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Abstract: We consider the problem of numerically approximating the solution of the coupling of the flow equation in a random porous medium, with the advection-diffusion equation. More precisely, we present and analyse a numerical method to compute the mean value of the spread of a solute introduced at the initial time, and the mean value of the macro-dispersion, defined at the temporal derivative of the spread. We propose a Monte-Carlo method to deal with the uncertainty, i.e. with the randomness of the permeability field. The flow equation is solved using finite element. The advection-diffusion equation is seen as a Fokker-Planck equation, and its solution is approximated thanks to a probabilistic particular method. The spread is indeed the expected value of a function of the solution of the corresponding stochastic differential equation, and is computed using an Euler scheme for the stochastic differential equation and a Monte-Carlo method. Error estimates on the mean spread and on the mean dispersion are established, under various assumptions, in particular on the permeability random field.

Key-words: uncertainty quantification, elliptic PDE with random coefficients, advection-diffusion equation, Monte-Carlo method, Euler scheme for SDE

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Analyse numérique de l'advection-diffusion d'un soluté dans des milieux aléatoires

Résumé : On s'intéresse à l'approximation numérique de la solution du couplage entre l'équation d'écoulement dans un milieu poreux aléatoire et l'équation d'advection-diffusion. Plus précisément, on présente et analyse une méthode numérique pour calculer la valeur moyenne de l'extension d'un soluté introduit au temps initial, et la valeur moyenne de la macro-dispersion, définie comme la dérivée temporelle de l'extension. On propose une méthode de Monte-Carlo pour tenir compte des incertitudes, c'est à dire du caractère aléatoire du champ de perméabilité. L'équation d'écoulement est alors résolue en utilisant des éléments finis. L'équation d'advection-diffusion est vue comme une équation de Fokker-Planck, et sa solution est donc approchée grâce à une méthode particulaire probabiliste. L'extension peut en effet être exprimée comme l'espérance d'une fonction de la solution de l'équation différentielle stochastique correspondante, et est calculée grâce à un schéma d'Euler pour l'équation différentielle stochastique et une méthode de Monte-Carlo. On donne des estimations d'erreur pour l'extension moyenne et la dispersion moyenne, sous différentes hypothèses, en particulier sur le champ de perméabilité.

Mots-clés : quantification des incertitudes, EDP elliptique à coefficients aléatoires, équation d'advection-diffusion, méthode de Monte-Carlo, schéma d'Euler pour les EDS

1 Introduction

Numerical modeling is an important key for the management and remediation of groundwater resources. The heterogeneity of natural geological formations has a major impact in the contamination of groundwater by migration of pollutants. In order to account for the limited knowledge of the geological characteristics and for the natural heterogeneity, stochastic models have been developed, see e.g. [7],[21]. The permeability of the porous media is then a random field. Our aim is then to study the migration of a contaminant in steady flow. The flow velocity is computed by solving an elliptic partial differential equation with random coefficients. The solute concentration is then the solution of an advection-diffusion equation, where the flow velocity, which is a random field, appears as a coefficient. The quantities we are interested in are finally the mean value of the spread of the solute, that is to say the mean value of the spatial variance of the solute, and the mean value of the dispersion, which is defined as the derivative of the spread with respect to the time. The determination of the large-scale dispersion coefficients has been widely debated in the last twenty five years, see e.g [3], [4], [8], [11], [25], [27] and [29].

Here we are interested in the case of a lognormal permeability field, which is a widely used model. Moreover we consider the case, physically pertinent, where the correlation length is small and the uncertainty important. Therefore methods based on the approximation of the coefficients in a finite dimensional stochastic space, such as stochastic galerkin methods and stochastic collocation method would be highly expensive, and hence do not seem to be suitable to deal with such cases. Neither seem perturbation methods, since we suppose the uncertainty to be important. As regards the advection-diffusion equation, we focus on the advection-dominated model. Therefore we choose not to consider an Eulerian method, in order to avoid numerical diffusion. The below described method has therefore being proposed and implemented by A.Beaudoin, J.R. de Dreuzy and J.Erhel to compute the mean dispersion in 2D, their numerical results can be found e.g. in [6]. A Monte-Carlo method is used to deal with the uncertainty. The solution of the steady flow equation is computed by using finite elements. The solution of the advection-diffusion equation is approximated using a probabilistic method. We consider the stochastic differential equation associated to this Fokker Planck equation and its solution is approximated with an Euler scheme. A Monte-Carlo method provides finally an approximation to the solution of the Fokker Planck equation. All these steps together lead to an approximation of the mean spread. The mean dispersion is then approximated by the numerical derivative of the computed mean spread.

The aim of this paper is to make the numerical analysis of the above described method. More precisely we furnish a priori error estimates for the approximations of the spread and of the dispersion. We focus on the spread and the dispersion, because of their physical interest, but the result given here are more general. A specifity of this work is to address the coupling of the flow equation with the advection-diffusion equation, whereas most of the existing numerical analysis of methods for uncertainty quantification are limited to the flow equation, see e.g [1], [2]. The main novelty of this work is the use of numerical analysis tools from two differents areas: finite elements method and weak error analysis for SDEs. Moreover, since we estimate the time derivative of the spread, we have to generalize the weak error to estimate the error for time derivatives of averages.

After describing the physical model in section 2, we describe in section 3 the numerical method mentioned above. Section 4 is devoted to the numerical analysis of this method in the case of a random permeability field, supposed to be almost surely periodic, uniformly coercive and bounded under various additional regularity assumptions. We first give preliminary results. The first one is a weak error result for the Euler scheme on a stochastic differential equation with additive noise and $\mathcal{C}^{1,\alpha}$ drift. The second one is a continuity result on the approximation of the solution of a stochastic differential equation using an Euler scheme, with respect to the drift endowed with uniform norm. This result is combined with a classical $W^{1,\infty}$ finite elements error result. After these preliminary results, we give the two main results of this paper, namely error results on the mean spread and on its time derivative, the mean dispersion.

2 Physical model

2.1 Steady flow equation

We consider an isotropic porous medium, we suppose the porosity to be constant, equal to 1. The domain O is a box in $\mathbb{R}^d$, with $d = 1, 2$, or 3. The heterogeneity of the natural geological formations and the lack of data led us to use a stochastic model. A classical case is the homogeneous lognormal permeability field with a correlation function of the following type:

$$a(\omega, x) = e^{g(\omega,x)},\ x \in O,\ \omega \in \Omega,$$

where g is a gaussian field characterized by its mean m and its covariance function:

$$cov[g](x, y) = \sigma^2 exp\left(-\frac{\|x - y\|^\delta}{l}\right), \tag{1}$$

for some $\delta > 0$. The random parameter is denoted ω. The case of an exponential covariance function corresponds to $\delta = 1$, and furnishes a model which is a reasonably good fit to some field data , see e.g [22] and [23]. Unfortunately, as we will see later, for technical reasons, we are not able to treat this case rigorously.

The variance of the log hydraulic conductivity σ^2 is typically in the interval $[1, 10]$, the correlation length l typically ranges between 0.1m and $100m$, whereas the size of the domain has to be at least hundred times the correlation length l. Classical laws governing the steady flow in porous media without source are mass conservation $\mathrm{div}(v) = 0$ and Darcy law $v = -a\nabla p$, where v is the Darcy velocity and p the hydraulic head. Boundary conditions are homogeneous Dirichlet condition, the hydraulic head on the boundary in denoted by p_0. Finally, the hydraulic head is the solution of the following elliptic PDE with a random coefficient : for almost all ω

$$\mathrm{div}(a(\omega, x)\nabla p(\omega, x)) = 0\ , x \in O. \tag{2}$$

this equation is subjected to mixed boundary conditions, and is imposed for almost all $\omega \in \Omega$.
Here, ω is then the parameter describing the randomness of the media. We recall that the Darcy velocity is then defined by

$$v(\omega, x) = -a(\omega, x)\nabla p(\omega, x).$$

2.2 Advection-diffusion equation

An inert solute is injected in the porous medium and transported by advection and diffusion. Here we consider only molecular diffusion, assumed homogeneous and isotropic. This type of solute migration is described by the advection-diffusion equation:

$$\frac{\partial c(\omega, x, t)}{\partial t} + v(\omega, x).\nabla_x(c(\omega, x, t)) - D\Delta c(\omega, x, t) = 0, \tag{3}$$

where $D > 0$ is the molecular diffusion coefficient, v the Darcy velocity defined previously and c the solute concentration. We consider the case of advection-dominated model, i.e. the case where the Peclet number $Pe = \frac{l\|v\|_{mean}}{D}$ is large (typically ≥ 100). The initial condition at $t = 0$ is the injection of the solute, i.e. $c(t = 0) = \mathbb{1}_R$ where R is a box included in O. Equation (3) should be supplemented with boundary conditions on ∂O.

2.3 Spread and dispersion

We now define the two quantities we want to compute.
First we introduce the center of mass of the solute distribution :

$$G(\omega, t) = \int_O c(\omega, x, t) x dx.$$

Our aim is then to compute $S(t)$ the mean spread of mass around G, and the mean macro-dispersion $\mathcal{D}(t)$, defined as its time derivative, i.e.

$$S(\omega, t) = \int_O c(\omega, x, t)(x - G(\omega, t))(x - G(\omega, t))^t dx, \quad S(t) = \mathbb{E}_\omega[S(\omega, t)]$$

and

$$\mathcal{D}(\omega, t) = \frac{dS(\omega, t)}{dt}, \quad \mathcal{D}(t) = \mathbb{E}_\omega[\mathcal{D}(\omega, t)].$$

3 Description of the numerical method

3.1 A Monte-Carlo method to deal with uncertainty

As precised above, we suppose the uncertainty to be large, typically $\sigma^2 \in [1, 10]$, therefore perturbation type methods [3], [8], [11], [24], [25], [26], [29] do not seem to be suitable. Moreover, since we suppose l to be small, σ^2 to be large and $cov[g]$ to be only lipschitz, stochastic galerkin and stochastic collocation methods (see e.g. [1], [2], [12], [28] and the references therein) do not seam to be adapted. Namely, in this case, the permeability field a cannot be approximated correctly with a reasonable number of random variable. In particular, the eigenvalues of the Karhunen-Loève development are explicit in this case (see [30]), and we know that the number of term in the truncated KL development should be much greater than 100, which is not possible on a practical point of view. Therefore we choose to use a Monte-Carlo method to deal with uncertainty.
More precisely, we consider N independent realizations of the permeability field $a(x, \omega_1), ...a(x, \omega_N)$. For each i from 1 to N, we compute approximations of the spread $S^i(t)$ and of the dispersion $\mathcal{D}^i(t)$ corresponding to the permeability field a^i as specified below, and we approximate $S(t)$ by $\frac{1}{N}\sum_{i=1}^N S^i(t)$ and $\mathcal{D}(t)$ by $\frac{1}{N}\sum_{i=1}^N \mathcal{D}^i(t)$. For simplicity, the index i will be omitted in the remainder of this section, which is devoted to the description of the numerical method used to compute the solution of a deterministic problem: the computation of spread and dispersion.

