PDE methods for multiscale control and differential games

Martino Bardi

PDE methods for multiscale control and differential games

Martino Bardi

Department of Pure and Applied Mathematics
University of Padua, Italy

Kickoff Meeting I.T.N. SADCO
ENSTA, Paris, March 3rd, 2011

Plan

- Two-scale systems: examples and motivations
- The Hamilton-Jacobi approach to Singular Perturbations
- Representation of the limit control problem:
  - Uncontrolled fast variables
  - Homogenization in low dimensions

Two-scale systems

Dynamical systems with two groups of variables evolving on different time-scales (x_s, y_τ), $\tau = s/\varepsilon$, $0 < \varepsilon << 1$, governed by ODEs

$$\dot{x}_s = f(x_s, y_s) \qquad x_s \in \mathbf{R}^n,$$
$$\dot{y}_s = \frac{1}{\varepsilon} g(x_s, y_s) \qquad y_s \in \mathbf{R}^m,$$

or by Stochastic DEs

$$dx_s = f(x_s, y_s)\, ds + \sigma(x_s, y_s)\, dW_s$$
$$dy_s = \frac{1}{\varepsilon} g(x_s, y_s)\, ds + \frac{1}{\sqrt{\varepsilon}} \nu(x_s, y_s)\, dW_s$$

Hope to simplify the model in the limit $\varepsilon \to 0$:

a Singular Perturbation problem.

Two-scale systems

Dynamical systems with two groups of variables evolving on different time-scales (x_s, y_τ), $\tau = s/\varepsilon$, $0 < \varepsilon << 1$, governed by ODEs

$$\dot{x}_s = f(x_s, y_s) \qquad x_s \in \mathbf{R}^n,$$
$$\dot{y}_s = \frac{1}{\varepsilon} g(x_s, y_s) \qquad y_s \in \mathbf{R}^m,$$

or by Stochastic DEs

$$dx_s = f(x_s, y_s)\, ds + \sigma(x_s, y_s)\, dW_s$$
$$dy_s = \frac{1}{\varepsilon} g(x_s, y_s)\, ds + \frac{1}{\sqrt{\varepsilon}} \nu(x_s, y_s)\, dW_s$$

Hope to simplify the model in the limit $\varepsilon \to 0$:

a Singular Perturbation problem.

The theory is classical for Ordinary Differential Equations, see Levinson and Tychonov 1952, O'Malley's book 1974, and has a large literature also for systems with controls,

$$\dot{x}_s = f(x_s, y_s, \alpha_s)$$
$$\dot{y}_s = \frac{1}{\varepsilon} g(x_s, y_s, \alpha_s),$$

α_s a measurable control function taking values in a given set A, see Kokotović - Khalil - O'Reilly book 1986 (deterministic case)
Bensoussan 1988, Kushner 1990 also stochastic case:

$$dx_s = f(x_s, y_s, \alpha_s)\, ds + \sigma(x_s, y_s, \alpha_s)\, dW_s$$
$$dy_s = \frac{1}{\varepsilon} g(x_s, y_s, \alpha_s)\, ds + \frac{1}{\sqrt{\varepsilon}} \nu(x_s, y_s, \alpha_s)\, dW_s$$

Main motivation: reducing the dimension of the state space.

There are many different models in Physics, Engineering, Finance,....

The theory is classical for Ordinary Differential Equations, see Levinson and Tychonov 1952, O'Malley's book 1974, and has a large literature also for systems with controls,

$$\dot{x}_s = f(x_s, y_s, \alpha_s)$$
$$\dot{y}_s = \frac{1}{\varepsilon} g(x_s, y_s, \alpha_s),$$

α_s a measurable control function taking values in a given set A, see Kokotović - Khalil - O'Reilly book 1986 (deterministic case)
Bensoussan 1988, Kushner 1990 also stochastic case:

$$dx_s = f(x_s, y_s, \alpha_s)\, ds + \sigma(x_s, y_s, \alpha_s)\, dW_s$$
$$dy_s = \frac{1}{\varepsilon} g(x_s, y_s, \alpha_s)\, ds + \frac{1}{\sqrt{\varepsilon}} \nu(x_s, y_s, \alpha_s)\, dW_s$$

Main motivation: reducing the dimension of the state space.

There are many different models in Physics, Engineering, Finance,....

Example 1: Mechanical system with Large Damping

The large time behavior of

$$\ddot{X} = F(X,t) - \frac{\dot{X}}{\varepsilon}$$

is described by $x(s) := X(s/\varepsilon)$ that solves

$$\varepsilon^2 \ddot{x} = F(x, s/\varepsilon) - \dot{x}.$$

In the autonomous case ($F = F(x)$) this writes

$$\dot{x}_s = y_s$$

$$\dot{y}_s = \frac{F(x_s) - y_s}{\varepsilon^2},$$

The limit is the Quasi-Static approximation

$$\dot{x} = F(x).$$

F. Hoppensteadt, "Quasi-Static state analysis...", Courant L. N. 2010

Example 1: Mechanical system with Large Damping

The large time behavior of

$$\ddot{X} = F(X,t) - \frac{\dot{X}}{\varepsilon}$$

is described by $x(s) := X(s/\varepsilon)$ that solves

$$\varepsilon^2 \ddot{x} = F(x, s/\varepsilon) - \dot{x}.$$

In the autonomous case ($F = F(x)$) this writes

$$\dot{x}_s = y_s$$
$$\dot{y}_s = \frac{F(x_s) - y_s}{\varepsilon^2},$$

The limit is the Quasi-Static approximation

$$\dot{x} = F(x).$$

F. Hoppensteadt, "Quasi-Static state analysis...", Courant L. N. 2010

Example 2: Control systems with stable fast variables

In Example 1 the formal limit

$$\dot{x}_s = f(x_s, y_s), \qquad 0 = g(x_s, y_s)$$

is correct: it fits in the Reduced Order Method. For control system the ROM gives in the limit the DIFFERENTIAL-ALGEBRAIC system

$$\dot{x}_s = f(x_s, y_s, \alpha_s), \qquad 0 = g(x_s, y_s, \alpha_s),$$

provided that (roughly speaking) the "fast subsystem" (with frozen x)

$$\dot{y}_\tau = g(x, y_\tau, \alpha_\tau)$$

has an equilibrium regime with an asymptotically stabilizing feedback, see Kokotović - Khalil - O'Reilly book 1986.
Examples in Engineering: high gain feedback, cheap control,....
BUT, many other models do NOT have this stability property!

