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► **To cite this version:**

Marc Quincampoix. Optimal control problems with horizon tending to infinity and lacking controllability assumption. SADCO Kick off, Mar 2011, Paris, France. <inria-00585688>

HAL Id: inria-00585688

<https://hal.inria.fr/inria-00585688>

Submitted on 13 Apr 2011

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Optimal control problems with horizon tending to infinity and lacking controllability assumption

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SADCO, March 2011

M. Q., & J. Renault, On Existence of a limit value in some non expansive optimal control problems, (*submitted*) (2009)

An Optimal Control Problem

$$V_t(y_0) := \inf_{u \in \mathcal{U}} \frac{1}{t} \int_{s=0}^t h(y(s, u, y_0), u(s)) ds,$$

where $s \mapsto y(s, u, y_0)$ denotes the solution to

$$y'(s) = g(y(s), u(s)), \quad y(0) = y_0.$$

$g : \mathbb{R}^d \times U \rightarrow \mathbb{R}^d$ Lipschitz, U compact, g h bounded.

PROBLEM : Existence of a limit of $V_t(y_0)$ as $t \rightarrow +\infty$.

No ergodicity condition here (**Lions-Papanicolaou-Varadhan, Arisawa-Lions, Bettiol, Alvarez-Bardi Capuzzo-Dolcetta, Artstein-Gaitsgory, Fathi...**) The limit may depend on the initial condition

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Introduction

Definition 1 *The problem $\Gamma(y_0) := (\Gamma_t(y_0))_{t>0}$ has a limit value if*

$$V(y_0) := \lim_{t \rightarrow \infty} V_t(y_0) = \lim_{t \rightarrow \infty} \inf_{u \in \mathcal{U}} \frac{1}{t} \int_{s=0}^t h(y(s, u, y_0), u(s)) ds.$$

Definition 2 *The problem $\Gamma(y_0)$ has a uniform value if it has a limit value $V(y_0)$ and if:*

$$\forall \varepsilon > 0, \exists u \in \mathcal{U}, \exists t_0, \forall t \geq t_0, \frac{1}{t} \int_{s=0}^t h(y(s, u, y_0), u(s)) ds \leq V(y_0) + \varepsilon.$$

Examples

- **Example 1:** here $y \in \mathbb{R}^2$ (seen as the complex plane $i^2 = -1$), there is no control

$$y'(t) = i y(t),$$

$$V_t(y_0) \xrightarrow{t \rightarrow \infty} \frac{1}{2\pi|y_0|} \int_{|z|=|y_0|} h(z) dz,$$

and since there is no control, the value is uniform.

- **Example 2:** in the complex plane again, but now $g(y, u) = i y u$, where $u \in U$ a given bounded subset of \mathbb{R} , and h is continuous in y .

Assumptions and Notations

The function $h : \mathbb{R}^d \times U \rightarrow \mathbb{R}$ is measurable and bounded

$$\left\{ \begin{array}{l} \exists L \geq 0, \forall (y, y') \in \mathbb{R}^{2d}, \forall u \in U, \|g(y, u) - g(y', u)\| \leq L\|y - y'\| \\ \exists a > 0, \forall (y, u) \in \mathbb{R}^d \times U, \|g(y, u)\| \leq a(1 + \|y\|) \end{array} \right.$$

(HK) \exists a compact invariant set K for the control system

Average cost induced by u between 0 and t by:

$$\gamma_t(y_0, u) := \frac{1}{t} \int_0^t h(y(s, u, y_0), u(s)) ds, \quad V_t(y_0) = \inf_{u \in \mathcal{U}} \gamma_t(y_0, u).$$

$$\text{for } m \geq 0, \quad \gamma_{m,t}(y_0, u) := \frac{1}{t} \int_m^{m+t} h(y(s, u, y_0), u(s)) ds,$$

A Classical Controlability Approach

Suppose that $\exists T > 0, \forall (y_1, y_2) \in K, \exists t \leq T, \forall u \in \mathcal{U}, \exists v \in \mathcal{U}, \|y(t, u, y_1) - y(t, v, y_2)\| = 0$.

Then for any $t \geq T$ and any $\Psi \in C(K)$ the maps

$$y_0 \mapsto V_t^\Psi(y_0) := \inf_{u \in \mathcal{U}} \int_{s=0}^t h(y(s, u, y_0)) ds + \Psi(y(t, u, y_0)),$$

are equicontinuous with a modulus of continuity which does not depend on t and Ψ (but only on the Lipschitz constants of h and f).

Thus V_t^Ψ is more regular than Ψ . This also could be obtained and generalized using HJB results with coercive concave hamiltonians.

• **Example 3:** $g(y, u) = -y + u$, where $u \in U$ a given bounded subset of IR^d , and h is continuous in y .