3.2 Approximation of the flow velocity

The hydraulic head is defined as the solution of the following elliptic partial differential equation:

$$\operatorname{div}(a(x)\nabla p(x)) = 0, \ x \in O,$$

submitted to mixed boundary conditions.
In what follows, we work with the equivalent PDE with homogeneous mixed boundary conditions and a right hand side f. We define then an approximation $\tilde{p}$ of p in a finite elements space of degree 1, with maximum space mesh h. The velocity v is then approximated by $\tilde{v} = -a\nabla\tilde{p}$. For readibility the maximum mesh size h will be omitted, and the tilde will account for the space discretization approximation in this paper.

3.3 A probabilistic particular method

The solute concentration is defined as the solution of (3). the domain O is chosen so that a very small amount of the solute reaches the boundary. Therefore, in practice, it is harmless to replace (3) by :

$$\left\{\begin{array}{rcll} \dfrac{\partial c}{\partial t}(x,t) + v(x).\nabla c(x,t) - D\Delta c(x,t) & = & 0, & x \in \mathbb{R}^d \text{ and } t \in [O,T] \\ c(x,0) & = & c_0(x), & x \in \mathbb{R}^d, \end{array}\right.$$

where v is extended to $\mathbb{R}^d$. Since $\operatorname{div}(v) = 0$, this is a Fokker-Planck equation. We choose to approximate the solution of this Fokker-Planck equation with a probabilistic particular method, we define the associated stochastic differential equation:

$$\left\{\begin{array}{rcl} dX(t) & = & v(X(t))dt + \sqrt{2D}dW(t) \\ X(0) & = & X_0, \end{array}\right.$$

where X_0 is a random variable with density c_0 with respect to the Lebesgue measure. It is classical that $X(t)$ admits then $c(x,t)dx$ as density. The law of X can be approximated by a Monte-Carlo method. We take M independent realizations of approximations of X using an Euler scheme and the approximated flow velocity $\tilde{v}$: $X_n^1, \ldots X_n^M$.

$$\left\{\begin{array}{rcl} \tilde{X}_n^j(t_{k+1}) & = & \tilde{X}_n^j(t_{k+1}) + \tilde{v}(\tilde{X}_n^j(t_k))\Delta t + \sqrt{2D\Delta t}N_k^j \text{ for } t \in [t_k, t_{k+1}], \\ \tilde{X}_n^j(0) & = & X_0^j, \end{array}\right.$$

where $T = n\Delta t$, $t_k = k\Delta t$ and N^j are independent d-dimensional mean-free gaussian random vector with identity as covariance. Finally we approximate $G(t)$ by $\tilde{G}_n^M(t) = \frac{1}{M}\sum_{j=1}^M \tilde{X}_n^j(t)$, the spread $S(t)$ by

$$\tilde{S}_n^M(t) = \frac{1}{M}\sum_{j=1}^M (\tilde{X}_n^j(t) - \tilde{G}_n^M(t))(\tilde{X}_n^j(t) - \tilde{G}_n^M(t))^t.$$

In order to approximate the dispersion, we introduce a new time step Δs, and approximate $\mathcal{D}(t)$ by

$$\frac{\tilde{S}_n^M(t+\Delta s) - \tilde{S}_n^M(t)}{\Delta s}.$$

Indeed we recall that we have $G(t) = E[X(t)]$, $S(t) = E[(X(t) - G(t))(X(t) - G(t))^t]$, and $\mathcal{D}(t) = \frac{d}{dt}S(t)$.

For more details on a possible numerical implementation, see [6].

4 Numerical analysis of the method with additional assumptions

We consider O a box of $\mathbb{R}^d$, and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. For $k \in \mathbb{N}$ and $0 \leq \alpha \leq 1$, we denote by $\mathcal{C}_b^{k,\alpha}$ the space of $\mathcal{C}^k$ functions with bounded derivatives and such that any k-th

derivative is α-hölder continuous. For $f \in \mathcal{C}_b^{k,\alpha}$ we introduce associated norm:
$\|f\|_{\mathcal{C}_b^{k,\alpha}} = \max\{\|f\|_\infty, \|f'\|_\infty, ..., \|f^{(k)}\|_\infty, \|f^{(k)}\|_{\mathcal{C}_b^{0,\alpha}}\}$.

The numerical analysis of the above algorithm requires the solution of (2) to be sufficiently regular with respect to $x \in O$. Unfortunately, this is not the case for two reasons. First, we are dealing with an elliptic equation on a rectangular domain with mixed boundary conditions. This limitates the smoothness of the solution. Also, note that the advection-diffusion equation is set on the full space $\mathbb{R}^d$ and it is not clear how to extend the velocity field on O to $\mathbb{R}^d$. We consider that this is a techincal problem and avoid it by replacing the mixed boundary conditions by periodic boundary conditions, so that the solutions have the smoothness naturally associated to the smoothness of the permeability, and the extension to $\mathbb{R}^d$ is trivial. Another way could be truncate the velocity field close to the boundary and to then extend it by zero. The final solution would not be very different, since in practice the domain O is chosen very large to the box R and a very small amount of the solute reaches the boundary. The solution of (2) with mixed boundary conditions being smooth inside the domain, the same analysis as below gives a similar result. We chose the periodic boundary conditions to simplify the presentation.

Second, and this is a much deeper problem, the permeability field associated to an exponential covariance (i.e. $\delta = 1$ in (1)), is only $\mathcal{C}^\beta$, for $\beta < 1/2$, yielding a velocity of the same regularity. Our analysis requires at least a $\mathcal{C}^{1,\alpha}$, $\alpha > 0$ regularity, which exludes exponential covariance and we are only able to deal with $\delta > 2+2\alpha$ (or $\delta = 2$). Furthermore, as it is often the case in theoritical papers, we assume that the permeability field is uniformly bounded and coercive. This is clearly not the case for a lognormal probability field, but using the argument and methodology of [16] (see also [5] and [10]), we obtain a similar result. Note that this would considerably complicate the proof. To sum up, these considerations lead to the following assumption used in all the results below.

Hypothesis 1. *The permeability field $a \in L^\infty(\Omega, \mathcal{C}_b^{1,\alpha}(\mathbb{R}^d))$ for some $0 < \alpha < 1$, such that for almost all ω $a(\omega, .)$ is O periodic and such that for almost all ω, for any $x \in \mathbb{R}^d$,*

$$0 < a_{min} < a(\omega, x) < a_{max} < +\infty,$$

4.1 Solution of the flow equation and its approximation using finite elements

We consider then the flow equation:

$$\left\{ \begin{array}{rcl} \operatorname{div}(a(\omega,x)\nabla p(\omega,x)) & = & f, \quad on\ \Omega \times \mathbb{R}^d, \\ \int_{\partial O} p(\omega,x) & = & 0 \quad on\ \Omega, \end{array} \right. \tag{4}$$

and p is O periodic. The right hand side f takes into account non homogeneous boundary conditions and is assmued to be smooth.

Proposition 1. *Equation* (4) *admits a unique solution $p \in L^\infty(\Omega, \mathcal{C}_b^{2,\alpha}(\mathbb{R}^d))$.*

Proof. For almost all ω, the application of the elliptic regularity theorem [13] yields that the equation admits a unique solution $p(\omega, \cdot) \in \mathcal{C}_b^{2,\alpha}(\mathbb{R}^d)$, with

$$||p(\omega, \cdot)||_{\mathcal{C}_b^{2,\alpha}(\mathbb{R}^d)} \leq C(a(\omega, \cdot))$$

where $C(a(\omega, \cdot))$ is a constant which depends only on the $\mathcal{C}_b^{1,\alpha}(\mathbb{R}^d)$ norm of $a(\omega, \cdot)$ and on $a_{\min}(\omega)$ and can therefore be chosen to be uniform with respect to ω. The application $\omega \mapsto p(\omega, \cdot) \in \mathcal{C}_b^{2,\alpha}(\mathbb{R}^d)$ being strongly measurable, the result follows. □

We consider p^h the approximation of p in a finite elements space V_h.

We define the Darcy velocity: $v(\omega, x) = -a(\omega, x)\nabla p(\omega, x)$ and its estimate $v^h(\omega, x) = -a(\omega, x)\nabla p^h(\omega, x)$. Then $v \in L^\infty(\Omega, \mathcal{C}_b^{1,\alpha}(\mathbb{R}^d))$. Hence, since V_h is a finite dimensional space, we also have $v^h \in L^\infty(\Omega \times \mathbb{R}^d)$.

From now on, we make the following assumption on the finite element space.

Hypothesis 2. $\|v - v_h\|_{L^\infty(\Omega\times\mathbb{R}^d)} \leq C_1 h|ln(h)|$.

Remark 1. *This assumption is in particular fullfiled in the case of a finite element space of piecewse linear functions on a regular triangulation. In [9] for instance, this is proved for a $\mathcal{C}^1$ permeability field with dirichlet boundary conditions. The proof extends to periodic boundary conditions. Note that we expect that if the permeability field is $\mathcal{C}^{0,\alpha}$, a similar estimate holds with $h^{\alpha-\varepsilon}$ for $\varepsilon > 0$, instead of $h|ln(h)|$. We have not found such a result in the litterature but it is possible that it could be proved mixing the argument of [9] and [15], chap. 8 and 9.*

4.2 The advection-diffusion equation

We define $c_0(x) = \mathbb{1}_R(x)$, where R is a box included in O, and take $v \in \mathcal{C}_b^{1,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$, we consider then the following advection-diffusion equation:

$$\left\{ \begin{array}{rcll} \frac{\partial c}{\partial t}(x,t) + v(x).\nabla c(x,t) - D\Delta c(x,t) &=& 0, & x \in \mathbb{R}^d \text{ and } t \in [O,T] \\ c(x,0) &=& c_0(x), & x \in \mathbb{R}^d. \end{array} \right. \tag{5}$$

We consider also the stochastic differential equation associated to this Fokker Planck equation:

$$\left\{ \begin{array}{rcl} dX(t) &=& v(X(t))dt + \sqrt{2D}dW(t), \\ X(0) &=& X_0, \end{array} \right. \tag{6}$$

We suppose that X_0 admits $c_0(x)dx$ as density. We recall the following well known result (see [17] e.g).

Proposition 2. *The equation* (5) *admits a unique solution $c \in \mathcal{C}(]0,T], \mathcal{C}^2(\mathbb{R}^d))$ and $X(t)$ admits $c(x,t)dx$ as density.*

4.3 Weak error of time discretization

We give now the first preliminary result. We consider $(\Omega', \mathcal{F}', \mathbb{P}')$ an another probability space, whose generic variable is denoted by ξ. Here we give weak error results for the time discretization of a stochastic differential equation with an additive noise and a $\mathcal{C}^{1,\alpha}$ drift.

Let $v \in \mathcal{C}_b^{1,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$ for some $0 < \alpha \leq 1$.