Example 2: Control systems with stable fast variables

In Example 1 the formal limit

$$\dot{x}_s = f(x_s, y_s), \qquad 0 = g(x_s, y_s)$$

is correct: it fits in the Reduced Order Method. For control system the ROM gives in the limit the DIFFERENTIAL-ALGEBRAIC system

$$\dot{x}_s = f(x_s, y_s, \alpha_s), \qquad 0 = g(x_s, y_s, \alpha_s),$$

provided that (roughly speaking) the "fast subsystem" (with frozen x)

$$\dot{y}_\tau = g(x, y_\tau, \alpha_\tau)$$

has an equilibrium regime with an asymptotically stabilizing feedback, see Kokotović - Khalil - O'Reilly book 1986.
Examples in Engineering: high gain feedback, cheap control,....
BUT, many other models do NOT have this stability property!

Example 3: Control systems in oscillating media

The homogenization problem

$$\dot{x}_s = f\left(x_s, \frac{x_s}{\varepsilon}, \alpha_s\right), \quad J = \int_0^t l\left(x_s, \frac{x_s}{\varepsilon}, \alpha_s\right) ds + h\left(x_t, \frac{x_t}{\varepsilon}\right)$$

by setting $y_s = x_s/\varepsilon$ can be written as

$$\dot{x}_s = f(x_s, y_s, \alpha_s)$$
$$\dot{y}_s = \frac{1}{\varepsilon} f(x_s, y_s, \alpha_s)$$
$$J = \int_0^t l\,(x_s, y_s, \alpha_s)\, ds + h\,(x_t, y_t)$$

that is a Singular Perturbation problem with $g = f$.

Example 4: Financial models

The evolution of a stock S with stochastic volatility σ is

$$d \log S_s = \gamma\, ds + \sigma(y_s)\, dW_s$$

$$dy_s = \frac{1}{\varepsilon}(m - y_s) + \frac{\nu}{\sqrt{\varepsilon}} d\tilde{W}_s$$

see Fouque - Papanicolaou - Sircar, book 2000, for empirical data and many examples.

Merton portfolio optimization problem with stochastic volatility: invest β_s in the stock S_s, $1 - \beta_s$ in a bond with interest rate r. Then the wealth x_s evolves as

$$dx_s = (r + (\gamma - r)\beta_s)x_s\, ds + x_s \beta_s\, \sigma(y_s)\, dW_s$$

$$dy_s = \frac{1}{\varepsilon}(m - y_s)\, ds + \frac{\nu}{\sqrt{\varepsilon}} d\tilde{W}_s$$

Problem: maximize the expected utility at time t, $E[h(x_t)]$, h increasing concave function.

Example 5: Control systems on thin structures

State constraint on the z variables

$$\dot{x}_s = f(x_s, z_s, \alpha_s)$$
$$\dot{z}_s = g(x_s, z_s, \alpha_s) \qquad |z_s| \le \varepsilon,$$

by setting $y_s = z_s/\varepsilon$ becomes

$$\dot{x}_s = f(x_s, \varepsilon y_s, \alpha_s)$$
$$\dot{y}_s = \frac{1}{\varepsilon} g(x_s, \varepsilon y_s, \alpha_s) \qquad |y_s| \le 1.$$

This can be used to justify models of optimal control on graphs or networks.

The H-J approach to Singular Perturbations

General control system with TWO controllers

$$dx_s = f(x_s, y_s, \alpha_s, \beta_s)\,ds + \sigma(x_s, y_s, \alpha_s, \beta_s)\,dW_s, \; x_s \in \mathbf{R}^n, \; \alpha_s \in A, \; \beta_s \in B,$$

$$dy_s = \frac{1}{\varepsilon} g(x_s, y_s, \alpha_s, \beta_s)\,ds + \frac{1}{\sqrt{\varepsilon}} \nu(x_s, y_s, \alpha_s, \beta_s)\,dW_s, \qquad y_s \in \mathbf{R}^m,$$

$$x_0 = x, \quad y_0 = y$$

Cost-payoff functional (α minimizes, β_s maximizes)

$$J^\varepsilon(t, x, y, \alpha, \beta) := \int_0^t l(x_s, y_s, \alpha_s, \beta_s)\,ds + h(x_t, y_t)$$

Lower value function, where $\Gamma(t)$ are the nonanticipating strategies, $\mathcal{B}(t)$ the open-loop controls

$$u^\varepsilon(t, x, y) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} E\left[J^\varepsilon(t, x, y, \alpha[\beta], \beta)\right]$$

The H-J approach to Singular Perturbations

General control system with TWO controllers

$$dx_s = f(x_s, y_s, \alpha_s, \beta_s)\, ds + \sigma(x_s, y_s, \alpha_s, \beta_s)\, dW_s, \; x_s \in \mathbf{R}^n, \; \alpha_s \in A, \; \beta_s \in B,$$

$$dy_s = \frac{1}{\varepsilon} g(x_s, y_s, \alpha_s, \beta_s)\, ds + \frac{1}{\sqrt{\varepsilon}} \nu(x_s, y_s, \alpha_s, \beta_s)\, dW_s, \qquad y_s \in \mathbf{R}^m,$$

$$x_0 = x, \quad y_0 = y$$

Cost-payoff functional (α minimizes, β_s maximizes)

$$J^\varepsilon(t, x, y, \alpha, \beta) := \int_0^t l(x_s, y_s, \alpha_s, \beta_s)\, ds + h(x_t, y_t)$$

Lower value function, where $\Gamma(t)$ are the nonanticipating strategies, $\mathcal{B}(t)$ the open-loop controls

$$u^\varepsilon(t, x, y) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} E\left[J^\varepsilon(t, x, y, \alpha[\beta], \beta)\right]$$

