# Similar Triangles and Orientation in Plane Elementary Geometry for Coq-based Proofs

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## ABSTRACT

In plane elementary geometry, the concept of similar triangles not only forms an important foundation for trigonometry, but it also can be used to solve many geometric problems. The notion of orientation allows us to remove the usual ambiguities in presentation of object. In this paper, we present the formalization of these notions in Coq. We also introduce their properties and how they are applied to the proof of two theorems: the Ptolemy's theorem and the Intersecting Chords theorem.

## Categories and Subject Descriptors

I.2.3 [ **Deduction and Theorem Proving**]:

## General Terms

Theory

## Keywords

geometric theorem proving, orientation, similar triangles, formalization, Coq

## 1. INTRODUCTION

Formalizing mathematics allows to verify machenically all of the steps of proofs by a proof assistant system. For elementary geometry, it is more important by the fact that correctness of a traditional proof is affected by the exactness of figures.

In the Coq, some approaches are based on the axiom systems of Hilbert, Tarski [3, 5] to formalize geometric constructions. For formalizing proofs, we can cite here work using the area method [5]. This method reduces many concepts to a single one, and thus somtimes obfuscates the reasoning process.

Another interesting formalization of F.Guilhot [1] for high school curriculum. Its formalized proofs are the traditional ones. However, it lacks many notions. In this context, we chose the notion of similar triangles and the notion of orientation to enrich it, we also deal with application of these notions in proof of two in the top hundred list [4], being the Ptolemy's theorem and the Product of Segments of Chords .



## 2. SIMILAR TRIANGLES

Two triangles are said to be similar (denoted by $\simeq$) when all corresponding angles are equal. We separate similarity into 2 types (direct and inverse) by the rotational sense.

### 2.1 Properties

A triangle and its image after an Euclidean transformations are congruent triangles (denoted by $\simeq$), which are a special case of similar triangles. Thanks to these properties, we have the proportionality in similar triangles.

*Lemma SimTriangles_Proportion : forall A B C A' B' C' :PO, ABC* $\sim$ *A'B'C'* $\rightarrow \frac{AB}{MN} = \frac{BC}{NP} = \frac{CA}{PM}$.

To prove this property, the case of inverse similarity is reduced to the one of direct similarity thanks to a reflection, so we continue in the last case. $triangle AB"C"$ is the image of $\triangle MNP$ after applying a rotation to have $MN \parallel AB$ and a translation to have $M \equiv A$. So we have $\triangle MNP \simeq \triangle AB"C"$ and we can prove that $BC \parallel B"C"$, B" lies on AB, C" lies on AC. The configuration of these point satisfies Thales' theorem proved in Coq. So we have $\frac{AB}{AB"} = \frac{BC}{B"C"} = \frac{AC}{AC"}$ . By replacing sides of $\triangle AB"C"$ with corresponding ones of $\triangle MNP$, we get $\frac{AB}{MN} = \frac{BC}{NP} = \frac{CA}{PM}$ .

## 3. ORIENTATION

We chose their enumerated order in counter-clockwise direction to talk about orientation (denoted by $\circlearrowleft$). We define it by the positive value of the signed area.

*Definition* $\circlearrowleft ABC := |\overrightarrow{AB}| * |\overrightarrow{AC}| * sin(\widehat{\overrightarrow{AB}\overrightarrow{AC}})$.

### 3.1 Properties

The first interesting property is about the equality of 2 inscribed angles which intercept the same arc and have the same orientation.

*Lemma InscirbedAngles_orient_equal:*
*forall (O A B C D: PO) (r: R), concyclic O r A B C D* $\rightarrow$ $\circlearrowleft ACD \rightarrow \circlearrowleft BCD \rightarrow \widehat{\overrightarrow{AC}\overrightarrow{AD}} = \widehat{\overrightarrow{BC}\overrightarrow{BD}}$ .

The second one relating to relative positions of points, obtained after verifying all properties given in [2, 3].

*Lemma Exists_Intersection:*
*forall (O A B C :PO), vecBetween2Vecs O A B C* $\rightarrow$

