Stability and robustness of nonlinear predictive control without stabilizing terminal constraints

Lars Grüne, Anders Rantzer, Nils Altmüller, Thomas Jahn, Jürgen Pannek, Karl Worthmann

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Mathematisches Institut, Universität Bayreuth

in collaboration with

Anders Rantzer (Lund)
Nils Altmüller (Bayreuth), Thomas Jahn (Bayreuth),
Jürgen Pannek (Perth), Karl Worthmann (Bayreuth)

supported by DFG priority research program 1305
"Control theory for digitally networked dynamical systems"

SADCO Kick-Off Meeting, Paris, March 2011

Setup

We consider nonlinear discrete time control systems

$$x(n+1) = f(x(n), u(n))$$

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possibly subject to state/control constraints

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We obtain a feedback law F_N by a moving horizon technique

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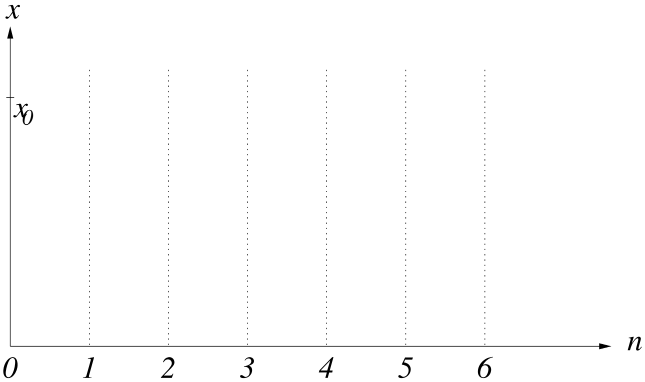
$\rightsquigarrow$ feedback controlled system ("closed loop")

$$x(n+1) = f(x(n), F_N(x(n))) = f(x^{opt}(0), u^{opt}(0)) = x^{opt}(1)$$

MPC from the trajectory point of view



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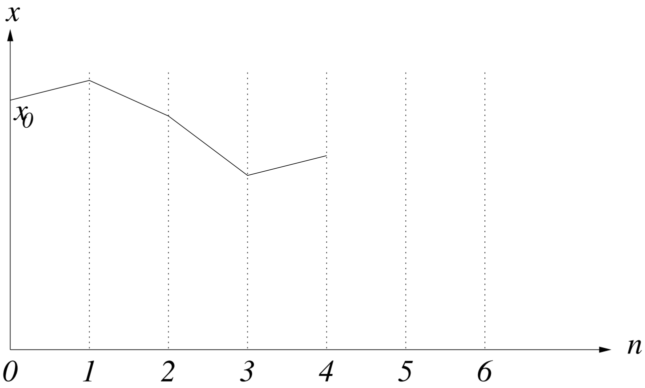

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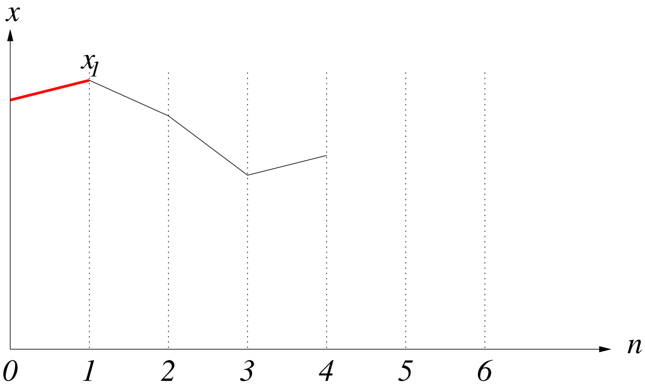

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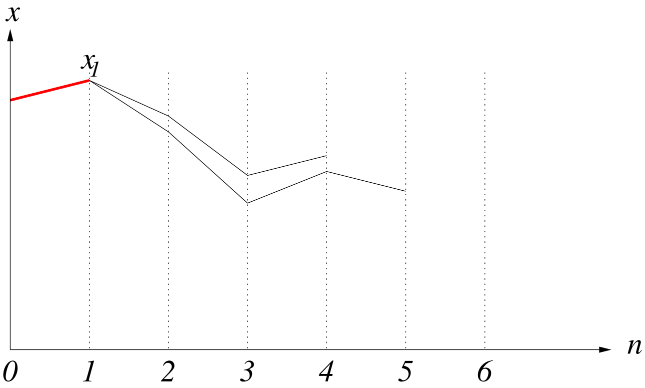

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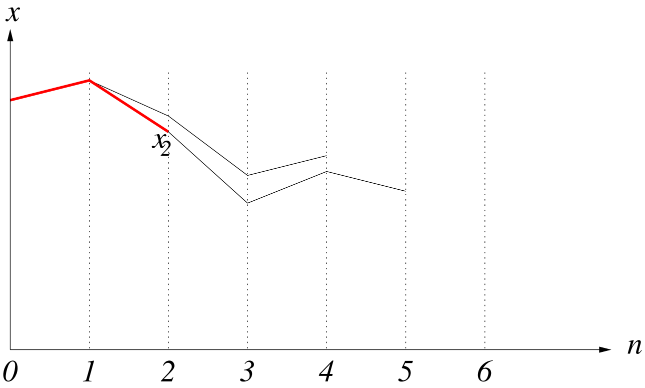

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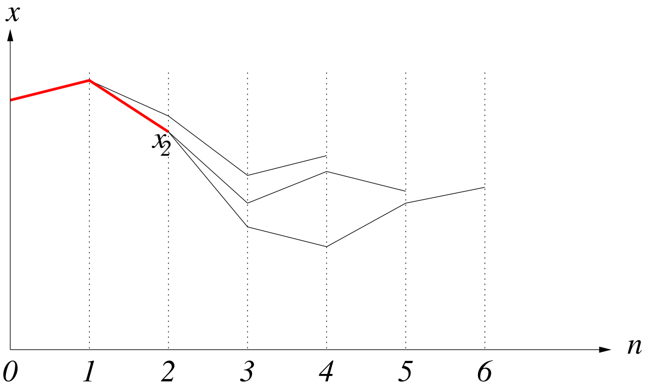

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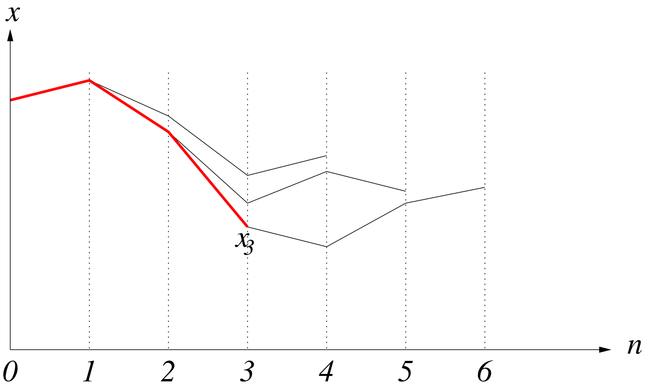

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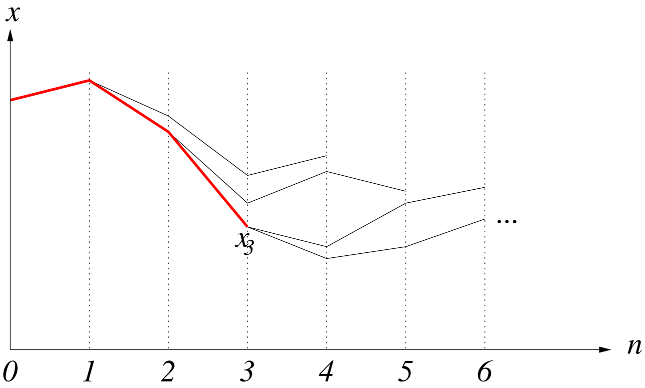

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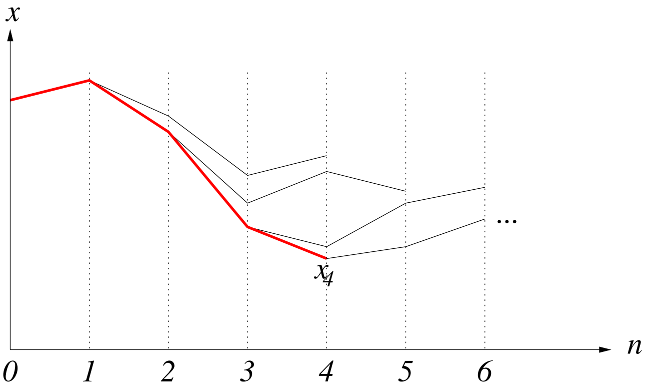

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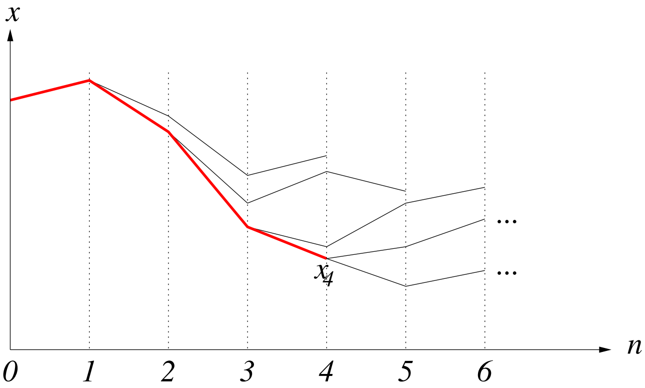

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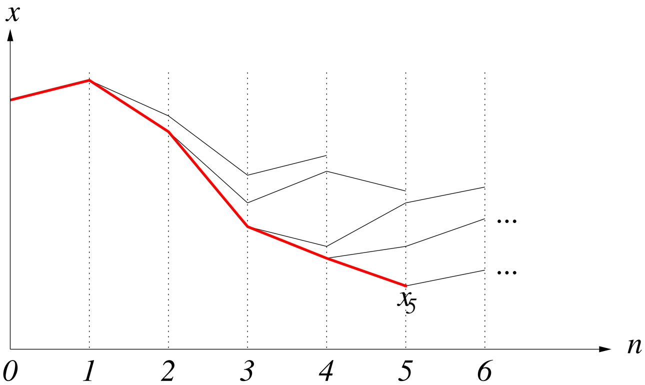

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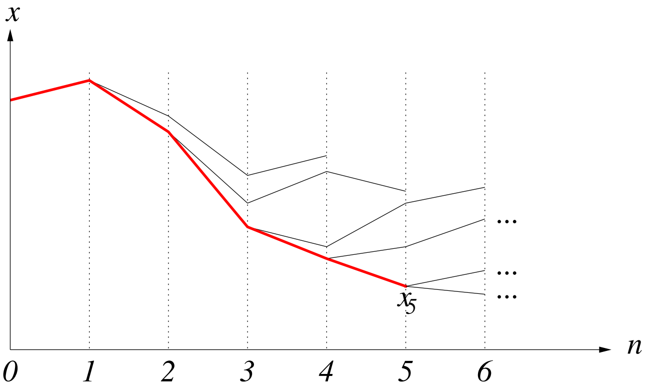

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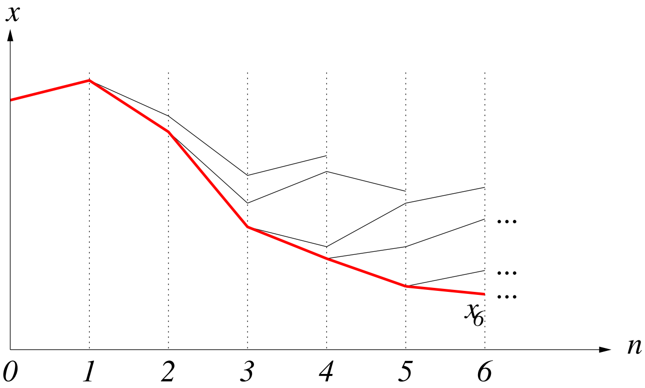

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How large is "sufficiently large"?

Estimating N

For obtaining a quantitative estimate we need quantitative information.

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A suitable condition is "exponential controllability through ℓ":

> there exist real numbers $C > 0$, $\sigma \in (0,1)$ such that for each $x(0) \in X$ there is $u(\cdot)$ with
>
> $$\ell(x(n), u(n)) \leq C\sigma^n \ell^*(x(0))$$
>
> with $\ell^*(x) = \min_u \ell(x,u)$

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Theorem: If $\alpha > 0$, then the MPC feedback F_N stabilizes all C, σ-exponentially controllable systems and we get $J_\infty(x, F_N) \leq \inf_{u \in U^\infty} J_\infty(x, u)/\alpha$

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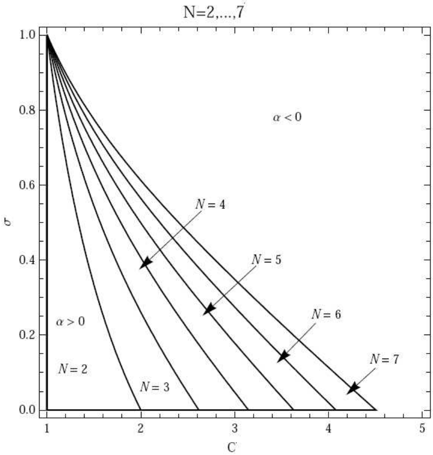
Moreover, $\alpha \to 1$ as $N \to \infty$

Stability chart for C and σ



(Figure: Harald Voit)

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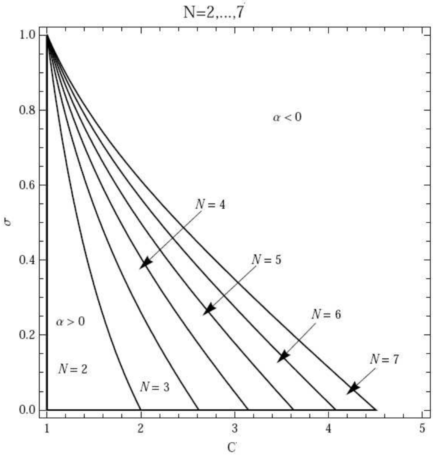

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Conclusion: try to reduce C, e.g., by choosing ℓ appropriately

A PDE example

We illustrate this with the 1d controlled PDE

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y) + u$$

with

domain $\Omega = [0, 1]$

solution $y = y(t, x)$

boundary conditions $y(t, 0) = y(t, 1) = 0$

parameters $\nu = 0.1$ and $\mu = 10$

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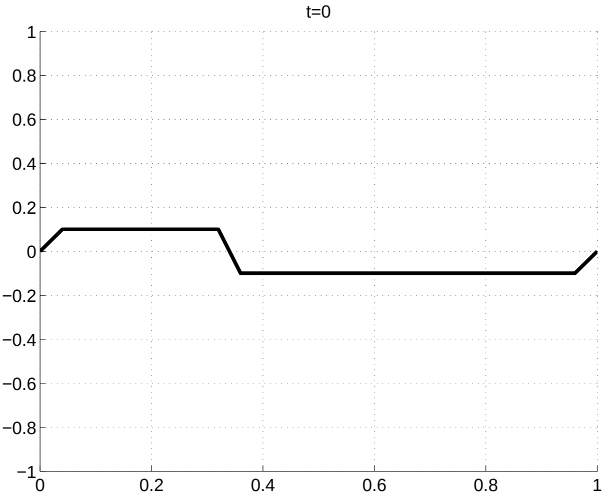
Discrete time system: $y(n) = y(nT, \cdot)$ for some $T > 0$

("sampled data system with sampling time T")

The uncontrolled PDE



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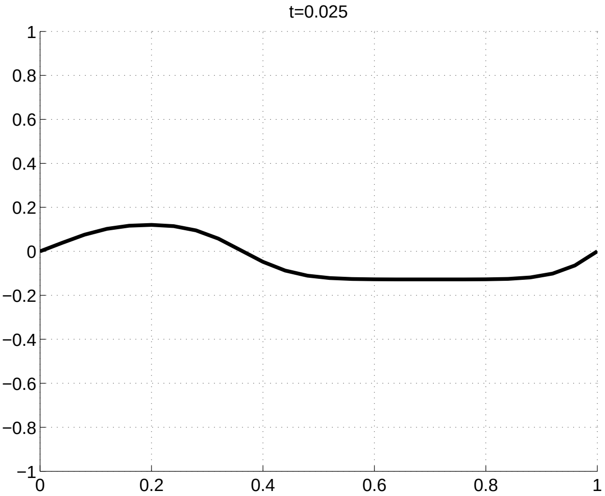