The H-J approach to Singular Perturbations

General control system with TWO controllers

$$dx_s = f(x_s, y_s, \alpha_s, \beta_s)\, ds + \sigma(x_s, y_s, \alpha_s, \beta_s)\, dW_s, \ x_s \in \mathbf{R}^n, \ \alpha_s \in A, \ \beta_s \in B,$$

$$dy_s = \frac{1}{\varepsilon} g(x_s, y_s, \alpha_s, \beta_s)\, ds + \frac{1}{\sqrt{\varepsilon}} \nu(x_s, y_s, \alpha_s, \beta_s)\, dW_s, \qquad y_s \in \mathbf{R}^m,$$

$$x_0 = x, \quad y_0 = y$$

Cost-payoff functional (α minimizes, β_s maximizes)

$$J^\varepsilon(t, x, y, \alpha, \beta) := \int_0^t l(x_s, y_s, \alpha_s, \beta_s)\, ds + h(x_t, y_t)$$

Lower value function, where $\Gamma(t)$ are the nonanticipating strategies, $\mathcal{B}(t)$ the open-loop controls

$$u^\varepsilon(t, x, y) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} E\left[J^\varepsilon(t, x, y, \alpha[\beta], \beta)\right]$$

H-J-Bellman-Isaacs equation for the SP problem

Dynamic Programming method:
in the deterministic case $\sigma, \nu \equiv 0$, the value function solves

$$
(\mathrm{CP}_\varepsilon) \qquad \begin{cases} \frac{\partial u^\varepsilon}{\partial t} + H\left(x, y, D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}\right) = 0 & \text{in } (0, +\infty) \times \mathbf{R}^n \times \mathbf{R}^m, \\ u^\varepsilon(0, x, y) = h(x, y) & \text{in } \mathbf{R}^n \times \mathbf{R}^m, \end{cases}
$$

$$
H(x, y, p, q) = \min_{\beta \in B} \max_{\alpha \in A} \{-p \cdot f(x, y, \alpha, \beta) - q \cdot g(x, y, \alpha, \beta) - l(x, y, \alpha, \beta)\}
$$

in the viscosity sense.

In the stochastic case $\sigma \neq 0$ or $\nu \neq 0$ the H-J-B-I equation is

$$\frac{\partial u^\varepsilon}{\partial t} + \min_{\beta \in B} \max_{\alpha \in A} \left[\mathcal{L}^\varepsilon_{\alpha,\beta} u^\varepsilon - l(x, y, \alpha, \beta) \right] = 0 \quad \text{in } (0, +\infty) \times \mathbf{R}^n \times \mathbf{R}^m,$$

$\mathcal{L}^\varepsilon_{\alpha,\beta}$ = infinitesimal generator of the process with constant controls α, β

it is of 2*nd* order involving also D^2_{xx}, D^2_{yy}/ε, $D^2_{xy}/\sqrt{\varepsilon}$.

This H-J-B-I PDE is (degenerate) parabolic.

Again, u^ε is the unique viscosity solution of the Cauchy problem.

In the stochastic case $\sigma \neq 0$ or $\nu \neq 0$ the H-J-B-I equation is

$$\frac{\partial u^\varepsilon}{\partial t} + \min_{\beta \in B} \max_{\alpha \in A} \left[\mathcal{L}^\varepsilon_{\alpha,\beta} u^\varepsilon - l(x, y, \alpha, \beta) \right] = 0 \quad \text{in } (0, +\infty) \times \mathbf{R}^n \times \mathbf{R}^m,$$

$\mathcal{L}^\varepsilon_{\alpha,\beta}$ = infinitesimal generator of the process with constant controls α, β

it is of 2*nd* order involving also $D^2_{xx}, D^2_{yy}/\varepsilon, D^2_{xy}/\sqrt{\varepsilon}$.

This H-J-B-I PDE is (degenerate) parabolic.

Again, u^ε is the unique viscosity solution of the Cauchy problem.

Plan of the method:

1. pass to the limit as $\varepsilon \to 0$ in the PDE
2. associate to the limit PDE a "limit control problem"

1. was developed in the papers

O. ALVAREZ - M. B. : SIAM J. Cont. '01, Arch. Rat. Mech. Anal. '03, Memoir A.M.S. '10;

O. A. - M. B. - C. MARCHI : J. D. E. '07, '08 (more than 2 scales)

under boundedness assumptions on the fast state variables,
and for unbounded but uncontrolled fast variables in

M. B. - A. CESARONI - L. MANCA : SIAM J. Financial Math. 2010

M. B. - A. CESARONI : European J. Control 2011

Methods are related to HOMOGENIZATION of H- J equations:

P.L.Lions- Papanicolaou- Varadhan 1986, L.C.Evans 1989, ...

Plan of the method:

1. pass to the limit as $\varepsilon \to 0$ in the PDE
2. associate to the limit PDE a "limit control problem"

1. was developed in the papers

O. ALVAREZ - M. B. : SIAM J. Cont. ’01, Arch. Rat. Mech. Anal. ’03, Memoir A.M.S. ’10;

O. A. - M. B. - C. MARCHI : J. D. E. ’07, ’08 (more than 2 scales)

under boundedness assumptions on the fast state variables,

and for unbounded but uncontrolled fast variables in

M. B. - A. CESARONI - L. MANCA : SIAM J. Financial Math. 2010

M. B. - A. CESARONI : European J. Control 2011

Methods are related to HOMOGENIZATION of H- J equations:

P.L.Lions- Papanicolaou- Varadhan 1986, L.C.Evans 1989, ...

Plan of the method:

1. pass to the limit as $\varepsilon \to 0$ in the PDE
2. associate to the limit PDE a "limit control problem"

1. was developed in the papers

O. ALVAREZ - M. B. : SIAM J. Cont. ’01, Arch. Rat. Mech. Anal. ’03, Memoir A.M.S. ’10;

O. A. - M. B. - C. MARCHI : J. D. E. ’07, ’08 (more than 2 scales)

under boundedness assumptions on the fast state variables, and for unbounded but uncontrolled fast variables in

M. B. - A. CESARONI - L. MANCA : SIAM J. Financial Math. 2010

M. B. - A. CESARONI : European J. Control 2011

Methods are related to HOMOGENIZATION of H- J equations:

P.L.Lions- Papanicolaou- Varadhan 1986, L.C.Evans 1989, ...

