Large time behavior of systems of fi rst-order Hamilton-Jacobi equations

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Large time behavior of systems of first-order Hamilton-Jacobi equations

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Time-dependent systems of HJ equations

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + F_i(x, Du_i) + \displaystyle\sum_{j=1}^{m} d_{ij}(x)u_j = f_i(x) & \mathbb{T}^N \times (0, +\infty) \\ u_i(x, 0) = u_{0,i}(x) & \mathbb{T}^N \end{cases}$$

for $i = 1, \dots, m$.

- Periodic setting
- Linear coupling
- More precise assumptions later

Aim : Study the behavior of $u(x, t) = (u_1(x, t), \dots, u_m(x, t))$ when $t \to +\infty$

Plan of the talk

1. Recall of the scalar case
2. Motivations from control
3. Assumptions and a result
4. Sketch of the proof

The scalar periodic case

$$\begin{cases} \dfrac{\partial u}{\partial t} + F(x, Du) = f(x) & \mathbb{T}^N \times (0, +\infty) \\ u(x, 0) = u_0(x) & \mathbb{T}^N \end{cases}$$

A lot of works : Lions 82, Fathi 98, Namah-Roquejoffre 99, Barles-Souganidis 00, Davini-Siconolfi 06, Ishii-Mitake 06,07,...

Namah-Roquejoffre Thm for $\frac{\partial u}{\partial t} + F(x, Du) = f(x)$

Theorem. [Namah-Roquejoffre 99]

- Periodicity : $F(\cdot, p), f, u_0$ are 1-periodic continuous
- convexity and coercivity of $F(x, \cdot)$
- $F(x,p) \geq F(x,0) = 0$
- $f(x) \geq 0, \quad \mathcal{A} = \{x \in \mathbb{T}^N : f(x) = 0\} \neq \emptyset$
- regularity : $|F(x,p) - F(y,p)| \leq \omega((1+|p|)|x-y|)$

Then, for every $u_0 \in \mathrm{Lip}(\mathbb{T}^N)$, there exists $(c, v) \in \mathbb{R} \times \mathrm{Lip}(\mathbb{T}^N)$ such that

- $u(x,t) - ct \to v(x)$ as $t \to +\infty$ uniformly in $\mathbb{T}^N$
- v is solution of $F(x, Dv) = f + c$ in $\mathbb{T}^N$
- c is the ergodic constant, unique,
  $c = -\min_{\mathbb{T}^N} f = \lim_{t\to+\infty} -\frac{u(x,t)}{t} = 0$

.

Goal

Is this theorem true in the case of systems ?

The Aubry set $\mathcal{A} := \{x \in \mathbb{T}^N : f(x) = 0\}$ plays a particular role. What is the equivalent for systems ?

Control interpretation (scalar case)

Dynamics : $\begin{cases} \dot{X}(s) = b(X(s), \alpha(s)) \quad s \geq 0 \\ X(0) = x \end{cases}$

$\alpha(s)$ control, takes its value in a compact space K.

Value function :

$$V(x,t) = \inf_{\alpha(\cdot)} \{ \int_0^t f(X(s))ds + u_0(X(t)) \}$$

Then V is the unique viscosity solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} + \sup_{\alpha \in K} \{ -b(x,\alpha) \cdot Du \} = f(x) \\ u(x,0) = u_0(x) \end{cases}$$

Optimal trajectories are attracted by $\mathcal{A} = \mathrm{argmin}\, f$ and $\dfrac{V(x,t)}{t} \underset{t \to +\infty}{\sim} -c = \min f$.

Control for systems : piecewise deterministic trajectories with random jumps (1)

Dynamics : $\begin{cases} \dot{X}(s) = b_{\nu(t)}(X(s), \alpha(s)) & s \geq 0 \\ X(0) = x \end{cases}$

solution : $(X(s), \nu(s))$ with $\nu(s)$ a Markov process with values in $\{1, 2, \ldots, m\}$

Transition probabilities :
$\mathbb{P}\left(\nu(t+h) = j \mid \nu(t) = i, X(t) = x\right) = \gamma_{ij}(x)h + o(h)$
for $j \neq i$.

