coefficient in the objective function still holds. With the same arguments, we get the almost everywhere Fréchet differentiability for the optimal value function of the quantization approximation problem (28). □

Therefore, we are able to obtain exact sensitivity values for both discrete approximation problems following Danskin's theorem almost everywhere, i.e. the inequalities (31)- (34) become equalities almost everywhere.

Lemma 4.6. *A subsequence of the optimal strategy $u^{*,m}_Q$ of the quantization approximation problem* (28) *converges weakly in $L^2(\Omega,(\mathcal{F}_t),\mathbf{P};\mathbb{R}^{n\times K})$ to an optimal strategy u^* of continuous problem* (2)*.*

Proof. We denote by v^*_Q the limit inferior of the associated value function $v^*_Q := \liminf_{m\to\infty} v^{*,m}_Q$, and the subsequence associated to this limit inferior $u^{*,m_{n1}}_Q$. The feasible set of (28) is a subset of U taking value in $\mathfrak{U}$ which is non empty, convex and bounded. The optimal solution $u^{*,m_{n1}}_Q$ is therefore a bounded sequence in a Hilbert space U. Therefore, we deduce the existence of a weakly convergent subsequence $u_Q^{*,m_{n2}}$ that converges in the weak topology to u_Q^* following [11, Corollary 3.30]. Since the feasible set is closed and convex, it is weakly closed and hence u_Q^* is feasible. Since $u_Q^{*,m}$ is always feasible for the original problem (2), we get $v \leq v_Q^{*,m}$ for every m. Let m go to ∞ and we obtain the first inequality $v \leq v_Q^*$.

In order to obtain the reverse inequality, let us build a sequence which is feasible for the quantization approximation problem (28) and which converges to an optimal solution of problem (2). The feasible solution set U is included in L^2, where C^∞ is dense. Then for one optimal solution $(u_k^*) \in U^*$ of the continuous problem (2), there exists one sequence $(u_k^{n,c}) \in C^\infty$ converging to (u_k^*) such that $\sum_{k=0}^{K-1} \|u_k^{n,c} - u^*\| \leq 1/n$. We build

$$u_k^{n,m}(\boldsymbol{\xi}_{[k]}) = \mathbb{E}\left[u_k^{n,c}(\boldsymbol{\xi}_{[k]}) \,|\, \hat{\mathcal{F}}_k^m\right],$$

which is one feasible solution of the quantization approximation problem (28). The convergence of $u_k^{n,m}$ to $u_k^{n,c}$ in L^2 is due to the convergence $\|\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}^m\| \to 0$ and the dominated convergence theorem. We denote by $v_Q^{n,m}$ the value associated with strategy $u_k^{n,m}$. Therefore, $v_Q^m \leq v_Q^{n,m}$. Let us consider the sequence $v_Q^{n,n}$, and let $n \to \infty$, we obtain the second inequality $v_Q^* \leq v$. Thus, we get $v_Q^* = v$ and hence prove the lemma. □

One immediate corollary is

Corollary 4.7. *The sensitivity estimates of the quantization approximation problem $\bar{\delta}_{Q,F}^m$ and $\bar{\delta}_{Q,\sigma}^m$ defined in* (31) *and* (32) *converge to the sensitivity values of continuous problem δ_F and δ_σ (defined in* (21) *and* (22)*) at almost every $F \in \mathbb{R}_+^{d\times K}$ and $\sigma \in \mathbb{R}_+^d$.*

Lemma 4.8. *A subsequence of the optimal strategy u_S^* of scenario approximation problem* (30) *converges weakly in $L^2(\Omega, (\mathcal{F}_t), \mathbf{P}; \mathbb{R}^{n\times K})$ to an optimal strategy u^* of continuous problem* (2).