1. Search *effective Hamiltonian* $\overline{H}$ and effective initial data $\overline{h}$ s. t.

$$u^\varepsilon(t,x,y) \to u(t,x) \quad \text{as } \varepsilon \to 0,$$

u solution of

$$(\overline{\mathrm{CP}}) \qquad \begin{cases} \frac{\partial u}{\partial t} + \overline{H}\left(x, D_x u, D^2_{xx} u\right) = 0 & \text{in} \quad (0,+\infty)\times \mathbf{R}^n, \\ u(0,x) = \overline{h}(x) & \text{in } \mathbf{R}^n \end{cases}$$

2. Intepret the effective Hamiltonian $\overline{H}$ as the Bellman-Isaacs Hamiltonian for a new *effective system*

$$\dot{x}_s = \overline{f}(x_s,\eta_s,\theta_s) \qquad x_s \in \mathbf{R}^n,\ \eta_s \in E(x_s),\ \theta_s \in \Theta(x_s)$$

and effective cost functional

$$\overline{J}(t,x,\eta,\theta) := \int_0^t \overline{l}(x_s,\eta_s,\theta_s)\,ds + \overline{h}(x_t).$$

This is a variational limit of the initial $n+m$-dimensional problem.
Step 2 is largely OPEN !

1. Search *effective Hamiltonian* $\overline{H}$ and effective initial data $\overline{h}$ s. t.

$$u^\varepsilon(t,x,y) \to u(t,x) \quad \text{as } \varepsilon \to 0,$$

u solution of

$$(\overline{\text{CP}}) \qquad \begin{cases} \frac{\partial u}{\partial t} + \overline{H}\left(x, D_x u, D^2_{xx} u\right) = 0 & \text{in} \quad (0,+\infty) \times \mathbf{R}^n, \\ u(0,x) = \overline{h}(x) & \text{in } \mathbf{R}^n \end{cases}$$

2. Intepret the effective Hamiltonian $\overline{H}$ as the Bellman-Isaacs Hamiltonian for a new *effective system*

$$\dot{x}_s = \overline{f}(x_s, \eta_s, \theta_s) \qquad x_s \in \mathbf{R}^n,\ \eta_s \in E(x_s),\ \theta_s \in \Theta(x_s)$$

and effective cost functional

$$\overline{J}(t,x,\eta,\theta) := \int_0^t \overline{l}(x_s, \eta_s, \theta_s)\, ds + \overline{h}(x_t).$$

This is a variational limit of the initial $n+m$-dimensional problem.
Step 2 is largely OPEN !

1. Search *effective Hamiltonian* $\overline{H}$ and effective initial data $\overline{h}$ s. t.

$$u^\varepsilon(t,x,y) \to u(t,x) \quad \text{as } \varepsilon \to 0,$$

u solution of

$$(\overline{\text{CP}}) \qquad \begin{cases} \frac{\partial u}{\partial t} + \overline{H}\left(x, D_x u, D^2_{xx} u\right) = 0 & \text{in} \quad (0,+\infty)\times \mathbf{R}^n, \\ u(0,x) = \overline{h}(x) & \text{in } \mathbf{R}^n \end{cases}$$

2. Intepret the effective Hamiltonian $\overline{H}$ as the Bellman-Isaacs Hamiltonian for a new *effective system*

$$\dot{x}_s = \overline{f}(x_s, \eta_s, \theta_s) \qquad x_s \in \mathbf{R}^n,\ \eta_s \in E(x_s),\ \theta_s \in \Theta(x_s)$$

and effective cost functional

$$\overline{J}(t,x,\eta,\theta) := \int_0^t \overline{l}(x_s, \eta_s, \theta_s)\, ds + \overline{h}(x_t).$$

This is a variational limit of the initial $n+m$-dimensional problem.
Step 2 is largely OPEN !

Definition of $\overline{H}$: deterministic case $\sigma \equiv 0, \nu \equiv 0$

Consider the fast subsystem with frozen x and $\varepsilon = 1$

$$\text{(FS)} \qquad \dot{y}_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau), \qquad y_0 = y$$

and the family of value functions in $\mathbf{R}^m$ with parameters $x, p \in \mathbf{R}^n$

$$w(t, y; x, p) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} \int_0^t L(y_\tau, \alpha[\beta]_\tau, \beta_\tau; x, p)\, d\tau,$$

$$L(y, \alpha, \beta; x, p) := p \cdot f(x, y, \alpha, \beta) + l(x, y, \alpha, \beta)$$

Say (FS) is ERGODIC if, for all x, p,

$$\lim_{t \to +\infty} \frac{w(t, y; x, p)}{t} = \text{constant (in } y\text{), uniformly in } y$$

$$=: -\overline{H}(x, p)$$

Definition of $\overline{H}$: deterministic case $\sigma \equiv 0, \nu \equiv 0$

Consider the fast subsystem with frozen x and $\varepsilon = 1$

(FS) $$\dot{y}_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau), \qquad y_0 = y$$

and the family of value functions in $\mathbf{R}^m$ with parameters $x, p \in \mathbf{R}^n$

$$w(t, y; x, p) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} \int_0^t L(y_\tau, \alpha[\beta]_\tau, \beta_\tau; x, p)\, d\tau,$$

$$L(y, \alpha, \beta; x, p) := p \cdot f(x, y, \alpha, \beta) + l(x, y, \alpha, \beta)$$

Say (FS) is ERGODIC if, for all x, p,

$$\lim_{t \to +\infty} \frac{w(t, y; x, p)}{t} = \text{constant (in } y\text{), uniformly in } y$$

$$=: -\overline{H}(x, p)$$

Definition of $\overline{H}$: deterministic case $\sigma \equiv 0, \nu \equiv 0$

Consider the fast subsystem with frozen x and $\varepsilon = 1$

(FS) $$\dot{y}_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau), \qquad y_0 = y$$

and the family of value functions in $\mathbf{R}^m$ with parameters $x, p \in \mathbf{R}^n$

$$w(t, y; x, p) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} \int_0^t L(y_\tau, \alpha[\beta]_\tau, \beta_\tau; x, p)\, d\tau,$$

$$L(y, \alpha, \beta; x, p) := p \cdot f(x, y, \alpha, \beta) + l(x, y, \alpha, \beta)$$

