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# Continuation from a flat to a round Earth model in the coplanar orbit transfer problem

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# The coplanar orbit transfer problem

- Spherical Earth
- Central gravitational field  $g(r) = \frac{\mu}{r^2}$

## System in cylindrical coordinates

$$\dot{r}(t) = v(t) \sin \gamma(t)$$

$$\dot{\phi}(t) = \frac{v(t)}{r(t)} \cos \gamma(t)$$

$$\dot{v}(t) = -g(r(t)) \sin \gamma(t) + \frac{T_{\max}}{m(t)} u_1(t)$$

$$\dot{\gamma}(t) = \left( \frac{v(t)}{r(t)} - \frac{g(r(t))}{v(t)} \right) \cos \gamma(t) + \frac{T_{\max}}{m(t)v(t)} u_2(t)$$

$$\dot{m}(t) = -\beta T_{\max} \|u(t)\|$$

- Thrust:  $T(t) = u(t) T_{\max}$  ( $T_{\max}$  large: strong thrust)
- Control:  $u(t) = (u_1(t), u_2(t))$  satisfying  $\|u(t)\| = \sqrt{u_1(t)^2 + u_2(t)^2} \leq 1$



Denis Polson

# The coplanar orbit transfer problem

## Initial conditions

$$r(0) = r_0, \varphi(0) = \varphi_0, v(0) = v_0, \gamma(0) = \gamma_0, m(0) = m_0,$$

## Final conditions

- a point of a specified orbit:  $r(t_f) = r_f, v(t_f) = v_f, \gamma(t_f) = \gamma_f$ ,  
or
- an elliptic orbit of energy  $K_f < 0$  and eccentricity  $e_f$ :

$$\xi_{K_f} = \frac{v(t_f)^2}{2} - \frac{\mu}{r(t_f)} - K_f = 0,$$

$$\xi_{e_f} = \sin^2 \gamma + \left(1 - \frac{r(t_f)v(t_f)^2}{\mu}\right)^2 \cos^2 \gamma - e_f^2 = 0.$$

(orientation of the final orbit not prescribed:  $\varphi(t_f)$  free; in other words: argument of the final perigee free)



## Optimization criterion

$$\max m(t_f) \quad (\text{note that } t_f \text{ has to be fixed})$$

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# Application of the Pontryagin Maximum Principle

## Hamiltonian

$$H(q, p, p^0, u) = p_r v \sin \gamma + p_\varphi \frac{v}{r} \cos \gamma + p_v \left( -g(r) \sin \gamma + \frac{T_{\max}}{m} u_1 \right) \\ + p_\gamma \left( \left( \frac{v}{r} - \frac{g(r)}{v} \right) \cos \gamma + \frac{T_{\max}}{mv} u_2 \right) - p_m \beta T_{\max} \|u\|,$$

## Extremal equations

$$\dot{q}(t) = \frac{\partial H}{\partial p}(q(t), p(t), p^0, u(t)), \quad \dot{p}(t) = -\frac{\partial H}{\partial q}(q(t), p(t), p^0, u(t)),$$

## Maximization condition

$$H(q(t), p(t), p^0, u(t)) = \max_{\|w\| \leq 1} H(q(t), p(t), p^0, w)$$



# Application of the Pontryagin Maximum Principle

## Hamiltonian

$$H(q, p, p^0, u) = p_r v \sin \gamma + p_\varphi \frac{v}{r} \cos \gamma + p_v \left( -g(r) \sin \gamma + \frac{T_{\max}}{m} u_1 \right) \\ + p_\gamma \left( \left( \frac{v}{r} - \frac{g(r)}{v} \right) \cos \gamma + \frac{T_{\max}}{mv} u_2 \right) - p_m \beta T_{\max} \|u\|,$$

## Maximization condition leads to

- $u(t) = (u_1(t), u_2(t)) = (0, 0)$  whenever  $\Phi(t) < 0$
- $u_1(t) = \frac{p_v(t)}{\sqrt{p_v(t)^2 + \frac{p_\gamma(t)^2}{v(t)^2}}}$ ,  $u_2(t) = \frac{p_\gamma(t)}{v(t) \sqrt{p_v(t)^2 + \frac{p_\gamma(t)^2}{v(t)^2}}}$  whenever  $\Phi(t) > 0$

where

$$\Phi(t) = \frac{1}{m(t)} \sqrt{p_v(t)^2 + \frac{p_\gamma(t)^2}{v(t)^2}} - \beta p_m(t) \quad (\text{switching function})$$



