Optimal control problems with horizon tending to infinity and lacking controllability assumption

Marc Quincampoix

Optimal control problems with horizon tending to infinity and lacking controllability assumption

Marc Quincampoix
Université de Brest

SADCO, March 2011

M. Q., & J. Renault, On Existence of a limit value in some non expansive optimal control problems,(*submitted*) (2009)

An Optimal Control Problem

$$V_t(y_0) := \inf_{u \in \mathcal{U}} \frac{1}{t} \int_{s=0}^{t} h(y(s,u,y_0), u(s))ds,$$

where $s \mapsto y(s,u,y_0)$ **denotes the solution to**

$$y'(s) = g(y(s), u(s)), \quad y(0) = y_0.$$

$g : I\!R^d \times U \to I\!R^d$ **Lipschitz,** U **compact,** g h **bounded.**

PROBLEM : Existence of a limit of $V_t(y_0)$ **as** $t \to +\infty$**.**

No ergodicity condition here (Lions-Papanicolaou- Varadhan, Arisawa-Lions, Bettiol, Alvarez-Bardi Capuzzo-Dolcetta, Artstein-Gaitsgory, Fathi...)The limit may depend on the initial condition

Contents

Introduction

Definition 1 ***The problem*** $\Gamma(y_0) := (\Gamma_t(y_0))_{t>0}$ ***has a*** *limit value* ***if***

$$V(y_0) := \lim_{t\to\infty} V_t(y_0) = \lim_{t\to\infty} \inf_{u\in\mathcal{U}} \frac{1}{t}\int_{s=0}^{t} h(y(s,u,y_0),u(s))ds.$$

Definition 2 ***The problem*** $\Gamma(y_0)$ ***has a*** *uniform value* ***if it has a limit value*** $V(y_0)$ ***and if:***

$$\forall\varepsilon>0, \exists u\in\mathcal{U}, \exists t_0, \forall t\geq t_0, \frac{1}{t}\int_{s=0}^{t} h(y(s,u,y_0),u(s))ds \leq V(y_0)+\varepsilon.$$

Examples

• **Example 1: here** $y \in I\!R^2$ **(seen as the complex plane** $i^2 = -1$**), there is no control**

$$y'(t) = i\, y(t),$$

$$V_t(y_0) \xrightarrow[t\to\infty]{} \frac{1}{2\pi|y_0|} \int_{|z|=|y_0|} h(z)dz,$$

and since there is no control, the value is uniform.

• **Example 2: in the complex plane again, but now** $g(y,u) = i\ y\ u$**, where** $u \in U$ **a given bounded subset of** $I\!R$**, and** h **is continuous in** y**.**

Assumptions and Notations

$$\begin{cases} \textbf{The function } h : I\!R^d \times U \longrightarrow I\!R \textbf{ is measurable and bounded} \\ \exists L \geq 0, \forall (y, y') \in I\!R^{2d}, \forall u \in U, \ \|g(y,u) - g(y',u)\| \leq L\|y - y'\| \\ \exists a > 0, \forall (y,u) \in I\!R^d \times U, \ \|g(y,u)\| \leq a(1 + \|y\|) \end{cases}$$

(HK) $\exists$ **a compact invariant set** K **for the control system**

Average cost induced by u **between** 0 **and** t **by:**

$$\gamma_t(y_0, u) := \frac{1}{t}\int_0^t h(y(s,u,y_0), u(s))ds, \ V_t(y_0) = \inf_{u \in \mathcal{U}} \gamma_t(y_0, u).$$

$$\textbf{for } m \geq 0\textbf{, } \gamma_{m,t}(y_0, u) := \frac{1}{t}\int_m^{m+t} h(y(s,u,y_0), u(s))ds,$$

A Classical Controlability Approach

Suppose that $\exists T > 0, \forall (y_1, y_2) \in K, \exists t \leq T, \forall u \in \mathcal{U}, \ \exists v \in \mathcal{U}, \|y(t,u,y_1) - y(t,v,y_2)\| = 0.$

Then for any $t \geq T$ **and any** $\Psi \in C(K)$ **the maps**

$$y_0 \mapsto V_t^{\Psi}(y_0) := \inf_{u \in \mathcal{U}} \int_{s=0}^{t} h(y(s,u,y_0))ds + \Psi(y(t,u,y_0)),$$

are equicontinuous with a modulus of continuity which does not depend on t **and** Ψ **(but only on the Lipschitz constants of** h **and** f**).**

Thus V_t^{Ψ} is more regular than Ψ. This also could be obtained and generalized using HJB results with coercive concave hamiltonians.