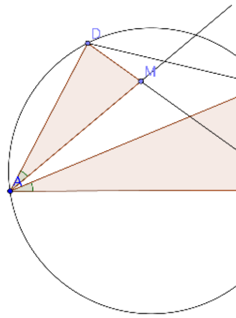


Figure 1: Demonstration of Ptolemy's theorem

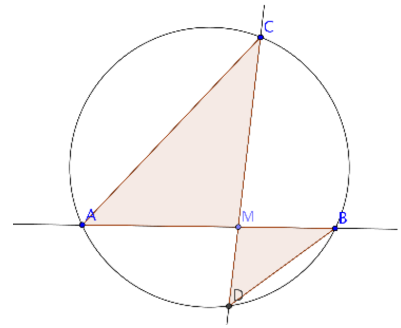


Figure 2: Product of Segments of Chords

*exists E :PO, isInSegment A B E* $\bigwedge$ *isSameSide O C E.*

Where *vecBetween2Vec O A B C* describes that $\overrightarrow{OC}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OB}$, it is defined by $\circlearrowleft OAC \bigwedge \circlearrowleft OCB \bigwedge \circlearrowleft OAB$, The notion *isSameSide* $P_0P_1P_2$ describes that $P_2$ is in the same side with $P_1$ with respect to $P_0$.

# 4. SOME APPLICATION

## 4.1 Ptolemy's Theorem

Let a convex quadrilateral ABCD be inscribed in a circle, Ptolemy's theorem is stated by $AD * BC + AB * CD = AC * BD$.

*Theorem Ptolemy: forall (O A B C D : PO) (r: R), convexQuad A B C D* $\rightarrow$ *concyclic O r A B C D* $\rightarrow$ $AD * BC + AB * CD = AC * BD$.

Where the convex quadrilateral is defined by using orientation.

*Definition convexQuad (A B C D :PO):=* $\circlearrowleft ABC \bigwedge \circlearrowleft BCD \bigwedge \circlearrowleft CDA \bigwedge \circlearrowleft DAB$

The traditional proof of this theorem is realized by locating a point M in the segment BD such that $\widehat{\overrightarrow{AB}\overrightarrow{AC}} = \widehat{\overrightarrow{AM}\overrightarrow{AD}}$ (see figure 1). With $\triangle ABC \sim \triangle AMD$, we have $BC*AD = MD*AC(1)$. With $\triangle ACD \sim \triangle ABM$, we have $AB * CD = BM * AC(2)$. The addition of (1)(2) give us $AB*CD+BC*AD = (BM+MD)*AC(3)$. By the fact that M is in the segment BC, which means $BM + MD = BD(4)$, we get the result.

The most important of this proof is (4). To have it, we need to prove the existence of M in the segment BD. By the definition of *convexQuad A B C D* , we have that $\overrightarrow{AC}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AD}$, so $\overrightarrow{AM}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AD}$. By applying the lemma *Exists_Intersection* with $\overrightarrow{AM}$, $\overrightarrow{AB}$, $\overrightarrow{AD}$, we get that M in the segment BD. (Q.e.d.)

## 4.2 Product of Segments of Chords

This theorem states that if two chords AB and CD intersect at a interior point M of a circle, so we have $MA*MB = MC * MD$.

*Theorem Chords: forall (O A B C D M :PO) (r :R), concyclic O r A B C D* $\rightarrow$*insideCircle O r M*$\rightarrow$*liesOn A B M*$\rightarrow$*liesOn C D M*$\rightarrow$ $MA * MB = MC * MD$.

Where *insideCircle O r M:= exists I J :PO, onCircle I O r* $\bigwedge$ *onCircle J O r* $\bigwedge$ *isInSegment I J M.*

The proof is simple, it uses the proportionality in similar triangles $\triangle MAC \sim \triangle MDB$.

However, to obtain this similarity by using the equality of inscribed angles, we need to have that M is interior to the both segments AB and CD. We consider the following property which is a consequence of lemma *Exists_Intersection* for the case of 4 concyclic points.

*Lemma Exists_Intersection_Concyclic: forall ( A B I J M O :PO) (r :R), concyclic O r A B I J* $\rightarrow$ *isInSegment I J M* $\rightarrow$ *liesOn A B M* $\rightarrow$ *isInSegment A B M.*

By the definition of *insideCircle O r M*, we have 2 point I, J on the circle and M lies in this segment. By applying the property *Exists_Intersection_Concyclic* with the pair IJ AB and the pair IJ CD, we get that M is interior to the both segments AB and CD. (Q.e.d.)

# 5. CONCLUSION

Developing a library to support the proof of geometric theorems in Coq for High School is useful. It helps us verify correctness of traditional proofs. The implementation in more 2000 code lines in Coq system with about 30 lemmas and theorems. New notions were formalized and their properties were verified, enough to prove 2 famous theorems which hadn't been formalized in Coq before.

A lot of work still remains to be done to verify if our notions are well formalized, specifically the notion of orientation. Moreover, there are still many notions not formalized in our library.

This work is only the first step in constructing an interactive proof system. We would like to build a system where each step in a drawing figure is translated into statement or a proof step. A proof library in Coq would be used as backend to verify steps and give us useful results or suggestions.

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