Proof. In view of lemma 4.6, we just need to compare the optimal value function v_S^m and v_Q^m. Both discretized approximation problems have the same feasible solution space $L^2(\Omega, (\hat{\mathcal{F}}_k^m), \mathbf{P}) \to (\mathfrak{U}_k)$. By a straightforward computation, we can obtain that

$$\begin{aligned}
v_Q^m - v_S^m &= \sum_{k=0}^{K-1} \mathbb{E}[c_k(\boldsymbol{\xi}_k)(u_Q^{*,m})_k] - \sum_{k=0}^{K-1} \mathbb{E}[c_k(\hat{\boldsymbol{\xi}}_k)(u_S^{*,m})_k] \\
&\leq \sum_{k=0}^{K-1} \mathbb{E}[c_k(\boldsymbol{\xi}_k)(u_S^{*,m})_k] - \sum_{k=0}^{K-1} \mathbb{E}[c_k(\hat{\boldsymbol{\xi}}_k)(u_S^{*,m})_k] \\
&\leq \sum_{k=0}^{K-1} [c_k]_{lip} \|\mathfrak{U}_k\|_\infty \|\boldsymbol{\xi}_t - \hat{\boldsymbol{\xi}}_t\|_2 \\
&\leq C_{QS} \sum_{k=0}^{K-1} \|\boldsymbol{\xi}_k - \hat{\boldsymbol{\xi}}_k\|_2 \leq 2C_{QS} \|\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}\|_2
\end{aligned}$$

where $\|\mathfrak{U}\|_\infty = \sup\{|u| \in \mathfrak{U}\}$, and C_{QS} is a constant which only depends on $[c_k]_{lip}$ and $\|\mathfrak{U}\|_\infty$. This error bound can be obtained on $(v_S^m - v_Q^m)$ by using same technique. Thus we have

$$|v_Q^m - v_S^m| \leq 2C_{QS}\|\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}\|_2$$

Following Zador's theorem (see below theorem 4.10), we get that $\lim_{m\to\infty} v_Q^m = \lim_{m\to\infty} v_S^m$. Then, the lemma can be proved by following the same argument as in the proof of 4.6. □

Then, we have a similar corollary as corollary 4.7 for the scenario approximation problem:

Corollary 4.9. *The sensitivity estimates of the scenario approximation problem $\bar{\delta}_{S,F}^m$ and $\bar{\delta}_{S,\sigma}^m$ defined in* (33) (34) *converge to the sensitivity values of continuous problem δ_F and δ_σ defined in* (21) *and* (22)*, at almost every $F \in \mathbb{R}_+^{d\times K}$ and $\sigma \in \mathbb{R}_+^d$.*

Proof. This corollary is obtained following the fact that $(\hat{\boldsymbol{\xi}}_k^m)$ converges to $(\boldsymbol{\xi}_k)$ and that $u_S^{*,m}$ converges weakly in U to u^* and the [11, Proposition 3.13]. □

Finally, we recall Zador's theorem:

Theorem 4.10. *[20, Theorem 6.2] If $\mathbb{E}[|\boldsymbol{\xi}|^{p+\eta}] < \infty$ for some $\eta > 0$, then,*

$$\lim_N \left(N^{p/d} \min_{|\Gamma|\leq N} \|\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}\|_p^p\right) = J_{p,d}\left(\int |g|^{d/(d+p)}(u)du\right)^{1+p/d}$$

where $\mathbf{P}(du) = g(u)\lambda_d(du) + \nu, \nu \perp \lambda_d$ (λ_d Lebesgue measure on $\mathbb{R}^d$). The constant $J_{p,d}$ corresponds to the case of the uniform distribution on $[0,1]^d$.

This theorem implies that $\min_{|\Gamma|\leq N} \|\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}\|_p = O(N^{1/d})$.

5 Algorithm and numerical tests

5.1 Algorithm

After getting one optimal (ϵ-optimal) strategy $(u_k^{*,tr-sddp})$ thanks to our discretization and SDDP algorithm described in section 2, we use the Monte Carlo method to compute the expectations in (31)-(32).