Say (FS) is ERGODIC if, for all x, p,

$$\lim_{t \to +\infty} \frac{w(t, y; x, p)}{t} = \text{constant (in } y\text{), uniformly in } y$$

$$=: -\overline{H}(x, p)$$

Definition of $\overline{H}$: deterministic case $\sigma \equiv 0, \nu \equiv 0$

Consider the fast subsystem with frozen x and $\varepsilon = 1$

(FS) $$\dot{y}_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau), \qquad y_0 = y$$

and the family of value functions in $\mathbf{R}^m$ with parameters $x, p \in \mathbf{R}^n$

$$w(t, y; x, p) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} \int_0^t L(y_\tau, \alpha[\beta]_\tau, \beta_\tau; x, p)\, d\tau,$$

$$L(y, \alpha, \beta; x, p) := p \cdot f(x, y, \alpha, \beta) + l(x, y, \alpha, \beta)$$

Say (FS) is ERGODIC if, for all x, p,

$$\lim_{t \to +\infty} \frac{w(t, y; x, p)}{t} = \text{constant (in } y\text{), uniformly in } y$$

$$=: -\overline{H}(x, p)$$

- Example: (FS) uncontrolled, i.e.

$$\dot{y}_\tau = g(x, y_\tau),$$

and ergodic in the classical sense with a UNIQUE INVARIANT MEASURE μ_x. Then

$$-\overline{H}(x,p) = \int_{\mathbf{R}^m} \min_{\alpha \in A} L(y, \alpha; x, p)\, d\mu_x(y).$$

- NO explicit formula for $\overline{H}$ in general!
- Definition of $\overline{H}$ in general non-deterministic case: fast subsystem

$$dy_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau)\, d\tau + \nu(x, y_\tau, \alpha_\tau, \beta_\tau)\, dW_\tau, \qquad y_0 = y,$$

$L = \text{trace}\left(M\sigma\sigma^T(x, y, \alpha, \beta)\right)/2 + \ldots$ (M a $n \times n$ symmetric matrix)

$$w(t, y; x, p, M) = \inf_\alpha \sup_\beta E\left[\int_0^t L d\tau\right], \quad \lim_{t \to +\infty} w/t =: -\overline{H}(x, p, M)$$

- Example: (FS) uncontrolled, i.e.

$$\dot{y}_\tau = g(x, y_\tau),$$

and ergodic in the classical sense with a UNIQUE INVARIANT MEASURE μ_x. Then

$$-\overline{H}(x,p) = \int_{\mathbf{R}^m} \min_{\alpha \in A} L(y, \alpha; x, p)\, d\mu_x(y).$$

- NO explicit formula for $\overline{H}$ in general!

- Definition of $\overline{H}$ in general non-deterministic case: fast subsystem

$$dy_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau)\, d\tau + \nu(x, y_\tau, \alpha_\tau, \beta_\tau)\, dW_\tau, \qquad y_0 = y,$$

$$L = \text{trace}\left(M\sigma\sigma^T(x, y, \alpha, \beta)\right)/2 + \ldots \ (M \text{ a } n \times n \text{ symmetric matrix})$$

$$w(t, y; x, p, M) = \inf_\alpha \sup_\beta E\left[\int_0^t L\, d\tau\right], \quad \lim_{t\to+\infty} w/t =: -\overline{H}(x, p, M)$$

- Example: (FS) uncontrolled, i.e.

$$\dot{y}_\tau = g(x, y_\tau),$$

and ergodic in the classical sense with a UNIQUE INVARIANT MEASURE μ_x. Then

$$-\overline{H}(x,p) = \int_{\mathbf{R}^m} \min_{\alpha \in A} L(y, \alpha; x, p)\, d\mu_x(y).$$

- NO explicit formula for $\overline{H}$ in general!

- Definition of $\overline{H}$ in general non-deterministic case: fast subsystem

$$dy_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau)\, d\tau + \nu(x, y_\tau, \alpha_\tau, \beta_\tau)\, dW_\tau, \qquad y_0 = y,$$

$L = \text{trace}\left(M\sigma\sigma^T(x, y, \alpha, \beta)\right)/2 + \ldots$ (M a $n \times n$ symmetric matrix)

$$w(t, y; x, p, M) = \inf_\alpha \sup_\beta E\left[\int_0^t L d\tau\right], \quad \lim_{t \to +\infty} w/t =: -\overline{H}(x, p, M)$$

Weak convergence theorem

Meta-Theorem

Fast subsystem (FS) ergodic $\implies$

$$u^\varepsilon(t,x,y) \to u(t,x) \quad \text{as } \varepsilon \to 0,$$

(in the sense of relaxed semilimits, or weak viscosity limits),
and u solves (in viscosity sense)

$$\frac{\partial u}{\partial t} + \overline{H}\left(x, D_x u, D^2_{xx} u\right) = 0$$

This is in fact a Theorem if the fast variables y live on the torus $\mathbb{T}^M$ (i.e., all data are $\mathbb{Z}^m$- periodic in y), or in all $\mathbf{R}^M$ but the process

$$dy_\tau = g(x, y_\tau)\, d\tau + \nu(x, y_\tau)\, dW_\tau$$

is an UN-controlled NON-degenerate diffusion.

The effective initial data $\overline{h}$

For the Fast Subsystem with frozen x

(FS) $\quad dy_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau)\, d\tau + \nu(x, y_\tau, \alpha_\tau, \beta_\tau)\, dW_\tau, \qquad y_0 = y,$

consider now the value function

$$v(t, y; x) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} E\,[h(x, y_t)]$$

Definition: (FS) is STABILIZING (to a constant) for the cost h if, $\forall\, x$,

$$\lim_{t \to +\infty} v(t, y; x) = \text{constant (in } y\text{), uniformly in } y \;\; =: \overline{h}(x)$$

Convergence Theorem at $t = 0$

Fast subsystem (FS) stabilizing for the cost h $\quad \Longrightarrow$

$$\lim_{t \to 0} \lim_{\varepsilon \to 0} u^\varepsilon(t, x, y) = \overline{h}(x)$$

(in the sense of relaxed semilimits, or weak viscosity limits).