# Application of the Pontryagin Maximum Principle

## Hamiltonian

$$H(q, p, p^0, u) = p_r v \sin \gamma + p_\varphi \frac{v}{r} \cos \gamma + p_v \left( -g(r) \sin \gamma + \frac{T_{\max}}{m} u_1 \right) + p_\gamma \left( \left( \frac{v}{r} - \frac{g(r)}{v} \right) \cos \gamma + \frac{T_{\max}}{mv} u_2 \right) - p_m \beta T_{\max} \|u\|,$$

## Transversality conditions

- case of a fixed point of a specified orbit:  $p_\varphi(t_f) = 0$ ,  $p_m(t_f) = -p^0$
- case of an orbit of given energy and eccentricity:

$$\partial_r \xi_{K_f} (p_\gamma \partial_v \xi_{E_f} - p_v \partial_\gamma \xi_{E_f}) + \partial_v \xi_{K_f} (p_r \partial_\gamma \xi_{E_f} - p_\gamma \partial_r \xi_{E_f}) = 0$$

## Remark

- $p^0 \neq 0$  (no abnormal)  $\Rightarrow p^0 = -1$
- no singular arc (Bonnard - Caillau - Faubourg - Gergaud - Haberkorn - Noailles - Trélat)

# Shooting method

Find a zero of

$$S(t_f, p_0) = \begin{pmatrix} r(t_f, p_0) - r_f \\ v(t_f, p_0) - v_f \\ \gamma(t_f, p_0) - \gamma_f \\ p_\varphi(t_f, p_0) \\ p_m(t_f, p_0) - 1 \end{pmatrix} \text{ or } \begin{pmatrix} \xi_{K_f}(p_0) \\ \xi_{e_f}(p_0) \\ *** \\ p_\varphi(t_f, p_0) \\ p_m(t_f, p_0) - 1 \end{pmatrix},$$

Main problem: how to make the shooting method converge?

- initialization of the shooting method
- discontinuities of the optimal control





# Shooting method

Main problem: how to make the shooting method converge?

- initialization of the shooting method
- discontinuities of the optimal control

Several methods:

- use first a direct method to provide a good initial guess, e.g. AMPL combined with IPOPT:



R. Fourer, D.M. Gay, B.W. Kernighan, *AMPL: A modeling language for mathematical programming*, Duxbury Press, Brooks-Cole Publishing Company (1993).



A. Wächter, L.T. Biegler *On the implementation of an interior-point iter line- search algorithm for large-scale nonlinear programming*, *Mathematical Programming* **106** (2006), 25–57.

but usual flaws of direct methods (computationally demanding, lack of numerical precision).



# Shooting method

Main problem: how to make the shooting method converge?

- initialization of the shooting method
- discontinuities of the optimal control

Several methods:

- use the impulse transfer solution to provide a good initial guess:



P. Augros, R. Delage, L. Perrot, *Computation of optimal coplanar orbit transfers*, AIAA 1999.

but valid only for nearly circular initial and final orbits. See also:



J. Gergaud, T. Haberkorn, *Orbital transfer: some links between the low-thrust and the impulse cases*, Acta Astronautica **60**, no. 6-9 (2007), 649–657.



L.W. Neustadt, *A general theory of minimum-fuel space trajectories*, SIAM Journal on Control **3**, no. 2 (1965), 317–356.



# Shooting method

Main problem: how to make the shooting method converge?

- initialization of the shooting method
- discontinuities of the optimal control

Several methods:

- multiple shooting method parameterized by the number of thrust arcs:



H. J. Oberle, K. Taubert, *Existence and multiple solutions of the minimum-fuel orbit transfer problem*, J. Optim. Theory Appl. **95** (1997), 243–262.



# Shooting method

Main problem: how to make the shooting method converge?

- initialization of the shooting method
- discontinuities of the optimal control

Several methods:

- differential or simplicial continuation method linking the minimization of the  $L^2$ -norm of the control to the minimization of the fuel consumption:



J. Gergaud, T. Haberkorn, P. Martinon, *Low thrust minimum fuel orbital transfer: an homotopic approach*, J. Guidance Cont. Dyn. **27**, 6 (2004), 1046–1060.