Value function :

$$V_i(x,t) = \inf_{\alpha(\cdot)} E_{x,i}\{\int_0^t f_{\nu(s)}(X(s))ds + u_{0,\nu(t)}(X(t))\}$$

Control for systems : piecewise deterministic trajectories with random jumps (2)

Then $V = (V_1, ..., V_m)$ is the unique viscosity solution of the system

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + \sup\limits_{\alpha \in K}\{-b_i(x,\alpha)\cdot Du_i\} + \sum\limits_{j=1}^{m} \gamma_{ij}(x)(u_i - u_j) = f_i(x) \\ u_i(x,0) = u_{0,i}(x) \end{cases}$$

for $i = 1, \ldots, m$.

For instance Fleming-Zhang 98

$$\sum_{j=1}^{m} \gamma_{ij}(x)(u_i - u_j) = \sum_{j=1}^{m} d_{ij}(x)u_j$$

with $d_{ii} = \sum\limits_{j \neq i} \gamma_{ij} \geq 0$ and $d_{ij} = -\gamma_{ij} \leq 0$ for $i \neq j$

Assumptions on the Hamiltonian and initial conditions

The same as in Namah-Roquejoffre Theorem.
For all $i = 1, \dots m$:

- Periodicity : $F_i(\cdot, p), f_i, u_{0,i}$ are 1-periodic continuous
- convexity and coercivity of $F_i(x, \cdot)$
- $F_i(x, p) \geq F_i(x, 0) = 0$
- $f_i(x) \geq 0$
- regularity : $|F_i(x, p) - F_i(y, p)| \leq \omega((1 + |p|)|x - y|)$

Assumptions on the coupling matrix $D(x) = (d_{ij}(x))_{1\le i,j\le m}$

For all $x \in \mathbb{T}^N$:

- $d_{ii} \ge 0, \quad d_{ij} \le 0 \text{ for } j \ne i, \quad \sum_{j=1}^{m} d_{ij} \ge 0.$

  ➪ D is a M-**matrix**
  Classical assumptions to have a monotone system
  $\Rightarrow$ maximum principle for the evolution problem
- d_{ij} are periodic in x
- $D(x)$ has non zero coefficients or :
  is an **irreducible** matrix in $\mathbb{T}^N$
  equivalent definitions :
  (i) $\forall \mathcal{I} \subsetneq \{1, ..., m\}$, $\exists i \in \mathcal{I}$ and $j \notin \mathcal{I}$ s.t. $d_{ij} \ne 0$
  (ii) $\forall i, j \in \{1, ..., m\}$, there exists
  a "chain" $i = i_0, i_1, ..., i_n = j$ s.t. $d_{i_{l-1} i_l} \ne 0$.
  ➪ means that the coupling is nontrivial

Introduction of a "Aubry set" and assumptions

Let $\mathcal{F} = \bigcap_{1\leq i\leq m} \operatorname{argmin} f_i$

$$\mathcal{D} = \bigcap_{1\leq i\leq m} \{x \in \mathbb{T}^N : \sum_{j=1}^{m} d_{ij}(x) = 0\}$$

We define

$$\mathcal{A} = \mathcal{F} \cap \mathcal{D}.$$

Assume that

- $\mathcal{A}$ is **non empty**
- all the f_i's have the **same** minimum $\overline{f}$ (➪ to simplify we take $\overline{\mathbf{f}} = \mathbf{0}$)
- $\mathcal{D} = \mathbb{T}^N$ (➪ to simplify)

With these assumptions, since $f_i \geq 0$,

$$\mathcal{A} = \mathcal{F} = \{x \in \mathbb{T}^N : \sum_{i=1}^{m} f_i(x) = 0\}$$

A Lemma on the coupling matrix

Lemma. Suppose that :

- $D(x)$ is an irreducible M-matrix on $\mathbb{T}^N$
- $\displaystyle\sum_{j=1}^{m} d_{ij} = 0$ (i.e., $\mathcal{D} = \mathbb{T}^N$)

Then, for all $x \in \mathbb{T}^N$, :

- $D(x)$ is degenerate of rank $m-1$
- the kernel of $D(x)$ is spanned with $(1, ..., 1)$
- there exists a **positive** function $\Lambda : \mathbb{T}^N \to \mathbb{R}^m$ such that

$$D(x)^T \Lambda(x) = 0 \quad \left(\text{i.e.}, \sum_{\mathbf{i}=1}^{m} \Lambda_i(x) d_{ij} = 0\right)$$

[Proof : Perron-Froebenius+continuous dependence]