We simulate a sample $\xi^s, s = 1, \ldots, S$ of trajectories and compute the value of sensitivities for each trajectory:

$$^s(\delta_F)_k^i = \frac{\partial c_k(\boldsymbol{\xi}_k^s)}{\partial (F_0^k)^i} \cdot {}^s u_k^{*,tr-sddp}, \quad i = 1, \ldots, d; \tag{35}$$

$$^s(\delta_\sigma)^i = \sum_{k=0}^{K-1} \frac{\partial c_k(\boldsymbol{\xi}_k^s)}{\partial \sigma^i} \cdot {}^s u_k^{*,tr-sddp}, \quad i = 1, \ldots, d. \tag{36}$$

Therefore we obtain the mean and standard deviation of this sample:

$$\mu^{tr-sddp}_{(\delta_F)^i_k} = \frac{1}{S}\sum_{s=1}^{S} {}^s(\delta_F)^i_k, \quad \sigma^{tr-sddp}_{(\delta_F)^i_k} = \left(\frac{1}{S-1}\sum_{s=1}^{S}({}^s(\delta_F)^i_k - \mu^{tr-sddp}_{(\delta_F)^i_k})^2\right)^{1/2}; \tag{37}$$

$$\mu^{tr-sddp}_{(\delta_\sigma)^i} = \frac{1}{S}\sum_{s=1}^{S} {}^s(\delta_\sigma)^i, \quad \sigma^{tr-sddp}_{(\delta_\sigma)^i} = \left(\frac{1}{S-1}\sum_{s=1}^{S}({}^s(\delta_\sigma)^i - \mu^{tr-sddp}_{(\delta_\sigma)^i})^2\right)^{1/2}. \tag{38}$$

5.2 Comparison of methods

We shall now address the question of the accuracy of the results obtained from the application of Danskin's theorem. In that perspective, we benchmark them against the sensitivities obtained by the mean of other algorithms.

The first comparison lies in the discretization of state variable (x_k). Indeed, many popular algorithms impose the discretization of (x_k) as a prerequisite whilst our SDDP based algorithm removes this constraint. Although more efficient than SDDP in low dimensions, the (x_k) discrete algorithms become numerically intractable when the dimension of (x_k) increases.

The second source of comparison concerns the discretization of the random process $(\boldsymbol{\xi}_k)$. In our framework, we have chosen the optimal quantization tree method as a way to discretize the randomness. As long as $(\boldsymbol{\xi}_k)$ remains in low dimension, we may also implement a PDE based method.

Finally, the last and probably most important criteria of comparison resides in the way the sensitivity values are computed. We challenge the numerical results obtained by the application of the Danskin's theorem against the finite difference method which is as simple as it is popular.

5.2.1 Danskin + quantization tree + state space discretization

The first comparison method is the algorithm proposed in [5] that combines the quantization tree method and the dynamic programming principle on discretized state space $(\hat{x}_k)$. Let us stress again that this method proves to be very efficient as long as the state variable (x_k) remains in low dimension. Let $v^{tr-disc}$ stands for the Bellman value and $u^{*,tr-disc}$ for the optimal strategy:

$$v^{tr-disc}(k,\hat{x}_k,\hat{\boldsymbol{\xi}}_k) = \min_{u_k\in\mathfrak{U}_k}\Big(c_k(\hat{\boldsymbol{\xi}}_k)u_k + \mathbb{E}[v^{tr-disc}(k+1,\hat{x}_k+u_k,\hat{\boldsymbol{\xi}}_{k+1})\,|\,\hat{\boldsymbol{\xi}}_k]\Big); \tag{39}$$

$$u^{*,tr-disc}(k,\hat{x}_k,\hat{\boldsymbol{\xi}}_k) = \operatorname*{argmin}_{u_k\in\mathfrak{U}_k}\Big(c_k(\hat{\boldsymbol{\xi}}_k)u_k + \mathbb{E}[v^{tr-disc}(k+1,\hat{x}_k+u_k,\hat{\boldsymbol{\xi}}_{k+1})\,|\,\hat{\boldsymbol{\xi}}_k]\Big); \tag{40}$$

where

$$\mathbb{E}[v^{tr-disc}(k+1,x,\hat{\boldsymbol{\xi}}_{k+1})\,|\,\hat{\boldsymbol{\xi}}_k=\xi^i_k] = \sum_{\xi^j_{k+1}\in\Gamma_{k+1}} p^{ij}_k v^{tr-disc}(k+1,x,\xi^j_{k+1}).$$

We can reuse the same formula as (35)-(38) with the optimal policy $u^{*,tr-disc}$ to compute the sensitivity values that will be denoted by $(\delta_F)^{tr-disc}$ and $(\delta_\sigma)^{tr-disc}$.