The effective initial data $\overline{h}$

For the Fast Subsystem with frozen x

(FS) $$dy_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau)\, d\tau + \nu(x, y_\tau, \alpha_\tau, \beta_\tau)\, dW_\tau, \qquad y_0 = y,$$

consider now the value function

$$v(t, y; x) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} E\left[h(x, y_t)\right]$$

Definition: (FS) is STABILIZING (to a constant) for the cost h if, $\forall\, x$,

$$\lim_{t \to +\infty} v(t, y; x) = \text{constant (in } y\text{), uniformly in } y \ =: \overline{h}(x)$$

Convergence Theorem at $t = 0$

Fast subsystem (FS) stabilizing for the cost h $\implies$

$$\lim_{t \to 0} \lim_{\varepsilon \to 0} u^\varepsilon(t, x, y) = \overline{h}(x)$$

(in the sense of relaxed semilimits, or weak viscosity limits).

The effective initial data $\overline{h}$

For the Fast Subsystem with frozen x

(FS) $\quad dy_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau)\, d\tau + \nu(x, y_\tau, \alpha_\tau, \beta_\tau)\, dW_\tau, \qquad y_0 = y,$

consider now the value function

$$v(t, y; x) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} E\left[h(x, y_t)\right]$$

Definition: (FS) is STABILIZING (to a constant) for the cost h if, $\forall\, x$,

$$\lim_{t \to +\infty} v(t, y; x) = \text{constant (in } y\text{), uniformly in } y \;\; =: \overline{h}(x)$$

Convergence Theorem at $t = 0$

Fast subsystem (FS) stabilizing for the cost $h \quad \Longrightarrow$

$$\lim_{t \to 0} \lim_{\varepsilon \to 0} u^\varepsilon(t, x, y) = \overline{h}(x)$$

(in the sense of relaxed semilimits, or weak viscosity limits).

Main convergence theorem

Corollary

Fast subsystem (FS) ERGODIC and STABILIZING $\implies$

$\exists\ \overline{H}$ and $\overline{h}$ continuous, $\overline{H}$ degenerate elliptic, such that

$u^\varepsilon(t,x,y) \to u(t,x)$ as $\varepsilon \to 0$, u solution of

$$(\overline{CP}) \qquad \begin{cases} \frac{\partial u}{\partial t} + \overline{H}\left(x, D_x u, D^2_{xx} u\right) = 0 & \text{in} \quad (0,+\infty) \times \mathbf{R}^n, \\ u(0,x) = \overline{h}(x) & \text{in } \mathbf{R}^n. \end{cases}$$

If moreover, $\overline{H}$ is regular enough w.r.t. x, then $(\overline{CP})$ has a unique solution and

$$u^\varepsilon \to u \qquad \text{locally uniformly.}$$

Conclusion

The initial $(n+m)$-dimensional H-J-B-I equation is split into

- two m-dimensional ergodic-type problems (one for $\overline{H}$ and one for $\overline{h}$),
- a n-dimensional "effective" PDE

$\Longrightarrow$ we got the desired SEPARATION OF SCALES for the H-J-B-I equation.

Conclusion

The initial $(n+m)$-dimensional H-J-B-I equation is split into

- two m-dimensional ergodic-type problems
  (one for $\overline{H}$ and one for $\overline{h}$),
- a n-dimensional "effective" PDE

$\Longrightarrow$ we got the desired SEPARATION OF SCALES for the H-J-B-I equation.

The next steps

Main remaining questions:

1. when is (FS) ergodic and stabilizing ?
2. can we find effective dynamics $\bar{f}, \overline{\sigma}$, running cost $\bar{l}$, and control constraints E, Θ associated to $\overline{H}$?

Some answers:

1. for bounded fast variables (related to homogenization):
   - uniformly nondegenerate fast subsystem (FS),
   - deterministic fast subsystem (FS) controllable by one player from each point to any other point in a uniformly bounded time,
   - under nonresonance conditions on the torus;

   for unbounded fast variables: uncontrolled diffusion processes with a unique invariant measure;
2. several examples but no general recipe.

The next steps

Main remaining questions:

1. when is (FS) ergodic and stabilizing ?
2. can we find effective dynamics $\bar{f}, \overline{\sigma}$, running cost $\bar{l}$, and control constraints E, Θ associated to $\overline{H}$?

Some answers:

1. for bounded fast variables (related to homogenization):
   - uniformly nondegenerate fast subsystem (FS),
   - deterministic fast subsystem (FS) controllable by one player from each point to any other point in a uniformly bounded time,
   - under nonresonance conditions on the torus;

   for unbounded fast variables: uncontrolled diffusion processes with a unique invariant measure;
2. several examples but no general recipe.

The next steps

Main remaining questions:

1. when is (FS) ergodic and stabilizing ?
2. can we find effective dynamics $\overline{f}, \overline{\sigma}$, running cost $\overline{l}$, and control constraints E, Θ associated to $\overline{H}$?

Some answers:

1. for bounded fast variables (related to homogenization):
   - uniformly nondegenerate fast subsystem (FS),
   - deterministic fast subsystem (FS) controllable by one player from each point to any other point in a uniformly bounded time,
   - under nonresonance conditions on the torus;

   for unbounded fast variables: uncontrolled diffusion processes with a unique invariant measure;
2. several examples but no general recipe.

The next steps

Main remaining questions:

1. when is (FS) ergodic and stabilizing ?
2. can we find effective dynamics $\overline{f}, \overline{\sigma}$, running cost $\overline{l}$, and control constraints E, Θ associated to $\overline{H}$?

Some answers:

1. for bounded fast variables (related to homogenization):
   - uniformly nondegenerate fast subsystem (FS),
   - deterministic fast subsystem (FS) controllable by one player from each point to any other point in a uniformly bounded time,
   - under nonresonance conditions on the torus;

   for unbounded fast variables: uncontrolled diffusion processes with a unique invariant measure;
2. several examples but no general recipe.