For simplicity, we suppose that $\Lambda(x) = (1, ..., 1)$

A result

Theorem. Under the previous assumptions, for every $u_0 = (u_{0,1}, ..., u_{0,m}) \in \mathrm{Lip}(\mathbb{T}^N)$, there exists $((c_1, ..., c_m), (v_1, ..., v_m)) \in \mathbb{R}^m \times \mathrm{Lip}(\mathbb{T}^N)$ such that

- $u(x,t) - ct \to v(x)$ as $t \to +\infty$ uniformly in $\mathbb{T}^N$
- v is solution of the stationary system

$$F_i(x, Dv_i) + \sum_{j=1}^{m} d_{ij}(x) v_j = f_i + c_i \quad \text{in } \mathbb{T}^N, i = 1, \ldots, m$$

- c is in the kernel of $D(x)$ so $c = (c_1, ..., c_1)$, with $c_1 = -\overline{f} = \lim_{t\to+\infty} -\frac{u_i(x,t)}{t} = \mathbf{0}$ for all i.
- $v_i = v_j$ on $\mathcal{A}$.

Comparison result for the stationary system

$$F_i(x, Dv_i) + \sum_{j=1}^{m} d_{ij}(x)v_j = f_i \quad \text{in } \mathbb{T}^N, i = 1, \dots, m$$

Theorem. Let u be a bounded subsolution and v a bounded supersolution.

- (Classical case) If, for all i, $\sum_{j=1}^{m} d_{ij} > 0 \ \ (\mathcal{D} = \emptyset)$
  then $u \leq v$ on $\mathbb{T}^N$.
- (Degenerate case) If $\sum_{j=1}^{m} u_j \leq \sum_{j=1}^{m} v_j$ on $\mathcal{A}$
  then $u \leq v$

➪ $\mathcal{A}$ may be empty

Existence & uniqueness with prescribed datas on $\mathcal{A}$

Theorem.

- (Classical case) there exists a unique viscosity solution to the stationary system.
- (Degenerate case) For all g continuous on $\mathcal{A}$ with compatibility conditions (see Fathi-Siconolfi 05, Ishii-Mitake 07), there exists a unique viscosity v solution such that $v = g$ on $\mathcal{A}$.

➪ $\mathcal{A}$ is a uniqueness set

Idea of the proof of the comparison theorem (1)

Let $0 < \mu < 1$

$M := \sup_{1 \le i \le m} \sup_{\mathbb{T}^N} \{\mu u_i - v_i\}$ is achieved at $\overline{x}$.

Let $\mathcal{I} = \{i : \max \text{ is achieved for index } i\}$

➪ Case 1 : $\mathcal{I} = \{1, ..., m\}$ and $\overline{x} \in \mathcal{A}$.

$$\sum_i u_i(\overline{x}) \le \sum_i v_i(\overline{x}) \quad \text{and} \quad (\mu u_i - v_i)(\overline{x}) = M \text{ for all } i$$

$$\Rightarrow mM = \sum_i (\mu u_i - v_i)(\overline{x}) \le (1-\mu) m |u|_\infty$$

$$\Rightarrow M \le 0 \text{ when } \mu \to 1$$

Idea of the proof of the comparison theorem (2)

➪ Case 2 : $\mathcal{I} = \{1, ..., m\}$ and $\overline{x} \notin \mathcal{A}$.
$\Rightarrow \exists i$ s.t. $f_i(\overline{x}) > 0$
Subsolution and supersolutions inequalities for equation i :

$$\mu F_i(\overline{x}, \tfrac{D\mu u_i(\overline{x})}{\mu}) + \textstyle\sum_j d_{ij}\mu u_j(\overline{x}) \leq \mu f_i(\overline{x})$$
$$F_i(\overline{x}, Dv_i(\overline{x})) + \textstyle\sum_j d_{ij} v_j(\overline{x}) \geq f_i(\overline{x})$$

Therefore

$$\underbrace{\mu F_i(\overline{x}, \frac{D\mu u_i(\overline{x})}{\mu}) - F_i(\overline{x}, Dv_i(\overline{x}))}_{\geq 0 \text{ (convexity and max.point)}} + \underbrace{\sum_j d_{ij}(\mu u_j - v_j)(\overline{x})}_{=(\sum_j d_{ij})M} \leq \underbrace{(\mu - 1) f_i(\overline{x})}_{<0}$$

Idea of the proof of the comparison theorem (3)

➪ Case 3 : $\mathcal{I} \neq \{1, ..., m\}$

$D(\overline{x})$ irreducible $\Rightarrow \exists i \in \mathcal{I}, k \notin \mathcal{I}$ s.t. $d_{ik}(\overline{x}) < 0$