Thanks to this procedure, we are able to compare two methods to compute the optimal solutions obtained on the same quantization tree as the one used in our SDDP based algorithm.

5.2.2 Danskin + PDE + state space discretization

If the random process $(\boldsymbol{\xi}_k)$ is in low dimension, one natural idea is to use a PDE method to compute conditional expectations. Therefore, the second comparison method presented in this paragraph combines a PDE method with the dynamic programming principle on a discretized state space $(\hat{x}_k)$. In order to use a PDE method, we need to build one continuous time process from the discrete time random process. To be consistent with our price model (9), we will use the continuous version of the Black model:

$$dF_t^{\cdot} = \sigma F_t^{\cdot} dW_t; \tag{41}$$

and make an interpolation of F_0^t between t_k and t_{k+1}:

$$F_0^t = \frac{t - t_k}{t_{k+1} - t_k} F_0^{k+1} + \frac{t - t_{k+1}}{t_k - t_{k+1}} F_0^k, \quad t_k \leq t \leq t_{k+1}. \tag{42}$$

Let $v^{pde-disc}(k, x_k, \boldsymbol{\xi}_k)$ stand for the Bellman value, and $u^{*,pde-disc}(k, x_k, \boldsymbol{\xi}_k)$ for the optimal strategy:

$$v^{pde-disc}(k, \hat{x}_k, \boldsymbol{\xi}_k) = \min_{u_k \in \mathfrak{U}_k} \left(c_k(\boldsymbol{\xi}_k) u_k + \mathbb{E}[v^{pde-disc}(k+1, \hat{x}_k + u_k, \boldsymbol{\xi}_{k+1}) \,|\, \boldsymbol{\xi}_k] \right); \tag{43}$$

$$u^{*,pde-disc}(k, \hat{x}_k, \boldsymbol{\xi}_k) = \operatorname*{argmin}_{u \in \mathfrak{U}_k} \left(c_k(\boldsymbol{\xi}_k) u_k + \mathbb{E}[v^{pde-disc}(k+1, \hat{x}_k + u_k, \boldsymbol{\xi}_{k+1}) \,|\, \boldsymbol{\xi}_k] \right). \tag{44}$$

Let $\tilde{v}(t, x, \boldsymbol{\xi}_t) := \mathbb{E}[v^{pde-disc}(k+1, x, \boldsymbol{\xi}_{k+1}) \,|\, \boldsymbol{\xi}_t]$, $t_k \leq t \leq t_{k+1}$. Combining the Feynman-Kac formula and Itô's lemma on the continuous extension of our price model (41) (42), we compute the conditional expectations by solving the following PDE backwards from t_{k+1} to t_k:

$$\begin{cases} \dfrac{\partial \tilde{v}}{\partial t} + \frac{1}{2} \sum_{i=1}^d (\sigma^i)^2 (\boldsymbol{\xi}^i)^2 \dfrac{\partial^2 \tilde{v}}{\partial (\boldsymbol{\xi}^i)^2} + \sum_{i \neq j} \rho_{ij} \sigma^i \sigma^j \boldsymbol{\xi}^i \boldsymbol{\xi}^j \dfrac{\partial^2 \tilde{v}}{\partial \boldsymbol{\xi}^i \partial \boldsymbol{\xi}^j} = 0, \quad t_k \leq t \leq t_{k+1}; \\ \tilde{v}(t_{k+1}, \cdot, \boldsymbol{\xi}_{t_{k+1}}) = v^{pde-disc}(t_{k+1}, \cdot, \boldsymbol{\xi}_{t_{k+1}}). \end{cases} \tag{45}$$

This parabolic partial differential equation is straightforward to solve numerically in dimension 1. For example, we may solve it by the use of a finite difference method with an implicit scheme, which is introduced in appendix A.