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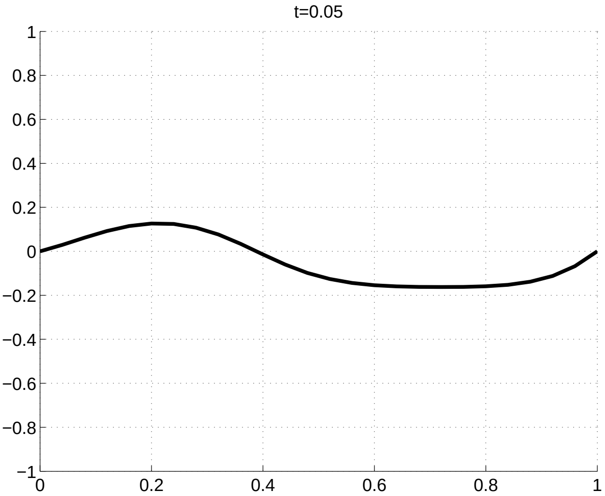

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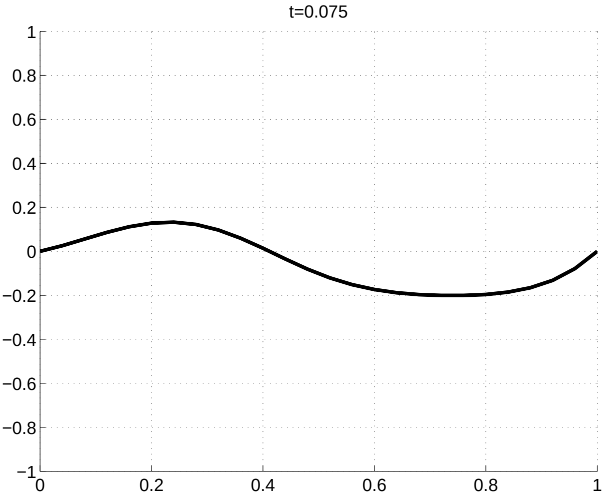

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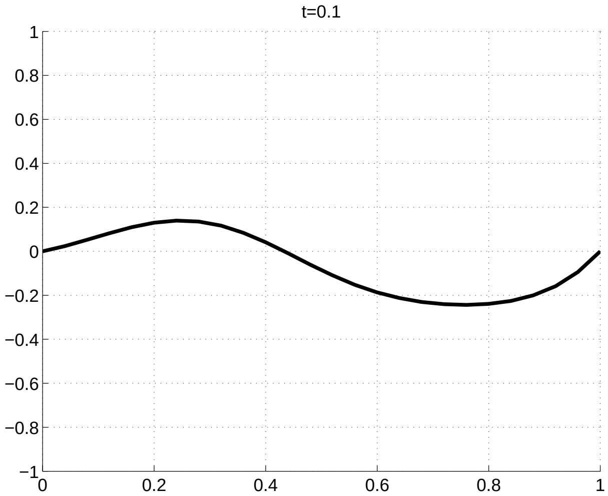

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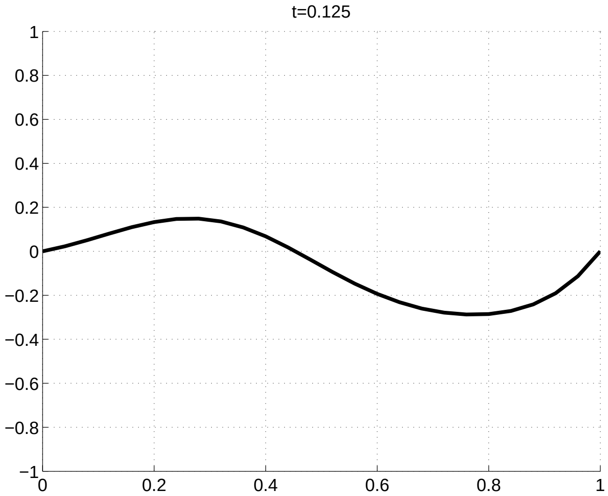

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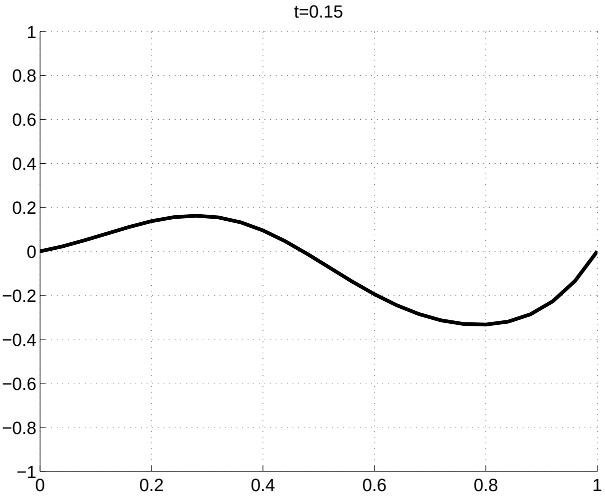

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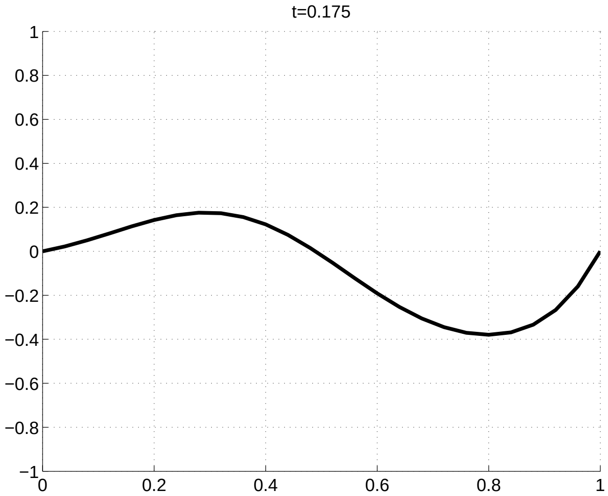

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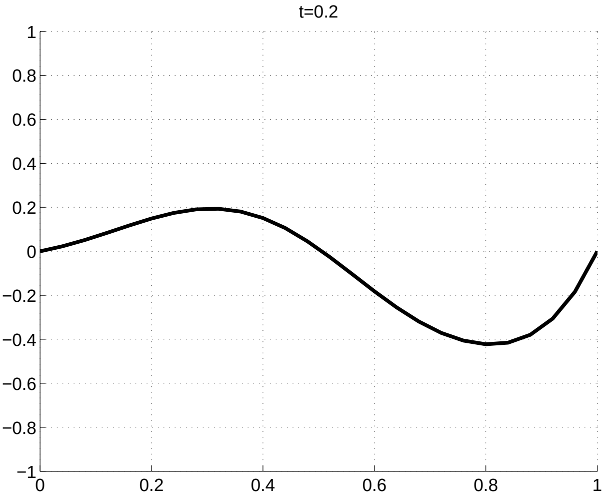

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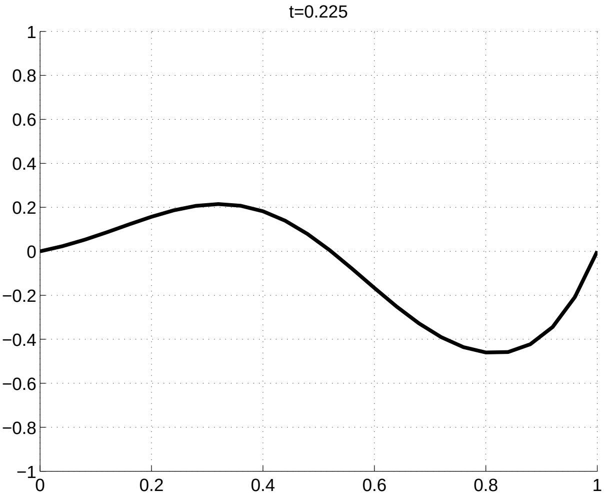

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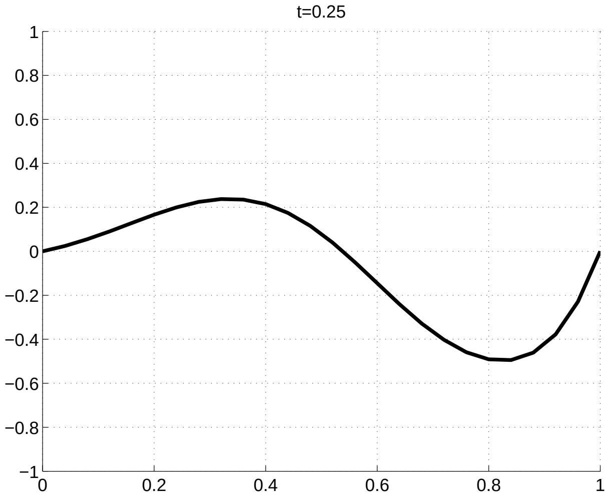

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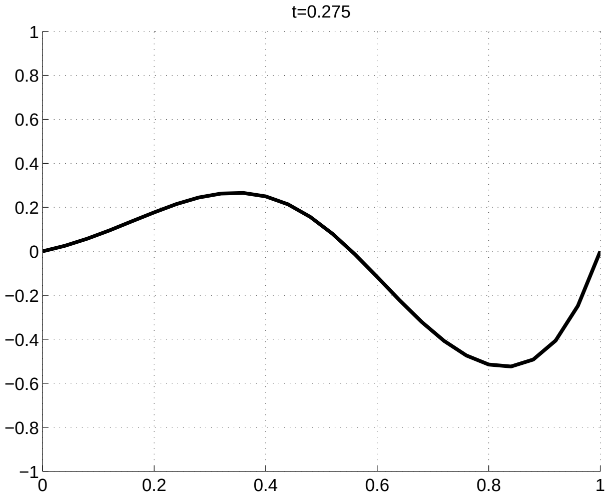

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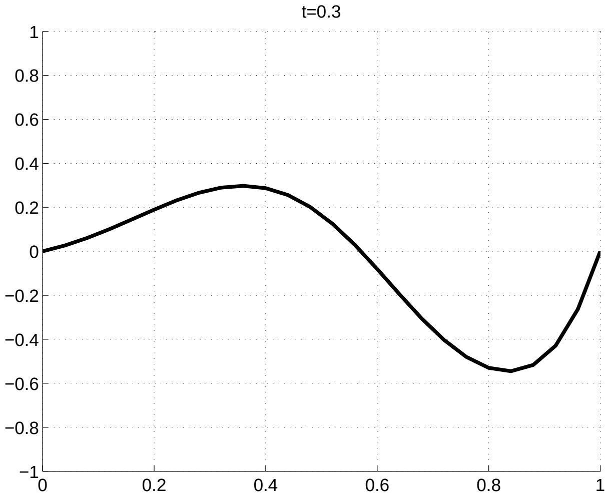

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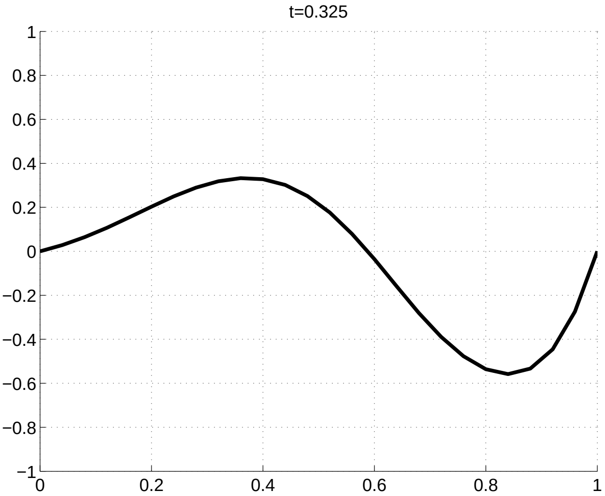

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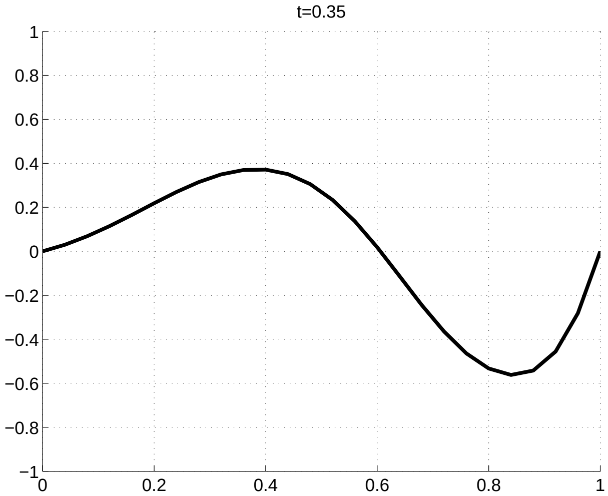

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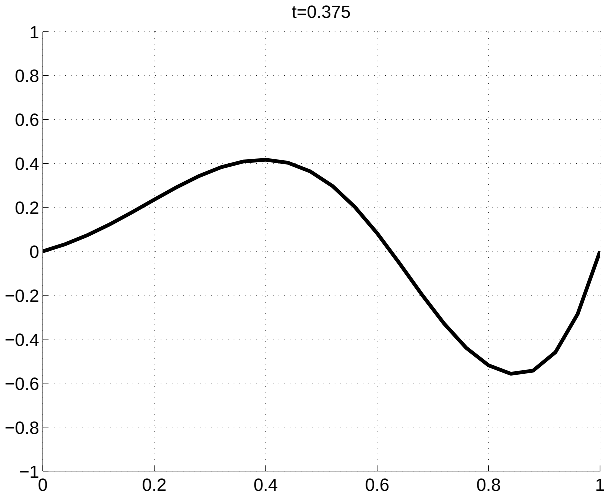

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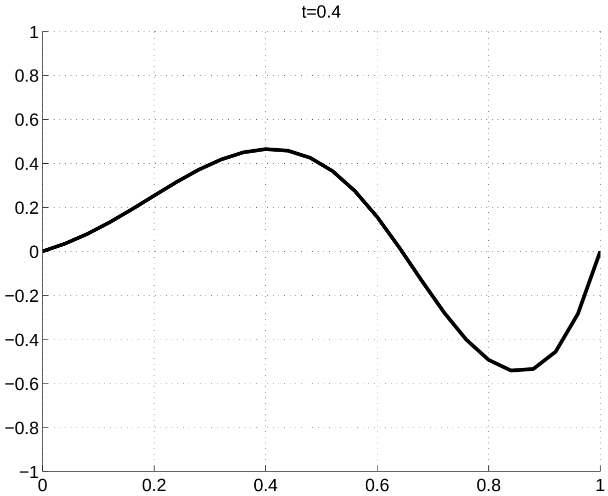

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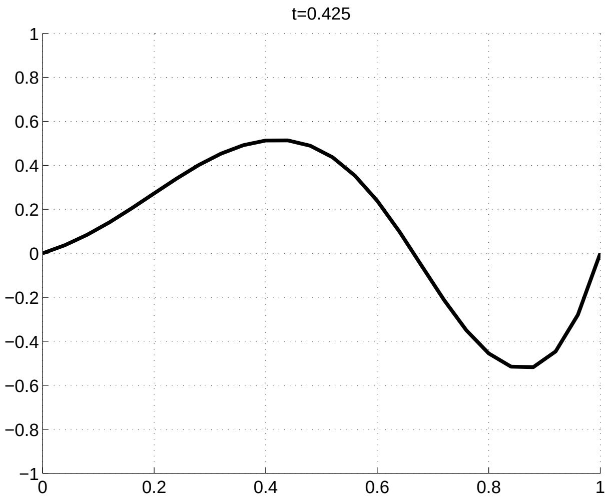

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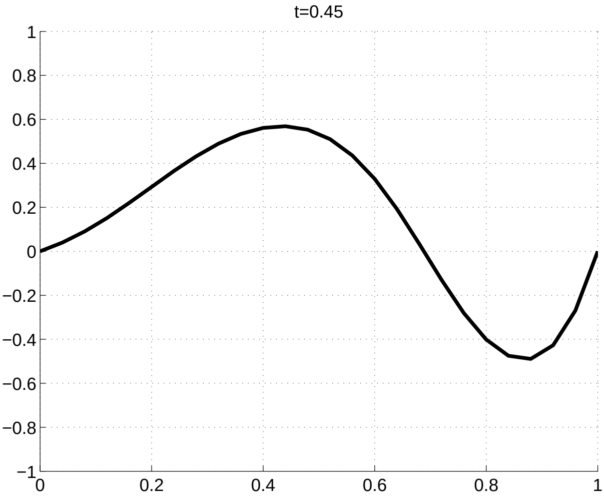