The case of uncontrolled fast variables

$$dx_s = f(x_s, y_s, \alpha_s, \beta_s)\, ds + \sigma(x_s, y_s) dW_s$$

$$dy_s = \frac{1}{\varepsilon} g(x_s, y_s)\, ds + \frac{1}{\sqrt{\varepsilon}} \nu(x_s, y_s)\, dW_s,$$

$$J^\varepsilon(t, x, y, \alpha, \beta) := \int_0^t l(x_s, y_s, \alpha_s, \beta_s)\, ds + h(x_t, y_t)$$

Assume there exists a unique invariant measure μ_x of

(FS) $$dy_\tau = g(x, y_\tau)\, d\tau + \nu(x, y_\tau)\, dW_\tau,$$

Denote $\langle\phi\rangle(x) := \int \phi(x, y) d\mu_x(y)$. Then effective H and h are

$$\overline{h}(x) = \langle h \rangle(x)$$

$$\overline{H}(x, p, M) = \langle \min_{\beta \in B} \max_{\alpha \in A} \left\{ -\text{trace}(M\sigma\sigma^T)/2 - f \cdot p - l \right\} \rangle$$

Corollary

For split systems and cost, i.e.,

$$f = f_0(x,y) + f_1(x,\alpha,\beta), \quad l = l_0(x,y) + l_1(x,\alpha,\beta),$$

the linear averaging of the data is the correct limit, i.e.,

$$\lim_{\varepsilon\to 0} u^\varepsilon(t,x,y) = u(t,x) :=$$

$$\inf_{\alpha\in\Gamma(t)} \sup_{\beta\in\mathcal{B}(t)} E\left[\int_0^t \langle l\rangle(x_s,\alpha[\beta]_s,\beta_s)\,ds + \langle h\rangle(x_t)\right],$$

$$dx_s = \langle f\rangle(x_s,\alpha[\beta]_s,\beta_s)\,ds + \langle\sigma\sigma^T\rangle^{1/2}(x_s)\,dW_s$$

Proof: $\overline{H}(x,p,M) = -\text{trace}(M\langle\sigma\sigma^T\rangle)/2 + \min_B \max_A\left\{-\langle f\rangle\cdot p - \langle l\rangle\right\}.$

A similar result was proved by Kushner (book, 1990) for a single controller by probabilistic methods.

In general, for system or cost NOT split,

$$\overline{H}(x,p,M) = \langle \min_B \max_A \{...\} \rangle \neq \min_B \max_A \langle \{...\} \rangle$$

and the limit control problem is not obvious.
For some classical problems we derived the explicit form of the effective control problem:

- Merton portfolio optimization with stochastic volatility,
- Ramsey model of optimal economic growth with (fast) random parameters,
- Vidale - Wolfe advertising model with random parameters,
- advertising game in a duopoly with Lanchester dynamics and random parameters.

Often they involve a nonlinear average of some parameter.
E.g., the limit of Merton problem is still a Merton problem with constant volatility the harmonic average $\overline{\sigma} := \langle \sigma^{-2} \rangle^{-1/2}$.

In general, for system or cost NOT split,

$$\overline{H}(x,p,M) = \langle \min_B \max_A \{...\} \rangle \neq \min_B \max_A \langle \{...\} \rangle$$

and the limit control problem is not obvious.
For some classical problems we derived the explicit form of the effective control problem:

- Merton portfolio optimization with stochastic volatility,
- Ramsey model of optimal economic growth with (fast) random parameters,
- Vidale - Wolfe advertising model with random parameters,
- advertising game in a duopoly with Lanchester dynamics and random parameters.

Often they involve a nonlinear average of some parameter.
E.g., the limit of Merton problem is still a Merton problem with constant volatility the harmonic average $\overline{\sigma} := \langle \sigma^{-2} \rangle^{-1/2}$.

In general, for system or cost NOT split,

$$\overline{H}(x,p,M) = \langle \min_B \max_A \{...\} \rangle \neq \min_B \max_A \langle \{...\} \rangle$$

and the limit control problem is not obvious.
For some classical problems we derived the explicit form of the effective control problem:

- Merton portfolio optimization with stochastic volatility,
- Ramsey model of optimal economic growth with (fast) random parameters,
- Vidale - Wolfe advertising model with random parameters,
- advertising game in a duopoly with Lanchester dynamics and random parameters.

Often they involve a nonlinear average of some parameter.
E.g., the limit of Merton problem is still a Merton problem with constant volatility the harmonic average $\overline{\sigma} := \langle \sigma^{-2} \rangle^{-1/2}$.

We can give a general representation formula for the effective control problem by enlarging the control set.
Assume for simplicity $\sigma \equiv 0$ (deterministic slow subsystem) and a single controller (B singleton).
Define $E_x := L^1((\mathbf{R}^m, \mu_x); A) \supset A$ and for $\tilde{\alpha} \in E_x$

$$\overline{f}(x, \tilde{\alpha}) := \int_{\mathbf{R}^m} f(x, y, \tilde{\alpha}(y))\, d\mu_x(y), \; \overline{l}(x, \tilde{\alpha}) := \int_{\mathbf{R}^m} l(x, y, \tilde{\alpha}(y))\, d\mu_x(y).$$

Then

$$\overline{H}(x, p) = \langle \max_{\alpha \in A} \{-f \cdot p - l\} \rangle = \max_{\tilde{\alpha} \in E_x} \left\{ -\overline{f}(x, \tilde{\alpha}) \cdot p - \overline{l}(x, \tilde{\alpha}) \right\}$$

and the effective control problem is

$$\dot{x}_s = \overline{f}(x_s, \tilde{\alpha}_s), \quad x_0 = x, \quad \tilde{\alpha}_s \in E_{x_s}$$
$$\min \quad \overline{J}(t, x, \tilde{\alpha}_.) = \int_0^t \overline{l}(x_s, \tilde{\alpha}_s)\, ds + \langle h \rangle (x_t).$$

A one-dimensional homogenization problem

$$\dot{x}_s = g(x_s)\,\alpha_s, \quad J = \int_0^t l\left(x_s, \frac{x_s}{\varepsilon}\right) ds + h\left(x_t, \frac{x_t}{\varepsilon}\right), \ -1 \le \alpha_s \le 1, \ g > 0.$$

Here $\dot{y}_s = \frac{1}{\varepsilon} g(x_s)\alpha_s$ depends on the control.