Equation for i (as in case 2) leads to
$\sum_j d_{ij}(\mu u_j - v_j)(\overline{x}) \leq (\mu - 1)f_i(\overline{x})$

But $k \notin \mathcal{I} \Rightarrow (\mu u_k - v_k)(\overline{x}) \leq M - \delta, \quad \delta > 0$

Therefore $(\sum_j d_{ij})M - \delta d_{ik}(\overline{x}) \leq (\mu - 1)f_i \leq 0$

contradiction

Ergodic problem (1)

$$\lambda v_i^\lambda + F_i(x, Dv_i^\lambda) + \sum_{j=1}^{m} d_{ij}(x) v_j^\lambda = f_i \quad \text{in } \mathbb{T}^N, i = 1, \dots, m$$

Theorem. There exists a unique solution which is Lipschitz continuous with constant L independent of λ.
Up to extract, as $\lambda \to 0$,

$$\begin{aligned} \lambda v^\lambda &\to -(c_1, ..., c_m) = -(c_1, ..., c_1) \in \ker D \\ v^\lambda - v^\lambda(x^*) &\to v \in \mathrm{Lip}(\mathbb{T}^N) \end{aligned}$$

and (c, v) is solution of

$$F_i(x, Dv_i) + \sum_{j=1}^{m} d_{ij}(x) v_j = f_i + c_1 \quad \text{in } \mathbb{T}^N, i = 1, \dots, m$$

➪ In fact $-c_1 = \min f_i = \overline{f} = \mathbf{0}$ here

Ergodic problem (2)

In a more general context

$$\sum_i \min f_i \le -mc_1 \le \min \sum_i f_i$$

Proof of the convergence theorem $u(x,t) - ct \to v(x)$ as $t \to +\infty$ (1)

Recall that $-c_1 = -\min f_i = 0$

Comparison for the evolution problem : $\exists C \geq |u_0|_\infty$ s.t. $v(x) - C \leq u(x,t) \leq v(x) + C$

The function $u^\varepsilon(x,t) = u(x, \frac{t}{\varepsilon})$ is solution to the system

$$\begin{cases} \varepsilon \dfrac{\partial u_i}{\partial t} + F_i(x, Du_i) + \displaystyle\sum_{j=1}^m d_{ij}(x) u_j = f_i(x) & \mathbb{T}^N \times (0,+\infty) \\ u_i(x,0) = u_{0,i}(x) & \mathbb{T}^N \end{cases}$$

for $i = 1, \ldots, m$.

Then, by stability, the half-relaxed limits

$$\overline{u}(x) = \limsup_{\varepsilon \to 0}{}^* u^\varepsilon(x,t) \quad \text{and} \quad \underline{u}(x) = \liminf_{\varepsilon \to 0}{}_* u^\varepsilon(x,t)$$

are respectively sub and supersolutions to the **stationary** system.

Proof of the convergence theorem $u(x,t) - ct \to v(x)$ as $t \to +\infty$ (2)

Summing the evolution equations for $1 \leq i \leq m$,

$$\frac{\partial}{\partial t}\Big(\sum_i u_i\Big) + \underbrace{\sum_i F_i(x, Du_i)}_{\geq 0} + \underbrace{\sum_i \sum_{j=1}^m d_{ij}(x) u_j}_{=\sum_j u_j \sum_{i=1}^m d_{ij}(x)=0} = \sum_i f_i(x)$$

But $\sum_i f_i(x) = 0$ on $\mathcal{A}$

Therefore

$$\frac{\partial}{\partial t}\Big(\sum_i u_i\Big) \leq 0 \quad \Rightarrow \quad \sum_i u_i(\cdot, t) \underset{t\to+\infty}{\to} \phi \text{ uniformly on } \mathcal{A}$$

Proof of the convergence theorem $u(x,t) - ct \to v(x)$ as $t \to +\infty$ (3)

Lemma. $\overline{u}_i = \underline{u}_i = \overline{u}_j = \underline{u}_j$ on $\mathcal{A}$

Therefore

$$\sum_i \overline{u}_i = \sum_i \underline{u}_i = \lim_{t \to +\infty} \sum_i u_i(\cdot, t) = \phi \quad \text{on } \mathcal{A}$$

Comparison theorem for the subsolution $\overline{u}$ and the supersolution $\underline{u}$ implies

$$\overline{u} = \underline{u} \quad \text{on } \mathbb{T}^N.$$