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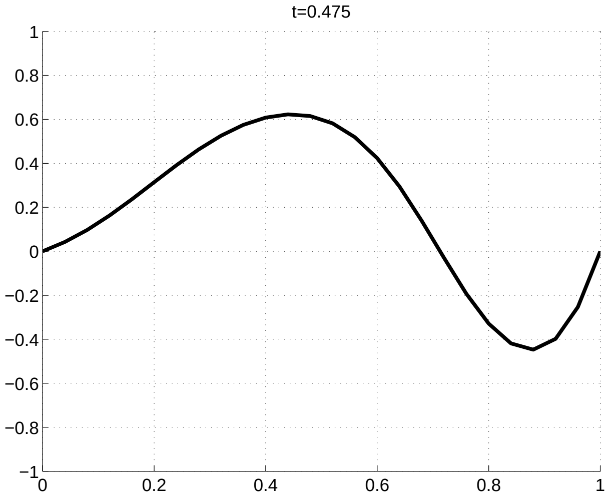

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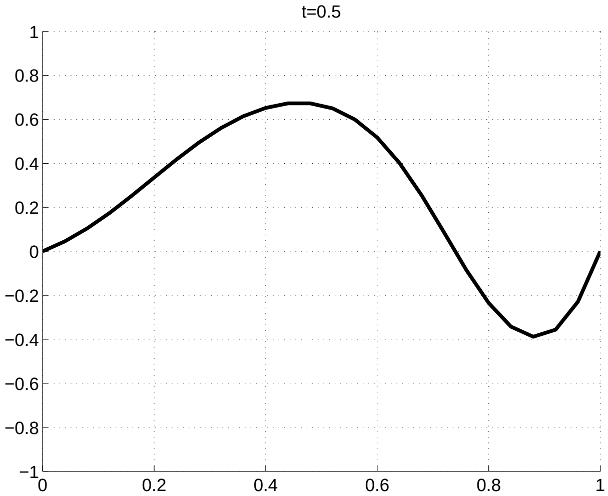

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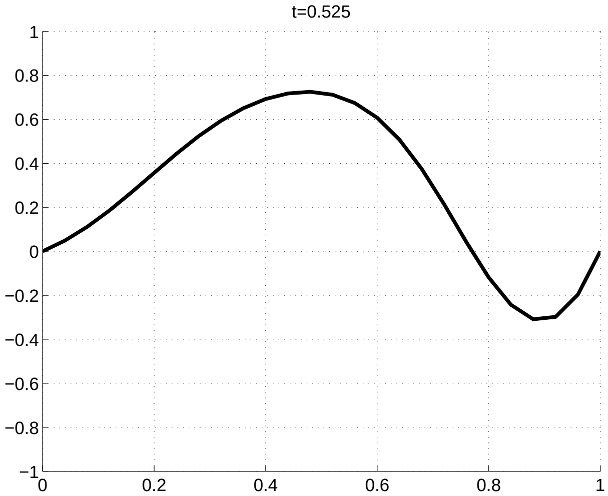

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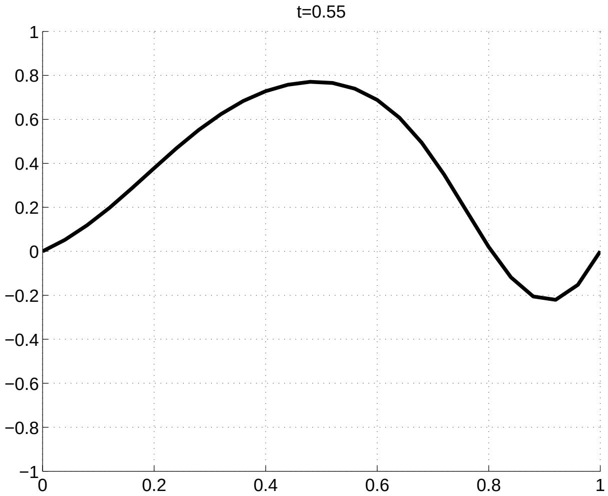

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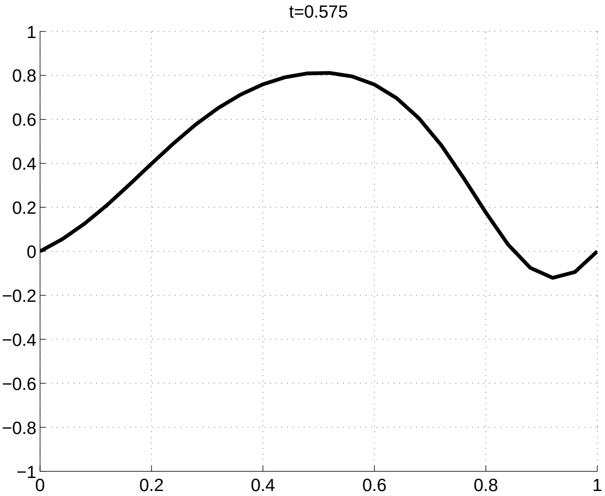

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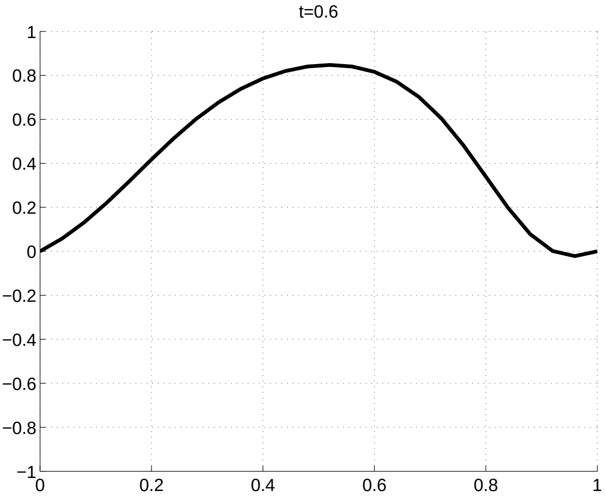

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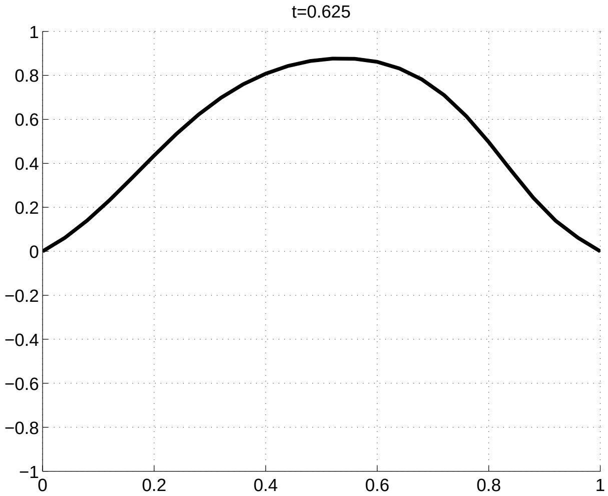

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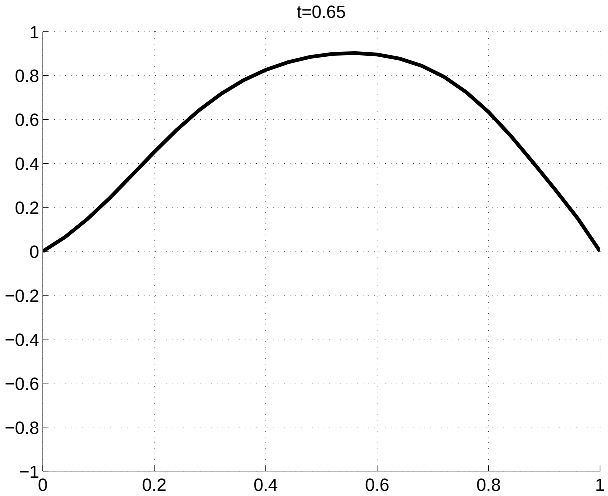

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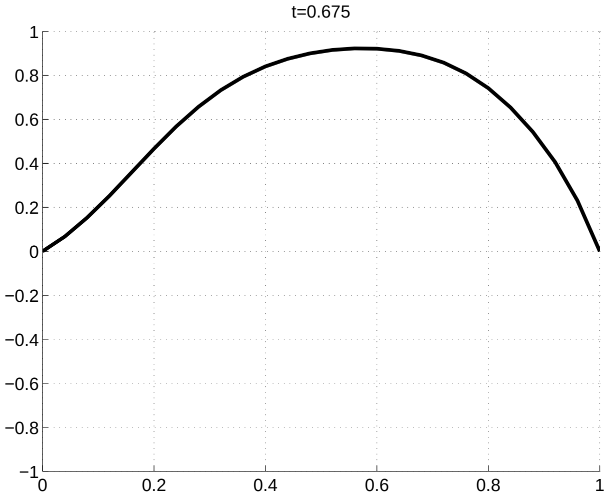

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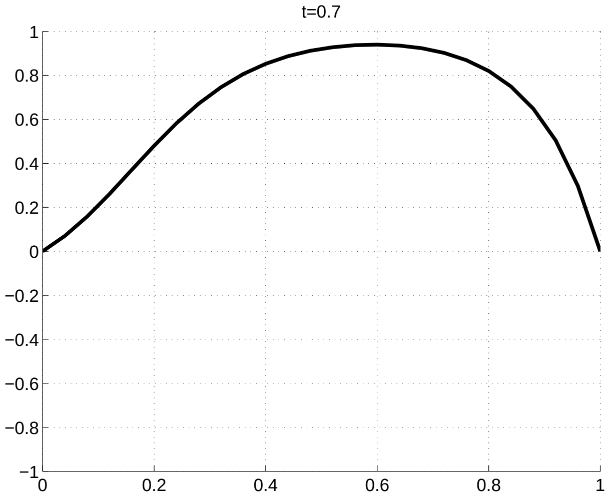

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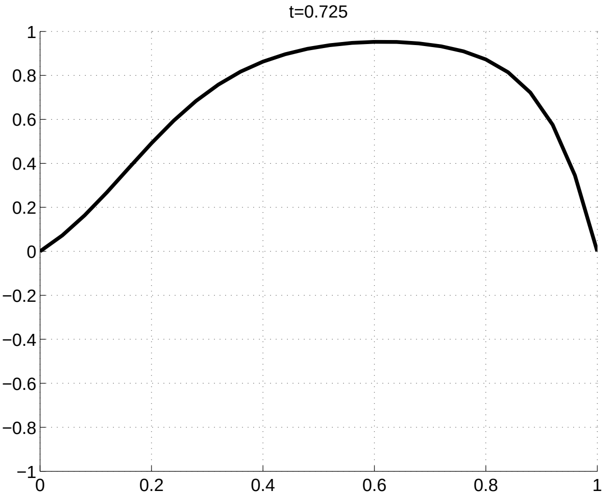

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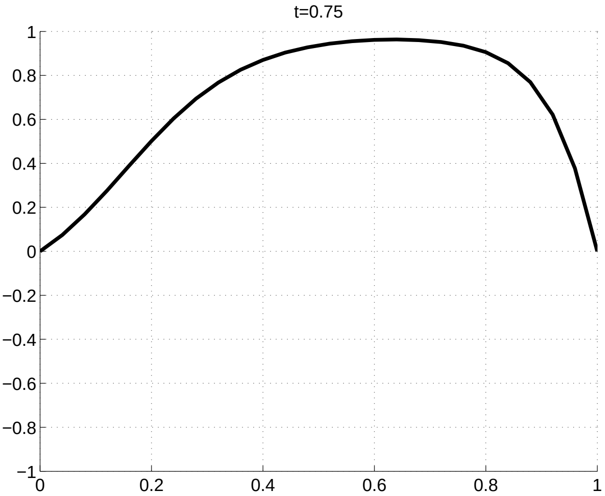

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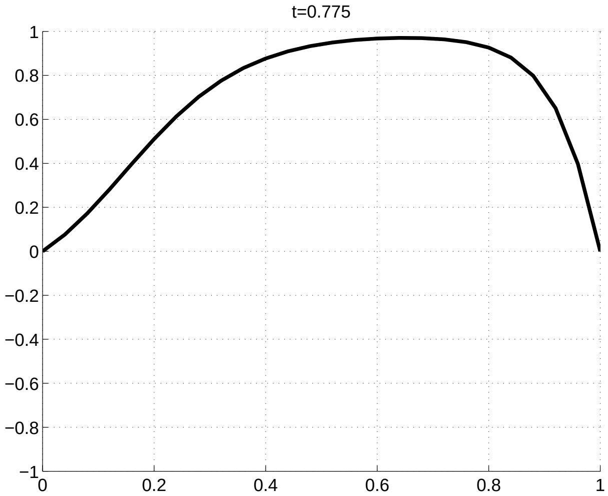

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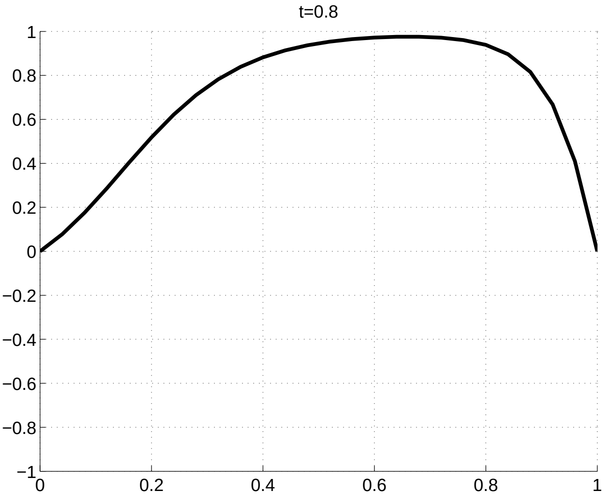

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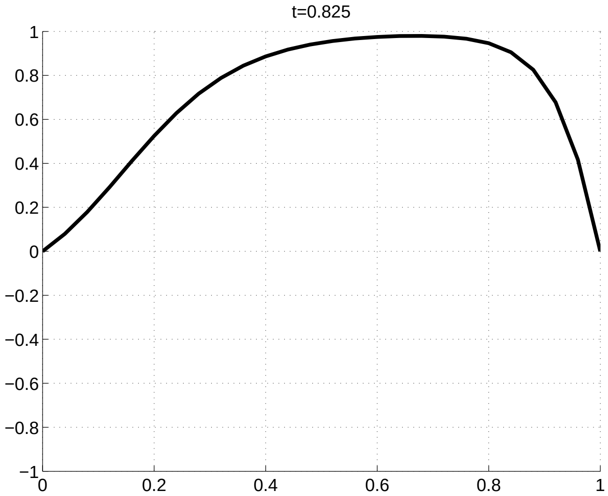

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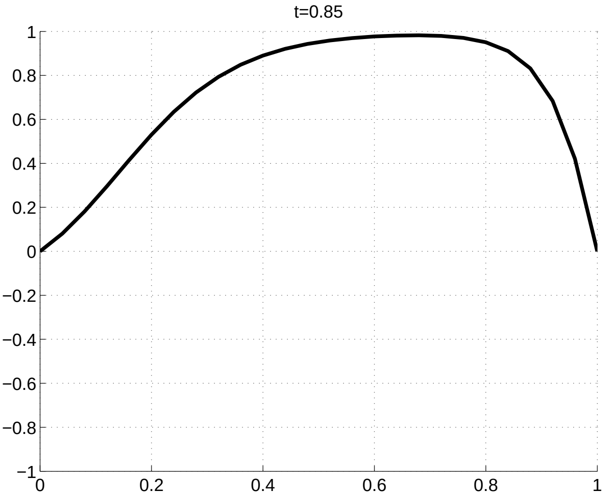

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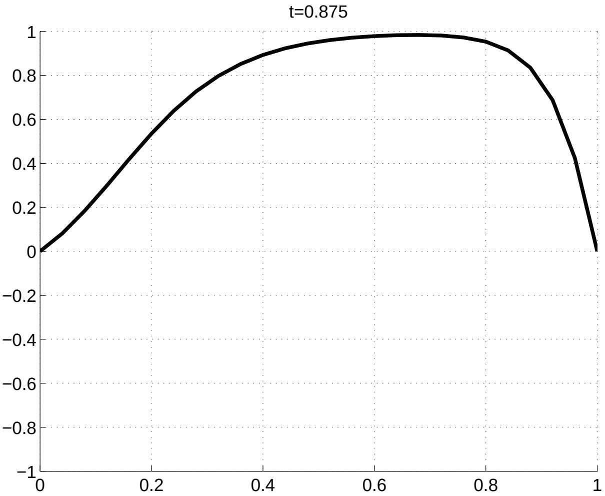