P. Martinon, J. Gergaud, *Using switching detection and variational equations for the shooting method*, Optimal Cont. Appl. Methods **28**, no. 2 (2007), 95–116.

but not adapted for high-thrust transfer.



# Flattening the Earth

## Observation:

Solving the optimal control problem for a flat Earth model with constant gravity is simple and algorithmically very efficient.

In view of that:

Continuation from this simple model to the initial round Earth model.



# Simplified flat Earth model

## System

$$\dot{x}(t) = v_x(t)$$

$$\dot{h}(t) = v_h(t)$$

$$\dot{v}_x(t) = \frac{T_{\max}}{m(t)} u_x(t)$$

$$\dot{v}_h(t) = \frac{T_{\max}}{m(t)} u_h(t) - g_0$$

$$\dot{m}(t) = -\beta T_{\max} \sqrt{u_x(t)^2 + u_h(t)^2}$$

max  $m(t_f)$   
 $t_f$  free

## Control

Control  $(u_x(\cdot), u_h(\cdot))$  such that  $u_x(\cdot)^2 + u_h(\cdot)^2 \leq 1$

- initial conditions:  $x(0) = x_0$ ,  $h(0) = h_0$ ,  $v_x(0) = v_{x0}$ ,  $v_h(0) = v_{h0}$ ,  $m(0) = m_0$
- final conditions:  $h(t_f) = h_f$ ,  $v_x(t_f) = v_{xf}$ ,  $v_h(t_f) = 0$

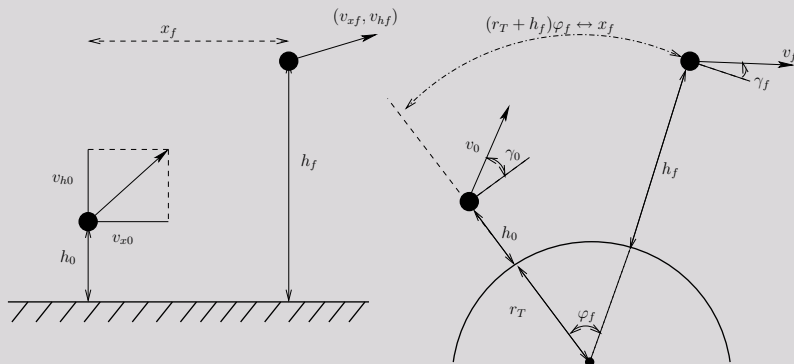


# Modified flat Earth model

Idea: mapping circular orbits to horizontal trajectories

$$\begin{cases} x = r\varphi \\ h = r - r_T \\ v_x = v \cos \gamma \\ v_h = v \sin \gamma \end{cases} \iff \begin{cases} r = r_T + h \\ \varphi = \frac{x}{r_T + h} \\ v = \sqrt{v_x^2 + v_h^2} \\ \gamma = \arctan \frac{v_h}{v_x} \end{cases}$$

$$\begin{pmatrix} u_x \\ u_h \end{pmatrix} = \begin{pmatrix} \cos \gamma & -\sin \gamma \\ \sin \gamma & \cos \gamma \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$



# Modified flat Earth model

Plugging this change of coordinates into the initial round Earth model:

$$\begin{aligned} \dot{r}(t) &= v(t) \sin \gamma(t) \\ \dot{\varphi}(t) &= \frac{v(t)}{r(t)} \cos \gamma(t) \\ \dot{v}(t) &= -g(r(t)) \sin \gamma(t) + \frac{T_{\max}}{m(t)} u_1(t) \\ \dot{\gamma}(t) &= \left( \frac{v(t)}{r(t)} - \frac{g(r(t))}{v(t)} \right) \cos \gamma(t) + \frac{T_{\max}}{m(t)v(t)} u_2(t) \\ \dot{m}(t) &= -\beta T_{\max} \|u(t)\| \end{aligned}$$

leads to...





# Modified flat Earth model

## Modified flat Earth model

$$\dot{x}(t) = v_x(t) + v_h(t) \frac{x(t)}{r_T + h(t)}$$

$$\dot{h}(t) = v_h(t)$$

$$\dot{v}_x(t) = \frac{T_{\max}}{m(t)} u_x(t) - \frac{v_x(t)v_h(t)}{r_T + h(t)}$$

$$\dot{v}_h(t) = \frac{T_{\max}}{m(t)} u_h(t) - g(r_T + h(t)) + \frac{v_x(t)^2}{r_T + h(t)}$$

$$\dot{m}(t) = -\beta T_{\max} \|u(t)\|$$

Differences with the simplified flat Earth model (with constant gravity):

- the term in green: variable (usual) gravity.
- the terms in red: "correcting terms" allowing the existence of horizontal (periodic up to translation in  $x$ ) trajectories with no thrust.



