

On the construction of Patchy Feedbacks for stabilization and optimization problems

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On the construction of Patchy Feedbacks for stabilization and optimization problems

Fabio ANCONA

University of Padova

ITN SADCO - Kick off Meeting

Ensta Paris Tech, Paris

March 3 - 4, 2011

Outline

1. Introduction
   - General Setting
   - Main Difficulties
2. Patchy Vector Fields
   - Definition and Basic Properties
   - Robustness of Patchy Vector Fields
3. Patchy Feedbacks
   - Definition
   - Stabilizing Patchy Feedback
   - Nearly Optimal Patchy Feedback
   - ISS Patchy Feedback

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Consider a control system

$$\dot{x} = f(x,u), \qquad x \in \mathbb{R}^n, \qquad u \in \mathbb{R}^m \tag{1}$$

- f smooth with sublinear growth

$$\big|f(x,u)\big| \leq c\big(1+|x|\big) \qquad \forall\, x,u$$

(I) Asymptotic Feedback Stabilization: Seek feedback control $u = U(x)$ s.t. all trajectories of

$$\dot{x} = g(x) \doteq f\big(x, U(x)\big) \tag{2}$$

approach the origin as $t \to \infty$ (and the origin is Lyapunov stable).

(II) Optimal Feedback Control: Given compact set $K \subset \mathbb{R}^n$, seek feedback control $u = U(x)$ s.t. all sol'ns of (1) reach given target set S in minimum time.

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- Smooth feedback controls that solve (I)-(II) in general may not exist

  (I) Topological obstructions may prevent the existence of smooth (or even continuous) stabilizing feedback
  (SUSSMANN '79; SONTAG & SUSSMANN '80; BROCKETT '83; CORON '90)

  Ex: Brockett's nonholonomic integrator

$$\left.\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1 \end{aligned}\right\} \doteq f(x,u) \qquad \begin{aligned} &(x_1, x_2, u_1, u_2) \to f(x,u) \\ &\text{not open at zero :} \\ &(0,0,x_3) \notin \mathrm{Im}(f) \quad \text{when} \quad x_3 \neq 0 \end{aligned}$$

  (II) Optimal feedback can be discontinuous with quite complicated structure, while optimal (open-loop) controls may have infinitely many switchings

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- Discontinuous stabilizing (or optimal) feedback in general are not robust w.r.t. (inner and outer) perturbations

$$\dot{x} = f\big(x,\, U(x + e_1(t))\big) + e_2(t) \tag{3}$$

(e_1 : measurement error, e_2 : external disturbance)

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Two possible approaches

in dealing with stabilization and optimization problems for

$$\dot{x} = f(x, U(x)).$$

1. Allow arbitrary discontinuous feedback controls $u = U(x)$ and work with weak or generalized solutions of discontinuous ODEs.

2. Use a particular class of feedback controls $U(x)$ whose discontinuities are sufficiently tame so to guarantee the existence of Carathéodory solutions

$$x(t) = \overline{x} + \int_0^t f\big(x(s), U(x(s))\big)\, ds.$$

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How to interpret solutions to discontinuous ODEs ?

- Define weaker concept of solution
  For example, one can consider Krasovskii sol'ns, i.e. trajectories of the differential inclusion

$$\dot{x} \in G(x) \doteq \bigcap_{\varepsilon>0} \overline{co}\Big\{g(y)\,;\ \ |y-x|<\varepsilon\Big\}, \qquad g(y) \doteq f(x,U(x))\,. \tag{4}$$

  Drawback: too many sol'ns are obtained. Not all enjoy desired properties.

- Introduce suitable generalized solutions:
  "Sample-and-hold-solutions" & "Euler solutions" : successfully implemented for problems (I)-(II)
  (CLARKE, LEDYAEV, SONTAG & SUBBOTIN '97; RIFFORD 2000, 2002)

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S-a-h solutions & Euler solutions of $\dot{x} = f(x, U(x))$

S-a-h solution $x(t)$ recursively defined as follows.

1. fix a time mesh $t_0 = 0 < t_1 < \ldots < t_i < \ldots$, with size $\delta \doteq \sup |t_{i+1} - t_i|$;
2. sample $U(x(t_i)) \doteq u_i$ and compute $x(t) \doteq x_{i+1}(t), t \in]t_i, t_{i+1}]$ solving

$$\dot{x}_{i+1} = f(x_{i+1}, u_i), \ x_{i+1}(t_i) = x_i(t_i) \qquad \text{on }]t_i, t_{i+1}].$$

Euler solution is a (uniform) limit of S-a-h solutions as size $\delta \to 0$.

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An alternative approach

1. Introduce a class of discontinuous, piecewise smooth vector fields "Patchy Vector Fields" (PVF) which admit (forward) Carathéodory solutions, and analyze their properties.
2. Construct piecewise smooth stabilizing or optimal feedback $U = U(x)$ so that corresponding vector field $g(x) \doteq f(x, U(x))$ belongs to such class of PVF.
3. Establish robustness properties of stabilizing or optimal feedback in this class, relative to external disturbances and internal measurement errors.

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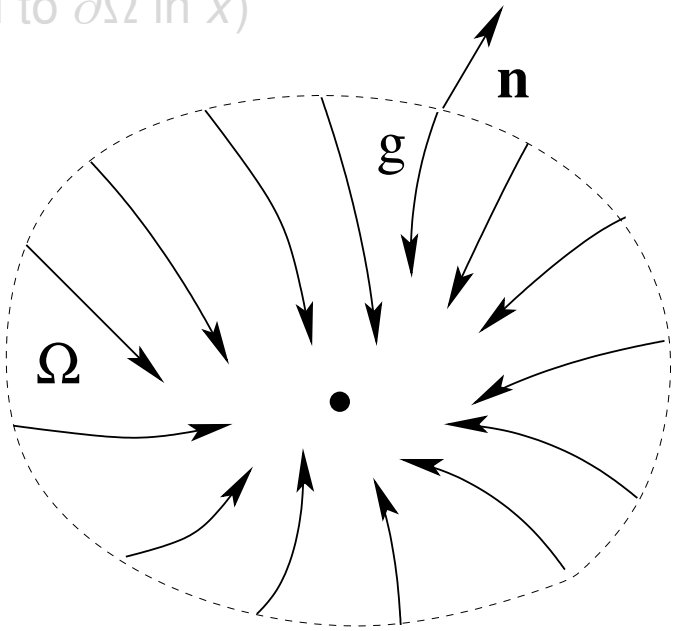
- Patches

  (Ω, g) is said to be a patch if:

  - $\Omega \subset \mathbb{R}^n$ is open domain with (piecewise) smooth boundary $\partial\Omega$
  - g smooth vector field on a neighborhood of $\partial\Omega$, that points strictly inward at each point $x \in \partial\Omega$

$$\langle g(x), \mathbf{n}(x) \rangle < 0 \qquad \forall\, x \in \partial\Omega$$