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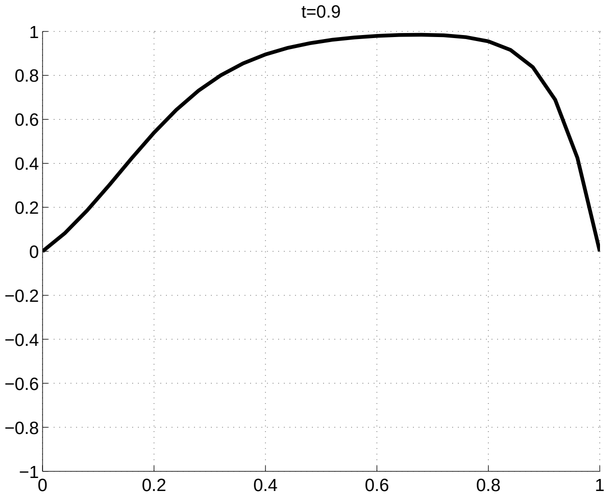

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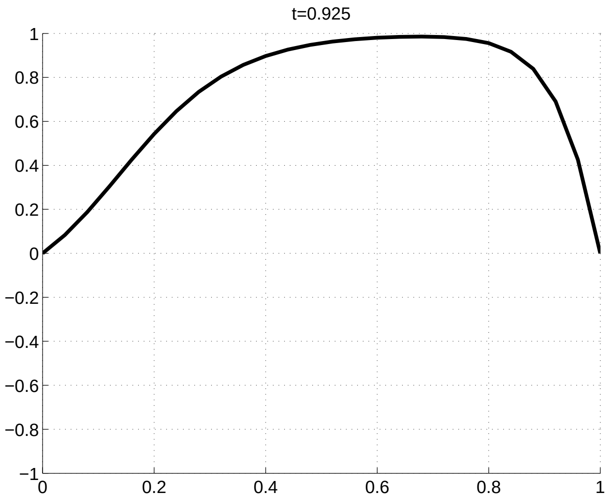

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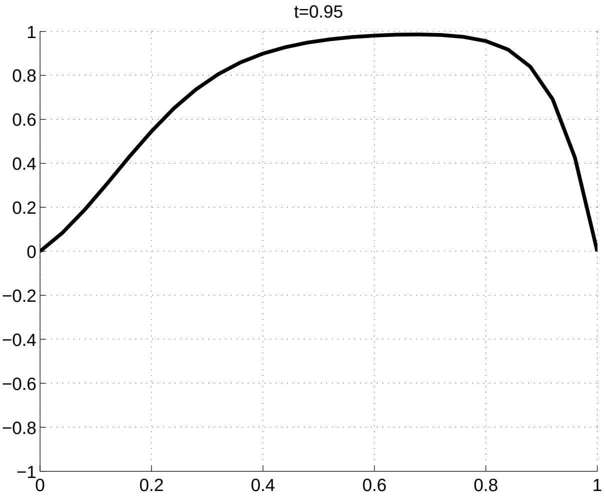

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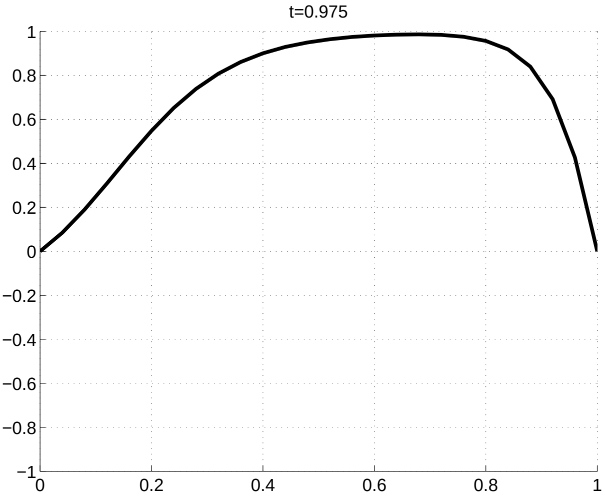

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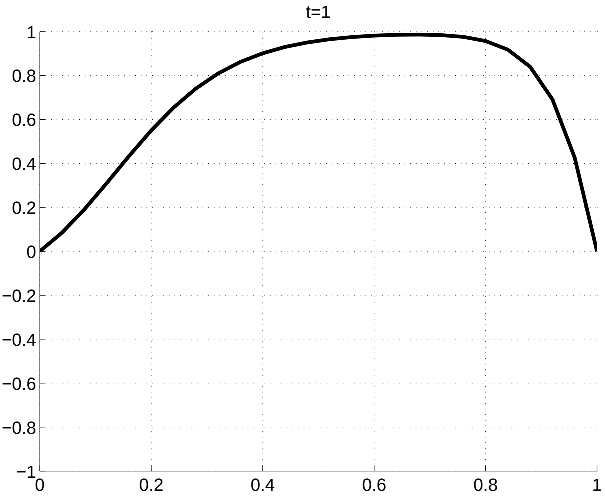

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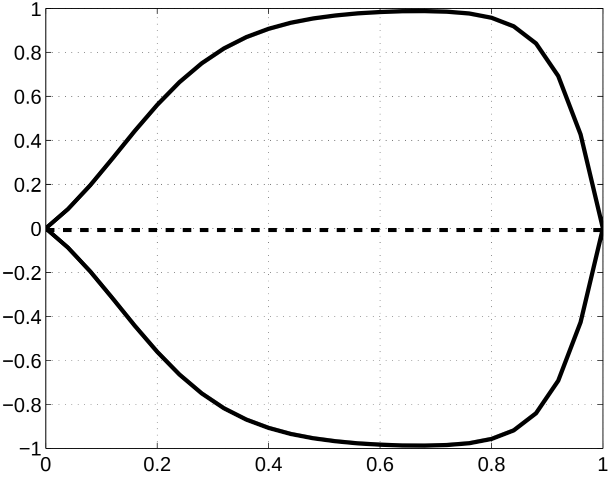

all equilibrium solutions

MPC for the PDE example

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For the (usual) quadratic L^2 cost

$$\ell(y(n), u(n)) = \|y(n)\|_{L^2}^2 + \lambda \|u(n)\|_{L^2}^2$$

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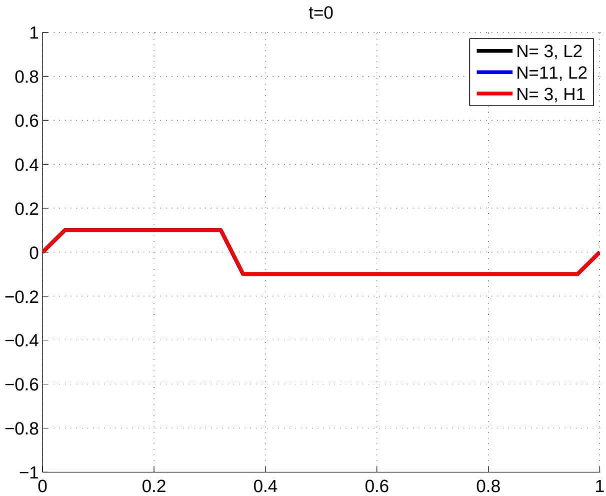
MPC with L_2 vs. H_1 cost



MPC with L_2 and H_1 cost, $\lambda = 0.1$, sampling time $T = 0.025$

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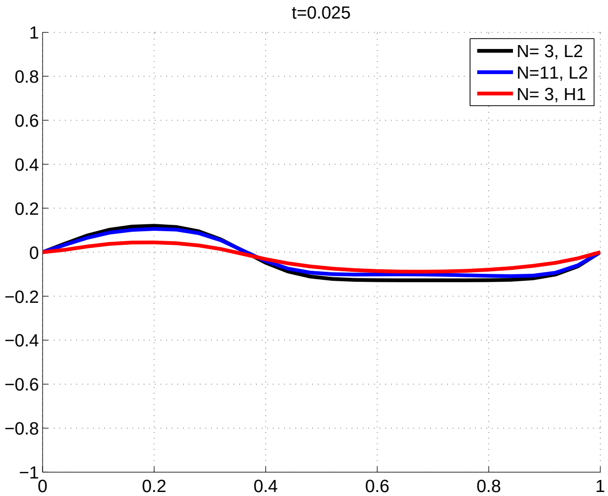

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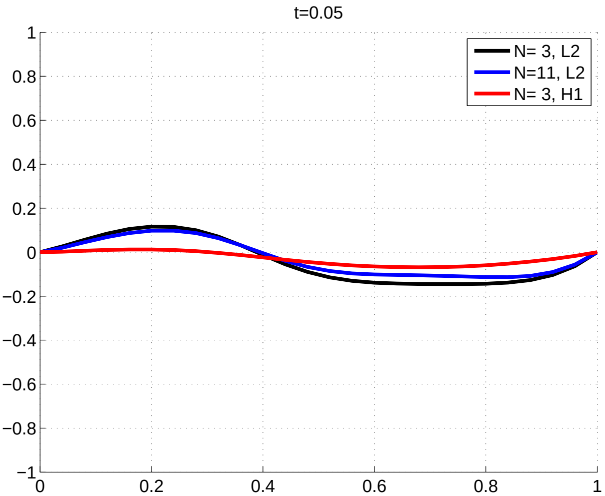

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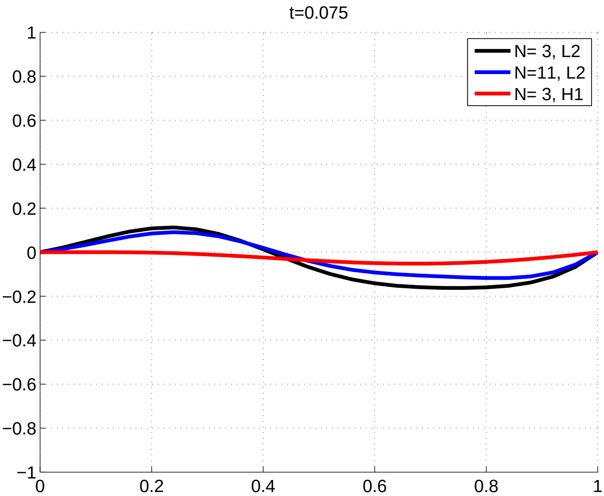

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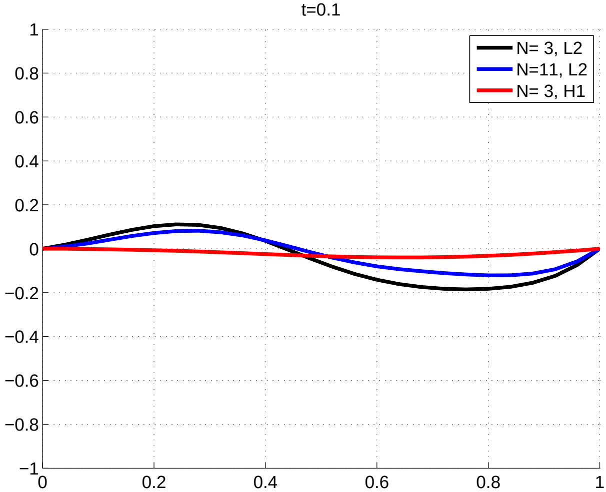

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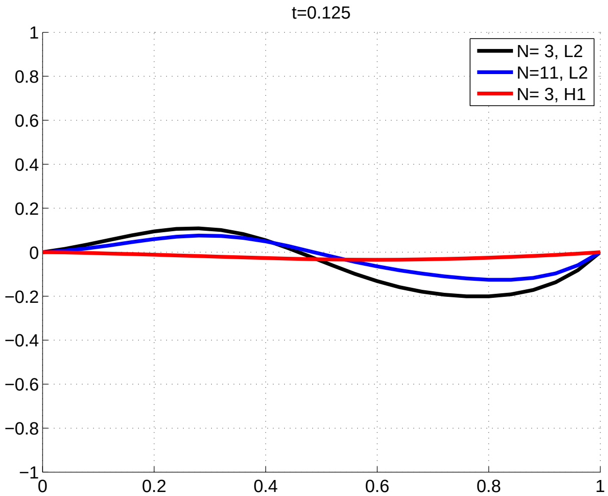

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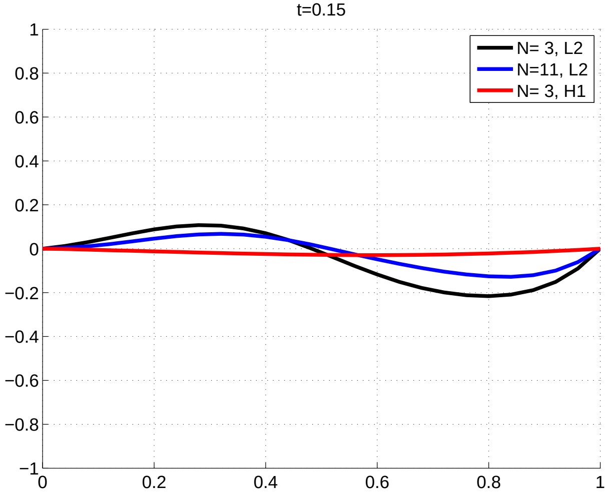

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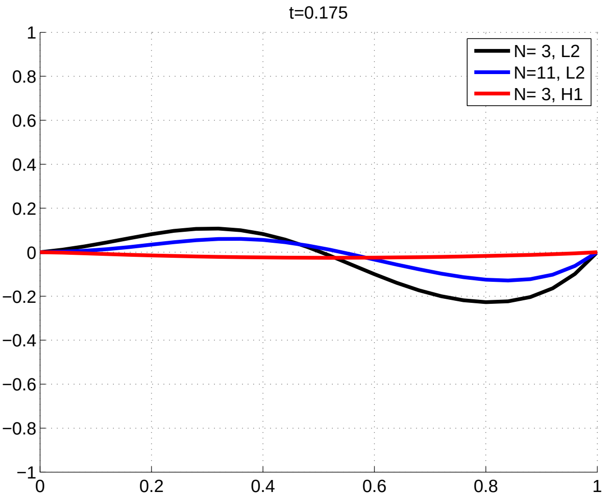

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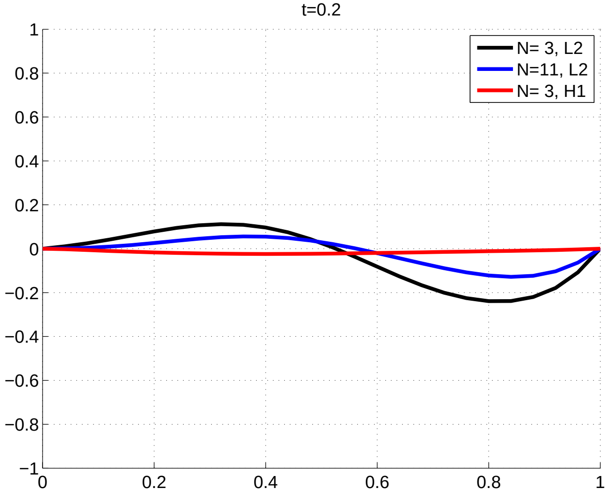

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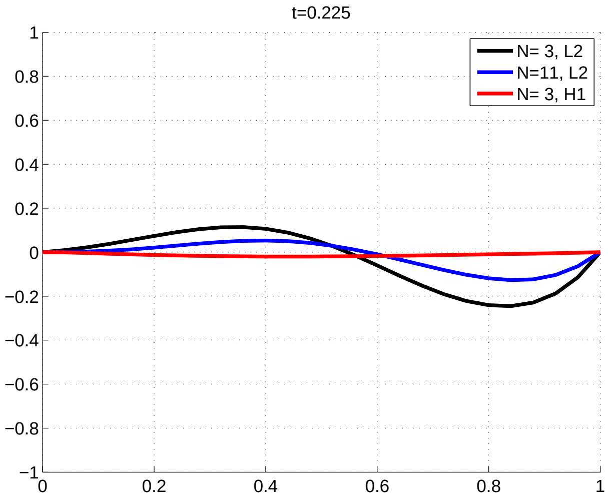