We denote by X^x the solution of the following stochastic differential equation:

$$\left\{ \begin{array}{rcl} dX^x(t) &=& v(X^x(t))dt + \sqrt{2D}dW(t), \\ X^x(0) &=& x. \end{array} \right. \tag{7}$$

We denote by X_n^x the numerical approximation of X^x using an Euler scheme as in section 3, where the mesh of the time discretization is $\Delta t = \frac{T}{n}$, and $t_k = k\Delta t$ for $0 \leq k \leq n$. We extend X_n^x to a function defined for all $t \geq 0$ by::

$$\left\{ \begin{array}{rcl} dX_n^x(t) &=& v(X_n^x(t_k))dt + \sqrt{2D}dW(t), \text{ for } t_k \leq t \leq t_{k+1}, \\ X_n^x(0) &=& x. \end{array} \right. \tag{8}$$

We need to deal with a drift less than $\mathcal{C}^2$, this has been studied in [19]. Nevertheless, we give a simple proof in the case of an additive noise for the sake of completness, which moreover yields a better weak convergence order in our particular case (namely $\frac{1+\alpha}{2}$) than the general result of [19] (namely $\frac{1}{2-\alpha}$).

Proposition 3. *Let $0 < \alpha \leq 1$, $0 < \beta < 1$, $V > 0$ and $\varphi \in \mathcal{C}_b^{2,\beta}(\mathbb{R}^d)$, then there exists a constant C_2 such that for any $x \in \mathbb{R}^d$ and $v \in \mathcal{C}_b^{1,\alpha}(\mathbb{R}^d)$ such that $\|v\|_{\mathcal{C}_b^{1,\alpha}(\mathbb{R}^d)} \leq V$, we have*

$$|\mathbb{E}[\varphi(X^x(T)) - \varphi(X_n^x(T))]| \leq C_2(\Delta t)^{\frac{1+\alpha}{2}}.$$

Proof. We denote by $\mathcal{C}_b^{1;2}(\mathbb{R}^d \times [0,T])$ the space of functions of (x,t) which admit derivatives with respect to t and two derivatives with respect to x, all these derivatives being continuous and bounded on $\mathbb{R}^d \times [0,T]$. For $u \in \mathcal{C}_b^{1;2}(\mathbb{R}^d \times [0,T])$, we introduce the natural norm

$$\|u\|_{\mathcal{C}_b^{1;2}(\mathbb{R}^d\times[0,T])} = \max\left\{\|u\|_\infty, \left\|\frac{\partial u(t,x)}{\partial t}\right\|_\infty, \left\|\frac{\partial u(t,x)}{\partial x}\right\|_\infty, \left\|\frac{\partial^2 u(t,x)}{\partial x^2}\right\|_\infty\right\}.$$

We recall now a classical result whose proof can be found in [18] page 184. We introduce the following Kolmogorov equation associated to the previous SDE :

$$\left\{\begin{array}{rcl} \frac{\partial u}{\partial t}(t,x) & = & D\Delta u(t,x) + v(x).\nabla u(t,x), \\ u(0,x) & = & \varphi(x). \end{array}\right. \tag{9}$$

Then for $0 < \alpha < 1$, $v \in \mathcal{C}_b^{0,\alpha}(\mathbb{R}^d)$ with $\|v\|_{\mathcal{C}_b^{0,\alpha}(\mathbb{R}^d)} \leq V$ and $\varphi \in \mathcal{C}_b^{2,\alpha}(\mathbb{R}^d)$, the equation (9) admits a unique solution $u \in \mathcal{C}_b^{1;2+\alpha}(\mathbb{R}^d \times [0,T])$ and $\|u\|_{\mathcal{C}_b^{1;2+\alpha}(\mathbb{R}^d\times[0,T])}$ can be bounded by a constant C_V depending on d, φ, T and depending only on v through V. In particular in our case, since we have $v \in \mathcal{C}_b^{1,\alpha}(\mathbb{R}^d)$ with $\|v\|_{\mathcal{C}_b^{1,\alpha}(\mathbb{R}^d)} \leq V$ and $\varphi \in \mathcal{C}_b^{2,\beta}(\mathbb{R}^d)$ then the solution u of (9) belongs to $C_b^{1;2}(\mathbb{R}^d \times [0,T])$, with $\|u\|_{\mathcal{C}_b^{1;2}(\mathbb{R}^d\times[0,T])} \leq C_V$

Applying Itô formula yields classically $u(t,x) = \mathbb{E}[\varphi(X^x(t))]$. Then the weak error

$$\begin{array}{rcl} E & = & \mathbb{E}[\varphi(X^x(T))] - \mathbb{E}[\varphi(X_n^x(T))] \\ & = & u(T,x) - \mathbb{E}[u(0, X_n^x(T))] \end{array}$$

can be split into $E = \sum_{i=0}^{n-1} E_i$, where:

$$\begin{array}{rcl} E_i & = & \mathbb{E}[u(T-t_i, X_n^x(t_i))] - \mathbb{E}[u(T-t_{i+1}, X_n^x(t_{i+1}))] \\ & = & \mathbb{E}[u(T-t_{i+1}, X^{X_n^x(t_i)}(\Delta t))] - \mathbb{E}[u(T-t_{i+1}, X_n^{X_n^x(t_i)}(\Delta t))] \\ & = & \mathbb{E}[\mathbb{E}[u(T-t_{i+1}, X^{X_n^x(t_i)}(\Delta t)) - u(T-t_{i+1}, X_n^{X_n^x(t_i)}(\Delta t))|X_n^x(t_i)]] \\ & = & \mathbb{E}[e_i(X_n^x(t_i))], \end{array}$$

where $e_i(y) = \mathbb{E}[u(T-t_{i+1}, X^y(\Delta t)) - u(T-t_{i+1}, X_n^y(\Delta t))]$, by using the Markov property of the solution X and of the discretized solution X_n. Using a Taylor expansion of u at order two with respect to x, we get:

$$\begin{array}{rcl} |e_i(y)| & \leq & |\mathbb{E}[D_x u(T-t_{i+1}, X^y(\Delta t)).(X_n^y(\Delta t) - X^y(\Delta t))]| \\ & + & \|D_x^2 u\|_\infty \mathbb{E}\left[|X^y(\Delta t) - X_n^y(\Delta t)|^2\right]. \end{array}$$

We first bound the second term, using Cauchy-Schwarz inequality:

$$\begin{aligned}\mathbb{E}\left[|X^y(\Delta t)-X_n^y(\Delta t)|^2\right] &\leq \Delta t\mathbb{E}\left[\int_0^{\Delta t}|v(X^y(s))-v(y)|^2 ds\right]\\ &\leq \|Dv\|_\infty^2\Delta t\mathbb{E}\left[\int_0^{\Delta t}|X^y(s)-y|^2 ds\right].\end{aligned}$$

The integrand can be bounded as follows:

$$\begin{aligned}|X^y(s)-y| &= \left|\int_0^s v(X^y(t))dt+\sqrt{2D}W(t)\right|\\ &\leq \|v\|_\infty s+\sqrt{2D}|W(s)|,\end{aligned}$$

which yields the following bound for the second term:

$$\begin{aligned}\mathbb{E}\left[|X^y(\Delta t)-X_n^y(\Delta t)|^2\right] &\leq \|Dv\|_\infty^2\Delta t\int_0^{\Delta t}\mathbb{E}\left[\left(\|v\|_\infty\Delta t+\sqrt{2D\Delta t}\left|\frac{W(s)}{\sqrt{s}}\right|\right)^2 ds\right]\\ &\leq \|Dv\|_\infty^2(\Delta t)^2\int_0^{\Delta t}\mathbb{E}\left[(\|v\|_\infty\sqrt{\Delta t}+\sqrt{2D}\,|W(1)|)^2 ds\right]\\ &\leq 2(\sqrt{2D}+\|v\|_\infty^2)\|Dv\|_\infty^2(\Delta t)^3.\end{aligned}$$

We now bound the first term, since $v\in\mathcal{C}^{1,\alpha}(\mathbb{R}^d)$, for any $x,y\in\mathbb{R}^d$

$$\begin{aligned}|v(x)-v(y)-Dv(x).(y-x)| &= \left|\int_0^1(Dv(x+t(y-x))-Dv(x)).(y-x))dt\right|\\ &\leq \|v\|_{\mathcal{C}_b^{1,\alpha}(\mathbb{R}^d)}|y-x|^{\alpha+1}.\end{aligned}$$

Therefore, using a Taylor expansions of v and $D_x u$, we get

$$\begin{aligned}&|\mathbb{E}\left[D_x u(T-t_{i+1},X^y(\Delta t)).(X_n^y(\Delta t)-X^y(\Delta t))\right]|\\ =\ &\left|\mathbb{E}\left[D_x u(T-t_{i+1},X^y(\Delta t)).\int_0^{\Delta t}(v(X^y(u))-v(y))du\right]\right|\\ \leq\ &\left|\mathbb{E}\left[D_x u(T-t_{i+1},X^y(\Delta t)).\int_0^{\Delta t}Dv(y).(X^y(s)-y)ds\right]\right|\\ +\ &\|v\|_{\mathcal{C}_b^{1,\alpha}(\mathbb{R}^d)}\|Du\|_\infty\mathbb{E}\left[\int_0^{\Delta t}|X^y(s)-y|^{\alpha+1}ds\right]\\ \leq\ &\left|\mathbb{E}\left[(D_x u(T-t_{i+1},X^y(\Delta t))-D_x u(T-t_{i+1},y)).\int_0^{\Delta t}Dv(y).(X^y(s)-y)ds\right]\right|\\ +\ &|D_x u(T-t_{i+1},y).\int_0^{\Delta t}Dv(y).\mathbb{E}\left[X^y(s)-y\right]ds|\\ +\ &\|v\|_{\mathcal{C}_b^{1,\alpha}(\mathbb{R}^d)}\|Du\|_\infty\mathbb{E}\left[\int_0^{\Delta t}|(\|v\|_\infty s+\sqrt{2D}|W(s)|)|^{\alpha+1}ds\right]\end{aligned}$$

Hence, we have

$$
\begin{aligned}
& \left|\mathbb{E}\left[D_x u(T-t_{i+1}, X^y(\Delta t)).(X_n^y(\Delta t) - X^y(\Delta t))\right]\right| \\
\leq\ & \|D_x^2 u\|_\infty \|Dv\|_\infty \mathbb{E}\left[|X^y(\Delta t) - y| \int_0^{\Delta t} |X^y(s) - y| ds\right] \\
+\ & \|D_x u\|_\infty \|Dv\|_\infty \int_0^{\Delta t} \left|\mathbb{E}\left[X^y(u) - y\right]\right| du \\
+\ & \|v\|_{\mathcal{C}_b^{1,\alpha}(\mathbb{R}^d)} \|Du\|_\infty \int_0^{\Delta t} \mathbb{E}\left[(\|v\|_\infty \Delta t + \sqrt{2D\Delta t}|W(1)|)^{\alpha+1}\right] ds \\
\leq\ & \|D_x^2 u\|_\infty \|Dv\|_\infty \left(\mathbb{E}[(X^y(\Delta t) - y)^2]\right)^{1/2} \left(\mathbb{E}\left[\left(\int_0^{\Delta t} |X^y(s) - y| ds\right)^2\right]\right)^{1/2} \\
+\ & \|D_x u\|_\infty \|Dv\|_\infty \int_0^{\Delta t} \|v\|_\infty u du \\
+\ & \|v\|_{\mathcal{C}_b^{1,\alpha}(\mathbb{R}^d)} \|Du\|_\infty \int_0^{\Delta t} \mathbb{E}\left[(\|v\|_\infty \Delta t + \sqrt{2D\Delta t}|W(1)|)^{\alpha+1}\right] ds \\
\leq\ & C((\Delta t)^2 + (\Delta t)^{1+\frac{\alpha+1}{2}}).
\end{aligned}
$$