($\mathbf{n}(x)$ outer normal to $\partial\Omega$ in x)



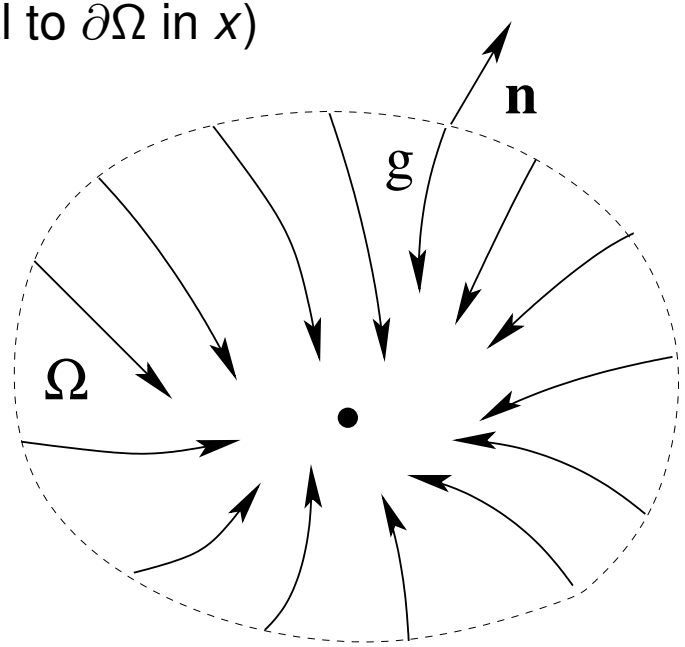
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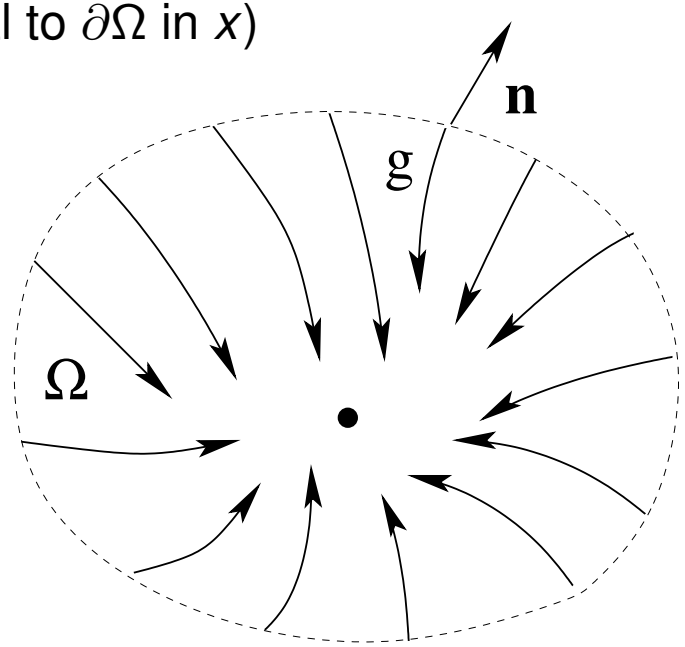
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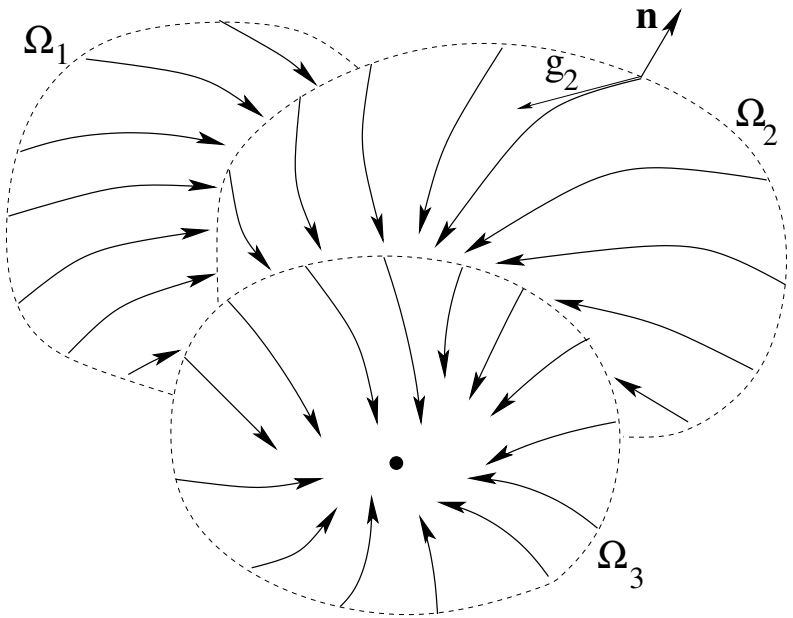
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- Patchy Vector Fields

  locally finite covering in terms of patches $(\Omega_i,\ g_i),\ i \in \mathbb{Z}$.
  $g : \Omega \to \mathbb{R}^n$ is said to be a patchy vector field if:

$$g(x) = g_i(x) \iff x \in \Omega_i \setminus \bigcup_{j>i} \Omega_j$$

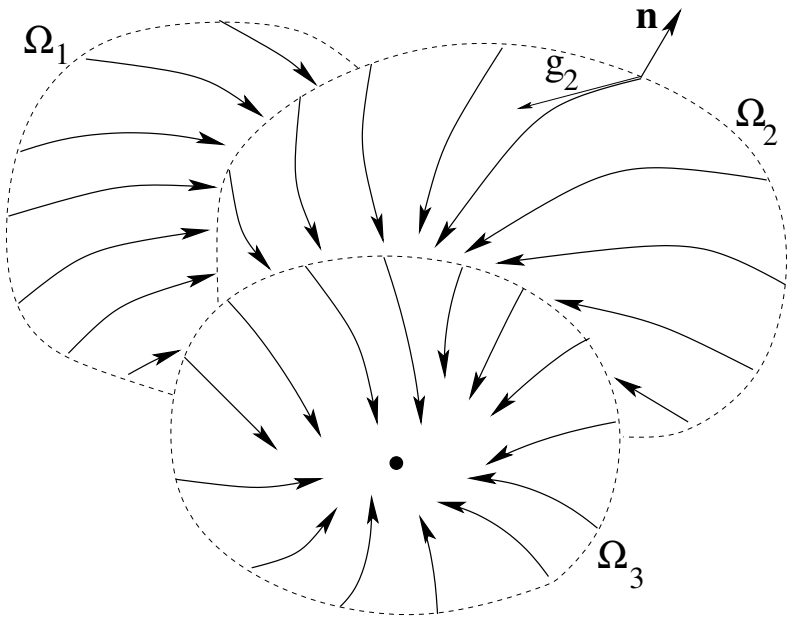


$$\Longrightarrow \quad g(x) = g_{i^*(x)}(x),\ \ i^*(x) \doteq \max\{i \in \mathbb{Z}\ :\ x \in \Omega_i\}$$