• Example 3: $g(y,u) = -y+u$, **where** $u \in U$ **a given bounded subset of** IR^d, **and** h **is continuous in** y.

• Example 4: in IR^2. **The initial state is** $y_0 = (0,0)$ **and** $U = [0,1]$, **and the cost is** $h(y) = 1 - y_1(1-y_2)$.

$$y'(s) = g(y(s), u(s)) = \begin{pmatrix} u(s)(1-y_1(s)) \\ u^2(s)(1-y_1(s)) \end{pmatrix}.$$

One can easily observe that the reachable set $G(y_0) \subset [0,1]^2$.

If $u = \varepsilon > 0$ **constant,** $y_1(t) = 1 - \exp(-\varepsilon t)$ **and** $y_2(t) = \varepsilon y_1(t)$. **So we have** $V_t(y_0) \xrightarrow[t\to\infty]{} 0$. **Existence of a Uniform Value**

No ergodicity :

$$\{y \in [0,1]^2, \lim_{t\to\infty} V_t(y) = \lim_{t\to\infty} V_t(y_0)\} = [0,1] \times \{0\},$$

and starting from y_0 it is possible to reach no point in $(0,1] \times \{0\}$.

A first result in Nonexpansive case

Denote by $G(y_0) := \{y(t,u,y_0), t \geq 0, u \in \mathcal{U}\}$ **the reachable set**

Theorem 3 $h(y,u) = h(y)$ ***only depends on the state,***
$G(y_0)$ ***is bounded (invariant),***
$\forall (y_1, y_2) \in G(y_0)^2$***,*** $\sup_{u \in U} \inf_{v \in U} < y_1 - y_2, g(y_1,u) - g(y_2,v) > \leq 0$.
Then $\Gamma(y_0)$ ***has a*** *limit value* $V_t(y_0) \xrightarrow[t \to +\infty]{} V^*(y_0)$***. The convergence of*** $(V_t)_t$ ***to*** V^* ***is uniform over*** $G(y_0)$***, and the value of*** $\Gamma(y_0)$ ***is uniform.***

A Crucial Technical Lemma

We define $V^-(y_0) := \liminf_{t\to+\infty} V_t(y_0)$**,**
$V^+(y_0) := \limsup_{t\to+\infty} V_t(y_0)$**.**

Lemma 4 ***For every*** m_0 ***in*** IR_+***, we have:***

$$\sup_{t>0} \inf_{m\leq m_0} V_{m,t}(y_0) \geq V^+(y_0) \geq V^-(y_0) \geq \sup_{t>0} \inf_{m\geq 0} V_{m,t}(y_0).$$

Definition 5

$$V^*(y_0) = \sup_{t>0} \inf_{m\geq 0} V_{m,t}(y_0).$$

Sketch of the proof of the first result

Lemma 6 $\forall T > 0, \forall \varepsilon > 0, \forall (y_1, y_2) \in G(y_0)^2, \forall u \in \mathcal{U},\ \exists v \in \mathcal{U},$

$$\forall t \in [0, T],\ \|y(t, u, y_1) - y(t, v, y_2)\| \leq \|y_1 - y_2\| + \varepsilon.$$

Proposition 7 $\forall \varepsilon > 0, \exists m_0, \sup_{t>0} \inf_{m \leq m_0} V_{m,t}(y_0) \leq \sup_{t>0} \inf_{m \geq 0} V_{m,t}(y_0) + 2\varepsilon$

• $(V_T(y_0))_{T>0}$ **is equicontinuous (Lemma 6 +continuity of** h**)**

• **Define** $G^m(y_0) := \{y(t,u,y_0), t \leq m, u \in \mathcal{U}\}$ **the reachable set in time** m**.**

$$\forall \varepsilon, \exists m_0, \forall z \in G(y_0), \exists z' \in G^{m_0}(y_0) \ \ \textbf{such that} \ \ \|z - z'\| \leq \varepsilon.$$

• **We have** $\inf_{m \geq 0} V_{m,t}(y_0) = \inf\{V_t(z), z \in G(y_0)\}$**, and** $\inf_{m \leq m_0} V_{m,t}(y_0$ $\inf\{V_t(z), z \in G^{m_0}(y_0)\}$**. By steps 1 and 2** $\inf\{V_t(z), z \in G^{m_0}(y_0)\} \leq \inf\{V_t(z), z \in G(y_0)\} + 2\varepsilon$.