Once we have obtained the optimal solution $u^{*,pde-disc}$, we may still apply Danskin's theorem by using the same formula (35)-(38). We denote the sensitivity result by $(\delta_F)^{pde-disc}$ and $(\delta_\sigma)^{pde-disc}$.

Note that this method does not depend on the quantization tree. Hence, the results obtained from the method presented in the current paragraph give evidence of the accuracy of discretizing $(\boldsymbol{\xi}_k)$ by the quantization tree method.

5.2.3 Finite difference (fd.) + PDE + state space discretizaton

The last method used to compare the accuracy of the sensitivity estimates is probably the most used in practice. Once an efficient numerical method to compute the optimal value $v(F, \sigma)$ is available, the finite difference method may be used to approximate the derivative value:

$$(\delta_F^{fd})_k^i = \frac{v(F + \epsilon e_k^i, \sigma) - v(F - \epsilon e_k^i, \sigma)}{2\epsilon}; \tag{46}$$

$$(\delta_\sigma^{fd})^i = \frac{v(F, \sigma + \epsilon e^i) - v(F, \sigma - \epsilon e^i)}{2\epsilon}. \tag{47}$$

In the following numerical tests, we use PDE and the state variable discretization to compute the optimal value $v(F, \sigma)$. We denote by $(\delta_F)^{fd-pde}$ and $(\delta_\sigma)^{fd-pde}$ the sensitivity estimates. This method allows us to compare the relevance of the values obtained from Danskin's theorem.

5.3 Swing option

The first test is performed on swing options which are fairly common in the energy markets. A swing option allows its holder to purchase a total amount of commodity during a predetermined period of time. A typical swing option contract has the following parameters: a feasible control set $\mathfrak{U}_k = [0, 1]$ for every time step t_k, a final state set $\mathfrak{X}_K = [L, U]$, and a time horizon $K = 50$ in our example. The running cost function $c(\boldsymbol{\xi}_k)$ is equal to $\boldsymbol{\xi}_k - K_k$, where the strike $K_k = F_0^k$.

Under certain circumstances, swing option featuring one dimension control possesses the bang-bang property. Bardou et al [5, 4] show that there always exists a bang-bang optimal strategy when $(L, U) \in \mathbb{Z}^2$. Therefore, the discretization on x_t takes values in $\mathbb{Z}$ only.

In the current test, we take the lower bound of the exercise right L equal to 20, and upper bound of the exercise right U equal to 30. We set F_0^k in price model curve to $1.0 + 0.2\sin(2\pi t_k/T)$, and volatility σ to $30\%/\sqrt{K}$. The quantization tree built contains $N = 5000$ vertices dispatched on $K = 50$ stages. The estimation of transition probability is carried out by Monte-Carlo method using 10^7 samples. The SDDP algorithm generates 5000 scenarios in forward pass and uses 20 scenarios in backward pass. The stopping criteria (8) takes $\rho = 1$. Concerning the parameters intervening in the PDE resolution (see section A), we set the lower bound $\underline{\boldsymbol{\xi}}$ to 0.1, the upper bound $\overline{\boldsymbol{\xi}}$ to 5.0, the number of discretization space steps J to 200, and finally the number of time discretization L to 5. Finally, we take $\epsilon = 0.01$ in the finite difference estimation. The program is written in C++ with GNU Linear Programming Kit(GLPK) 4.44, and the test was performed on a PC with a 3.07GHz Intel Xeon CPU and 12GByte main memory. The results obtained for optimal values are given in table 1.

	tree+sddp.			tree+discret.	pde.+discret.
	ub.	σ(ub.)	lb.		
optimal value	-2.41646	0.105759	-2.3741	-2.37197	-2.36187

Table 1: Optimal values for the 3 methods.

The forward value (by Monte Carlo) is a random variable whose expectation is a value associated to a suboptimal policy, i.e. an upper bound of optimal value. Furthermore, our backward value is a lower bound of discretized problem, which is not the lower bound of the original problem. Therefore, it is possible that the forward value is smaller than the backward value.