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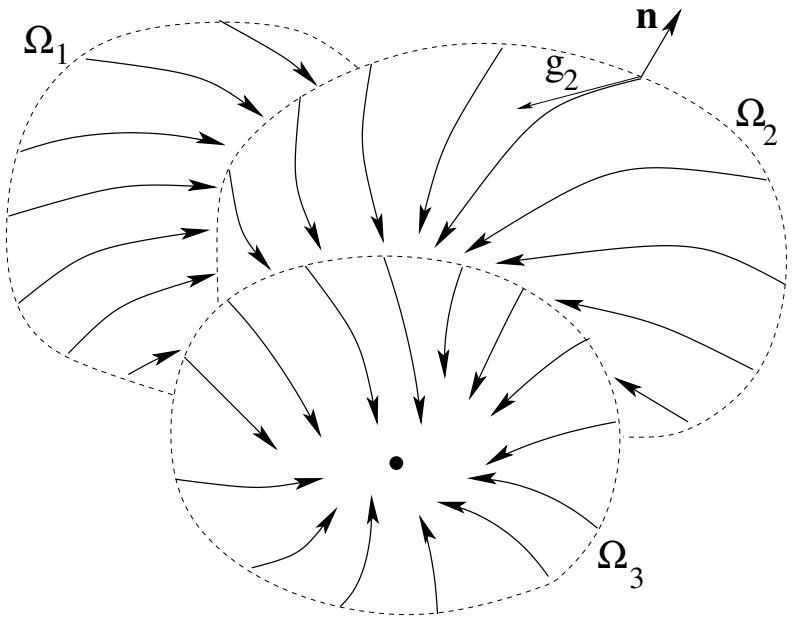


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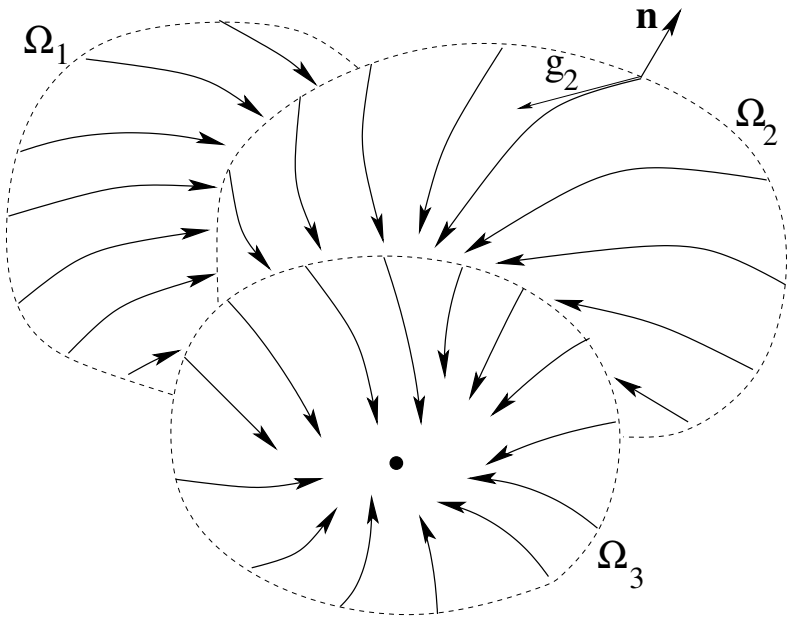


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Properties of Patchy Vector Fields

$$\dot{x} = g(x) = g_{i^*(x)}(x)$$

ODEs with patchy right hand side (A. & BRESSAN '99)

Let g be a patchy vector field on $\Omega \subset \mathbb{R}^n$. Then the following holds.

i) For every $\overline{x} \in \Omega$

$$\begin{cases} \dot{x} = g(x) \\ x(0) = \overline{x} \end{cases} \quad \text{has} \quad \boxed{\begin{array}{l} \text{at least 1 (local) forward} \\ \text{at most 1 backward} \end{array}} \quad \text{Carathéodory soln}$$

ii) sol'ns $x(t)$ are piecewise smooth, with derivatives having finite set of jumps on any compact time interval. The map $t \mapsto i_*(x(t))$ is piecewise constant, non-decreasing.

iii) Set of Carathéodory sol'ns is closed in the topology of uniform convergence.

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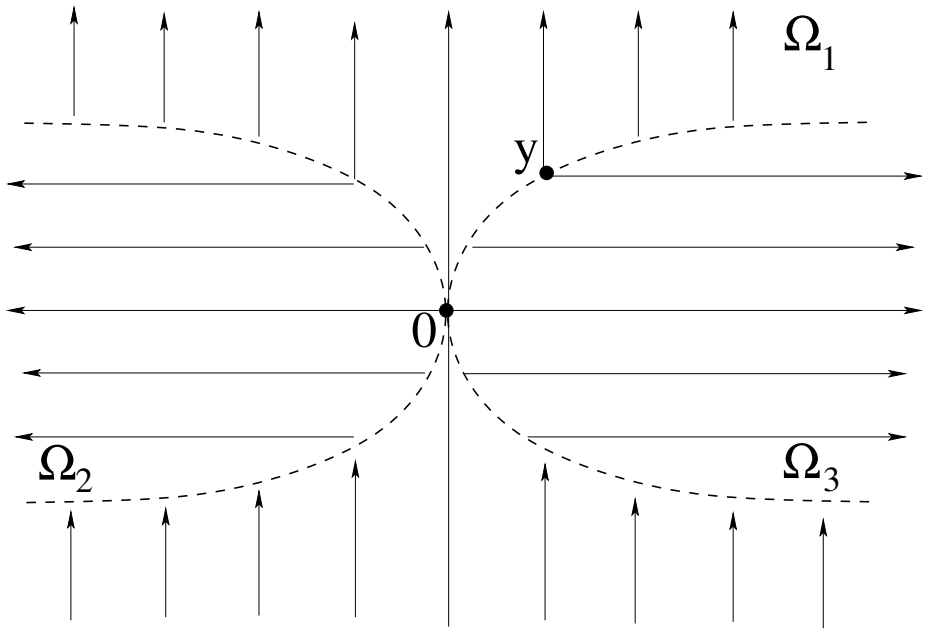
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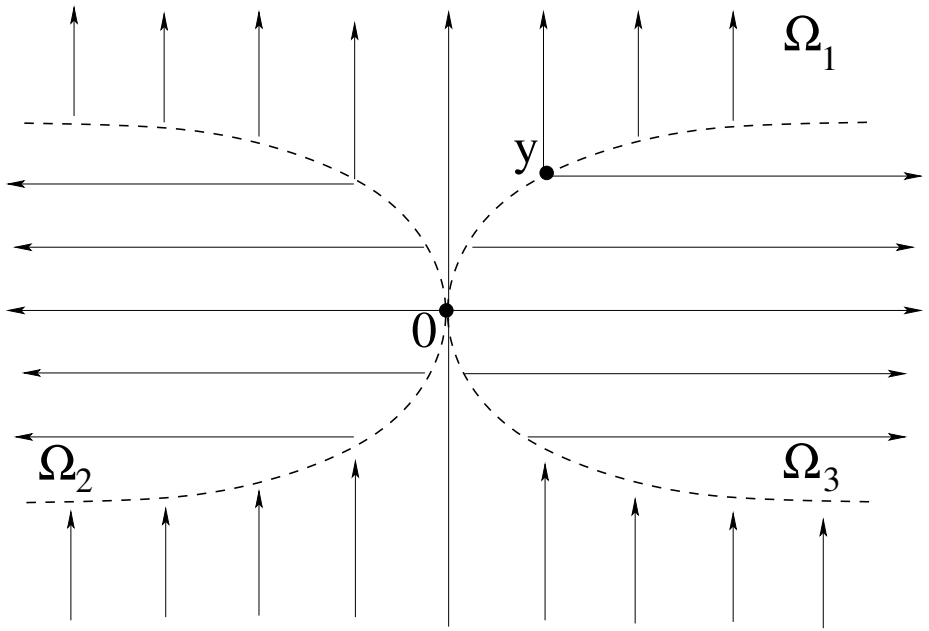
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Robustness w.r.t. (small) inner and outer perturbations