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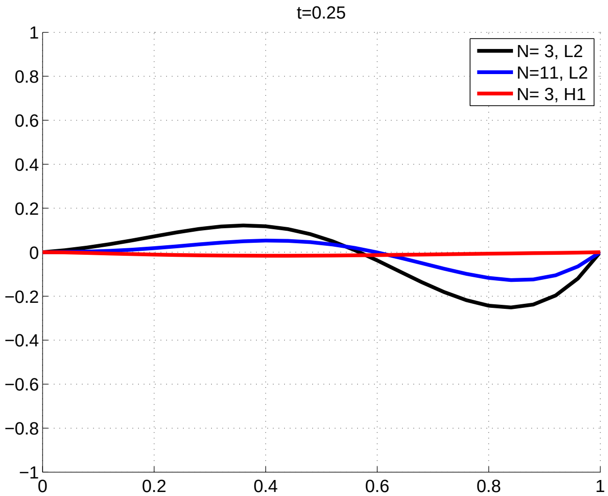

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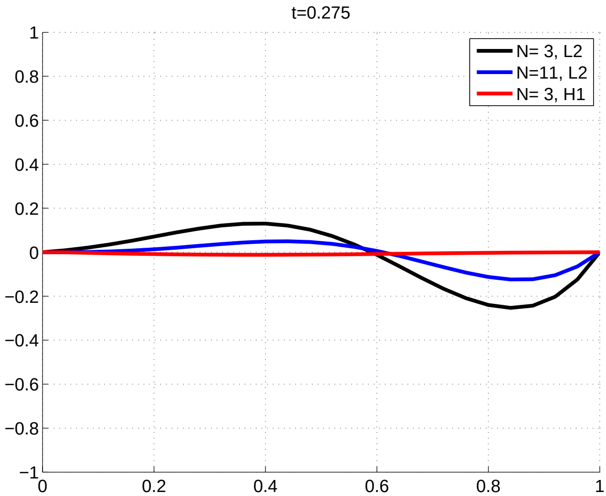

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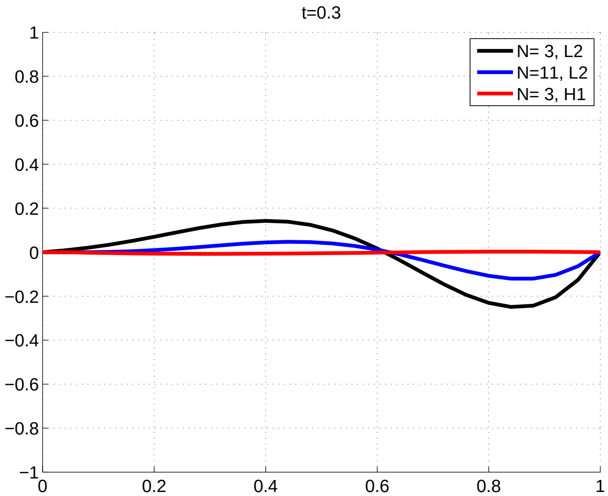

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Boundary Control

Now we change our PDE from distributed to (Dirichlet-) boundary control, i.e.

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y)$$

with

domain $\Omega = [0,1]$

solution $y = y(t,x)$

boundary conditions $y(t,0) = u_0(t)$, $y(t,1) = u_1(t)$

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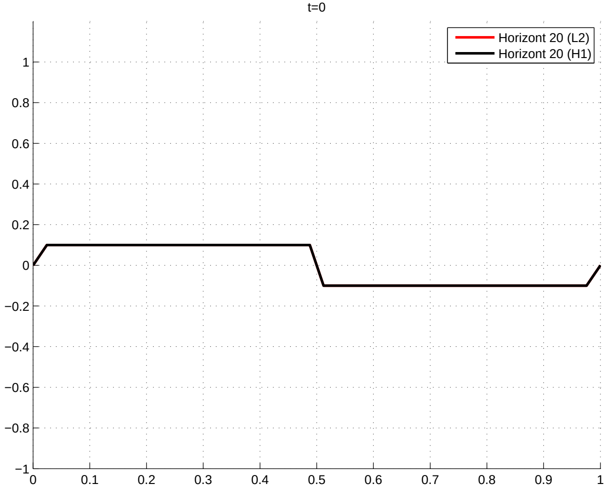
Boundary control, L_2 vs. H_1, $N = 20$



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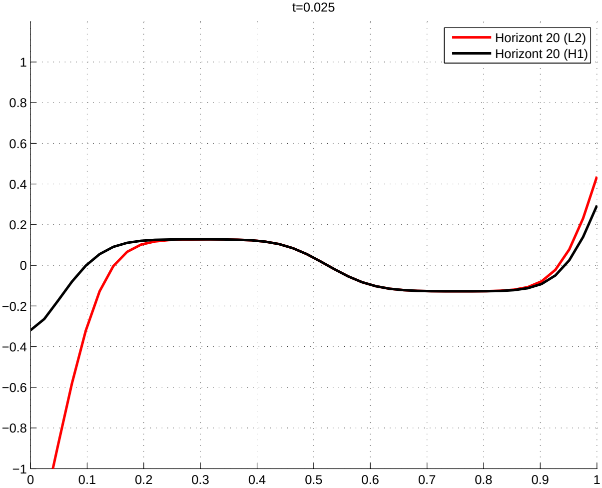

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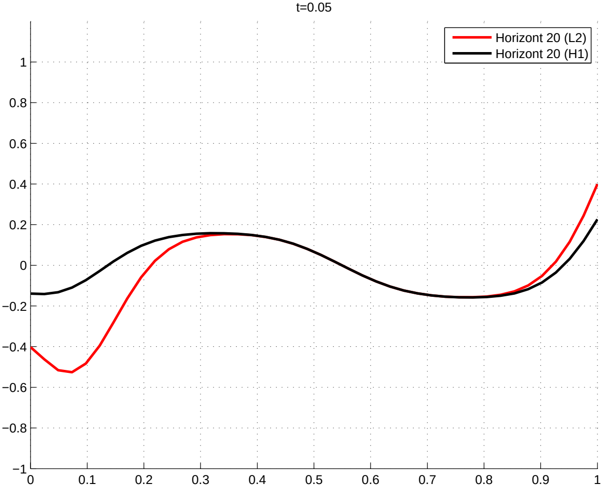

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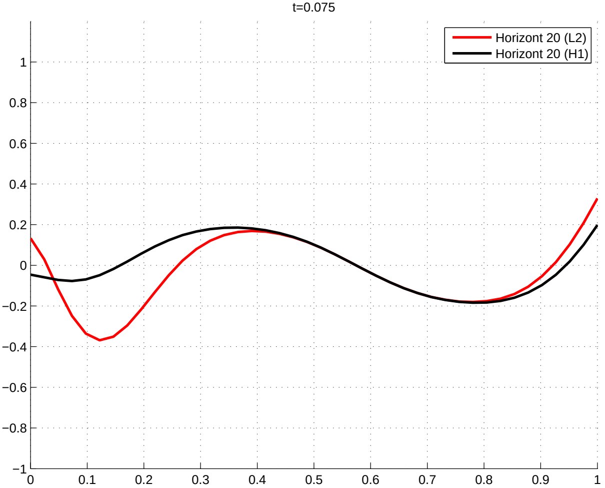

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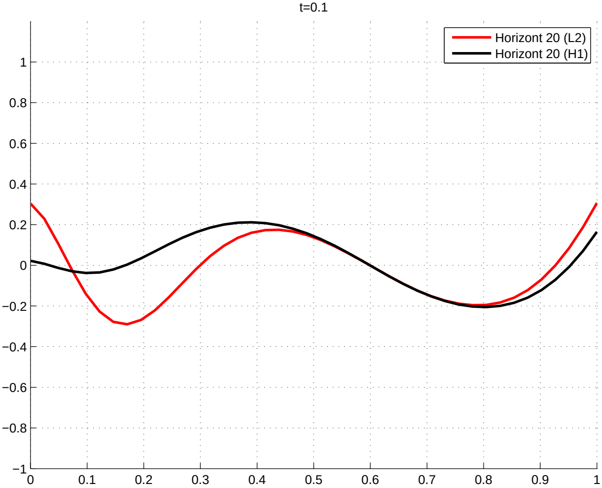

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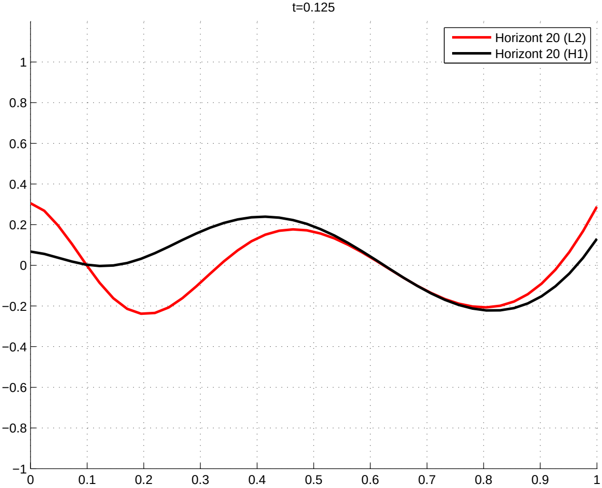

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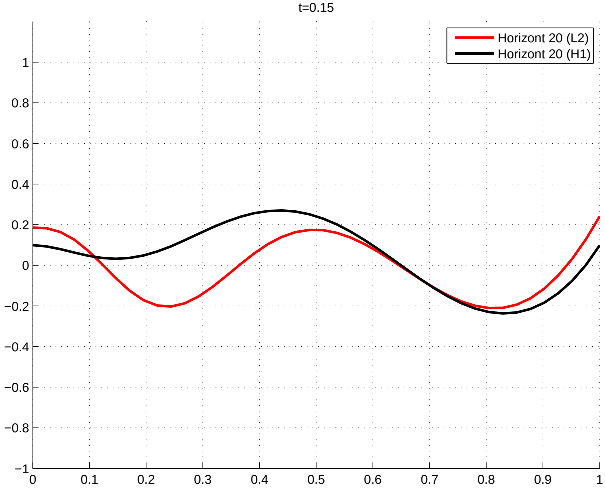

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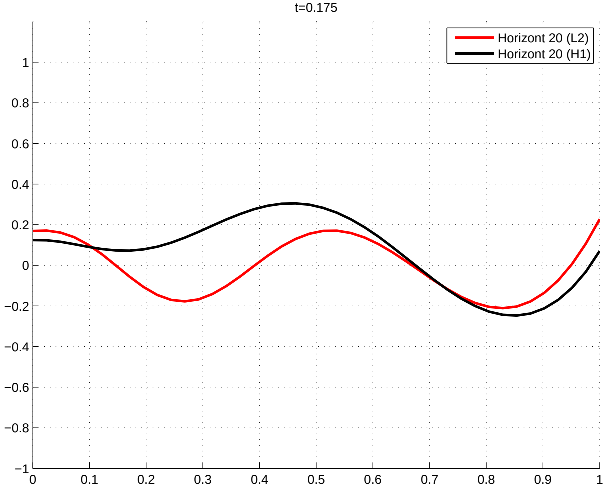

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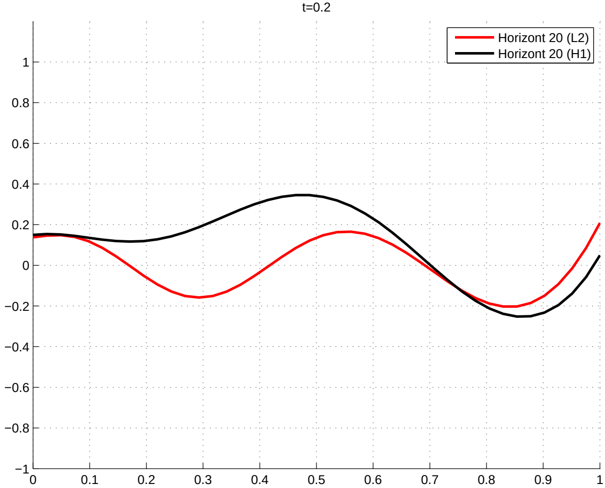

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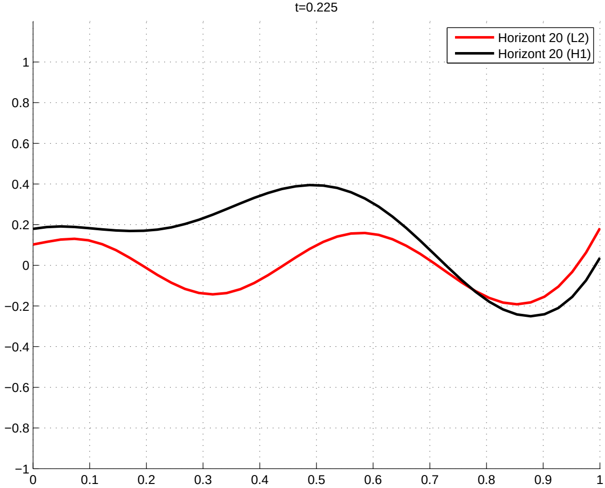

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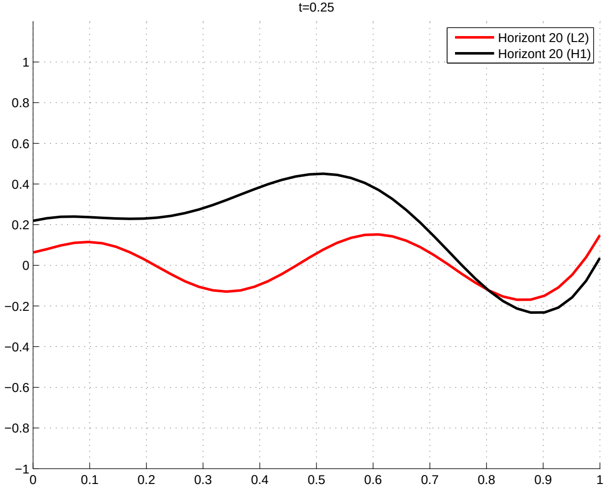

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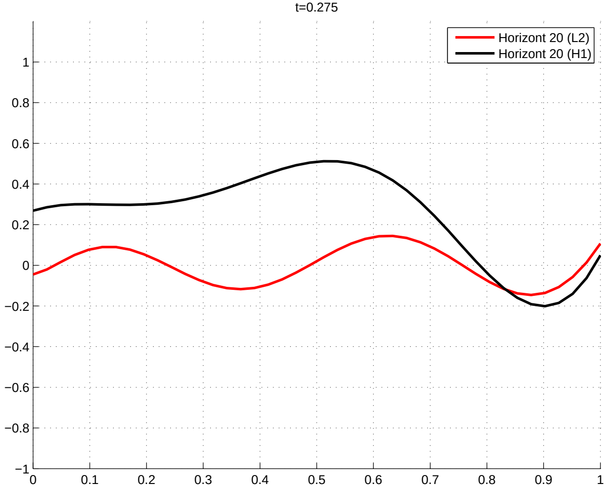

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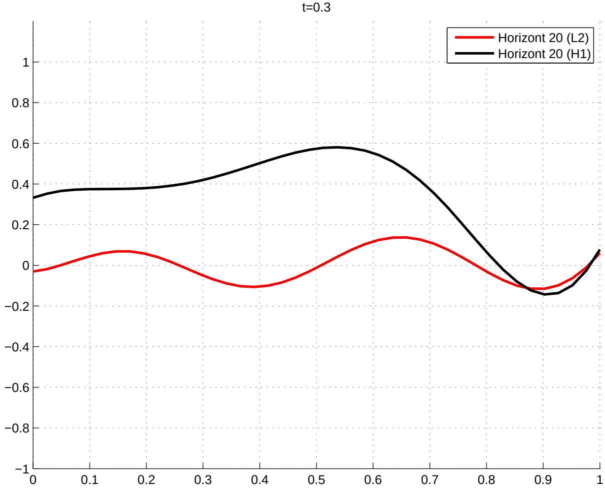

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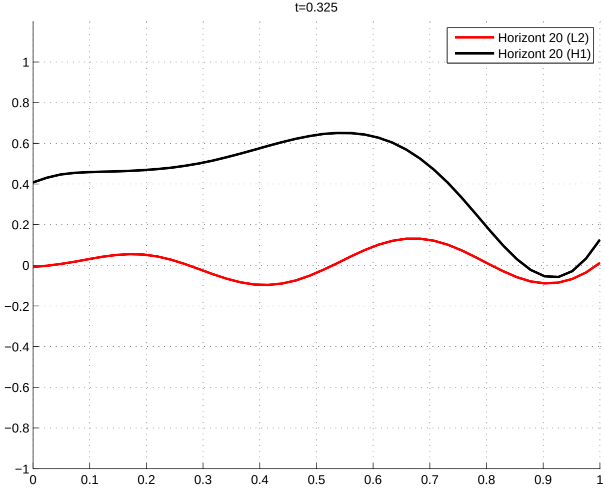