The constant C which appears in this bound depends only on V, φ, d and T. These two estimates lead to the following bound for e_i:

$$|e_i(y)| \leq C(\Delta t)^{1+\frac{1+\alpha}{2}}.$$

The final result follows by taking the sum over i of these inequalities, recalling that $n\Delta t = T$. □

Remark 2. *If $\alpha = 0$, the result on the weak order holds as a consequence of the result on the strong order, namely in this case, the drift is lipschitz and therefore it is classical that the strong order of the Euler scheme is $1/2$ (see [20] and [14]).*

Remark 3. *If the drift only belongs to $\mathcal{C}_b^{0,\alpha}(\mathbb{R}^d)$, we have a similar result.*

Let $0 < \alpha \leq 1$, $0 < \beta < 1$, $\varphi \in \mathcal{C}_b^{2,\beta}(\mathbb{R}^d)$, and $v \in \mathcal{C}_b^{0,\alpha}$ such that $\|v\|_{\mathcal{C}^{0,\alpha}} \leq V$, then there exists a constant C depending only on φ, α, d, T and V such that

$$|\mathbb{E}[\varphi(X^x(T)) - \varphi(X_n^x(T))]| \leq C(\Delta t)^{\frac{\alpha}{2}}.$$

4.4 Space discretization error on the solution of the SDE

We give now the second preliminary result. We consider the Euler scheme with an exact velocity v, where $v \in \mathcal{C}_b^1(\mathbb{R}^d)$:

$$
\left\{
\begin{array}{rcl}
dX_n(t) & = & v(X_n(t_k))dt + \sqrt{2D}dW(t), \text{ for } t \in [t_k, t_{k+1}], \\
X_n(0) & = & X_0,
\end{array}
\right.
$$

and the Euler scheme with an approximated velocity $\tilde{v}$, where $\tilde{v} \in L^\infty(\mathbb{R}^d)$:

$$
\left\{
\begin{array}{rcl}
d\tilde{X}_n(t) & = & \tilde{v}(\tilde{X}_n(t_k))dt + \sqrt{2D}dW(t), \text{ for } t \in [t_k, t_{k+1}], \\
\tilde{X}_n(0) & = & X_0.
\end{array}
\right.
$$

We give here a bound of the error between these two stochastic processes.

Proposition 4. *For any T and V, there exists a constant C_3 such that for any v such that $\|Dv\|_\infty \leq V$ we have almost surely*

$$\sup_{t\in[0,T]} |\tilde{X}_n(t) - X_n(t)| \leq C_3 ||v - \tilde{v}||_{L^\infty(\mathbb{R}^d)}.$$

Proof. For any $0 \leq k \leq N-1$ we define $u_k = \underset{t\in[t_k,t_{k+1}]}{sup} |\tilde{X}_n(t) - X_n(t)|$, and we also define $u_{-1} = t_{-1} = 0$.
Take $-1 \leq k \leq N-1$, $t \in [t_k, t_{k+1}]$, the difference between the two Euler schemes is then

$$\begin{aligned}
\tilde{X}_n(t) - X_n(t) &= \tilde{X}_n(t_k) - X_n(t_k) + (\tilde{v}(\tilde{X}_n(t_k)) - v(X_n(t_k)))(t - t_k), \\
|\tilde{X}_n(t) - X_n(t)| &\leq |\tilde{X}_n(t_k) - X_n(t_k)| \\
&+ (|\tilde{v}(\tilde{X}_n(t_k)) - v(\tilde{X}_n(t_k))| + |v(\tilde{X}_n(t_k) - v(X_n(t_k)|)\Delta t
\end{aligned}$$

$$\begin{aligned}
u_{k+1} &\leq u_k + (||v - \tilde{v}||_{L^\infty} + Vu_k)\Delta t \\
u_{k+1} &\leq (1 + V\Delta t)u_k + \Delta t||v - \tilde{v}||_{L^\infty}.
\end{aligned}$$

Hence, the discrete Gronwall lemma implies that for any $0 \leq k \leq N-1$:

$$\begin{aligned}
u_k &\leq \frac{(1 + \Delta t V)^{k+1}}{V} ||v - \tilde{v}||_{L^\infty} \\
&\leq \frac{e^{VT}}{V} ||v - \tilde{v}||_{L^\infty}.
\end{aligned}$$

□

4.5 Total error on the spread

We recall that $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$ are two probability spaces, with generic variables $\omega \in \Omega$ and $\xi \in \Omega'$. Let $\varphi \in \mathcal{C}_b^3(\mathbb{R}^d, \mathbb{R}^p)$ and $\psi \in \mathcal{C}_b^1(\mathbb{R}^p, \mathbb{R}^q)$ for some $p, q \in \mathbb{N}^*$. Take a constant κ such that φ, ψ and their derivatives are bounded by κ. For almost all $\omega \in \Omega$ we define $X(\omega, \xi, t)$ as the solution of the following stochastic differential equation:

$$\begin{cases} dX(\omega, \xi, t) = v(\omega, X(\omega, \xi, t))dt + \sqrt{2D}dW(\xi, t), \ x \in \mathbb{R}^d, t \geq 0, \\ X(\omega, \xi, 0) = X_0(\xi), \end{cases} \tag{10}$$

where v is defined as in subsection 4.1, W is a d-dimensionnal brownian motion on $(\Omega', \mathcal{F}', \mathbb{P}')$ and X_0 admits c_0 as density, as defined in section 4.2. Then we define for any $1 \leq i \leq N$, $1 \leq j \leq M$ and almost all ω its approximations $\tilde{X}_n^{i,j}(\omega, \xi, t)$ by:

$$\begin{cases} d\tilde{X}_n^{i,j}(\omega, \xi, t) = \tilde{v}^i(\omega, \tilde{X}_n^{i,j}(\omega, \xi, t_k))dt + \sqrt{2D}dW^{i,j}(\xi, t), \ \text{for } t \in [t_k, t_{k+1}] \\ \tilde{X}_n^{i,j}(\omega, \xi, 0) = X_0^{i,j}(\xi), \end{cases} \tag{11}$$

where $\tilde{v}^i$ is the finite element approximation of v^i as defined in subsection 4.1. We now define the error:

Definition 1.

$$Er(\omega, \xi) = \mathbb{E}_\omega[\psi(\mathbb{E}_\xi[\varphi(X(\omega, \xi, T))])] - \frac{1}{N}\sum_{i=1}^N \psi\left(\frac{1}{M}\sum_{j=1}^M \varphi(\tilde{X}_n^{i,j}(\omega, \xi, T))\right).$$

Theorem 1. *There exists a constant C such that*

$$\|Er(\omega,\xi)\|_{L^2_{\omega,\xi}} \leq C\left((\Delta t)^{\frac{1+\alpha}{2}} + h|ln(h)| + \frac{1}{\sqrt{M}} + \frac{1}{\sqrt{N}}\right).$$

Remark 4. *We expect that for a less regular permeability field, that is assumed to be only $\mathcal{C}^{0,\alpha}$, a similar result holds with the right hand side replaced by*

$$\Delta t^{\frac{\alpha}{2}} + h^{\alpha-\varepsilon} + \frac{1}{\sqrt{M}} + \frac{1}{\sqrt{N}}.$$

Reminding Remark 1 and Remark 3, the finite element error and the bound error due to time discretization generalize. The major difficulty is to the generalization of Proposition 4. Note that Proposition 4 gives a strong error, i.e. for each $\xi \in \Omega'$. In the proof below, only a weak error is needed. We believe that such a weak error of the form

$$|\mathbb{E}_\xi[\varphi(\tilde{X}_n(t))] - \mathbb{E}_\xi[\varphi(X_n(t))]| \leq C\|v-\tilde{v}\|_{L^\infty(\Omega\times\mathbb{R}^d)}$$

is true for a permeability field in $\mathcal{C}^{0,\alpha}$, but we are not able to prove this.

Remark 5. *An estimate of the error on the spread follows from the cases where $\varphi(x) = xx^t$, $\psi(x) = x$ and $\varphi(x) = x$, $\psi(x) = xx^t$. For simplicity, we treat only the case where φ and ψ are bounded with bounded derivatives. The result can however be generalized to the case where φ and ψ are respectively $\mathcal{C}^3$ and $\mathcal{C}^1$ with at most polynomial growth.*

Proof. Take $V = \|v\|_{\infty(\Omega,\mathcal{C}_b^{1,\alpha}(\mathbb{R}^d))}$.
We split this error into three terms:

$$Er(\omega,\xi) = Er1 + Er2(\omega) + Er3(\omega,\xi).$$

Where we define:

$$\begin{aligned}
Er1 &= \mathbb{E}_\omega[\psi(\mathbb{E}_\xi[\varphi(X(\omega,\xi,T))])] - \mathbb{E}_\omega[\psi(\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))])]\\
Er2(\omega) &= \mathbb{E}_\omega[\psi(\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))])] - \frac{1}{N}\sum_{i=1}^N \psi(\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))])\\
Er3(\omega,\xi) &= \frac{1}{N}\sum_{i=1}^N\left(\psi(\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))]) - \psi\left(\frac{1}{M}\sum_{j=1}^M \varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))\right)\right).
\end{aligned}$$

The first error term $Er1$ takes account for the space discretization error and the time discretization error, and can thus be split into two terms: for almost all ω we have

$$\begin{aligned}
&|\mathbb{E}_\xi[\varphi(X(\omega,\xi,T)] - \mathbb{E}_\xi[\varphi(\tilde{X}_n(\omega,\xi,T))]|\\
&\leq |\mathbb{E}_\xi[\varphi(X(\omega,\xi,T)] - \mathbb{E}_\xi[\varphi(X_n(\omega,\xi,T))]|\\
&+|\mathbb{E}_\xi[\varphi(X_n(\omega,\xi,T)] - \mathbb{E}_\xi[\varphi(\tilde{X}_n(\omega,\xi,T))]|\\
&\leq C_2(\Delta t)^{\frac{\alpha+1}{2}} + C_3\|v-\tilde{v}\|_{\infty(\Omega\times\mathbb{R}^d)}\\
&\leq C_2(\Delta t)^{\frac{\alpha+1}{2}} + C_4 h|ln(h)|,
\end{aligned}$$

where we have used Proposition 3 to bound the first term and Propositions 4 and 2 to bound the second term. The constant V which appears in Propositions 4 and 3 is then the constant