$$\dot{y} = g(y + e_1(t)) + e_2(t) \tag{5}$$

Flow stability of a Patchy Vector Field (A. & BRESSAN 2003, 2004)

Let g be a patchy vector field on $\Omega \subset \mathbb{R}^n$. Given any compact subset $C_0 \subset \Omega$, and any $T, \varepsilon > 0$, there exists $\delta > 0$ such that the following holds.
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$$\|e_1\|_{BV} \leq \delta\,, \qquad \|e_2\|_{\mathbf{L}^1} \leq \delta\,,$$

then there exists a solution $x : [0, T] \mapsto \Omega$ of the unperturbed equation

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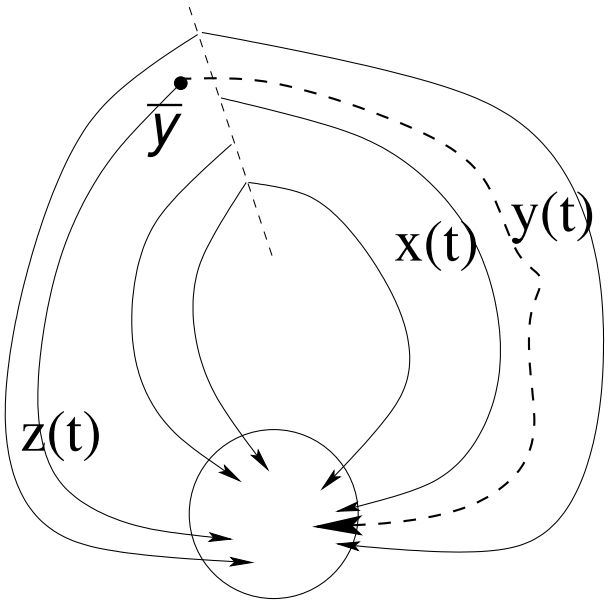
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Note: Since Cauchy problem for PVF lacks of forward uniqueness & continuous dependence, one cannot expect to have single sol'ns of PVF be stable w.r.t. perturbations.

Here we have stability property involving whole Carathéodory solution set: upper semicontinuity property of the solution set.



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$$\dot{x} = f(x,u), \qquad x \in \mathbb{R}^n, \quad u \in K \subset \mathbb{R}^m$$

A piecewise smooth map $U : \Omega \to \mathbb{R}^m$ is a patchy feedback control on open domain $\Omega \subset \mathbb{R}^n$ if:

- there exist $U_i : \Omega_i \to \mathbb{R}^m$, $i \in \mathbb{Z}$, smooth maps, with $\{\Omega_i\}_{i\in\mathbb{Z}}$ locally finite covering of Ω;
- (Ω_i, g_i), with

$$g_i(x) \doteq f(x, U_i(x)), \qquad x \in \Omega_i,$$

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$$U(x) = U_i(x) \qquad \forall\, x \in \Omega_i \setminus \bigcup_{j>i} \Omega_j.$$

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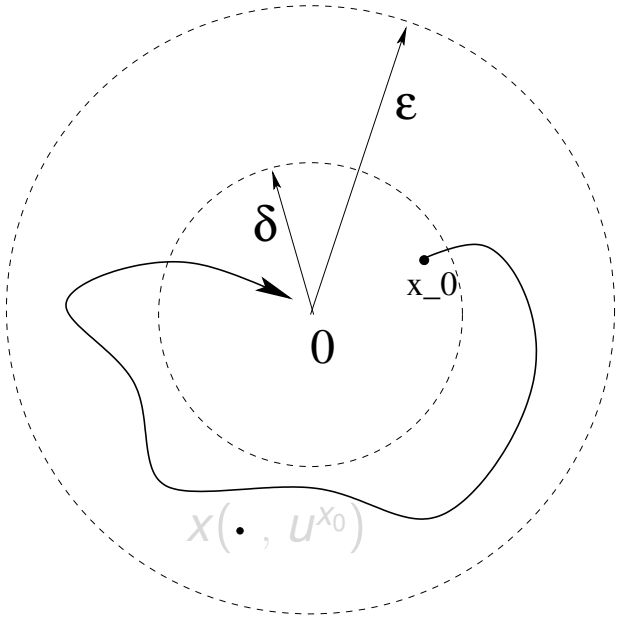
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Stabilization by Patchy Feedback (A. & BRESSAN 1999)

Assume: (6) is globally asymptotically controllable to the origin:

1. Attractiveness: $\forall\ x_0 \in \mathbb{R}^n\ \exists$ admissible open-loop control $u^{x_0}(t)$ s.t. the trajectory $x(\cdot, u^{x_0})$ starting at x_0 is defined $\forall\ t \geq 0$ and satisfies $x(t, u^{x_0}) \to 0$ as $t \to \infty$.
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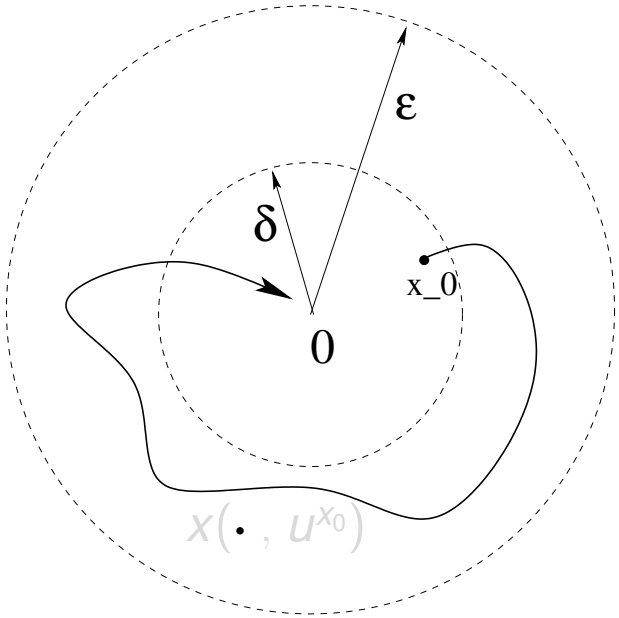