Generalizations

Theorem 8 $\exists$ C^1 $\Delta : I\!R^d \times I\!R^d \longrightarrow I\!R_+$, ***vanishing on the diagonal*** ($\Delta(y,y) = 0$) ***and symmetric*** ($\Delta(y_1, y_2) = \Delta(y_2, y_1)$)
$h(y,u) = h(y)$ ***only depends on the state,***
$G(y_0)$ ***is bounded (invariant),***
$\forall (y_1, y_2) \in G(y_0)^2$, $\forall u \in U, \exists v \in U.$

$$< g(y_1,u), \frac{\partial}{\partial y_1}\Delta(y_1,y_2) > + < g(y_2,v), \frac{\partial}{\partial y_2}\Delta(y_1,y_2) > \leq 0$$

Then $\Gamma(y_0)$ ***has a*** *limit value* $V_t(y_0) \xrightarrow[t\to+\infty]{} V^*(y_0)$. ***The convergence of*** $(V_t)_t$ ***to*** V^* ***is uniform over*** $G(y_0)$, ***and the value of*** $\Gamma(y_0)$ ***is uniform.***

• **This result can be applied to example 4, with** $\Delta(y_1, y_2) = \|y_1 - y_2\|_1$ **(**L^1**-norm). In this example, we have for each** y_1**,** y_2 **and** u**:** $\Delta(y_1 + tg(y_1, u), y_2 + tg(y_2, u)) \leq \Delta(y_1, y_2)$ **as soon as** $t \geq 0$ **is small enough.**

Example 4: in IR^2**. The initial state is** $y_0 = (0, 0)$ **and** $U = [0, 1]$**, and the cost is** $h(y) = 1 - y_1(1 - y_2)$**.**

$$y'(s) = g(y(s), u(s)) = \begin{pmatrix} u(s)(1 - y_1(s)) \\ u^2(s)(1 - y_1(s)) \end{pmatrix}.$$

Further Generalizations

Theorem 9 ***(H1)*** h ***is uniformly continuous in*** y ***on*** $\bar{Z}$ ***uniformly in*** u. ***And for each*** y ***in*** $\bar{Z}$, ***either*** h ***does not depend on*** u ***or the set*** $\{(g(y,u),h(y,u)) \in I\!R^d \times [0,1],\ u \in U\}$ ***is closed.*** ***(H2):*** $\exists \Delta : I\!R^d \times I\!R^d \longrightarrow I\!R_+$, ***vanishing on the diagonal*** ***(***$\Delta(y,y) = 0$***) and symmetric (***$\Delta(y_1,y_2) = \Delta(y_2,y_1)$ ***), and a unniformly continuous function*** $\hat{\alpha} : I\!R_+ \longrightarrow I\!R_+$ ***s.t.*** $\hat{\alpha}(t) \xrightarrow[t \to 0]{} 0$ ***satisfying:***

$a)$ $\forall$ ***sequence*** $(z_n)_n \subset Z$, $\forall \varepsilon > 0$, $\exists n$, $\liminf_p \Delta(z_n, z_p) \leq \varepsilon$.

$b)$ $\forall (y_1,y_2) \in \bar{Z}^2$, $\forall u \in U$, $\exists v \in U$ ***such that***

$$D \uparrow \Delta(y_1,y_2)(g(y_1,u),g(y_2,v)) \leq 0,\ h(y_2,v) - h(y_1,u) \leq \hat{\alpha}(\Delta(y_1,y_2)).$$

Then $\Gamma(y_0)$ ***has a uniform value*** $\lim_{t\to\infty} V_t = V^*$.

Remarks

• **Although** Δ **may not satisfy the triangular inequality nor the separation property, it may be seen as a "distance" adapted to the problem** $\Gamma(y_0)$**.**

• $D\uparrow$ **is the contingent epi-derivative (which reduces to the upper Dini derivative if** Δ **is Lipschitz)** $D\uparrow\Delta(z)(\alpha) = \liminf_{t\to 0^+,\alpha'\to\alpha} \frac{1}{t}(\Delta(z+t\alpha') - \Delta(z))$**. If** Δ **is differentiable, the condition** $D\uparrow\Delta(y_1,y_2)(g(y_1,u),g(y_2,v)) \le 0$ **just reads:** $< g(y_1,u), \frac{\partial}{\partial y_1}\Delta(y_1,y_2) > + < g(y_2,v), \frac{\partial}{\partial y_2}\Delta(y_1,y_2) > \le 0$**.**