V defined above, i.e. $V = \|v\|_{\infty(\Omega,\mathcal{C}_b^{1,\alpha}(\mathbb{R}^d))}$. This inequality holds for almost all ω, then by taking the expected value of the image by ψ we obtain:

$$\begin{aligned} |Er1| &\leq \mathbb{E}[\|\psi'\|_\infty (C_2(\Delta t)^{\frac{\alpha+1}{2}} + C_4 h|ln(h)|)] \\ &\leq \kappa(C_2(\Delta t)^{\frac{\alpha+1}{2}} + C_4 h|ln(h)|). \end{aligned}$$

The random variables $(Y_i)_{1\leq i\leq N}$ defined by $Y_i(\omega) = \psi(\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))]$ being independent, identically distributed and belonging to $L^2(\Omega)$, we have:

$$\begin{aligned} \|Er2(\omega)\|_{L^2_\omega} &\leq \frac{\|Y_i - \mathbb{E}[Y_i]\|_{L^2_\omega}}{\sqrt{N}} \\ &\leq \frac{2\kappa}{\sqrt{N}}. \end{aligned}$$

Indeed, for almost all ω, we have $|Y_i(\omega)| \leq \|\psi\|_\infty$. Analogously, for any $1 \leq i \leq N$ and almost all ω, the random variables $(Z_j)_{1\leq j\leq M}$ defined by $Z_j(\xi) = \mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))]$ are independent, identically distributed $L^2(\Omega)$ random variables, therefore:

$$\begin{aligned} \left\| \mathbb{E}[Z_j] - \frac{1}{M}\sum_{j=1}^{M} Z_j(\xi) \right\|_{L^2_\xi} &\leq \frac{\|Z_j - \mathbb{E}[Z_j]\|_{L^2_\xi}}{\sqrt{M}} \\ &\leq \frac{2\kappa}{\sqrt{M}}. \end{aligned}$$

For all $1 \leq i \leq N$ and almost all ω,

$$\begin{aligned} &\left| \psi(\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))]) - \psi\left(\frac{1}{M}\sum_{j=1}^{M}\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))\right)\right| \\ &\leq \|\psi'\|_\infty \left| \mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))] - \frac{1}{M}\sum_{j=1}^{M}\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))\right|, \end{aligned}$$

thus

$$\begin{aligned} &\left\| \psi(\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))]) - \psi\left(\frac{1}{M}\sum_{j=1}^{M}\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))\right)\right\|_{L^2_\xi} \\ &\leq \frac{2\kappa^2}{\sqrt{M}}. \end{aligned}$$

This bound holds for any $1 \leq i \leq N$ and almost all ω, therefore taking the sum over i and the L^2_ω norm yields finally the following bound for $Er3$:

$$\|Er3(\omega,\xi)\|_{L^2_\omega L^2_\xi} \leq \frac{2\kappa^2}{\sqrt{M}}.$$

□

4.6 Total error on the dispersion

We recall that $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$ are two probability spaces. Let $\varphi \in \mathcal{C}_b^5(\mathbb{R}^d, \mathbb{R}^p)$ and $\psi \in \mathcal{C}_b^2(\mathbb{R}^p, \mathbb{R}^q)$ for some $p, q \in \mathbb{N}^*$. Take a constant κ such that φ, ψ and their derivatives are bounded by κ. $X(\omega, \xi, t)$ and its approximations $\tilde{X}_n^{i,j}(\omega, \xi, t)$ are defined as previously by respectively (10) and (11).

In this section we give a bound for the below defined error.

Definition 2.

$$\begin{aligned} E(\omega,\xi) &= \frac{d}{dt}\mathbb{E}_\omega[\psi(\mathbb{E}_\xi[\varphi(X(\omega,\xi,T))])] \\ &- \frac{1}{N}\sum_{i=1}^{N} \frac{\psi\left(\frac{1}{M}\sum_{j=1}^{M}\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T+\Delta s)\right) - \psi\left(\frac{1}{M}\sum_{j=1}^{M}\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T)\right)}{\Delta s}, \end{aligned}$$

where we have naturally defined

$$\tilde{X}_n^{i,j}(\omega,\xi,T+\Delta s) = \tilde{X}_n^{i,j}(T) + \tilde{v}(\tilde{X}_n^{i,j}(T))\Delta s + \sqrt{2D\Delta s}(W^{i,j}(T+\Delta s) - W^{i,j}(T)).$$

To bound the error, we make the following additional assumption:

Hypothesis 3. $a \in L^\infty(\Omega, \mathcal{C}_b^{2,\alpha}(\mathbb{R}^d))$ *for some* $0 < \alpha < 1$.

Theorem 2. *There exists a constant C such that*

$$\|E(\omega,\xi)\|_{L^2_{\omega,\xi}} \leq C\left(\Delta t + \Delta s + h|ln(h)| + \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{M\Delta s}}\right).$$

Remark 6. *The last term in the right hand side (namely $\frac{1}{\sqrt{M\Delta s}}$) is due to the fact that we approximate the derivative of the expected value of a function of the solution of a SDE through a Monte-Carlo method (it comes from the term $E4$ in the following proof), and is optimal (as we can see with the case of a constant drift).*

Remark 7. *An estimate of the error on the dispersion follows from the cases where $\varphi(x) = xx^t$, $\psi(x) = x$ and $\varphi(x) = x$, $\psi(x) = xx^t$. For simplicity, we treat only the case where φ and ψ are bounded with bounded derivatives. The result can however be generalized to the case where φ and ψ are respectively $\mathcal{C}^5$ and $\mathcal{C}^2$ with growth at most polynomial.*

Proof. We split this error into four terms:

$$E(\omega,\xi) = E1 + E2 + E3(\omega) + E4(\omega,\xi).$$

These four terms are defined by:

$$
\begin{aligned}
E1 &= \frac{d}{dt}\mathbb{E}_\omega[\psi(\mathbb{E}_\xi[\varphi(X(\omega,\xi,T))])] \\
&- \mathbb{E}_\omega\left[D\psi(\mathbb{E}_\xi[\varphi(X(\omega,\xi,T)]).\frac{\mathbb{E}_\xi[\varphi(X(\omega,\xi,T+\Delta s))]-\mathbb{E}_\xi[\varphi(X(\omega,\xi,T))]}{\Delta s})\right], \\
E2 &= \mathbb{E}_\omega\left[D\psi(\mathbb{E}_\xi[\varphi(X(\omega,\xi,T))].\frac{\mathbb{E}_\xi[\varphi(X(\omega,\xi,T+\Delta s))]-\mathbb{E}_\xi[\varphi(X(\omega,\xi,T))]}{\Delta s}\right] \\
&- \frac{\mathbb{E}_\omega[\psi(\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T+\Delta s))]]-\mathbb{E}_\omega[\psi(\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))]]}{\Delta s}, \\
E3(\omega) &= \frac{\mathbb{E}_\omega[\psi(\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T+\Delta s))]]-\mathbb{E}_\omega[\psi(\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))]]}{\Delta s} \\
&- \frac{\frac{1}{N}\sum_{i=1}^{N}\psi(\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T+\Delta s))])-\psi(\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))])}{\Delta s}, \\
E4(\omega,\xi) &= \frac{\frac{1}{N}\sum_{i=1}^{N}\psi(\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T+\Delta s))])-\psi(\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))])}{\Delta s} \\
&- \frac{\frac{1}{N}\sum_{i=1}^{N}\psi\left(\frac{1}{M}\sum_{j=1}^{M}\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T+\Delta s))\right)-\psi\left(\frac{1}{M}\sum_{j=1}^{M}\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))\right)}{\Delta s}.
\end{aligned}
$$

E1 The first term $E1$ accounts for the approximation of the time derivative. To bound this term, we first prove the following lemma:

Lemma 1. *Take a constant V, $0<\alpha<1$, then there exists a constant C_V such that for any $v\in\mathcal{C}_b^{2,\alpha}(\mathbb{R}^d)$ with $\|v\|_{\mathcal{C}_b^{2,\alpha}(\mathbb{R}^d)}\leq V$, the solution Y^x of the stochastic differential equation* (8) *is such that for any $x\in\mathbb{R}^d$ we have*

$$\left|\frac{\mathbb{E}[\varphi(Y^x(T+\Delta s))]-\mathbb{E}[\varphi(Y^x(T))]}{\Delta s}-\frac{d}{dt}\mathbb{E}[\varphi(Y^x(t))]\right|\leq C_V\Delta s.$$

Proof. As previously we define u as the solution of (9), then we have $u\in\mathcal{C}_b^{1;2}(\mathbb{R}^d\times[0,2T])$ as previously, and thanks to the additionnal assumptions, applying the result of [18] to $\frac{\partial u}{\partial t}$, we have $\frac{\partial u}{\partial t}\in\mathcal{C}_b^{1;2}(\mathbb{R}^d\times[0,2T])$ with $\|\frac{\partial u}{\partial t}\|_{\mathcal{C}^{1;2}(\mathbb{R}^d\times[0,2T])}\leq C_V$. Therefore, we have for any $x\in\mathbb{R}^d$

$$
\begin{aligned}
&\left|\frac{\mathbb{E}[\varphi(Y^x(T+\Delta s))]-\mathbb{E}[\varphi(Y^x(T))]}{\Delta s}-\frac{d}{dt}\mathbb{E}[\varphi(Y^x(t))]\right| \\
&=\left|\frac{u(T+\Delta s,x)-u(T,x)}{\Delta s}-\frac{\partial}{\partial t}u(T,x)\right| \\
&\leq \Delta s\int_0^1(1-t)\left|\frac{\partial^2 u}{\partial t^2}(T+u\Delta s,x)\right|du \\
&\leq \Delta s\sup_{x\in\mathbb{R}^d,s\in[T,T+\Delta s]}\left|\frac{\partial^2 u}{\partial t^2}(s,x)\right| \\
&\leq C_V\Delta s.
\end{aligned}
$$

□

We can now bound $E1$, first we notice that since we have supposed that $a \in L^\infty(\Omega, \mathcal{C}_b^{2,\alpha}(\mathbb{R}^d))$, we have $v \in L^\infty(\Omega, \mathcal{C}_b^{2,\alpha}(\mathbb{R}^d))$ by [13], as in the proof of Proposition 1. Therefore, setting $V = \|v\|_{L^\infty(\Omega, \mathcal{C}_b^{2,\alpha}(\mathbb{R}^d))}$, then we can apply the previous lemma for almost all ω, and since the bound $C_V \Delta s$ is independent from the deterministic initial condition, the Markov property yields that for almost all ω:

$$\left| \frac{d}{dt}\mathbb{E}_\xi[\varphi(X(\omega,\xi,T))] - \frac{\mathbb{E}_\xi[\varphi(X(\omega,\xi,T+\Delta s))] - \mathbb{E}_\xi[\varphi(X(\omega,\xi,T))]}{\Delta s} \right| \\ \leq C_V \Delta s.$$

$E1$ can be rewritten as

$$\begin{aligned} E1 &= \mathbb{E}_\omega[D\psi(\mathbb{E}_\xi[\varphi(X(\omega,\xi,T))]).\frac{d}{dt}(\mathbb{E}_\xi[\varphi(X(\omega,\xi,T))]) \\ &- \mathbb{E}_\omega\left[D\psi(\mathbb{E}_\xi[\varphi(X(\omega,\xi,T)]).\frac{\mathbb{E}_\xi[\varphi(X(\omega,\xi,T+\Delta s))] - \mathbb{E}_\xi[\varphi(X(\omega,\xi,T))]}{\Delta s}\right] \end{aligned}$$