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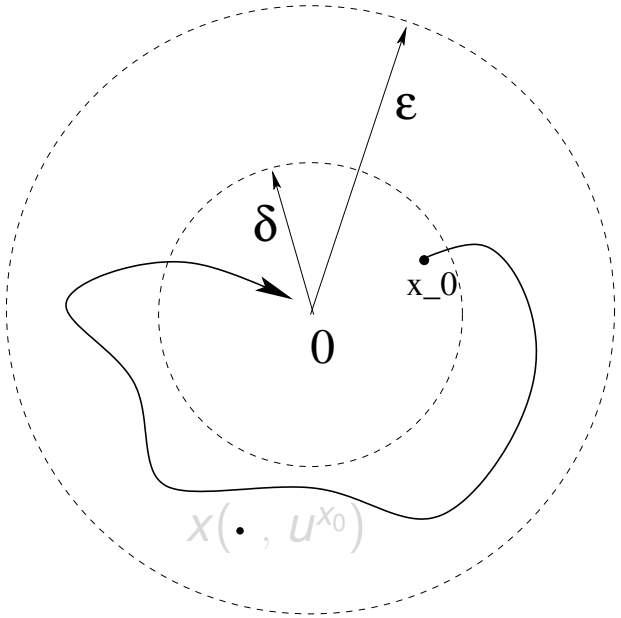


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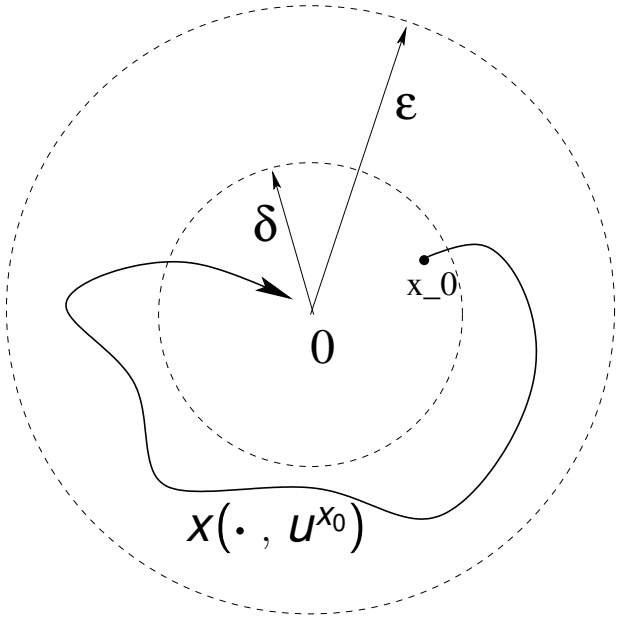


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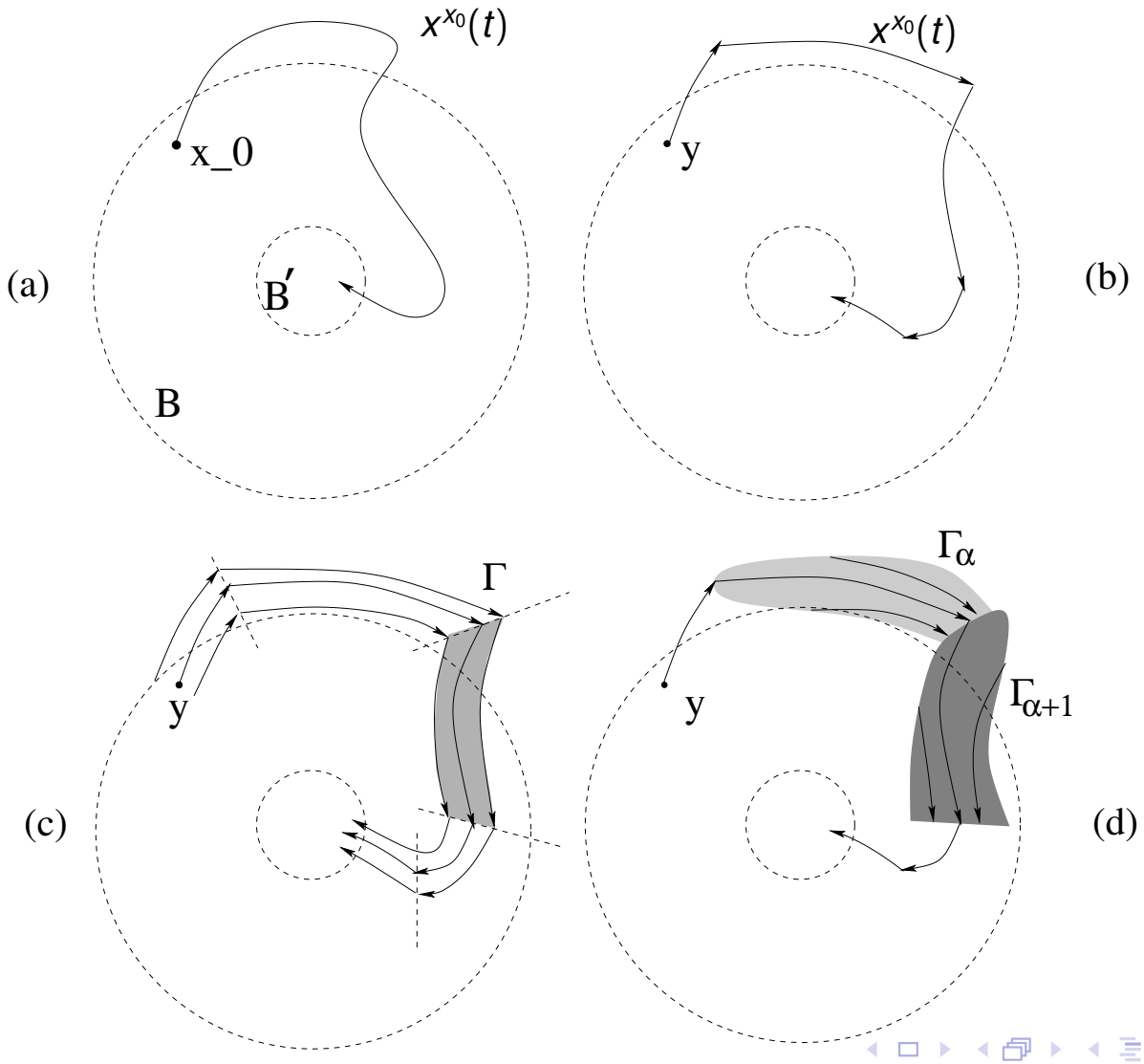
Stabilization by Patchy Feedback ... continued