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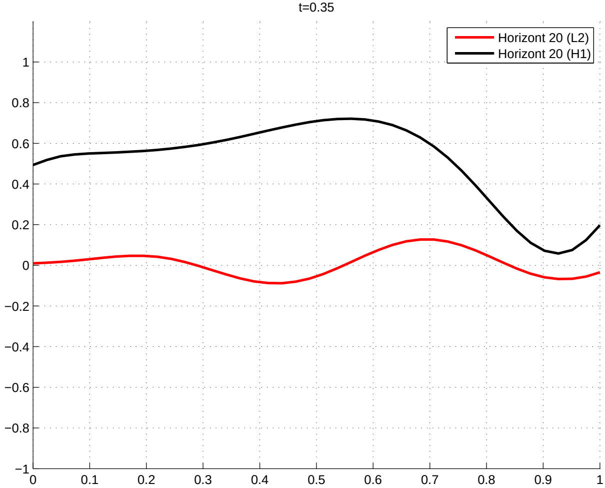

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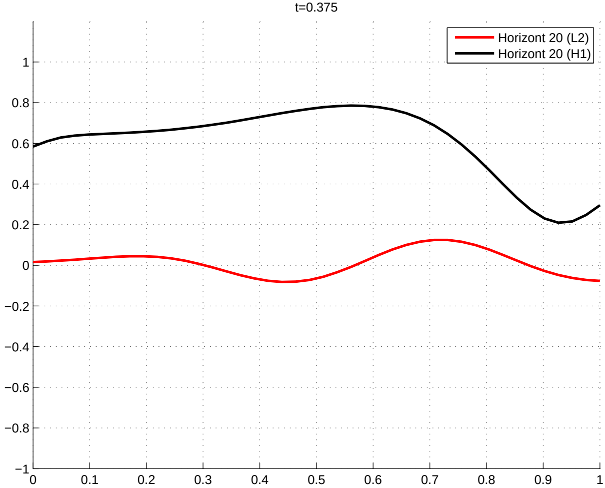

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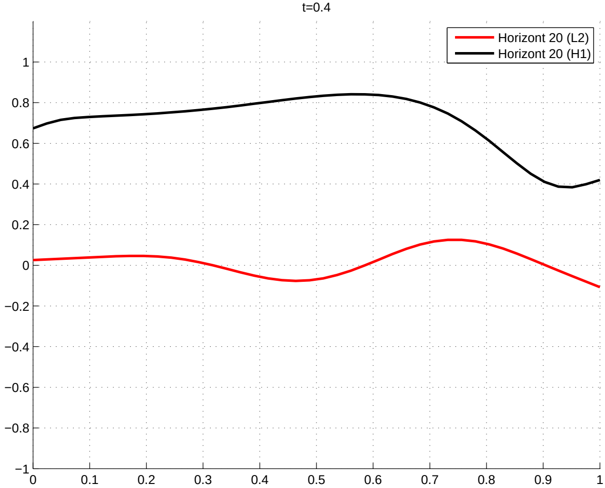

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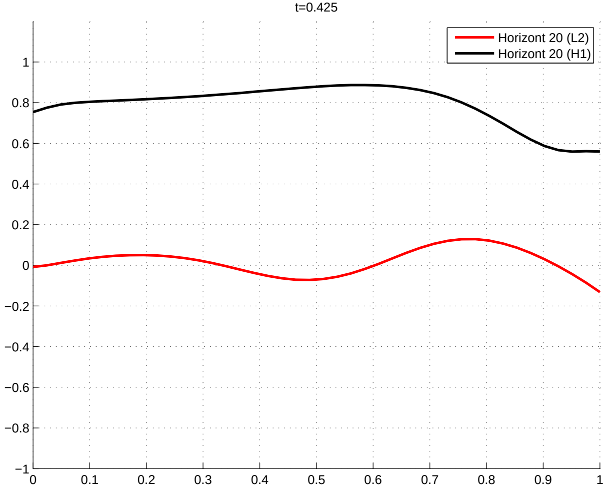
Boundary control, L_2 vs. H_1, $N = 20$



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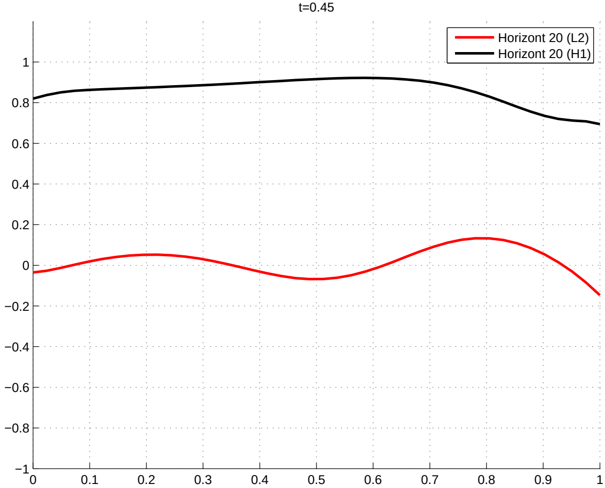

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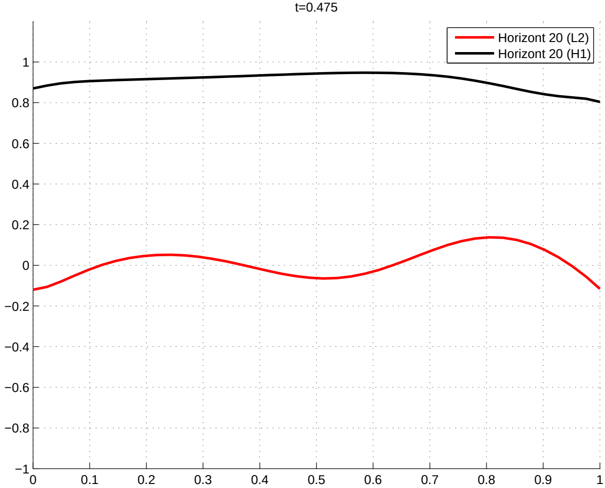

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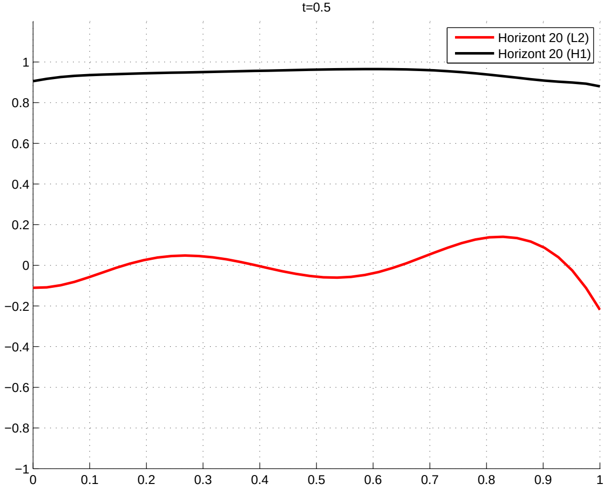

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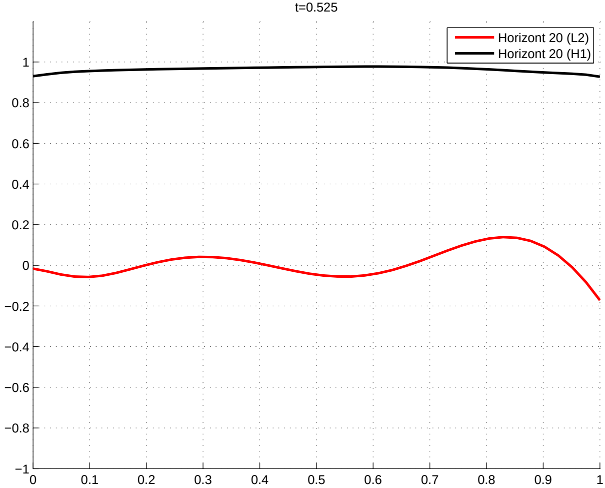

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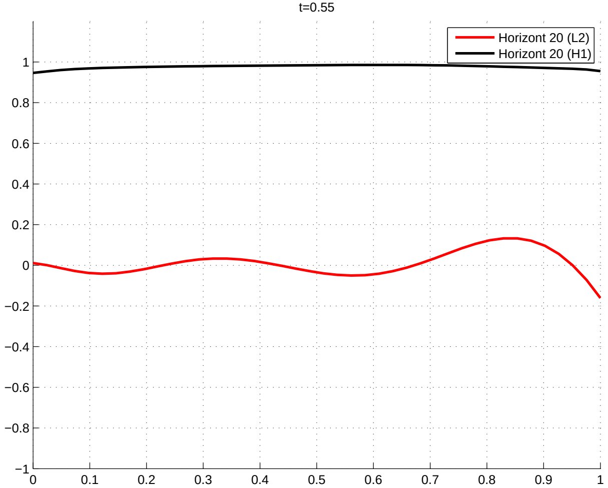

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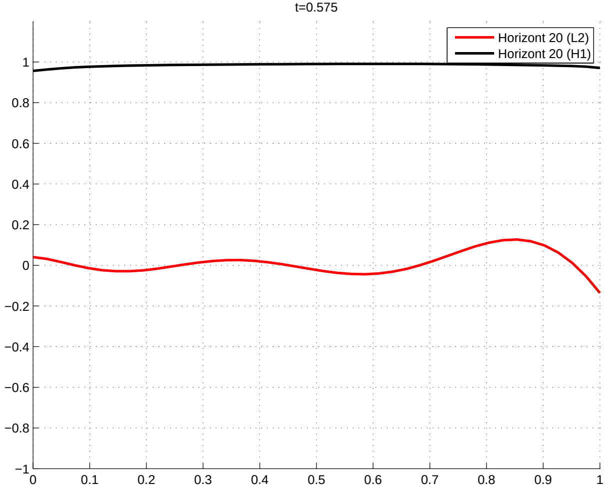

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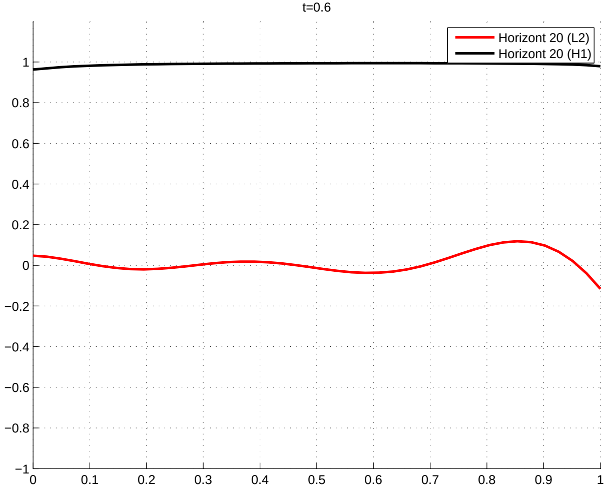

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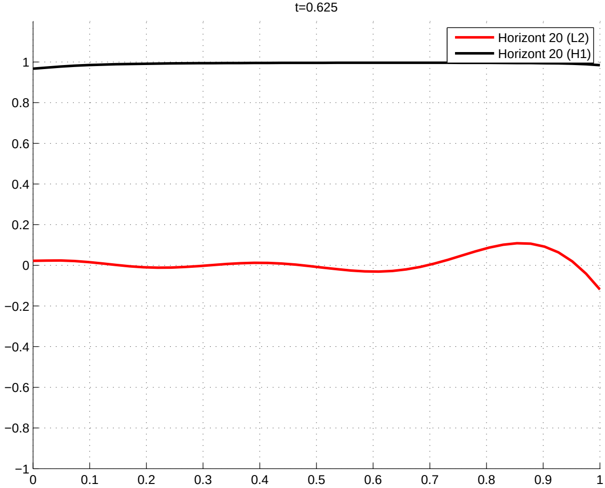

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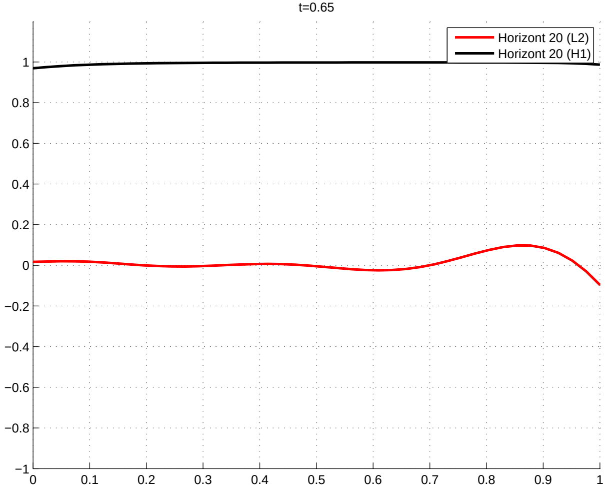

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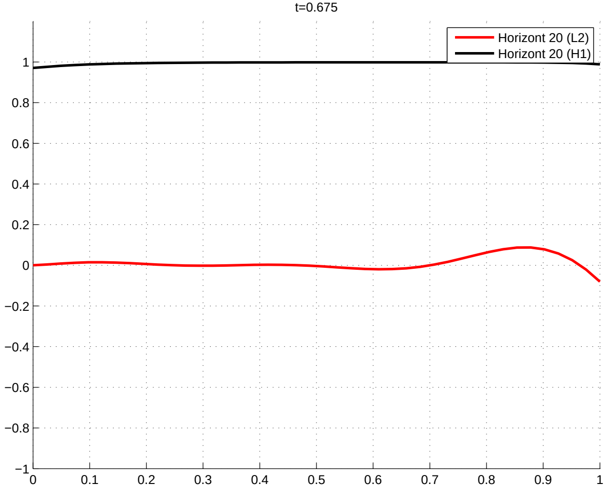

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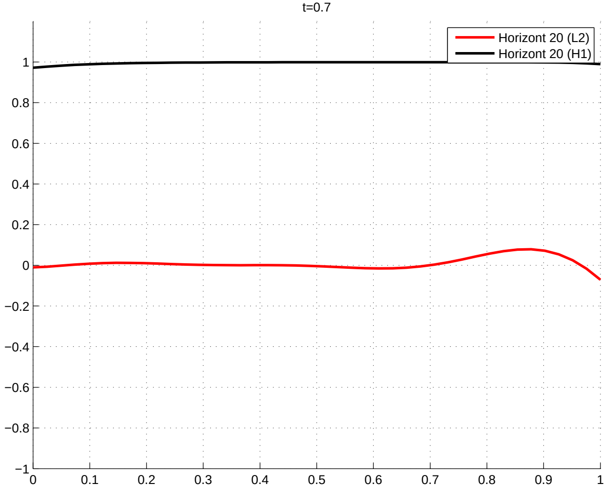

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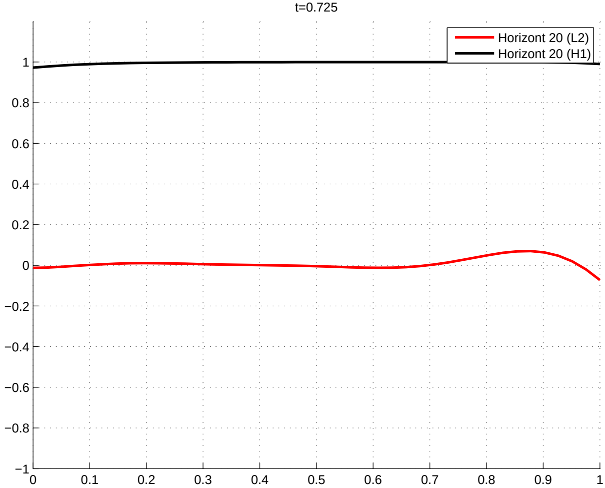

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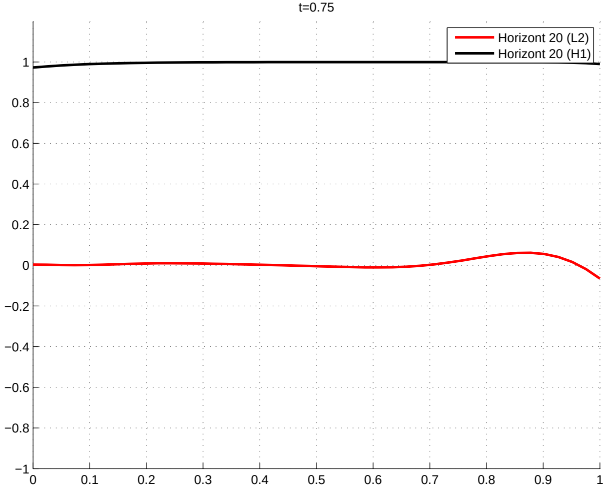