Using the previous bound, bounding $\|\psi'\|_\infty$ by κ and taking the expected value with respect to ω yields the following bound for $E1$:

$$|E1| \leq \kappa C_V \Delta s.$$

E2 The term $E2$ contains the space and time discretizations errors. We define once again $V = \|v\|_{L^\infty(\Omega, \mathcal{C}_b^{2,\alpha}(\mathbb{R}^d))}$. The estimate of $E2$ will follow from an estimate of:

$$\begin{aligned} A &= \frac{\mathbb{E}_\xi[\varphi(\tilde{X}_n(\omega,\xi,T+\Delta s))] - \mathbb{E}_\xi[\varphi(\tilde{X}_n(\omega,\xi,T))]}{\Delta s} \\ &- \frac{\mathbb{E}_\xi[\varphi(X(\omega,\xi,T+\Delta t))] - \mathbb{E}_\xi[\varphi(X(\omega,\xi,T))]}{\Delta s}. \end{aligned}$$

Actually A corresponds to the case where $\psi(x) = x$. To simplify the variable ω will be fixed and omitted in this paragraph, except in the final bound, and all bounds are uniform with respect to ω, the index i, j will be also omitted, since the random processes are identically distributed and the variable ξ will be also omitted, all expected values are implicitly taken with respect to ξ.
We split the error term A into a time discretization error B and a space discretization error C, i.e. $A = B + C$ where

$$B = \frac{\mathbb{E}[\varphi(X_n(T+\Delta s))] - \mathbb{E}[\varphi(X_n(T))]}{\Delta s} - \frac{\mathbb{E}[\varphi(X(T+\Delta t))] - \mathbb{E}[\varphi(X(T))]}{\Delta s},$$

$$C = \frac{\mathbb{E}[\varphi(\tilde{X}_n(T+\Delta s))] - \mathbb{E}[\varphi(\tilde{X}_n(T))]}{\Delta s} - \frac{\mathbb{E}[\varphi(X_n(T+\Delta t))] - \mathbb{E}[\varphi(X_n(T))]}{\Delta s}.$$

We first bound the time discretization error B. For any $\phi \in \mathcal{C}_b(\mathbb{R}^d)$ we introduce the following notations: $P^t(\phi)(x) = \mathbb{E}[\phi(X^x(t))]$ and $P_n^t(\phi)(x) = \mathbb{E}[\phi(X_n^x(t))]$ where X^x and X_n^x are defined by (7) and (8) respectively. For $v \in \mathcal{C}_b^2(\mathbb{R}_d)$, Proposition 3 in the case $\alpha = 1$ can be reformulated as follows: for any $\phi \in \mathcal{C}_b^{2,\beta}(\mathbb{R}^d)$ for some $0 < \beta < 1$ we have, recalling that $T = n\Delta t$,

$$|P^{\Delta s}(\phi)(x) - P_n^{\Delta s}(\phi)(x)| \leq C_2(\Delta s)^2 \text{ and } |P^T(\phi)(x) - P_n^T(\phi)(x)| \leq C_2 \Delta t \qquad (12)$$

where the constant C_2 depends only on v through V. The following inequality clearly hold true: for any t, and $\phi \in \mathcal{C}_b$,
$\|P^t(\phi)\|_\infty \leq \|\phi\|_\infty$ and $\|P_n^t(\phi)\|_\infty \leq \|\phi\|_\infty$.
We can now bound B, using these notations and inequalities. The Markov property of X and X_n yields:

$$\begin{aligned} B &= \mathbb{E}\left[P_n^T\left(\frac{P_n^{\Delta s}-Id}{\Delta s}\right)\varphi(X_0) - P^T\left(\frac{P^{\Delta s}-Id}{\Delta s}\right)\varphi(X_0)\right] \\ &= \mathbb{E}\left[P_n^T\left(\frac{P_n^{\Delta s}-P^{\Delta s}}{\Delta s}\right)\varphi(X_0) + (P_n^T - P^T)\left(\frac{P^{\Delta s}-Id}{\Delta s}\right)\varphi(X_0)\right]. \end{aligned}$$

For any $x \in \mathbb{R}^d$,

$$\left|\left(\frac{P_n^{\Delta s}-P^{\Delta s}}{\Delta s}\right)\varphi(x)\right| \leq C_2\Delta s,$$

whence for any $x \in \mathbb{R}^d$,

$$\left|P_n^T\left(\frac{P_n^{\Delta s}-P^{\Delta s}}{\Delta s}\right)\varphi(x)\right| \leq C_2\Delta s,$$

and

$$\left|\mathbb{E}\left[P_n^T\left(\frac{P_n^{\Delta s}-P^{\Delta s}}{\Delta s}\right)\varphi(X_0)\right]\right| \leq C_2\Delta s.$$

For any $x \in \mathbb{R}^d$,

$$\frac{P^{\Delta s}-Id}{\Delta s}\varphi(x) = \frac{u(\Delta s,x)-u(0,x)}{\Delta s},$$

therefore

$$\left|\frac{P^{\Delta s}-Id}{\Delta s}\varphi(x) - \frac{\partial u}{\partial t}(0,x)\right| \leq \Delta s\left|\sup_{s\in[0,T],x\in\mathbb{R}^d}\frac{\partial^2 u}{\partial t^2}\right|,$$

where u is still defined by (9). And finally, for any $x \in \mathbb{R}^d$,

$$\begin{aligned} &\left|(P_n^T - P^T)\left(\frac{P^{\Delta s}-Id}{\Delta s}\right)\varphi(x)\right| \\ &\leq \left|(P_n^T - P^T)\left(\frac{\partial u}{\partial t}(0,.)\right)(x)\right| \\ &+2\Delta s \sup_{s\in[0,T],x\in\mathbb{R}^d}\left|\frac{\partial^2 u}{\partial t^2}(s,x)\right| \\ &\leq C_2\Delta t + 2\Delta s C_V, \end{aligned}$$

by applying (12) with $\phi = \frac{\partial u}{\partial t}(0,.) = D\Delta\varphi + v.\nabla\varphi \in \mathcal{C}_b^{2,\alpha}(\mathbb{R}^d)$.
From the two previous bound, we deduce a bound for B:

$$|B| \leq C_2\Delta t + (2C_V + C_2)\Delta s.$$

We now give a bound for the term of space discretization error C.
We first introduce the notation $\tilde{P}_n^t(\varphi)(x) = \mathbb{E}[\varphi(\tilde{X}_n(t))]$ and use the Markov property of X_n and $\tilde{X}_n$:

$$\begin{aligned} C &= \frac{\mathbb{E}[\varphi(\tilde{X}_n(T+\Delta s))] - \mathbb{E}[\varphi(\tilde{X}_n(T))]}{\Delta s} - \frac{\mathbb{E}[\varphi(X_n(T+\Delta t))] - \mathbb{E}[\varphi(X_n(T))]}{\Delta s} \\ &= \mathbb{E}\left[\frac{(\tilde{P}_n^{T+\Delta s}(\varphi)(X_0) - \tilde{P}_n^T(\varphi)(X_0)) - (P_n^{T+\Delta s}(\varphi)(X_0) - P_n^T(\varphi)(X_0))}{\Delta s}\right]. \end{aligned}$$

The Markov property enables us to treat the case of a deterministic initial condition. We have now to bound the below defined function $F(x)$, uniformly with respect to x:

$$F(x) = \frac{\mathbb{E}[\varphi(\tilde{X}_n^x(T+\Delta s))] - \mathbb{E}[\varphi(\tilde{X}_n^x(T))]}{\Delta s} - \frac{\mathbb{E}[\varphi(X_n^x(T+\Delta t))] - \mathbb{E}[\varphi(X_n^x(T))]}{\Delta s},$$

since

$$C = \mathbb{E}[F(X_0)].$$

The following inequalities will be useful, and should be kept in mind to understand what will follow.

Lemma 2. *[i]*

1. *For almost all ξ, and all x we have:*

$$|\tilde{X}_n^x(T) - X_n^x(T)| \leq C_3 \|v - \tilde{v}\|_\infty.$$

2. *For almost all ξ, and all x we have:*

$$|Y_n^x| \leq \|v - \tilde{v}\|_\infty (1 + C_3 V)\Delta s.$$

where we use the notation

$$Y_n^x = X_n^x(T+\Delta s) - X_n^x(T) - (\tilde{X}_n^x(T+\Delta s) - \tilde{X}_n^x(T)).$$

3. *There exists a constant C_5 depending only on d and V such that for all x, for $p = 1$ and $p = 2$ we have:*

$$\|\tilde{X}_n^x(T+\Delta s) - \tilde{X}_n^x(T)\|_{L^p} \leq C_5\sqrt{\Delta s} \text{ and } \|X_n^x(T+\Delta s) - X_n^x(T)\|_{L^p} \leq C_5\sqrt{\Delta s}.$$

Proof. [i]

It is a direct application of Proposition 4.

2. For almost all ξ, and all x we have, using Proposition 4:

$$\begin{aligned}
&|(X_n^x(T+\Delta s) - X_n^x(T)) - (\tilde{X}_n^x(T+\Delta s) - \tilde{X}_n^x(T))| \\
&= |v(X_n^x(T)) - \tilde{v}(\tilde{X}_n^x)|\Delta s \\
&= |v(X_n^x(T)) - v(\tilde{X}_n^x(T)) + v(\tilde{X}_n^x(T)) - \tilde{v}(\tilde{X}_n^x)|\Delta s \\
&\leq \|Dv\|_\infty C_3 \|v - \tilde{v}\|_\infty \Delta s + \|v - \tilde{v}\|_\infty \Delta s,
\end{aligned}$$

where we have used the previous point.

3.

$$\begin{aligned}
|X_n^x(T+\Delta s) - X_n^x(T)| &= |v(X_n^x(T))\Delta s + \sqrt{2D\Delta s}N| \\
|\tilde{X}_n^x(T+\Delta s) - \tilde{X}_n^x(T)| &= |\tilde{v}(\tilde{X}_n^x(T))\Delta s + \sqrt{2D\Delta s}N|,
\end{aligned}$$

where N is a mean-free gaussian whose covariance is equal to identity.