• **The assumption: "**$\{(g(y,u),h(y,u)) \in IR^d \times [0,1],\ \ u \in U\}$ **closed" could be checked for instance if** U **is compact and if** h **and** g **are continuous with respect to** (y,u)**.**

• $H2a)$ **is a precompacity condition. It is satisfied as soon as** $G(y_0)$ **is bounded. cf Renault 2008**

• **Notice that** $H2$ **is satisfied with** $\Delta = 0$ **if we are in the trivial case where** $\inf_u h(y,u)$ **is constant.**

Definition 10 $\Gamma(y_0)$ ***has a*** *uniform value* ***if*** $\exists V(y_0)$ ***and if:***

$$\forall \varepsilon > 0, \exists u \in \mathcal{U}, \exists t_0, \forall t \geq t_0, \frac{1}{t}\int_{s=0}^{t} h(y(s,u,y_0),u(s))ds \leq V(y_0)+\varepsilon.$$

- **Example 5: in** IR^2, $y_0 = (0,0)$, **control set** $U = [0,1]$, $y'(t) = (y_2(t), u(t))$, **and** $h(y_1,y_2) = 0$ **if** $y_1 \in [1,2]$, $= 1$ **otherwise.**

We have $u(s) = y_2'(s) = y_1''(s)$,

Interpretation: u **"acceleration",** y_2 **"speed",** y_1 **the "position".**

If $u = \varepsilon$ **constant, then** $y_2(t) = \sqrt{2\varepsilon y_1(t)}$ $\forall t \geq 0$.

Limit Value: $V_T(y_0) \xrightarrow[T\to\infty]{} 1/2$

No Uniform Value.

Optimal control with discounted facteur $\lambda \to 0^+$

We define $\Theta_\lambda(y_0) := \inf_{u \in \mathcal{U}} \int_{s=0}^{+\infty} \lambda e^{-\lambda s} h(y(s,u,y_0), u(s))ds,$

Theorem 11 ***(Oliu-Barton Vigeral 2010) the following uniform limit in*** K ***exists*** $\lim_{\lambda \to 0^+} \Theta_\lambda(y_0)$

iff

the following uniform limit in K ***exists*** $\lim_{t \to \infty} V_t(y_0)$

Question **Application to different concepts of means**

Open Problems

Differential Game at horizon t:

$$V_t(y_0) := \text{"} \inf_{u\in\mathcal{U}} \sup_{v\in\mathcal{V}} \text{"} \frac{1}{t}\int_{s=0}^{t} h(y(s,u,v,y_0),u(s),v(s))ds,$$

where $s \mapsto y(s,u,y_0)$ denotes the solution to

$$y'(s) = g(y(s),u(s),v(s))), \quad y(0) = y_0.$$

OPEN PROBLEM : Existence of a limit of $V_t(y_0)$ as $t \to \infty$.

Only Partial results:

- **When the Hamiltonian is coercive (hence ergodicity and the limit is y independent) Alvarez-Bardi ...**
- **For nonconvex and non coercive Hamiltonian in IR^2 Cardaliagu**

Averaging Problem for singularly perturbed system

$$\begin{cases} i) \;\; x'(s) = f\big(x(s), y(s), u(s)\big), \;\; x(0) = x, \;\; s \in [0, T] \\ ii) \; \varepsilon y'(s) = g\big(x(s), y(s), u(s)\big) \; y(0) = y, \end{cases} \tag{1}$$

Change of variable $\tau = \frac{t}{\varepsilon}$**,** $(X(\tau), Y(\tau), U(\tau)) = (x(\varepsilon\tau), y(\varepsilon\tau), u(\varepsilon\tau))$

$$\begin{cases} X'(\tau) = \varepsilon f\big(X(\tau), Y(\tau), U(\tau)\big), \;\; X(0) = x, \tau \in [0, \frac{T}{\varepsilon}] \\ Y'(\tau) = g\big(X(\tau), Y(\tau), U(\tau)\big), \;\; Y(0) = y. \end{cases} \tag{2}$$

Take $\varepsilon = 0$ **in (2).We have the following associated system:**

$$y'(\tau) = g\big(x, y(\tau), u(\tau)\big), \;\; y(0) = y, \tag{3}$$

$y_x(\cdot, u, y)$ **denotes the unique solution of (3).**

Averaging method

We suppose that f and g are Lipschitz and there is a compact set $M \times N$ which is is invariant by (1) for all ε.