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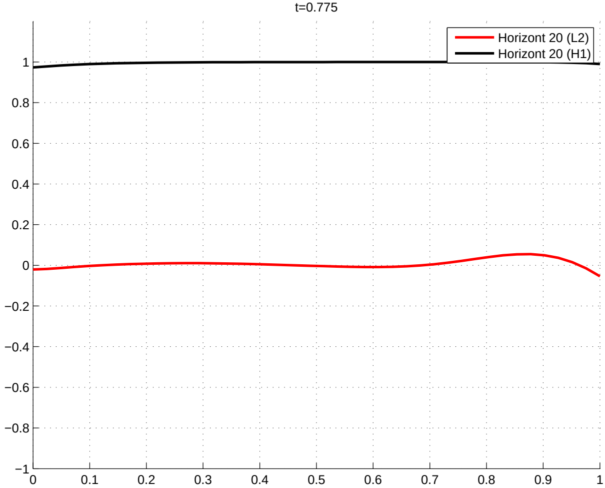

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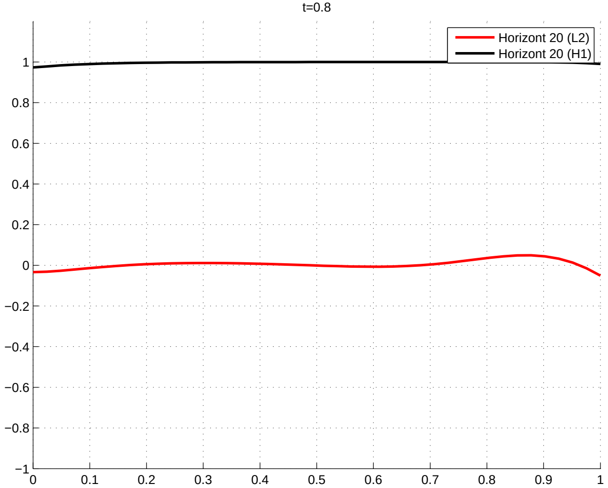

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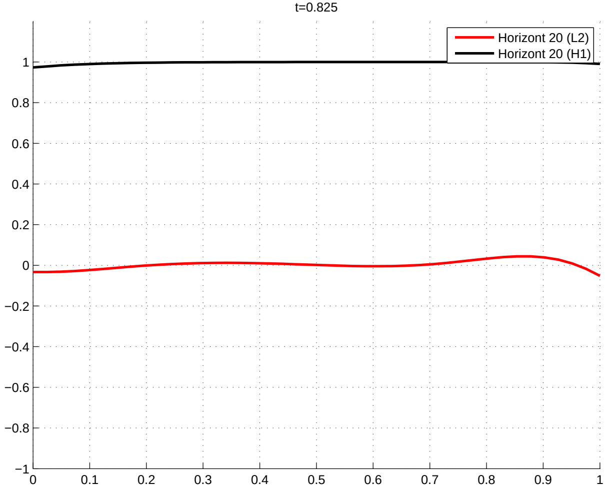

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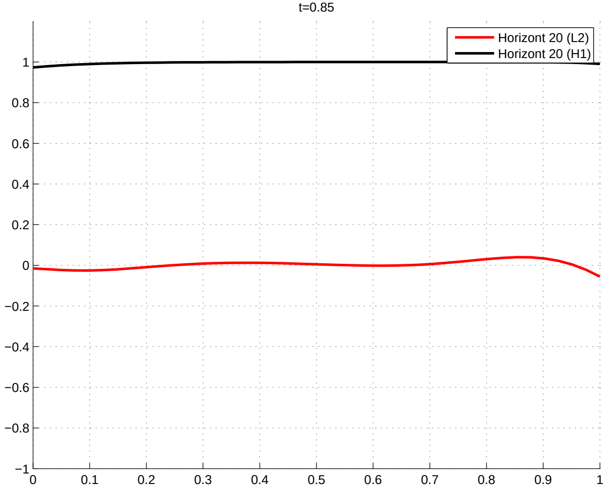

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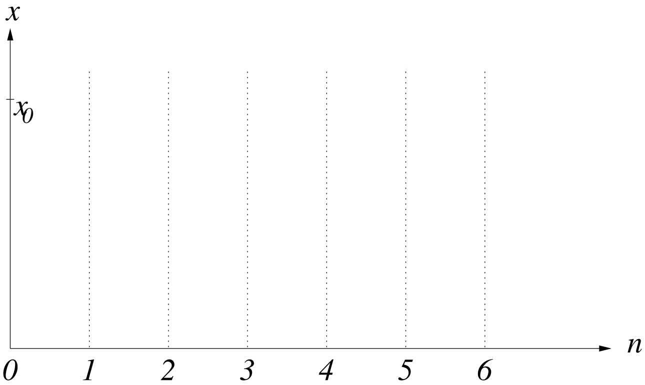
Robustness $:\Leftrightarrow$ the system still approaches/stays within a neighborhood of the stable equilibrium for small $d(n)$

Perturbations in MPC scheme



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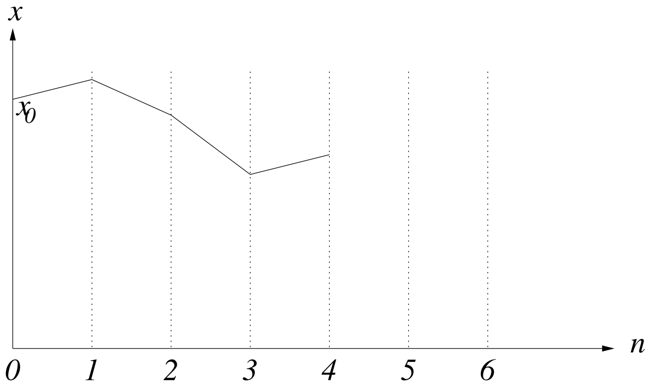

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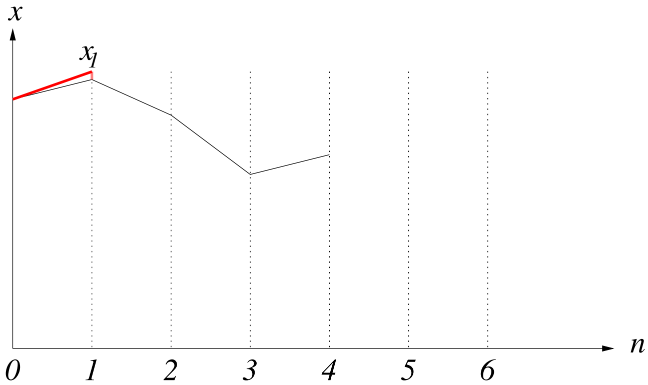

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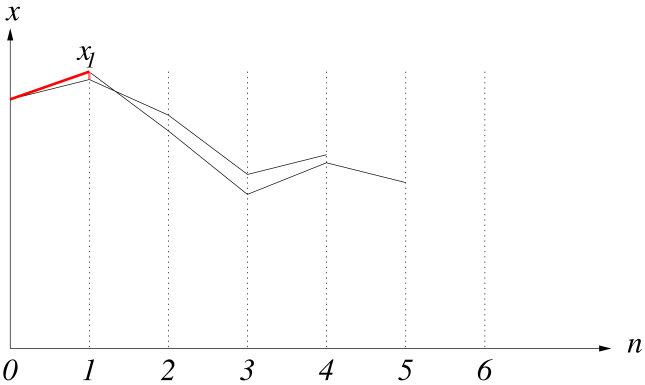

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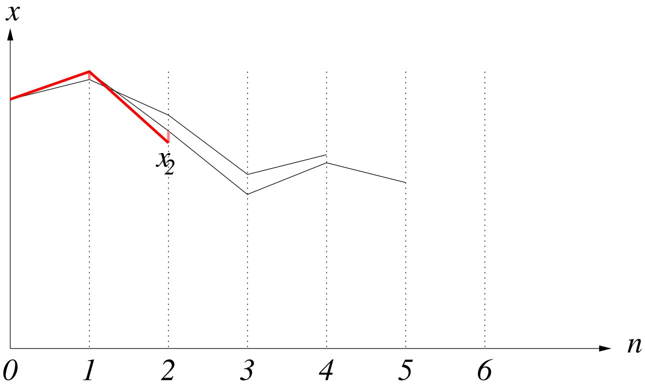

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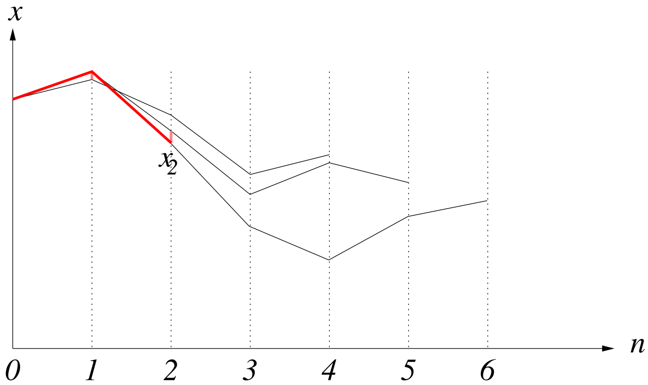

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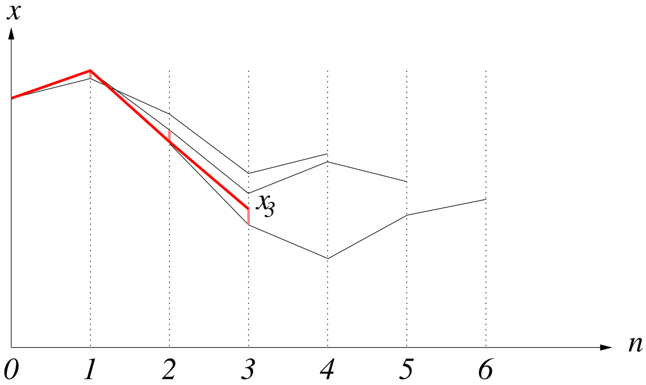

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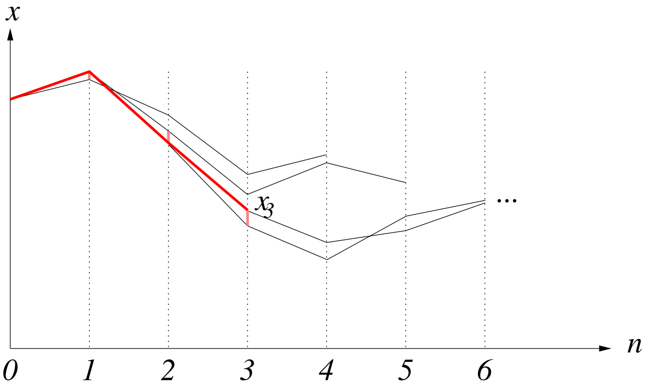

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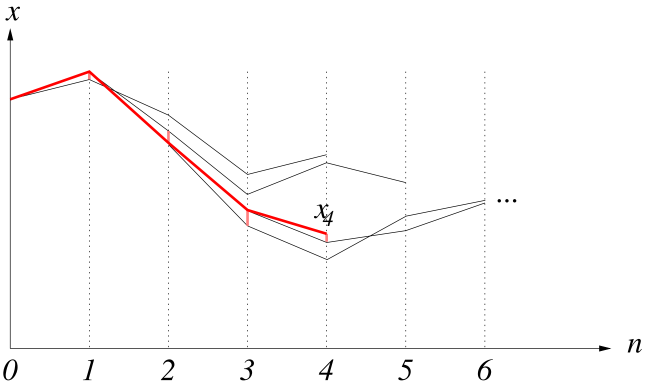

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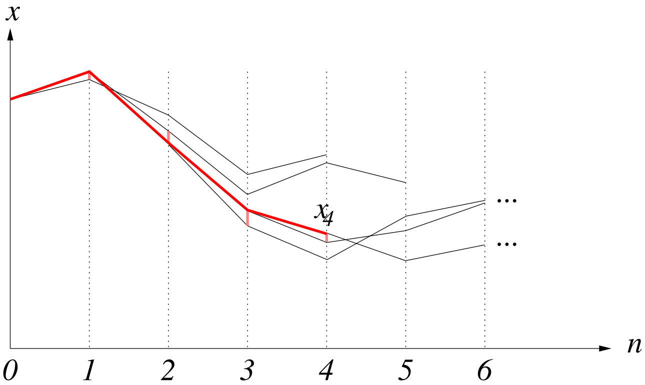

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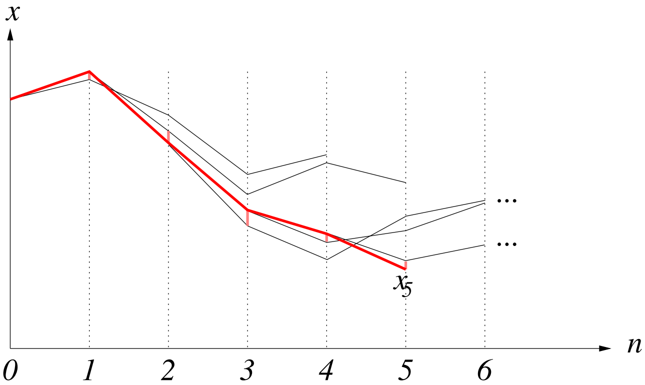

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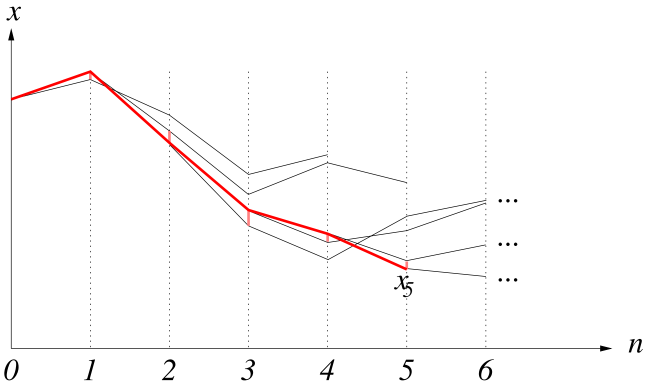

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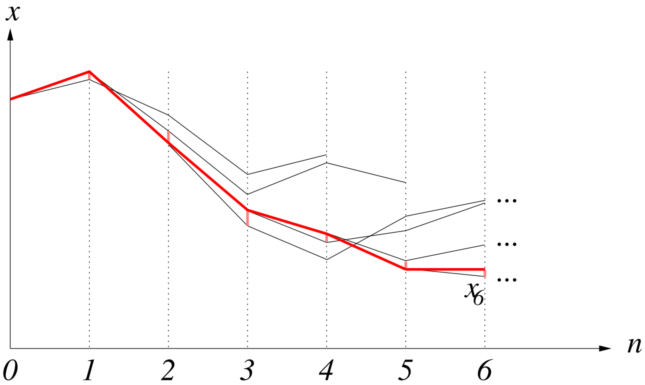

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Robustness can be ensured, e.g., by

- (uniform) continuity of the optimal value function $V_N(x) = \inf_u J_N(x,u)$, which serves as a Lyapunov function [De Nicolao/Magni/Scattolini '96; Nešić/Teel/Kokotović '99; Gr./Pannek '11]

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In the latter case, stability and robustness analysis must be carried out in an integrated way

Reducing the computational load

Back to the unperturbed case:

The computationally most expensive part of an MPC controller is the optimization

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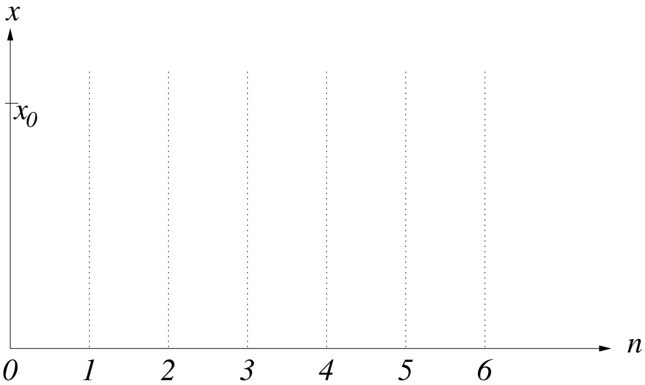
A more systems theoretic approach: perform re-optimization less often