• **Example 4:** in IR^2 . The initial state is $y_0 = (0, 0)$ and $U = [0, 1]$, and the cost is $h(y) = 1 - y_1(1 - y_2)$.

$$y'(s) = g(y(s), u(s)) = \begin{pmatrix} u(s)(1 - y_1(s)) \\ u^2(s)(1 - y_1(s)) \end{pmatrix}.$$

One can easily observe that the reachable set $G(y_0) \subset [0, 1]^2$.

If $u = \varepsilon > 0$ constant, $y_1(t) = 1 - \exp(-\varepsilon t)$ and $y_2(t) = \varepsilon y_1(t)$. So we have $V_t(y_0) \xrightarrow[t \rightarrow \infty]{} 0$. **Existence of a Uniform Value**

No ergodicity :

$$\{y \in [0, 1]^2, \lim_{t \rightarrow \infty} V_t(y) = \lim_{t \rightarrow \infty} V_t(y_0)\} = [0, 1] \times \{0\},$$

and starting from y_0 it is possible to reach no point in $(0, 1] \times \{0\}$.

A first result in Nonexpansive case

Denote by $G(y_0) := \{y(t, u, y_0), t \geq 0, u \in \mathcal{U}\}$ the reachable set

Theorem 3 $h(y, u) = h(y)$ *only depends on the state,*

$G(y_0)$ *is bounded (invariant),*

$\forall (y_1, y_2) \in G(y_0)^2, \sup_{u \in U} \inf_{v \in U} \langle y_1 - y_2, g(y_1, u) - g(y_2, v) \rangle \leq 0.$

Then $\Gamma(y_0)$ *has a limit value* $V_t(y_0) \xrightarrow[t \rightarrow +\infty]{} V^*(y_0).$ **The convergence of** $(V_t)_t$ *to* V^* **is uniform over** $G(y_0)$, **and the value of** $\Gamma(y_0)$ *is uniform.*

A Crucial Technical Lemma

We define $V^-(y_0) := \liminf_{t \rightarrow +\infty} V_t(y_0)$,

$V^+(y_0) := \limsup_{t \rightarrow +\infty} V_t(y_0)$.

Lemma 4 *For every m_0 in \mathbb{R}_+ , we have:*

$$\sup_{t>0} \inf_{m \leq m_0} V_{m,t}(y_0) \geq V^+(y_0) \geq V^-(y_0) \geq \sup_{t>0} \inf_{m \geq 0} V_{m,t}(y_0).$$

Definition 5

$$V^*(y_0) = \sup_{t>0} \inf_{m \geq 0} V_{m,t}(y_0).$$

Sketch of the proof of the first result

Lemma 6 $\forall T > 0, \forall \varepsilon > 0, \forall (y_1, y_2) \in G(y_0)^2, \forall u \in \mathcal{U}, \exists v \in \mathcal{U},$

$$\forall t \in [0, T], \|y(t, u, y_1) - y(t, v, y_2)\| \leq \|y_1 - y_2\| + \varepsilon.$$

Proposition 7 $\forall \varepsilon > 0, \exists m_0, \sup_{t>0} \inf_{m \leq m_0} V_{m,t}(y_0) \leq \sup_{t>0} \inf_{m \geq 0} V_{m,t}(y_0) + 2\varepsilon$

- $(V_T(y_0))_{T>0}$ is equicontinuous (Lemma 6 + continuity of h)
- Define $G^m(y_0) := \{y(t, u, y_0), t \leq m, u \in \mathcal{U}\}$ the reachable set in time m .

$\forall \varepsilon, \exists m_0, \forall z \in G(y_0), \exists z' \in G^{m_0}(y_0)$ such that $\|z - z'\| \leq \varepsilon$.

- We have $\inf_{m \geq 0} V_{m,t}(y_0) = \inf\{V_t(z), z \in G(y_0)\}$, and $\inf_{m \leq m_0} V_{m,t}(y_0) = \inf\{V_t(z), z \in G^{m_0}(y_0)\}$. By steps 1 and 2 $\inf\{V_t(z), z \in G^{m_0}(y_0)\} \leq \inf\{V_t(z), z \in G(y_0)\} + 2\varepsilon$.