Then: there exists a patchy feedback control $U(x)$ that makes asymptotically stable the closed-loop system

$$\dot{x} = f(x, U(x)) \tag{7}$$

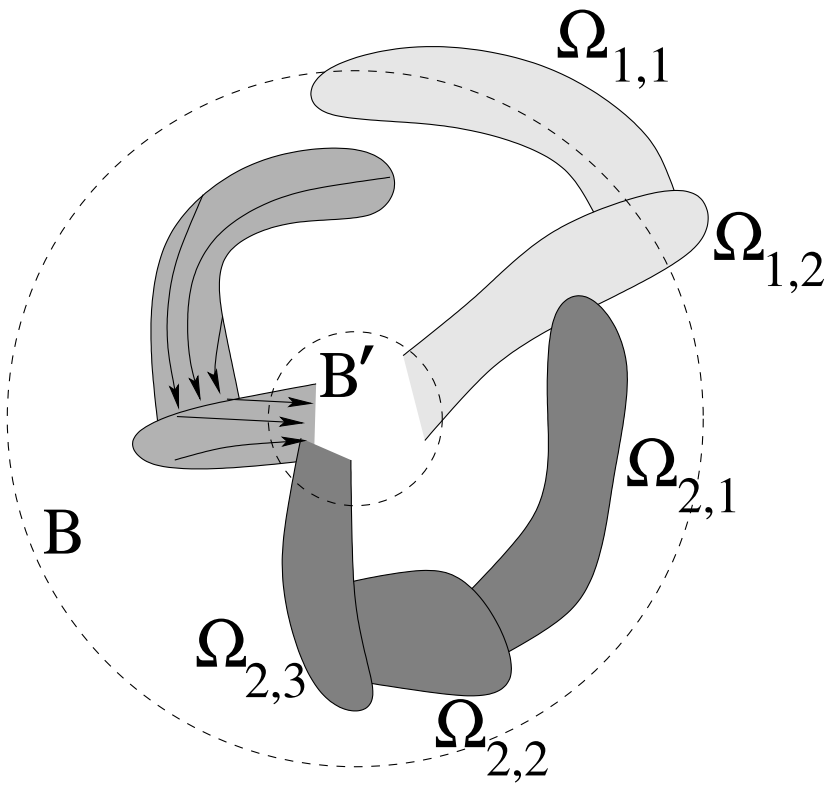
every Carathéodory solution of (7) either tends asymptotically to the origin, or reaches the origin in finite time. System enjoys (7) Lyapunov stability.

Construction of a stabilizing patchy feedback by patching together open-loop controls



Semi-global practical stabilization by a patchy feedback

Construct a patchy feedback that steers every point of a ball B inside a smaller ball $B' \subsetneq B$, within finite time.



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- f smooth with sublinear growth
- $T(y) \doteq$ minimum time to steer the system from y to the origin

Nearly Time Optimal Patchy Feedback

(A. & Bressan 2007; Bressan & Priuli 2008)

Given $\varepsilon > 0$, $\tau > 0$, there exists a sub-optimal patchy feedback $u = U(x)$ defined on sub-level set

$$R_\tau = \{y \,:\, T(y) \leq \tau\},$$

such that $\forall\, y \in R_\tau$, $|y| > \varepsilon$, any (Carathéodory) solution of

$$\dot{x} = f(x, U(x)) \qquad x(0) = y,$$

reaches the ε-ball $B_\varepsilon = \{|x| \leq \varepsilon\}$ within time $T(y) + \varepsilon$.

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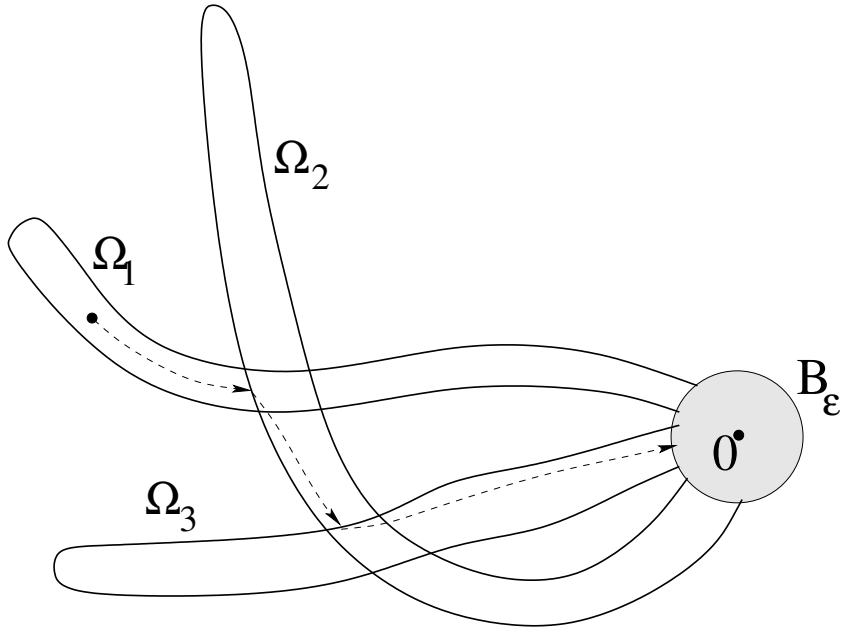
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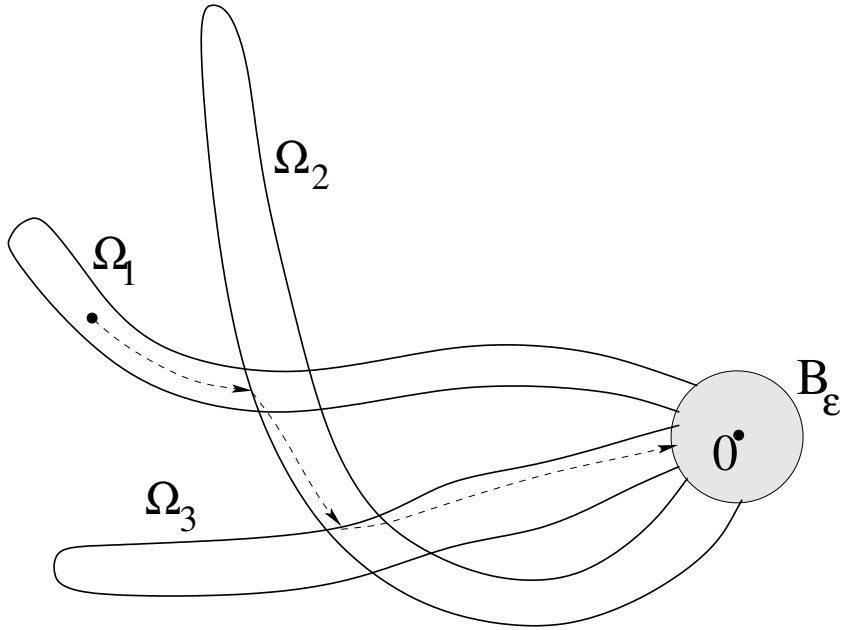
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Alternative construction: work with the value function