# Continuation procedure

Simplified flat Earth model (with constant gravity)  $\xrightarrow[\text{procedure}]{\text{continuation}}$  modified flat Earth model:

$$\dot{x}(t) = v_x(t) + \lambda_2 v_h(t) \frac{x(t)}{r_T + h(t)}$$

$$\dot{h}(t) = v_h(t)$$

$$\dot{v}_x(t) = \frac{T_{\max}}{m(t)} u_x(t) - \lambda_2 \frac{v_x(t) v_h(t)}{r_T + h(t)}$$

$$\dot{v}_h(t) = \frac{T_{\max}}{m(t)} u_h(t) - \frac{\mu}{(r_T + \lambda_1 h(t))^2} + \lambda_2 \frac{v_x(t)^2}{r_T + h(t)}$$

$$\dot{m}(t) = -\beta T_{\max} \sqrt{u_x(t)^2 + u_h(t)^2}$$

2 parameters:

$$0 \leq \lambda_1 \leq 1$$

$$0 \leq \lambda_2 \leq 1$$

- $\lambda_1 = \lambda_2 = 0$ : simplified flat Earth model with constant gravity
- $\lambda_1 = 1, \lambda_2 = 0$ : simplified flat Earth model with usual gravity
- $\lambda_1 = \lambda_2 = 1$ : modified flat Earth model (equivalent to usual round Earth)



# Continuation procedure

Simplified flat Earth model (with constant gravity)  $\xrightarrow[\text{procedure}]{\text{continuation}}$  modified flat Earth model:

$$\dot{x}(t) = v_x(t) + \lambda_2 v_h(t) \frac{x(t)}{r_T + h(t)}$$

$$\dot{h}(t) = v_h(t)$$

$$\dot{v}_x(t) = \frac{T_{\max}}{m(t)} u_x(t) - \lambda_2 \frac{v_x(t) v_h(t)}{r_T + h(t)}$$

$$\dot{v}_h(t) = \frac{T_{\max}}{m(t)} u_h(t) - \frac{\mu}{(r_T + \lambda_1 h(t))^2} + \lambda_2 \frac{v_x(t)^2}{r_T + h(t)}$$

$$\dot{m}(t) = -\beta T_{\max} \sqrt{u_x(t)^2 + u_h(t)^2}$$

2 parameters:

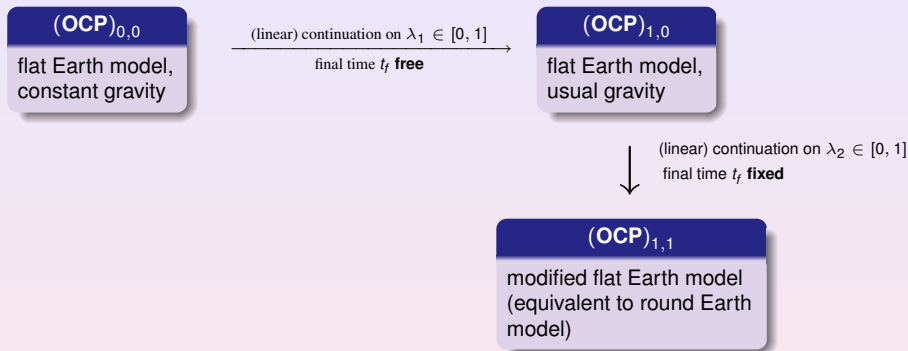
$$0 \leq \lambda_1 \leq 1$$

$$0 \leq \lambda_2 \leq 1$$

$\rightsquigarrow$  Two-parameters family of optimal control problems:  $(\text{OCP})_{\lambda_1, \lambda_2}$



# Continuation procedure



Application of the PMP to  $(\text{OCP})_{\lambda_1, \lambda_2} \Rightarrow$  series of shooting problems.



# Change of coordinates

**Remark:** Once the continuation process has converged, we obtain the initial adjoint vector for  $(\mathbf{OCP})_{1,1}$  in the modified coordinates.