The sensitivity values with respect to each contract of the forward curve are displayed in figure 1.

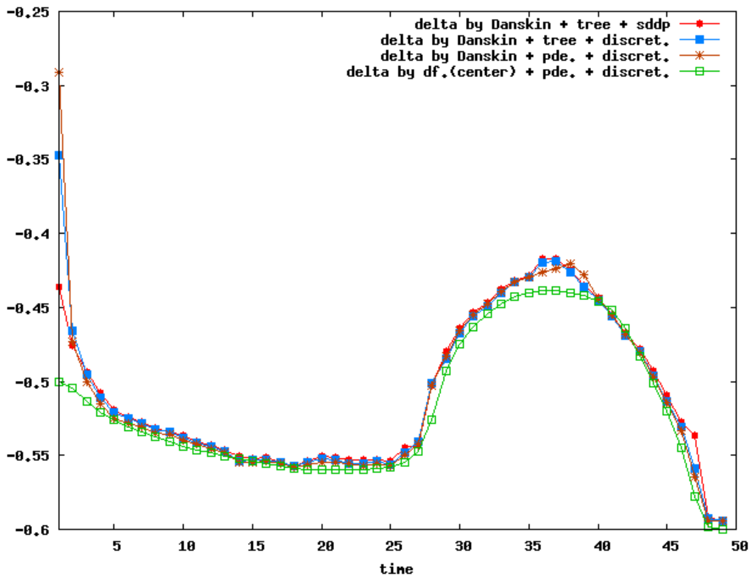

Figure 1: Sensitivity values with respect to F obtained by the 4 methods. Since the standard deviations of the first 3 methods represent less than 1% of the sensitivity values, we will not plot them for sake of clarity.

Finally, the sensitivity values with respect to σ are given in table 2.

	tree+sddp.		tree+discret.		pde.+discret.		fd.+ pde.
	mean	std. dev.	mean	std. dev.	mean	std. dev.	
δ_σ	-32.181	0.695	-32.198	0.698	-32.207	0.700	-35.558

Table 2: Sensitivity value with respect to σ obtained by the 4 methods.

We remark that the sensitivity values obtained by all 4 methods are close to each other. This result proves the accuracy of the sensitivity values obtained by our original algorithm (Danskin + quantization tree + SDDP). Furthermore, we also observe the closeness of the deltas obtained by finite differences to those obtained by Danskin for the case where the discretization methods are the same (i.e. PDE+discretization).

Compared to other methods, our algorithm does not procure any advantage in CPU time for such a low dimension problem. In order to show the advantage of our algorithm, we consider in next section the second numerical test in our previous article [9].

.1 Small commodity portfolio case study

We now consider a gas trading portfolio. A trading company purchases natural gas from a set of producing countries indexed at a price formula and sells it to consuming countries at another other price formula (see figure 2 for the main market and table 3 for the price formulae). Annual quantity and price formulae have been agreed contractually, the latter are functions of the future prices of major energy markets $\boldsymbol{\xi}_t$, such as crude oil price (OIL), north American natural gas price (NA NG), and Europe natural gas price (EU NG). The objective is to analyse the sensitivity of the optimal value of the portfolio with respect to the forward price 4 and volatility of the random process.

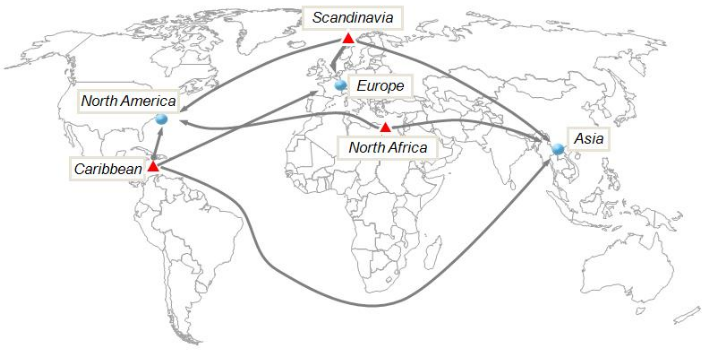

Figure 2: A fictive supply and demand portfolio, as well as the possible routes. ▲: producing country; ●: consuming country.