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Alternative construction: work with the value function

Construction of Nearly Time Optimal Patchy Feedback

- Approximate the minimum time function $T(\cdot)$ with a semi-concave, piecewise quadratic function:

$$\widetilde{T}(x) = \min_{j=1,\dots,N} \left\{ a_j + \mathbf{b}_j \cdot x \right\} + M \cdot |x|^2.$$

- Set of points where $\widetilde{T}$ not differentiable is contained in finite union of hyperplanes

$$\Gamma_{i,j} \doteq \left\{ a_i + \mathbf{b}_i \cdot x = a_j + \mathbf{b}_j \cdot x \right\}.$$

- Level sets $\widetilde{T}(x) = \text{const.}$ contained in union of spherical surfaces.

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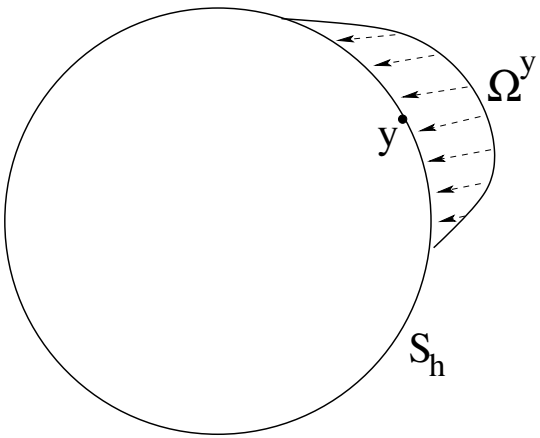
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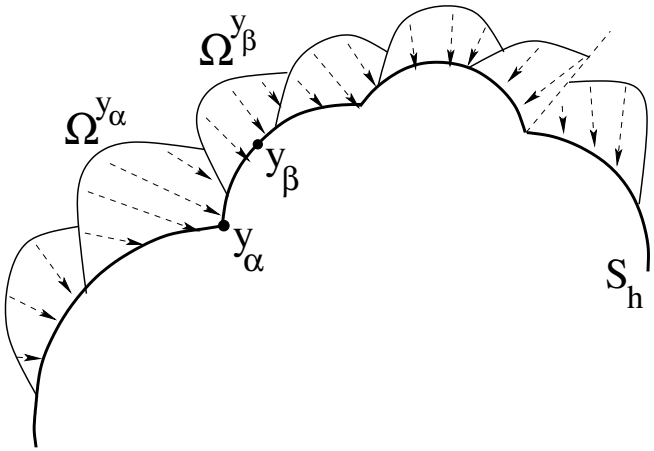
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Construction of Nearly Time Optimal Patchy Feedback ... continued

- Construct the patchy feedback by induction, progressively working on larger sub-level sets of $\widetilde{T}$





Outline

1. Introduction
   - General Setting
   - Main Difficulties
2. Patchy Vector Fields
   - Definition and Basic Properties
   - Robustness of Patchy Vector Fields
3. Patchy Feedbacks
   - Definition
   - Stabilizing Patchy Feedback
   - Nearly Optimal Patchy Feedback
   - ISS Patchy Feedback

ISS Stability: robustness of stabilization in presence of large external perturbations (SONTAG 1989)

- In presence of large external disturbances, solutions remain in a ball around the origin of the same size of the $\mathbf{L}^\infty$-strength of the perturbation
- If there exists a continuous stabilizing feedback control $U(x)$, then one can easily modify U so that the resulting system becomes ISS
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Consider affine system

$$f_0(x) + \sum_{i=1}^{m} u_i\, f_i(x) \qquad\qquad x \in \mathbb{R}^n$$

ISS Stability S-a-h solns (MALISOFF, RIFFORD & SONTAG 2004)

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} \Big(U_i(x + \zeta(t)) + d_i(t) \Big) f_i(x)$$

ISS Stabilizing Patchy Feedback for Carathéodory solns
(A. & PRIULI 2011)