$A(x,y,S,u) = \frac{1}{S}\int_0^S f(x, y_x(\tau,u,y), u(\tau))d\tau,$

$F(x,y,S) = \{A(x,y,S,u); u \in \mathcal{U}\}$

Theorem 12 ***Gaitsgory, Grammel*** ***If*** $\exists\gamma : I\!R \to I\!R_+$ ***with*** $\lim_{S\to+\infty}\gamma(S) = 0$ ***and a Lipschitz set-valued map*** $\bar{F} : M \to I\!R^b b$ ***with compact convex nonempty values such that***

$$d(co\, cl F(x,y,S), \overline{F}(x)) \leq \gamma(S),\ \forall (x,y) \in M \times N, \forall S > 0,$$

then $\forall x, y$ ***the solutions of the differential inclusion***

$$x'(s) \in \overline{F}\big(x(s)\big),\ x(0) = x. \tag{4}$$

approximate the solutions of the singularly perturbed system (1) in the following sense:

For any $\varepsilon > 0$, and any $T > 0$ there exists $M(T, \varepsilon) > 0$ with $\lim_{\varepsilon \to 0} M(T, \varepsilon) = 0$ such that

a) For any family of solutions $\big(x_\varepsilon(\cdot), y_\varepsilon(\cdot)\big)$ to (1) there exists a solution $x(\cdot)$ to (4) such that

$$\sup_{t \in [0,T]} \|x_\varepsilon(t) - x(t)\| \leq M(T, \varepsilon).$$

b) Conversely fix $x(\cdot)$ a solution to (4) then for any ε small enough there exists a solution $\big(x_\varepsilon(\cdot), y_\varepsilon(\cdot)\big)$ to (1) such that

$$\sup_{t \in [0,T]} \|x_\varepsilon(t) - x(t)\| \leq M(T, \varepsilon).$$

cf also Wattbled, M.Q Wattbled ...

QUESTION ases and conditions where $\overline{F}$ may depend on y.

[1] O. Alvarez, M. Bardi *Ergodicity, Stabilization, and singular perturbations for Bellman-Isaacs equations* To appear in Mem AMS.

[2] Z. Artstein, and V. Gaitsgory, *The value function of singularly perturbed control systems*, Appl. Math. Optim., 41 (2000), 425-445.

[3] Arisawa, M. and P.L. Lions (1998) Ergodic problem for the Hamilton Jacobi Belmann equations II, *Ann. Inst. Henri Poincaré, Analyse Nonlinéaire*, 15 ,1 , 1–24.

[4] Arisawa, M. and P.L. Lions (1998) On ergodic stochastic control. *Com. in partial differential equations*, 23, 2187–2217.

[5] Bettiol, P. (2005) On ergodic problem for Hamilton-Jacobi-Isaacs equations *ESAIM: Cocv*, 11, 522–541.

[6] Cardaliaguet P. Ergodicity of Hamilton-Jacobi equations with a non coercive non convex Hamiltonian in IR^2/Z^2 preprint [hal-00348219 - version 1] (18/12/2008)

[7] Frankowska, H., Plaskacz, S. and Rzezuchowski T. (1995): Measurable Viability Theorems and Hamilton-Jacobi-Bellman Equation, J. Diff. Eqs., 116, 265-305.

[8] V. Gaitsgory, *Use of Averaging Method in Control Problems*, Differential Equations (Translated from Russian), 46 (1986) pp. 1081-1088.

[9] V. Gaitsgory, M. Quincampoix *Linear programming analysis of deterministic infinite horizon optimal con-*

trol problems (discounting and time averaging cases), to appear in Siam Journal of Control and Opt.

[10] G. Grammel, *Averaging of singularly perturbed systems*, Nonlinear Analysis Theory, 28, 11 (1997), 1851-1865.

[11] Lions P.-L. , Papanicolaou G. , Varadhan S.R.S., Homogenization of Hamilton- Jacobi Equations, unpublished work.

[12] Quincampoix, M. and F. Watbled (2003) Averaging methods for discontinuous Mayer's problem of singularly perturbed control systems. *Nonlinear analysis*, 54, 819–837.

[13] Quincampoix, M. and J. Renault (submitted) On Existence of a limit value in some non expansive optimal control problems,

[14] Renault, J. (2007) Uniform value in Dynamic Programming. Cahier du Ceremade 2007-1. arXiv : 0803.2758.

Thank You for your Attention