Generalizations

Theorem 8 $\exists C^1 \Delta : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}_+$, *vanishing on the diagonal* ($\Delta(y, y) = 0$) *and symmetric* ($\Delta(y_1, y_2) = \Delta(y_2, y_1)$)
 $h(y, u) = h(y)$ *only depends on the state,*
 $G(y_0)$ *is bounded (invariant),*
 $\forall (y_1, y_2) \in G(y_0)^2, \forall u \in U, \exists v \in U.$

$$\langle g(y_1, u), \frac{\partial}{\partial y_1} \Delta(y_1, y_2) \rangle + \langle g(y_2, v), \frac{\partial}{\partial y_2} \Delta(y_1, y_2) \rangle \leq 0$$

Then $\Gamma(y_0)$ *has a limit value* $V_t(y_0) \xrightarrow[t \rightarrow +\infty]{} V^*(y_0)$. **The convergence of** $(V_t)_t$ *to* V^* *is uniform over* $G(y_0)$, *and the value of* $\Gamma(y_0)$ *is uniform.*

• This result can be applied to example 4, with $\Delta(y_1, y_2) = \|y_1 - y_2\|_1$ (L^1 -norm). In this example, we have for each y_1, y_2 and u : $\Delta(y_1 + tg(y_1, u), y_2 + tg(y_2, u)) \leq \Delta(y_1, y_2)$ as soon as $t \geq 0$ is small enough.

Example 4: in \mathbb{R}^2 . The initial state is $y_0 = (0, 0)$ and $U = [0, 1]$, and the cost is $h(y) = 1 - y_1(1 - y_2)$.

$$y'(s) = g(y(s), u(s)) = \begin{pmatrix} u(s)(1 - y_1(s)) \\ u^2(s)(1 - y_1(s)) \end{pmatrix}.$$

Further Generalizations

Theorem 9 (H1) *h is uniformly continuous in y on \bar{Z} uniformly in u . And for each y in \bar{Z} , either h does not depend on u or the set $\{(g(y, u), h(y, u)) \in \mathbb{R}^d \times [0, 1], u \in U\}$ is closed.*

(H2): *$\exists \Delta : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+$, vanishing on the diagonal ($\Delta(y, y) = 0$) and symmetric ($\Delta(y_1, y_2) = \Delta(y_2, y_1)$), and a uniformly continuous function $\hat{\alpha} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ s.t. $\hat{\alpha}(t) \xrightarrow{t \rightarrow 0} 0$ satisfying:*

a) *\forall sequence $(z_n)_n \subset Z$, $\forall \varepsilon > 0$, $\exists n$, $\liminf_p \Delta(z_n, z_p) \leq \varepsilon$.*

b) *$\forall (y_1, y_2) \in \bar{Z}^2$, $\forall u \in U$, $\exists v \in U$ such that*

$D \uparrow \Delta(y_1, y_2)(g(y_1, u), g(y_2, v)) \leq 0$, $h(y_2, v) - h(y_1, u) \leq \hat{\alpha}(\Delta(y_1, y_2))$.

Then $\Gamma(y_0)$ has a uniform value $\lim_{t \rightarrow \infty} V_t = V^$.*

Remarks

- Although Δ may not satisfy the triangular inequality nor the separation property, it may be seen as a “distance” adapted to the problem $\Gamma(y_0)$.
- $D \uparrow$ is the contingent epi-derivative (which reduces to the upper Dini derivative if Δ is Lipschitz) $D\uparrow\Delta(z)(\alpha) = \liminf_{t \rightarrow 0^+, \alpha' \rightarrow \alpha} \frac{1}{t}(\Delta(z + t\alpha') - \Delta(z))$. If Δ is differentiable, the condition $D \uparrow \Delta(y_1, y_2)(g(y_1, u), g(y_2, v)) \leq 0$ just reads:
 $\langle g(y_1, u), \frac{\partial}{\partial y_1} \Delta(y_1, y_2) \rangle + \langle g(y_2, v), \frac{\partial}{\partial y_2} \Delta(y_1, y_2) \rangle \leq 0$.

• The assumption: “ $\{(g(y, u), h(y, u)) \in \mathbb{R}^d \times [0, 1], u \in U\}$ closed” could be checked for instance if U is compact and if h and g are continuous with respect to (y, u) .

• $H2a)$ is a precompactness condition. It is satisfied as soon as $G(y_0)$ is bounded. **cf Renault 2008**

• Notice that $H2$ is satisfied with $\Delta = 0$ if we are in the trivial case where $\inf_u h(y, u)$ is constant.

On Uniform Value

Definition 10 $\Gamma(y_0)$ *has a uniform value if* $\exists V(y_0)$ *and if:*

$$\forall \varepsilon > 0, \exists u \in \mathcal{U}, \exists t_0, \forall t \geq t_0, \frac{1}{t} \int_{s=0}^t h(y(s, u, y_0), u(s)) ds \leq V(y_0) + \varepsilon.$$

• **Example 5:** in \mathbb{R}^2 , $y_0 = (0, 0)$, control set $U = [0, 1]$, $y'(t) = (y_2(t), u(t))$, and $h(y_1, y_2) = 0$ if $y_1 \in [1, 2]$, = 1 otherwise.

We have $u(s) = y_2'(s) = y_1''(s)$,

Interpretation: u "acceleration", y_2 "speed", y_1 the "position".