To recover the adjoint vector in the usual cylindrical coordinates, we use the general fact:

## Lemma

Change of coordinates  $x_1 = \phi(x)$  and  $u_1 = \psi(u)$   
 $\Rightarrow$  dynamics  $f_1(x_1, u_1) = d\phi(x) \cdot f(\phi^{-1}(x_1), \psi^{-1}(u_1))$   
 and for the adjoint vectors:

$$p_1(\cdot) = {}^t d\phi(x(\cdot))^{-1} p(\cdot).$$

Here, this yields:

$$p_r = \frac{x}{r_T + h} p_x + p_h$$

$$p_\varphi = (r_T + h) p_x$$

$$p_v = \cos \gamma p_{v_x} + \sin \gamma p_{v_h}$$

$$p_\gamma = v(-\sin \gamma p_{v_x} + \cos \gamma p_{v_h}).$$



# Analysis of the flat Earth model

## System

$$\dot{x} = v_x$$

$$\dot{h} = v_h$$

$$\dot{v}_x = \frac{T_{\max}}{m} u_x$$

$$\dot{v}_h = \frac{T_{\max}}{m} u_h - g_0$$

$$\dot{m} = -\beta T_{\max} \sqrt{u_x^2 + u_h^2}$$

## Initial conditions

$$x(0) = x_0$$

$$h(0) = h_0$$

$$v_x(0) = v_{x0}$$

$$v_h(0) = v_{h0}$$

$$m(0) = m_0$$

## Final conditions

$$x(t_f) \text{ free}$$

$$h(t_f) = h_f$$

$$v_x(t_f) = v_{xf}$$

$$v_h(t_f) = 0$$

$$m(t_f) \text{ free}$$

$$t_f \text{ free}$$

$$\max m(t_f)$$

## Theorem

If  $h_f > h_0 + \frac{v_{h0}^2}{2g_0}$ , then the optimal trajectory is a succession of at most two arcs, and the thrust  $\|u(\cdot)\| T_{\max}$  is

- either constant on  $[0, t_f]$  and equal to  $T_{\max}$ ,
- or of the type  $T_{\max} - 0$ ,
- or of the type  $0 - T_{\max}$ .

# Analysis of the flat Earth model

Main ideas of the proof:

- Application of the PMP
- The switching function  $\Phi = \frac{1}{m} \sqrt{p_{v_x}^2 + p_{v_h}^2} - \beta p_m$  satisfies:

$$\dot{\Phi} = \frac{-p_h p_{v_h}}{m \sqrt{p_{v_x}^2 + p_{v_h}^2}}$$

$$\ddot{\Phi} = \frac{\beta \|u\|}{m} \dot{\Phi} - \frac{m}{\sqrt{p_{v_x}^2 + p_{v_h}^2}} \dot{\Phi}^2 + \frac{p_h^2}{m \sqrt{p_{v_x}^2 + p_{v_h}^2}}$$

$\Rightarrow \Phi$  has at most one minimum

$\Rightarrow$  strategies  $T_{\max}$ ,  $T_{\max} - 0$ ,  $0 - T_{\max}$ , or  $T_{\max} - 0 - T_{\max}$

- The strategy  $T_{\max} - 0 - T_{\max}$  cannot occur



# Shooting method in the flat Earth model

A priori, we have:

5 unknowns

$\rho_h, \rho_{v_x}, \rho_{v_h}(0), \rho_m(0),$  and  $t_f$

5 equations

$h(t_f) = h_f, v_x(t_f) = v_{xf}, v_h(t_f) = 0, \rho_m(t_f) = 1, H(t_f) = 0$

but using several tricks and some system analysis, the shooting method can be simplified to:

3 unknowns

$\rho_{v_x}, \rho_{v_h}(0),$  and the first switching time  $t_1$

3 equations

$h(t_f) = h_f, v_x(t_f) = v_{xf}, v_h(t_1) + g_0 t_1 = g_0 \rho_{v_h}(0)$

⇒ very easy and efficient (instantaneous) algorithm

and **the initialization of the shooting method is automatic**  
(CV for any initial adjoint vector)