The value function $u^\varepsilon(t,x)$ solves the H-J equation

$$\frac{\partial u^\varepsilon}{\partial t} + g(x)\left|\frac{\partial u^\varepsilon}{\partial t}\right| = l\left(x, \frac{x_s}{\varepsilon}\right), \qquad u^\varepsilon(0,x) = h\left(x, \frac{x_s}{\varepsilon}\right).$$

For $l(x,\cdot), h(x,\cdot)$ 1-periodic and $\min l(x,\cdot) = 0$ the limit PDE is

$$\frac{\partial u}{\partial t} + \left(g(x)\left|\frac{\partial u}{\partial t}\right| - \langle l\rangle(x)\right)^+ = 0, \quad u(0,x) = \min_{y\in[0,1]} h(x,y).$$

The effective control problem is

$$\dot{x}_s = g(x_s)\alpha_s, \quad \overline{J} = \int_0^t |\alpha_s|\langle l\rangle(x_s)ds + \min_{y\in[0,1]} h(x_t,y), \ -1 \le \alpha_s \le 1.$$

A one-dimensional homogenization problem

$$\dot{x}_s = g(x_s)\,\alpha_s, \quad J = \int_0^t l\left(x_s, \frac{x_s}{\varepsilon}\right) ds + h\left(x_t, \frac{x_t}{\varepsilon}\right), \; -1 \le \alpha_s \le 1, \; g > 0.$$

Here $\dot{y}_s = \frac{1}{\varepsilon} g(x_s)\alpha_s$ depends on the control.

The value function $u^\varepsilon(t,x)$ solves the H-J equation

$$\frac{\partial u^\varepsilon}{\partial t} + g(x)\left|\frac{\partial u^\varepsilon}{\partial t}\right| = l\left(x, \frac{x_s}{\varepsilon}\right), \qquad u^\varepsilon(0,x) = h\left(x, \frac{x_s}{\varepsilon}\right).$$

For $l(x,\cdot), h(x,\cdot)$ 1-periodic and $\min l(x,\cdot) = 0$ the limit PDE is

$$\frac{\partial u}{\partial t} + \left(g(x)\left|\frac{\partial u}{\partial t}\right| - \langle l\rangle(x)\right)^+ = 0, \quad u(0,x) = \min_{y\in[0,1]} h(x,y).$$

The effective control problem is

$$\dot{x}_s = g(x_s)\alpha_s, \quad \bar{J} = \int_0^t |\alpha_s|\langle l\rangle(x_s)ds + \min_{y\in[0,1]} h(x_t,y), \; -1 \le \alpha_s \le 1.$$

A one-dimensional homogenization problem

$$\dot{x}_s = g(x_s)\,\alpha_s, \quad J = \int_0^t l\left(x_s, \frac{x_s}{\varepsilon}\right) ds + h\left(x_t, \frac{x_t}{\varepsilon}\right), \ -1 \le \alpha_s \le 1,\ g > 0.$$

Here $\dot{y}_s = \frac{1}{\varepsilon} g(x_s)\alpha_s$ depends on the control.

The value function $u^\varepsilon(t,x)$ solves the H-J equation

$$\frac{\partial u^\varepsilon}{\partial t} + g(x)\left|\frac{\partial u^\varepsilon}{\partial t}\right| = l\left(x, \frac{x_s}{\varepsilon}\right), \qquad u^\varepsilon(0,x) = h\left(x, \frac{x_s}{\varepsilon}\right).$$

For $l(x,\cdot), h(x,\cdot)$ 1-periodic and $\min l(x,\cdot) = 0$ the limit PDE is

$$\frac{\partial u}{\partial t} + \left(g(x)\left|\frac{\partial u}{\partial t}\right| - \langle l\rangle(x)\right)^{+} = 0, \quad u(0,x) = \min_{y\in[0,1]} h(x,y).$$

The effective control problem is

$$\dot{x}_s = g(x_s)\alpha_s, \quad \overline{J} = \int_0^t |\alpha_s|\langle l\rangle(x_s)ds + \min_{y\in[0,1]} h(x_t,y), \ -1 \le \alpha_s \le 1.$$

Homogenization of a 2-D differential game

$$\dot{x}_s = g_1(x_s, z_s)\alpha_s, \quad -1 \le \alpha_s \le 1,\ g_1 > 0$$

$$\dot{z}_s = g_2(x_s, z_s)\beta_s, \quad -1 \le \beta_s \le 1,\ g_2 > 0$$

$$J^\varepsilon(t, x, z, \alpha, \beta) := \int_0^t \left[l_1\left(x_s, z_s, \frac{x_s}{\varepsilon}\right) + l_2\left(x_s, z_s, \frac{z_s}{\varepsilon}\right)\right] ds + h\left(x_t, z_t, \frac{x_s}{\varepsilon}, \frac{z_s}{\varepsilon}\right)$$

$$\min_{[0,1]} l_1(x, z, \cdot) = 0,\ \max_{[0,1]} l_2(x, z, \cdot) = 0$$

and assume h has a saddle

$$\min_{\xi\in[0,1]} \max_{\eta\in[0,1]} h(x, z, \xi, \eta) = \max_{\eta\in[0,1]} \min_{\xi\in[0,1]} h(x, z, \xi, \eta) =: h_S(x, z)$$

The limit differential game has the same system and controls, and the effective cost-payoff

$$\overline{J} = \int_0^t [|\alpha_s|\langle l_1\rangle(x_s, z_s) + |\beta_s|\langle l_2\rangle(x_s, z_s)]\, ds + h_S(x_t, z_t)$$

joint work with G.Terrone.

Work in progress on homogenization of control systems and games:

- the last two examples extend to n-dimensional systems provided the oscillations are at 1-dimensional scale;
- an "abstract" representation of some effective control problems can be obtained by the limit occupational measures studied by Artstein, Gaitsgory, Borkar,..., (joint work with Gabriele TERRONE);
- homogenization of deterministic differential games is wide open: there are easy examples of non-convergence of the value functions and only a few known cases of convergence (see also Cardaliaguet '09)

Thanks for your attention !