If $u = \varepsilon$ constant, then $y_2(t) = \sqrt{2\varepsilon y_1(t)} \quad \forall t \geq 0$.

Limit Value: $V_T(y_0) \xrightarrow{T \rightarrow \infty} 1/2$

No Uniform Value.

Optimal control with discounted factor $\lambda \rightarrow 0^+$

We define $\Theta_\lambda(y_0) := \inf_{u \in \mathcal{U}} \int_{s=0}^{+\infty} \lambda e^{-\lambda s} h(y(s, u, y_0), u(s)) ds,$

Theorem 11 (*Oliu-Barton Vigerol 2010*) *the following uniform limit in K exists* $\lim_{\lambda \rightarrow 0^+} \Theta_\lambda(y_0)$

iff

the following uniform limit in K exists $\lim_{t \rightarrow \infty} V_t(y_0)$

Question Application to different concepts of means

Open Problems

Differential Game at horizon t :

$$V_t(y_0) := \inf_{u \in \mathcal{U}} \sup_{v \in \mathcal{V}} \frac{1}{t} \int_{s=0}^t h(y(s, u, v, y_0), u(s), v(s)) ds,$$

where $s \mapsto y(s, u, y_0)$ denotes the solution to

$$y'(s) = g(y(s), u(s), v(s)), \quad y(0) = y_0.$$

OPEN PROBLEM : Existence of a limit of $V_t(y_0)$ as $t \rightarrow \infty$.

Only Partial results:

- When the Hamiltonian is coercive (hence ergodicity and the limit is y independent) **Alvarez-Bardi ...**
- For nonconvex and non coercive Hamiltonian in \mathbb{R}^2 **Cardaliagu**

Averaging Problem for singularly perturbed system

$$\begin{cases} i) & x'(s) = f(x(s), y(s), u(s)), \quad x(0) = x, \quad s \in [0, T] \\ ii) & \varepsilon y'(s) = g(x(s), y(s), u(s)) \quad y(0) = y, \end{cases} \quad (1)$$

Change of variable $\tau = \frac{t}{\varepsilon}$, $(X(\tau), Y(\tau), U(\tau)) = (x(\varepsilon\tau), y(\varepsilon\tau), u(\varepsilon\tau))$

$$\begin{cases} X'(\tau) = \varepsilon f(X(\tau), Y(\tau), U(\tau)), \quad X(0) = x, \quad \tau \in [0, \frac{T}{\varepsilon}] \\ Y'(\tau) = g(X(\tau), Y(\tau), U(\tau)), \quad Y(0) = y. \end{cases} \quad (2)$$

Take $\varepsilon = 0$ in (2). We have the following associated system:

$$y'(\tau) = g(x, y(\tau), u(\tau)), \quad y(0) = y, \quad (3)$$

$y_x(\cdot, u, y)$ denotes the unique solution of (3).

Averaging method

We suppose that f and g are Lipschitz and there is a compact set $M \times N$ which is invariant by (1) for all ε .

$$A(x, y, S, u) = \frac{1}{S} \int_0^S f(x, y_x(\tau, u, y), u(\tau)) d\tau,$$

$$F(x, y, S) = \{A(x, y, S, u); u \in \mathcal{U}\}$$

Theorem 12 *Gaitsgory, Grammel* **If** $\exists \gamma : \mathbb{R} \rightarrow \mathbb{R}_+$ **with** $\lim_{S \rightarrow +\infty} \gamma(S) = 0$ **and a Lipschitz set-valued map** $\bar{F} : M \rightarrow \mathbb{R}^b$ **with compact convex nonempty values such that**

$$d(\text{co cl} F(x, y, S), \bar{F}(x)) \leq \gamma(S), \quad \forall (x, y) \in M \times N, \quad \forall S > 0,$$

then $\forall x, y$ **the solutions of the differential inclusion**

$$x'(s) \in \bar{F}(x(s)), \quad x(0) = x. \quad (4)$$

approximate the solutions of the singularly perturbed system (1) in the following sense:

For any $\varepsilon > 0$, and any $T > 0$ there exists $M(T, \varepsilon) > 0$ with $\lim_{\varepsilon \rightarrow 0} M(T, \varepsilon) = 0$ such that

a) For any family of solutions $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ to (1) there exists a solution $x(\cdot)$ to (4) such that

$$\sup_{t \in [0, T]} \|x_\varepsilon(t) - x(t)\| \leq M(T, \varepsilon).$$

b) Conversely fix $x(\cdot)$ a solution to (4) then for any ε small enough there exists a solution $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ to (1) such that

$$\sup_{t \in [0, T]} \|x_\varepsilon(t) - x(t)\| \leq M(T, \varepsilon).$$

cf also **Wattbled, M.Q Wattbled ...**

QUESTION ases and conditions where \overline{F} may depend on y .

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Thank You for your Attention