⇒ automatic tool for initializing the continuation procedure





# Numerical simulations

$$T_{\max} = 180 \text{ kN}$$

$$I_{sp} = 450 \text{ s}$$

## Initial conditions

$$\varphi_0 = 0 \quad (\text{SSO})$$

$$h_0 = 200 \text{ km}$$

$$v_0 = 5.5 \text{ km/s}$$

$$\gamma_0 = 2 \text{ deg}$$

$$m_0 = 40000 \text{ kg}$$

## Final conditions

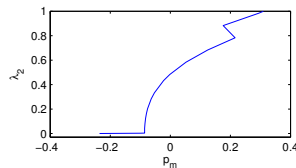
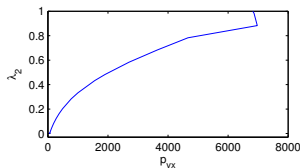
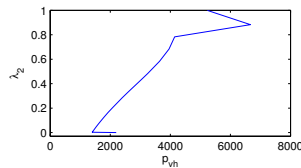
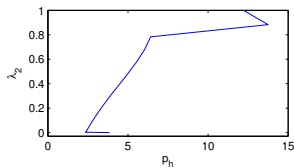
$$h_f = 800 \text{ km}$$

$$v_f = 7.5 \text{ km/s}$$

$$\gamma_f = 0 \text{ deg}$$

(nearly circular

final orbit)



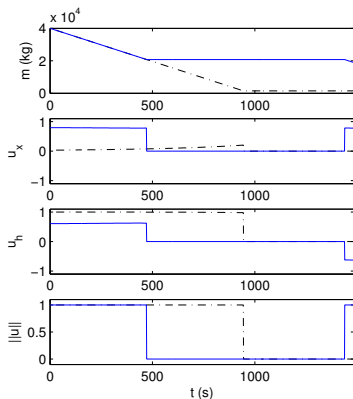
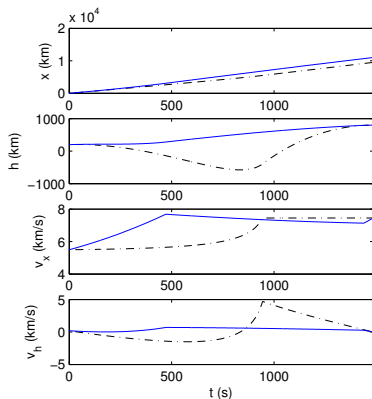
Evolution of the shooting function unknowns ( $p_h, p_{v_x}, p_{v_h}, p_m$ ) (abscissa) with respect to homotopic parameter  $\lambda_2$  (ordinate)

→ continuous but not  $C^1$  path:  $\lambda_2 \approx 0.01$ ,  $\lambda_2 \approx 0.8$ , and  $\lambda_2 \approx 0.82$ :

- $0 \leq \lambda_2 \lesssim 0.01$ :  $T_{\max} - 0$
- $0.01 \lesssim \lambda_2 \lesssim 0.8$ :  $T_{\max} - 0 - T_{\max}$
- $0.8 \lesssim \lambda_2 \lesssim 0.82$ :  $T_{\max} - 0$
- $0.82 \lesssim \lambda_2 \leq 1$ :  $T_{\max} - 0 - T_{\max}$



# Numerical simulations



Trajectory and control strategy of  $(\text{OCP})_{1,0}$  (dashed) and  $(\text{OCP})_{1,1}$  (plain).  $t_f \approx 1483\text{s}$

## Remark

In the case of a final orbit (no injecting point): additional continuation on transversality conditions.



# Numerical simulations

Comparison with a direct method:

- Heun (RK2) discretization with  $N$  points
- combination of AMPL with IPOPT
- needs however a careful initial guess

## Continuation method

3 seconds:

- $(\mathbf{OCP})_{0,0}$ : instantaneous
- from  $(\mathbf{OCP})_{0,0}$  to  $(\mathbf{OCP})_{1,0}$ : 0.5 second
- from  $(\mathbf{OCP})_{1,0}$  to  $(\mathbf{OCP})_{1,1}$ : 2.5 seconds

→ Accuracy:  $10^{-12}$

## Direct method

- $N = 100$ : 5 seconds
- $N = 1000$ : 165 seconds  
→ Accuracy:  $10^{-6}$



# Conclusion

- Algorithmic procedure to solve the problem of minimization of fuel consumption for the coplanar orbit transfer problem by shooting method approach
- Does not require any careful initial guess

## Open questions

- Is this procedure systematically efficient, for any possible coplanar orbit transfer?
- Extension to 3D