Therefore, using that for h small enough $\|\tilde{v}\| \leq 2V$ by Proposition 2, we get

$$\begin{aligned}
\|X_n^x(T+\Delta s) - X_n^x(T)\|_{L^1} &\leq V\Delta s + \sqrt{2D\Delta s} \\
\|\tilde{X}_n^x(T+\Delta s) - \tilde{X}_n^x(T)\|_{L^1} &\leq 2V\Delta s + \sqrt{2D\Delta s} \\
\|X_n^x(T+\Delta s) - X_n^x(T)\|_{L^2} &\leq V\Delta s + \sqrt{2D\Delta s} \\
\|\tilde{X}_n^x(T+\Delta s) - \tilde{X}_n^x(T)\|_{L^2} &\leq 2V\Delta s + \sqrt{2D\Delta s}
\end{aligned}$$

□

We can now bound F.

$$\begin{aligned}
F(x) &= \frac{\mathbb{E}[\varphi(\tilde{X}_n^x(T+\Delta s))] - \mathbb{E}[\varphi(\tilde{X}_n(T))]}{\Delta s} - \frac{\mathbb{E}[\varphi(X_n^x(T+\Delta t))] - \mathbb{E}[\varphi(X_n(T)^x)]}{\Delta s} \\
&= \frac{1}{\Delta s}\mathbb{E}\left[\int_0^1 D\varphi(\tilde{X}_n^x(T)) + u(\tilde{X}_n^x(T+\Delta s) - \tilde{X}_n^x(T))).(\tilde{X}_n^x(T+\Delta s) - \tilde{X}_n^x(T))du\right] \\
&- \frac{1}{\Delta s}\mathbb{E}\left[\int_0^1 D\varphi(X_n^x(T)) + u(X_n^x(T+\Delta s) - X_n^x(T))).(X_n^x(T+\Delta s) - X_n(T))du\right] \\
&= f_1(x) + f_2(x),
\end{aligned}$$

where f_1 and f_2 are defined by

$$\begin{aligned}
f_1(x) &= \frac{1}{\Delta s}\mathbb{E}\left[\int_0^1 D\varphi(\tilde{X}_n^x(T)) + u(\tilde{X}_n^x(T+\Delta s) - \tilde{X}_n^x(T))).(\tilde{X}_n^x(T+\Delta s) - \tilde{X}_n^x(T))du\right] \\
&- \frac{1}{\Delta s}\mathbb{E}\left[\int_0^1 D\varphi(X_n^x(T)) + u(X_n^x(T+\Delta s) - X_n^x(T))).(\tilde{X}_n^x(T+\Delta s) - \tilde{X}_n(T))du\right].
\end{aligned}$$

and

$$f_2(x) = \frac{1}{\Delta s}\mathbb{E}\left[\int_0^1 D\varphi[X_n^x(T) + u(X_n^x(T+\Delta s) - X_n^x(T))].Y_n^x du\right]$$

We first treat f_2: the application of the second point of Lemma 2 gives that for all x

$$|f_2(x)| \leq \kappa\|v - \tilde{v}\|_\infty(1 + C_3 V).$$

We now treat f_1

$$f_1(x) = \frac{1}{\Delta s}\mathbb{E}\left[\int_{[0,1]^2} D^2\varphi(Z_n^x(u,\lambda)).(\tilde{X}_n^x(T) - X_n^x(T) + uY_n^x, \tilde{X}_n^x(T+\Delta s) - \tilde{X}_n^x(T))dud\lambda\right],$$

where we use the notation:

$$\begin{aligned}
Z_n^x(u,\lambda) &= \lambda[\tilde{X}_n^x(T) + u(\tilde{X}_n^x(T+\Delta s) - \tilde{X}_n^x(T)] \\
&+ (1-\lambda)[X_n^x(T) + u(X_n^x(T+\Delta s) - X_n^x(T)].
\end{aligned}$$

We split f_1 into two terms, i.e. $f_1(x) = f_3(x) + f_4(x)$ where we define:

$$f_3(x) = \frac{1}{\Delta s}\mathbb{E}\left[\int_{[0,1]^2} D^2\varphi(Z_n^x(u,\lambda)).(\tilde{X}_n^x(T) - X_n^x(T), \tilde{X}_n^x(T+\Delta s) - \tilde{X}_n^x(T))dud\lambda\right],$$

and

$$f_4(x) = \frac{1}{\Delta s}\mathbb{E}\left[\int_{[0,1]^2} D^2\varphi(Z_n^x(u,\lambda)).(uY_n^x, \tilde{X}_n^x(T+\Delta s) - \tilde{X}_n^x(T))dud\lambda\right].$$

We first bound f_4 using the second and third points of Lemma 2.

$$|f_4(x)| \leq \kappa\|v-\tilde{v}\|_\infty(1+C_3V)C_5\sqrt{\Delta s} \leq \kappa\|v-\tilde{v}\|_\infty(1+C_3V)C_5\sqrt{T}.$$

We now bound f_3, by splitting it into two terms, i.e. $f_3 = f_5 + f_6$, where

$$f_5(x) = \frac{\mathbb{E}\left[\int_{[0,1]^2} D^2\varphi(\lambda\tilde{X}_n^x(T)+(1-\lambda)X_n^x(T)).(\tilde{X}_n^x(T)-X_n^x(T), \tilde{X}_n^x(T+\Delta s)-\tilde{X}_n^x(T))dud\lambda\right]}{\Delta s},$$

and

$$f_6(x) = \frac{\mathbb{E}[\int_{[0,1]^3} D^3\varphi(W_n^x(\lambda,\mu,u)).(Q_n^x(\lambda,u), \tilde{X}_n^x(T)-X_n^x(T), (\tilde{X}_n^x(T+\Delta s-\tilde{X}_n^x(T))))d\mu d\lambda du]}{\Delta s},$$

where

$$Q_n^x(\lambda,u) = \lambda s(\tilde{X}_n^x(T+\Delta s)-\tilde{X}_n^x(T)) + (1-\lambda)u(X_n^x(T+\Delta s)-X_n^x(T)),$$

and

$$W_n^x(\lambda,\mu) = \lambda\tilde{X}_n^x(T)+(1-\lambda)X_n^x(T)+\mu Q_n^x(\lambda,u).$$

We first bound f_5, by using the independence of $(\lambda\tilde{X}_n^x(T)+(1-\lambda)X_n^x(T), \tilde{X}_n^x(T)-X_n^x(T))$, which is $\mathcal{F}_T$-measurable and $(W(T+\Delta T)-W(T))$ together with the fact that $\mathbb{E}[W(T+\Delta T)-W(T)]=0$.

$$\begin{aligned} f_5(x) &= \frac{\mathbb{E}\left[\int_{[0,1]^2} D^2\varphi(\lambda\tilde{X}_n^x(T)+(1-\lambda)X_n^x(T)).(\tilde{X}_n^x(T)-X_n^x(T), \tilde{v}(\tilde{X}_n^x(T))\Delta s)dud\lambda\right]}{\Delta s} \\ &+ \frac{\mathbb{E}\left[\int_{[0,1]^2} D^2\varphi(\lambda\tilde{X}_n^x(T)+(1-\lambda)X_n^x(T)).(\tilde{X}_n^x(T)-X_n^x(T), \sqrt{2D}(W(T+\Delta T)-W(T)))dud\lambda\right]}{\Delta s} \\ &= \frac{\mathbb{E}\left[\int_{[0,1]^2} D^2\varphi(\lambda\tilde{X}_n^x(T)+(1-\lambda)X_n^x(T)).(\tilde{X}_n^x(T)-X_n^x(T), \tilde{v}(\tilde{X}_n^x(T))\Delta s)dud\lambda\right]}{\Delta s} \\ &+0. \end{aligned}$$

Whence, the first point of Lemma 2 yields

$$|f_5(x)| \leq \kappa VC_3\|v-\tilde{v}\|_\infty.$$

Besides this, using the first and third points of Lemma 2 and Cauchy-Schwarz inequality we get

$$\begin{aligned} |f_6(x)| &\leq \kappa C_3\|v-\tilde{v}\|_\infty \frac{\int_{[0,1]^3}\mathbb{E}\left[|Q_n^x(\lambda,u)||\tilde{X}_n^x(T+\Delta s)-\tilde{X}_n^x(T)\|\right]d\mu d\lambda du}{\Delta s} \\ &\leq \kappa C_3\|v-\tilde{v}\|_\infty \\ &\quad \frac{(\|\tilde{X}_n^x(T+\Delta s)-\tilde{X}_n^x(T)\|_{L^2}+\|X_n^x(T+\Delta s)-X_n^x(T)\|_{L^2})\|\tilde{X}_n^x(T+\Delta s)-\tilde{X}_n^x(T)\|_{L^2}}{\Delta s} \\ &\leq 2\kappa C_3\|v-\tilde{v}\|_\infty C_5^2. \end{aligned}$$

Taking the sum of all these bounds yields:

$$\begin{aligned} |F(x)| &\leq \kappa\|v-\tilde{v}\|_\infty(1+C_3V)+\kappa\|v-\tilde{v}\|_\infty(1+C_3V)C_5\sqrt{T} \\ &+ \kappa VC_3\|v-\tilde{v}\|_\infty+2\kappa C_3\|v-\tilde{v}\|_\infty C_5^2. \end{aligned}$$

Thus,

$$|C| \leq C_6\|v-\tilde{v}\|_\infty.$$

And therefore there exists a constant C_7 such that for almost all ω, we have:

$$|A| \leq C_7(\|v-\tilde{v}\|_\infty+\Delta s+\Delta t). \tag{13}$$

We can now bound $E2$.

$$\begin{aligned} E2 &= D\psi(\mathbb{E}[\varphi(X(T))]).\frac{\mathbb{E}[\varphi(X(T+\Delta s))-\varphi(X(T))]}{\Delta s} \\ &- \frac{\psi(\mathbb{E}[\varphi(\tilde{X}_n(T+\Delta s))])-\psi(\mathbb{E}[\varphi(\tilde{X}_n(T))])}{\Delta s} \\ &= D\psi(\mathbb{E}[\varphi(X(T))]).\frac{\mathbb{E}[\varphi(X(T+\Delta s))-\varphi(X(T))]}{\Delta s} \\ &- D\psi(\mathbb{E}[\varphi(\tilde{X}_n(T))]).\frac{\mathbb{E}[\varphi(\tilde{X}_n(T+\Delta s))-\varphi(\tilde{X}_n(T))]}{\Delta s} \\ &- \int_0^1 D^2\psi(M_n(u)).\frac{(\mathbb{E}[\varphi(\tilde{X}_n(T+\Delta s))-\varphi(\tilde{X}_n(T))])^2}{\Delta s}du \end{aligned}$$

where $M_n(u)=\mathbb{E}[\varphi(\tilde{X}_n(T)]+u(\mathbb{E}[\varphi(\tilde{X}_n(T+\Delta s))-\varphi(\tilde{X}_n(T))]])$. Hence, we have

$$\begin{aligned} E2 &= D\psi(\mathbb{E}[\varphi(X(T))]).A \\ &+ (D\psi(\mathbb{E}[\varphi(X(T))])-D\psi(\mathbb{E}[\varphi(\tilde{X}_n(T))])).\frac{(\mathbb{E}[\varphi(\tilde{X}_n(T+\Delta s))-\varphi(\tilde{X}_n(T))])}{\Delta s} \\ &- \int_0^1 D^2\psi(M_n(u)).\frac{(\mathbb{E}[\varphi(\tilde{X}_n(T+\Delta s))-\varphi(\tilde{X}_n(T))])^2}{\Delta s}du. \end{aligned}$$

In order to bound $E2$, we prove the following lemma:

Lemma 3. *There exists a constant C_8 such that for h small enough, we have for almost all ω*

$$|\mathbb{E}[\varphi(\tilde{X}_n(T+\Delta s))]-\mathbb{E}[\varphi(\tilde{X}_n(T))]| \leq C_8\Delta s.$$