Port	Annual QC.*	Monthly QC.*	Price formula**
Caribbean	[48.0, 54.0]	[0.0, 6.0]	$\text{NA NG} - 0.1$
Scandinavia	[24.0, 30.0]	[0.0, 3.0]	$\begin{cases} 0.05\text{OIL} + 2.5 & \text{if OIL} \leq 75 \\ 0.07\text{OIL} + 1.0 & \text{otherwise} \end{cases}$
North Africa	[100.0, 100.0]	[0.0, 12.0]	$\begin{cases} 0.9\text{NA NG} + 0.4 & \text{if NA NG} \leq 5 \\ 0.8\text{NA NG} + 0.9 & \text{otherwise} \end{cases}$
North American	[84.0, 88.0]	[0.0, 8.0]	NA NG
Europe	[68.0, 76.0]	[0.0, 8.0]	EU NG
Asia	[20.0, 20.0]	[0.0, 4.0]	$0.08\text{OIL} - 0.8$

Table 3: Constraints and price formulae

* The quantity unit is in TBtu. MMBtu stands for a million British thermal unit, a TBtu is a trillion British thermal unit thus equivalent to a Million MMBtu.
** The price unit is \$/MMBtu.

The forward price F_0^t in (9) is read from the energy market. We take the following values in the numerical test (see in table 4).

Time	OIL*	EU NG*	NA NG*
0	73.0	5.2	5.15
1	73.5	5.3	5.30
2	74.0	5.2	5.22
3	74.5	4.5	4.51
4	75.0	4.4	4.35
5	75.5	4.3	4.33
6	76.0	4.2	4.18
7	76.5	4.3	4.32
8	77.0	4.4	4.37
9	77.5	5.2	5.21
10	78.0	5.3	5.25
11	78.5	5.4	5.42

Table 4: Forward price

* $\boldsymbol{\xi}^1 = \text{OIL}$, $\boldsymbol{\xi}^2 = \text{EU NG}$, and $\boldsymbol{\xi}^3 = \text{NA NG}$.

The quantization tree is processed with $N = 36000$ $\mathbb{R}^3$-valued elementary quantizers dispatched on $T = 12$ stages. The estimation of transition probability is carried out by Monte-Carlo method using 10^9 samples.

The parameters driving the stochastic processes in the test are set as follows: volatility $\sigma_1 = \sigma_2 = \sigma_3 = 40\%$, and correlations $\rho_{12} = 0.7$ $\rho_{13} = 0.2$ $\rho_{23} = 0.4$. And the parameters driving the SDDP part of the algorithm are forward pass samples $M_f = 3000$, backward pass sample $M_b = 10$. The stopping criteria (8) takes $\rho = 1$. The program is written in C++ with Cplex 10.1, and the tests were performed on a PC with a 2.2 GHz Dual Core AMD CPU and 16GByte main memory.

In this numerical test, the other comparison methods are not adapted because of the large dimension of the state variable $m = 6$. Discretizing both state variable and the random space will make the problem numerically intractable. Therefore, we can only provide results based on Danskin's theorem combined with SDDP algorithm in section 2. The optimal value is presented in table 5,

and sensitivity values are in tables 6 and 7.

	upper bound	std. dev. of upper bound	lower bound
optimal value	-39.867	2.101	-41.900