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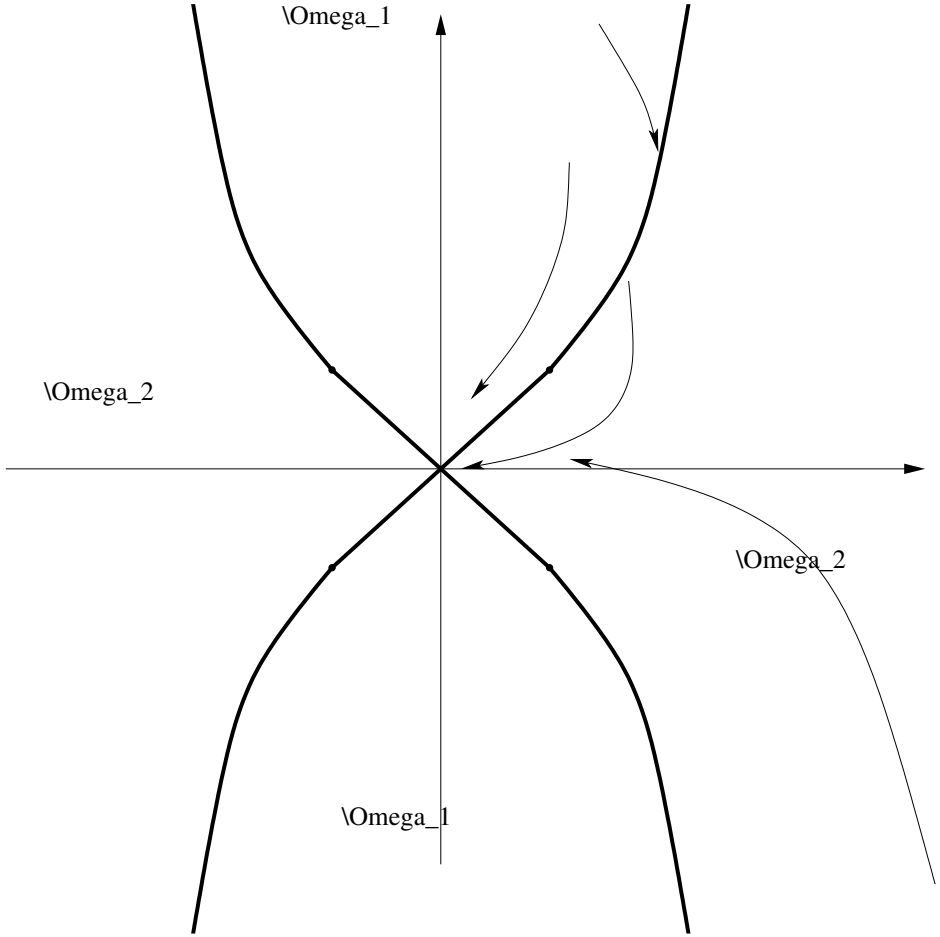
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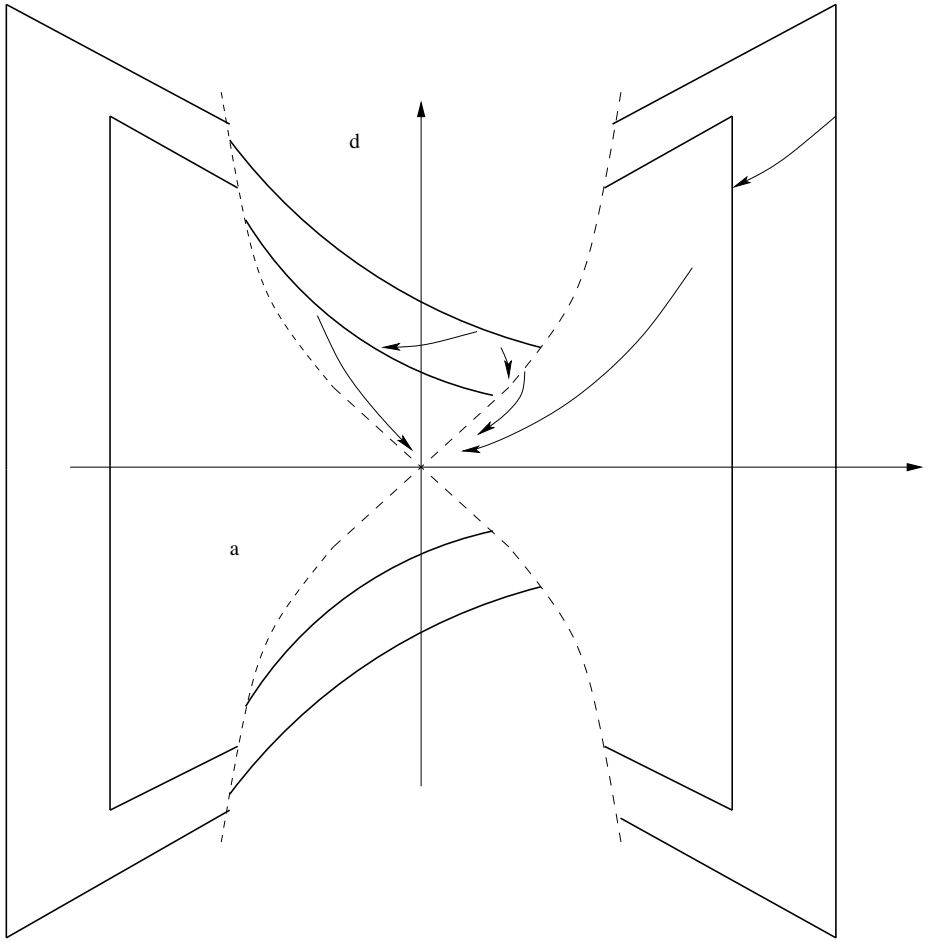
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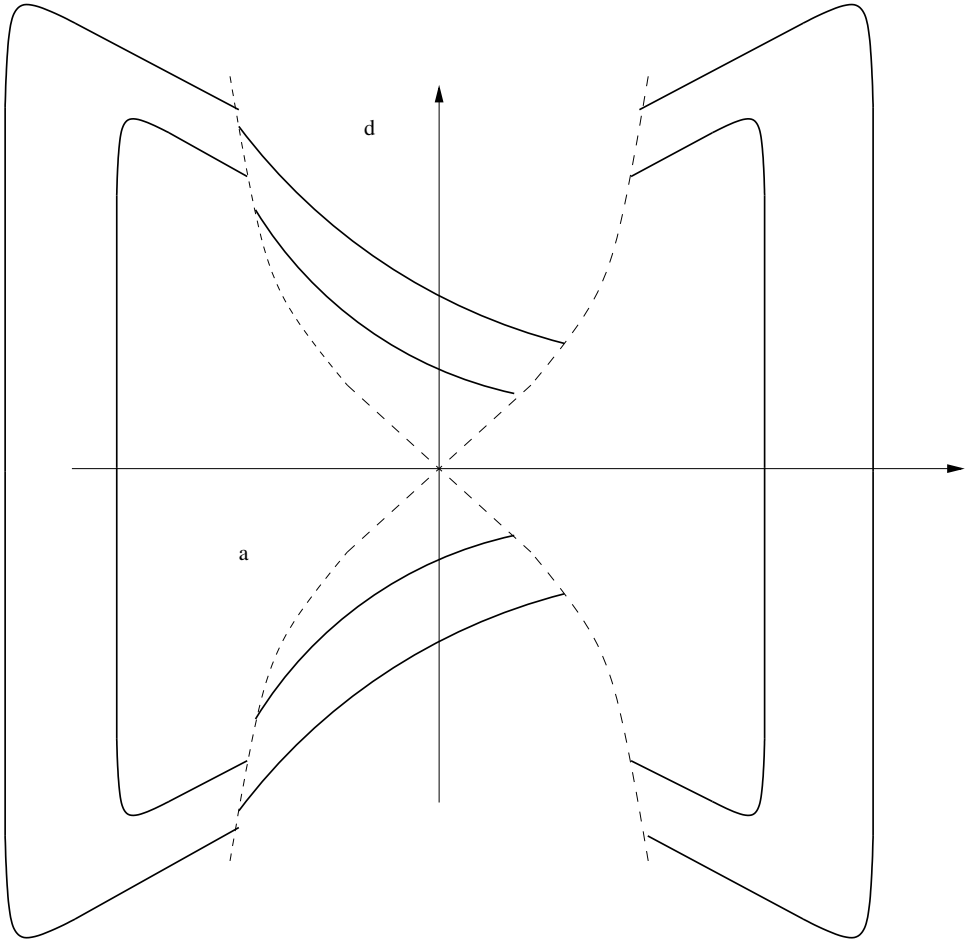
Stabilizing Patchy Feedback for Nonholonomic Integrator



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Future directions

- Costruction of stabilizing or sub-optimal patchy feedback for control problems with constraints.
- Costruction of stabilizing or sub-optimal hybrid patchy feedback. (GOEBEL, PRIEUR, TEEL)
- Introduction of numeric schemes to solve HJ or HJB equations, based on patchy decomposition technique.

(NEVASCA & KRENER 2007-2008; FALCONE & PICARELLI 2011)

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et finalement....

Merci de votre attention!