Proof. We recall that

$$\tilde{X}_n(T+\Delta s)-\tilde{X}_n(T)=\tilde{v}(\tilde{X}_n(T))\Delta s+\sqrt{2D\Delta s}N.$$

$$\begin{aligned} &\mathbb{E}[\varphi(\tilde{X}_n(T+\Delta s))]-\mathbb{E}[\varphi(\tilde{X}_n(T))] \\ &= \mathbb{E}[D\varphi(\tilde{X}_n(T)).(\tilde{v}(\tilde{X}_n(T))\Delta s+\sqrt{2D\Delta s}N)] \\ &+\mathbb{E}\left[\int_0^1 D^2\varphi(\tilde{X}_n(T)+u(\tilde{X}_n(T+\Delta s)-\tilde{X}_n(T))).(\tilde{v}(\tilde{X}_n(T))\Delta s+\sqrt{2D\Delta s}N)^2du\right] \\ &= \mathbb{E}[D\varphi(\tilde{X}_n(T)).(\tilde{v}(\tilde{X}_n(T))\Delta s)]+0 \\ &+\mathbb{E}\left[\int_0^1 D^2\varphi(\tilde{X}_n(T)+u(\tilde{X}_n(T+\Delta s)-\tilde{X}_n(T))).(\tilde{v}(\tilde{X}_n(T))\Delta s+\sqrt{2D\Delta s}N)^2du\right] \end{aligned}$$

where we have used once again the independence of N and $\tilde{X}_n^x(T)$, whence for h small enough :

$$|\mathbb{E}[\varphi(\tilde{X}_n(T+\Delta s))] - \mathbb{E}[\varphi(\tilde{X}_n(T))]| \leq 2\kappa V\Delta s + \kappa\mathbb{E}[(2V\sqrt{T} + \sqrt{2D}N)^2]\Delta s.$$

□

We can now bound $E2$, using the Lemma 3, the bound (13) of A, the bound (12) and the first point of Lemma 2

$$\begin{aligned} |E2| &\leq \kappa A + \kappa C_8|\mathbb{E}[\varphi(X(T)) - \mathbb{E}[\varphi(\tilde{X}_n(T))]| + \kappa C_8^2\Delta s \\ &\leq \kappa C_7(\|v-\tilde{v}\|_\infty + \Delta s + \Delta t) + \kappa C_8(C_2\Delta s + \kappa C_3\|v-\tilde{v}\|_\infty) + \kappa C_8^2\Delta s. \end{aligned}$$

We introduce a new constant C_9 which depends only on v through V and which is therefore independent of ω, and we have then finally

$$|E2| \leq C_9(\Delta s + \Delta t + \|v-\tilde{v}\|_\infty).$$

E3 For $1 \leq i \leq N$ we introduce the notation

$$P_i(\omega) = \frac{\psi(\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T+\Delta s))]) - \psi(\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))])}{\Delta s}.$$

The $(P_i)_{1\leq i\leq N}$ are independent and identically distributed L^2 random variables, therefore

$$\begin{aligned} \|E3(\omega)\|_{L^2_\omega} &= \left\| \mathbb{E}[P_i] - \frac{1}{N}\sum_{i=1}^N P_i \right\|_{L^2_\omega} \\ &\leq \frac{\|P_i - \mathbb{E}[P_i]\|_{L^2_\omega}}{\sqrt{N}} \end{aligned}$$

For almost all ω, we have, using Markov property and Lemma 3

$$\begin{aligned} |P_i(\omega)| &\leq \|D\psi\|_\infty \frac{|\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T+\Delta s))]) - \mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))]|}{\Delta s} \\ &\leq \|D\psi\|_\infty C_8. \end{aligned}$$

Thus

$$\|E3(\omega)\|_{L^2_\omega} \leq \frac{2\kappa C_8}{\sqrt{N}}.$$

E4 We use some preliminary results to bound $E4$.
Let $1 \leq i \leq N$ and ω be fixed, then for $1 \leq j \leq M$, we introduce the random variables

$$Q_{i,j}(\omega,\xi)) = \varphi(\tilde{X}_n^{i,j}(\omega,\xi,T+\Delta s)) - \varphi(\tilde{X}_n^{i,j}(\omega,\xi,T)).$$

We recall that if $(U_j)_{1\leq j\leq M}$ are idependant and identically distributed random variables in L^4, then

$$\left\| \mathbb{E}[U_j] - \frac{1}{M}\sum_{j=1}^M U^j \right\|_{L^4_\xi} \leq \frac{2}{\sqrt{M}}(\|U_j\|_{L^4} + \|U_j\|_{L^2}).$$

The $(\varphi(\tilde{X}_n^{i,j}(\omega,\cdot))_{1\le j\le M}$ being independent and identically distributed L^4 random variables, we have

$$\left\| \mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T))] - \frac{1}{M}\sum_{j=1}^{M}\varphi(\tilde{X}_n^{i,j}(\omega,\xi,T)) \right\|_{L^4_\xi} \le \frac{4\kappa}{\sqrt{M}}. \tag{14}$$

For almost all ω,ξ and all i,j we have

$$\begin{aligned} |Q_{i,j}(\omega,\xi)| &\le \kappa\left(V\Delta s + \sqrt{2D}(W^{i,j}(T+\Delta s) - W^{i,j}(T)))\right) \\ &\le \kappa\sqrt{\Delta s}\left(V\sqrt{T} + \sqrt{2D}\frac{(W^{i,j}(T+\Delta s)-W^{i,j}(T))}{\sqrt{\Delta s}}\right). \end{aligned}$$

Therefore, for almost all ω and all i we have

$$\|Q_{i,j}(\omega,\xi)\|_{L^2_\xi} \le C_9\kappa\sqrt{\Delta s},$$

and

$$\|Q_{i,j}(\omega,\xi)\|_{L^4_\xi} \le C_{10}\kappa\sqrt{\Delta s}.$$

Besides this, the $(Q_{i,j}(\omega,\cdot)_{1\le j\le M}$ are idependant and identically distributed random variables in L^4, therefore

$$\left\| \mathbb{E}_\xi[Q_{i,j}(\omega,\xi)] - \frac{1}{M}\sum_{j=1}^{M} Q_{i,j}(\omega,\xi) \right\|_{L^4_\xi} \le \frac{2}{\sqrt{M}}(C_9+C_{10})\kappa\sqrt{\Delta s}, \tag{15}$$

and

$$\left\| \mathbb{E}_\xi[Q_{i,j}(\omega,\xi)] - \frac{1}{M}\sum_{j=1}^{M} Q_{i,j}(\omega,\xi) \right\|_{L^2_\xi} \le \frac{1}{\sqrt{M}}C_9\kappa\sqrt{\Delta s}, \tag{16}$$

Finally, for almost all ω, ξ and all $1\le i\le N$,

$$\begin{aligned} &\frac{\psi(\mathbb{E}[\varphi(\tilde{X}_n^{i,j}(T+\Delta s))]) - \psi(\mathbb{E}[\varphi(\tilde{X}_n^{i,j}(T))])}{\Delta s} \\ &-\frac{\psi(\frac{1}{M}\sum_{j=1}^{M}\varphi(\tilde{X}_n^{i,j}(T+\Delta s))) - \psi(\frac{1}{M}\sum_{j=1}^{M}\varphi(\tilde{X}_n^{i,j}(T)))}{\Delta s} \\ &= \int_0^1 D\psi(\mathbb{E}[\varphi(\tilde{X}_n^{i,j}(T))] + u\mathbb{E}[Q_{i,j}]).\frac{\mathbb{E}_\xi[Q_{i,j}]}{\Delta s}du \\ &- \int_0^1 D\psi\left(\frac{1}{M}\sum_{j=1}^{M}\varphi(\tilde{X}_n^{i,j}(,T)) + \frac{1}{M}\sum_{j=1}^{M}Q_{i,j}\right).\frac{\frac{1}{M}\sum_{j=1}^{M}Q_{i,j}}{\Delta s}du \\ &= \int_0^1 D\psi(\mathbb{E}[\varphi(\tilde{X}_n^{i,j}(T))] + u\mathbb{E}[Q_{i,j}]).\left(\frac{\mathbb{E}_\xi[Q_{i,j}] - \frac{1}{M}\sum_{j=1}^{M}Q_{i,j}}{\Delta s}\right)du \\ &- \int_0^1 \left(D\psi(\mathbb{E}[\varphi(\tilde{X}_n^{i,j}(T)) + uQ_{i,j}]) - D\psi\left(\frac{1}{M}\sum_{j=1}^{M}\varphi(\tilde{X}_n^{i,j}(T)) + uQ_{i,j})\right)\right).\frac{\sum_{j=1}^{M}Q_{i,j}}{M\Delta s}du. \end{aligned}$$

Thus, thanks to Cauchy-Schwarz inequality and the above preliminary results (14), (16) and (15), we have for almost all ω, ξ and all $1 \leq i \leq N$,

$$\begin{aligned}
&\| \frac{\psi(\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(T+\Delta s))]) - \psi(\mathbb{E}_\xi[\varphi(\tilde{X}_n^{i,j}(T))])}{\Delta s} \\
&- \frac{\psi(\frac{1}{M}\sum_{j=1}^M \varphi(\tilde{X}_n^{i,j}(T+\Delta s))) - \psi(\frac{1}{M}\sum_{j=1}^M \varphi(\tilde{X}_n^{i,j}(T)))}{\Delta s} \|_{L^2_\xi} \\
&\leq \frac{\kappa}{\Delta s} \left\| \mathbb{E}_\xi[Q_{i,j}] - \frac{1}{M}\sum_{j=1}^M Q_{i,j} \right\|_{L^2_\xi} \\
&+ \frac{\kappa \, \|Q_{i,j}\|_{L^4_\xi}}{\Delta s} \left(\left\| \mathbb{E}[\varphi(\tilde{X}_n^{i,j}(T))] - \frac{1}{M}\sum_{j=1}^M \varphi(\tilde{X}_n^{i,j}(T)) \right\|_{L^4_\xi} + \left\| \mathbb{E}[Q_{i,j}] - \frac{1}{M}\sum_{j=1}^M Q_{i,j}) \right\|_{L^4_\xi} \right) \\
&\leq \frac{\kappa}{\Delta s} \frac{C_9 \kappa \sqrt{\Delta s}}{\sqrt{M}} + \kappa \left(\frac{4\kappa}{\sqrt{M}} + \frac{2}{\sqrt{M}}(C_9 + C_{10})\kappa\sqrt{\Delta s} \right) \frac{C_{10}\kappa\sqrt{\Delta s}}{\Delta s} \\
&\leq C_{11}\kappa^2(1+\kappa)\frac{1}{\sqrt{M\Delta s}}.
\end{aligned}$$

Taking the sum over i of these inequalities and taking the expected value with respect to ω yields

$$\|E4(\omega,\xi)\|_{L^2_{\omega,\xi}} \leq C_{11}\kappa^2(1+\kappa)\frac{1}{\sqrt{M\Delta s}}.$$

The final result on E follows, taking the sum of the bounds of $E1, E2, E3, E4$.

□

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