Table 5: optimal value by quantization tree + SDDP method

time (k)	$(\delta_F)_k^1$	$\sigma(\delta_F)_k^1$*	$(\delta_F)_k^2$	$\sigma(\delta_F)_k^2$*	$(\delta_F)_k^3$	$\sigma(\delta_F)_k^3$*
1	0.18109	0.001100	-6.84792	0.03678	1.71388	0.05419
2	0.17694	0.001921	-6.46495	0.04078	0.74796	0.05983
3	0.11645	0.003923	-7.61937	0.02690	3.99273	0.04073
4	-0.13737	0.003147	-7.71228	0.03440	7.34259	0.04771
5	-0.19478	0.003637	-6.62139	0.05821	7.02937	0.04985
6	-0.28043	0.003461	-3.50367	0.08464	7.23409	0.08504
7	-0.25886	0.004282	-3.55706	0.10215	6.41680	0.10195
8	-0.20894	0.009255	-4.75887	0.09313	6.74792	0.13136
9	0.15477	0.004703	-6.25490	0.07210	-0.02695	0.09731
10	0.13744	0.003497	-6.66237	0.10089	1.13770	0.14961
11	0.11403	0.003813	-5.71724	0.08349	-0.80012	0.13695

Table 6: sensitivity value with respect to forward price

* σ here means standard derivation.

	δ_σ^1	δ_σ^2	δ_σ^3
value	31.6862	-38.9346	-95.6475
std. dev.	10.7139	11.7742	8.4706

Table 7: sensitivity values with respect to volatility

A Appendix – Implicit scheme of finite difference for 1 dimension PDE

Generally, the parabolic partial differential equation is very difficult to solve numerically in high dimension. We refer to Allaire [1] and Ern and Guermand [17] for more details.

In the case that $\boldsymbol{\xi}_t$ is in dimension one, the equation (45) is written as

$$\begin{cases} \dfrac{\partial \tilde{v}}{\partial t} + \frac{1}{2}\sigma^2 \boldsymbol{\xi}^2 \dfrac{\partial^2 \tilde{v}}{\partial \boldsymbol{\xi}^2} = 0, \quad t_k \le t \le t_{k+1}, \ \boldsymbol{\xi} \in [\underline{\boldsymbol{\xi}}, \overline{\boldsymbol{\xi}}], \\ \tilde{v}(t_{k+1}, \cdot, \boldsymbol{\xi}_{t_{k+1}}) = g(t_{k+1}, \cdot, \boldsymbol{\xi}_{t_{k+1}}). \end{cases} \tag{48}$$

To apply the finite difference scheme, we need to discretize both time t and space $\boldsymbol{\xi}$. Let the time step be $h_t = (t_{k+1} - t_k)/L$, and the space step be $h_{\boldsymbol{\xi}} = (\overline{\boldsymbol{\xi}} - \underline{\boldsymbol{\xi}})/J$, where L, J are positive integers. We denote by $\tilde{v}_j^l$ an approximation of $\tilde{v}(t_k + lh_t, \underline{\boldsymbol{\xi}} + jh_{\boldsymbol{\xi}})$ where $0 \le l \le L$ and $0 \le j \le J$. The standard implicit finite difference scheme is

$$\begin{cases} \dfrac{\tilde{v}_j^{l+1} - \tilde{v}_j^l}{h_t} + \frac{1}{2}\sigma^2 \boldsymbol{\xi}_j^l \dfrac{\tilde{v}_{j+1}^l - 2\tilde{v}_j^l + \tilde{v}_{j-1}^l}{h_{\boldsymbol{\xi}}^2} = 0, \\ \tilde{v}_j^L(\cdot) = g(t_{k+1}, \cdot, \underline{\boldsymbol{\xi}} + jh_{\boldsymbol{\xi}}). \end{cases} \tag{49}$$

Thus, we need the boundary condition at bounds of $\boldsymbol{\xi}$. By the arguments of the problem context, we take the following boundary conditions:

$$\begin{cases} \dfrac{\partial^2 \tilde{v}}{\partial \boldsymbol{\xi}^2} = 0, \boldsymbol{\xi} = \underline{\boldsymbol{\xi}}, \\ \dfrac{\partial^2 \tilde{v}}{\partial \boldsymbol{\xi}^2} = 0, \boldsymbol{\xi} = \overline{\boldsymbol{\xi}}. \end{cases} \tag{50}$$

Thus, the finite difference scheme at bounds are:

$$\begin{cases} \tilde{v}_0^{l+1} - \tilde{v}_0^l = 0, \\ \tilde{v}_J^{l+1} - \tilde{v}_J^l = 0. \end{cases} \tag{51}$$

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