Schematic illustration of the idea



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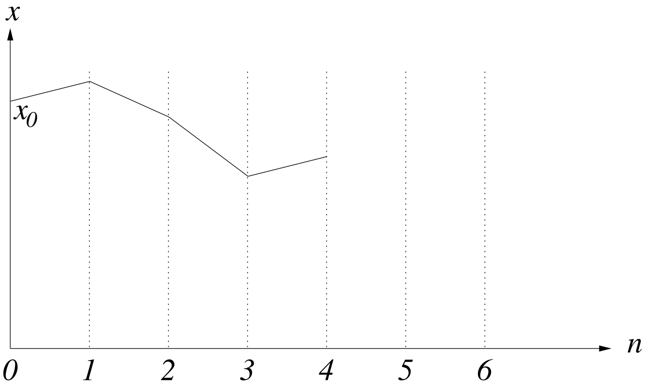

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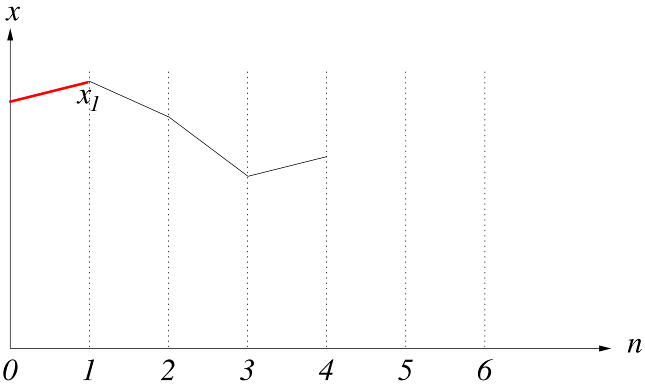

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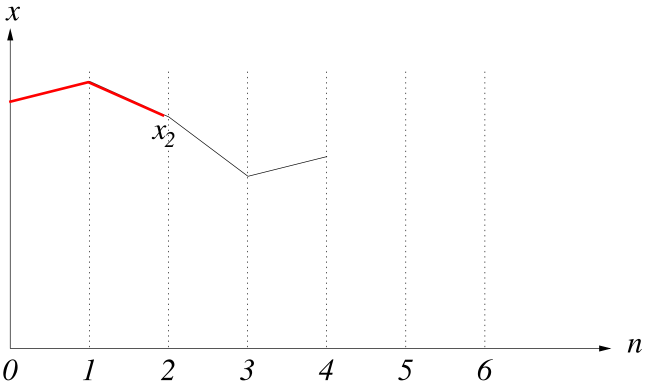

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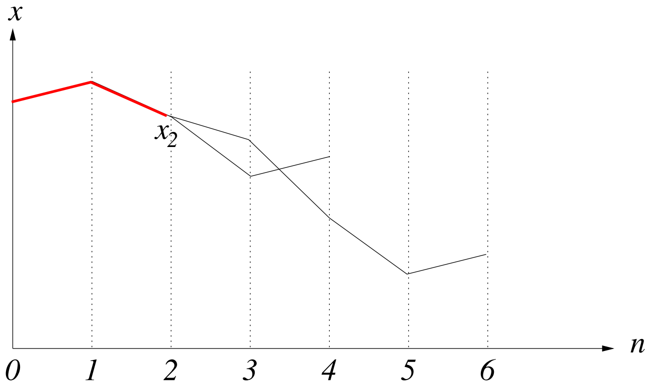

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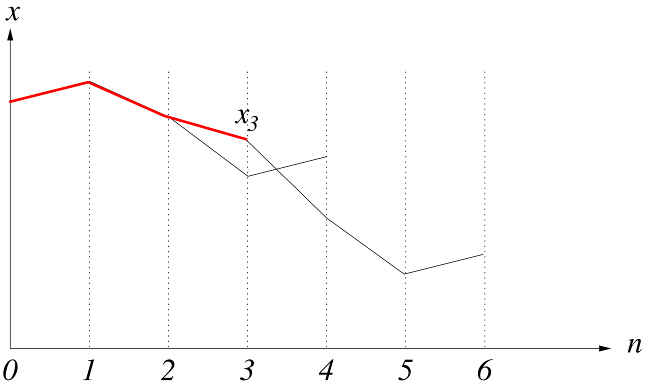

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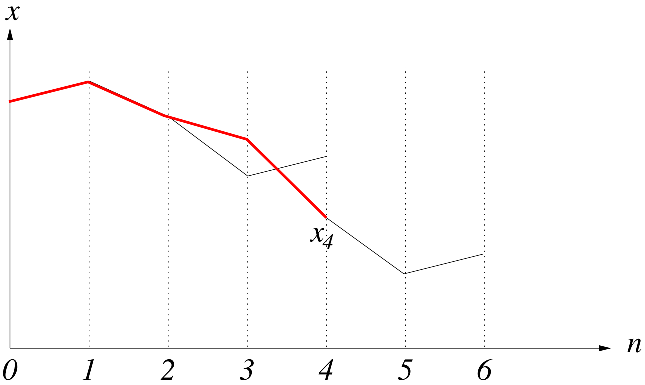

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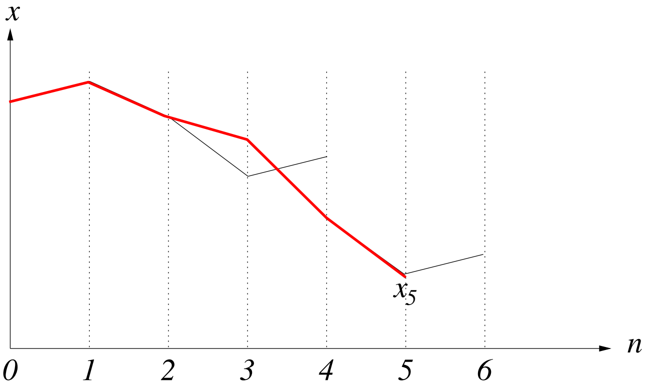

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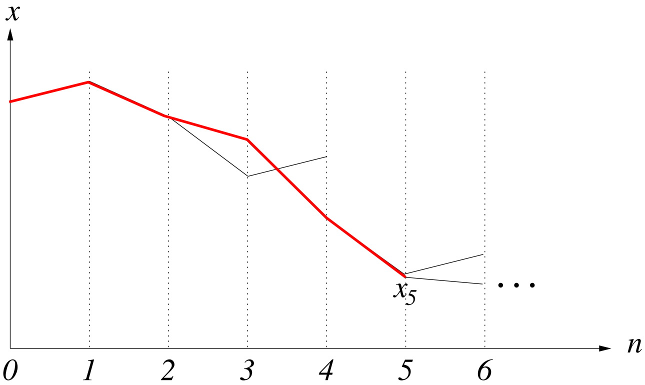

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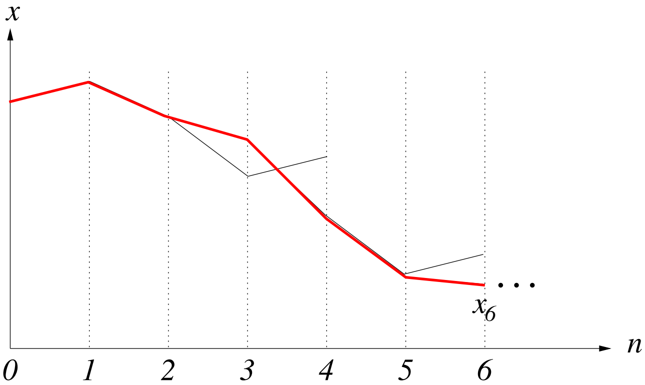

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Stability analysis

Denote the by m_j the number of elements used from the j-th control sequence, called the "control horizon"

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Then the stability and performance analysis extends to time-varying control horizons if we use $\alpha = \min_{m_j} \alpha(m_j)$ where

$$\alpha(m) = 1 - \frac{\prod_{i=m+1}^{N} (\gamma_i - 1) \prod_{i=N-m+1}^{N} (\gamma_i - 1)}{\left(\prod_{i=m+1}^{N} \gamma_i - \prod_{i=m+1}^{N} (\gamma_i - 1)\right)\left(\prod_{i=N-m+1}^{N} \gamma_i - \prod_{i=N-m+1}^{N} (\gamma_i - 1)\right)}$$

with $\gamma_i = \sum_{k=0}^{i-1} C\sigma^k$

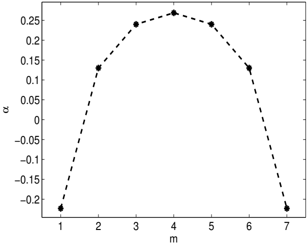
Property of $\alpha(m)$

Theorem: The values $\alpha(m)$ satisfy

$$\alpha(m) = \alpha(N-m), \quad m = 1, \ldots, N-1$$

and

$$\alpha(m) \leq \alpha(m+1), \quad m = 1, \ldots \lceil N/2 \rceil$$



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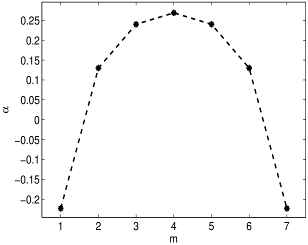

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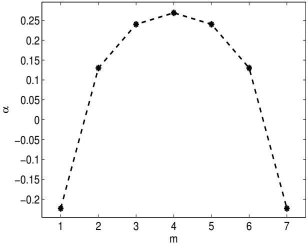

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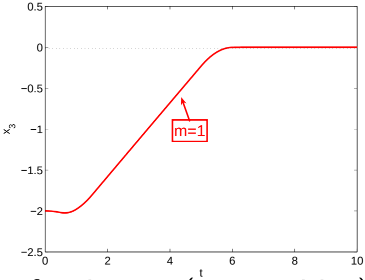
How does $\alpha(m)$ look like for a single system?

Example: linearized inverted pendulum

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ g & -k & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} u, \qquad x_0 = \begin{pmatrix} 0 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

sampling time $T = 0.5$, $\ell(x,u) = 2\|x\|_1 + 4\|u\|_1$, $N = 11$



x_3 component of trajectory (cart position) for different m

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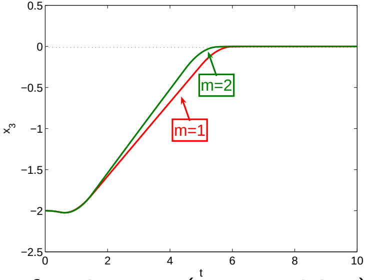

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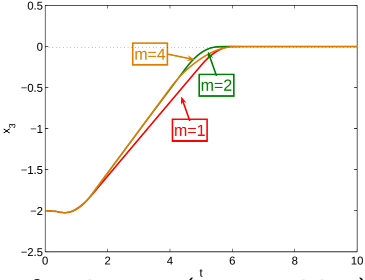
sampling time $T = 0.5$, $\ell(x,u) = 2\|x\|_1 + 4\|u\|_1$, $N = 11$



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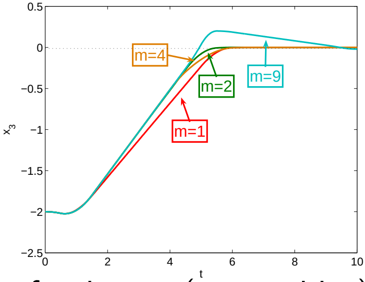

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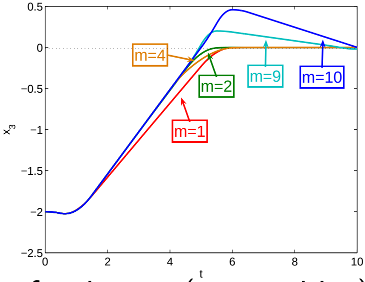

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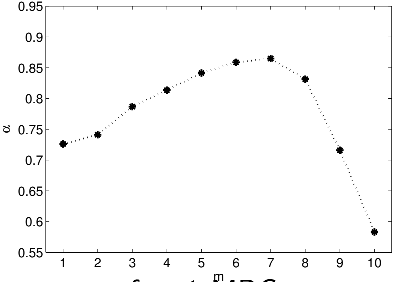
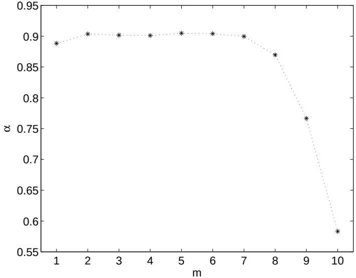


α after 1 MPC step



α at time $n = 20$

Discussion of the approach

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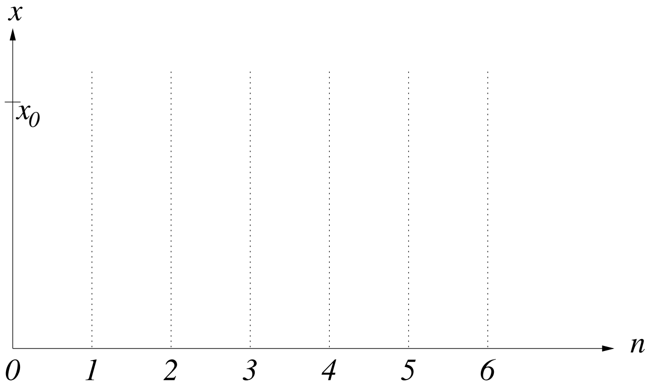

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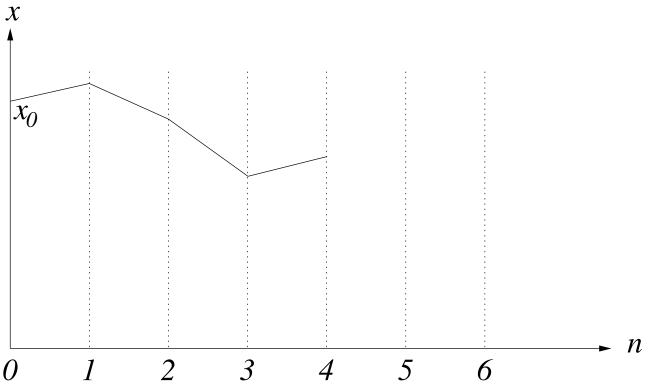

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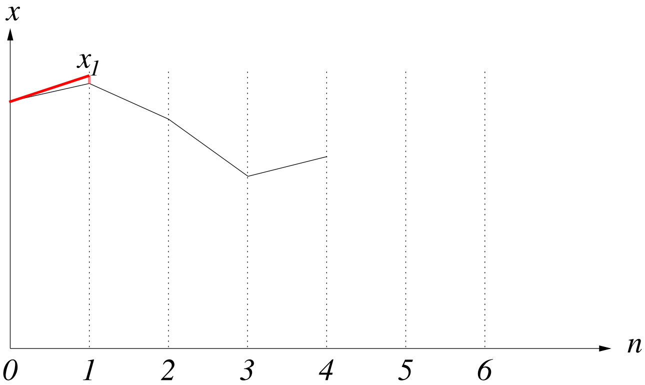

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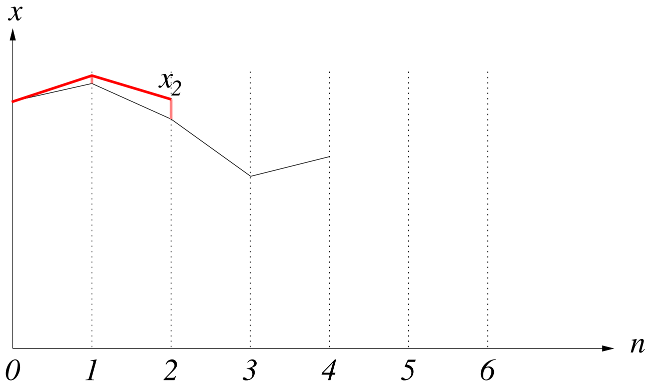

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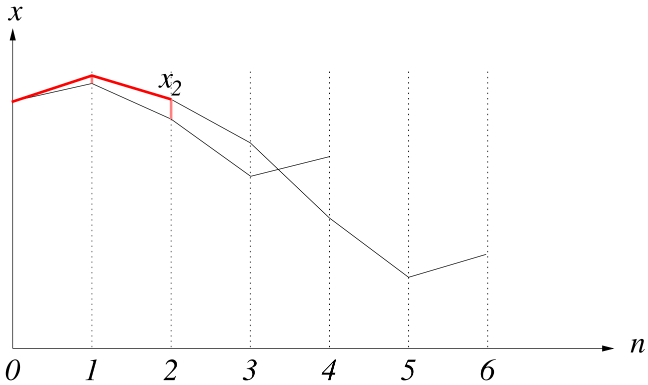

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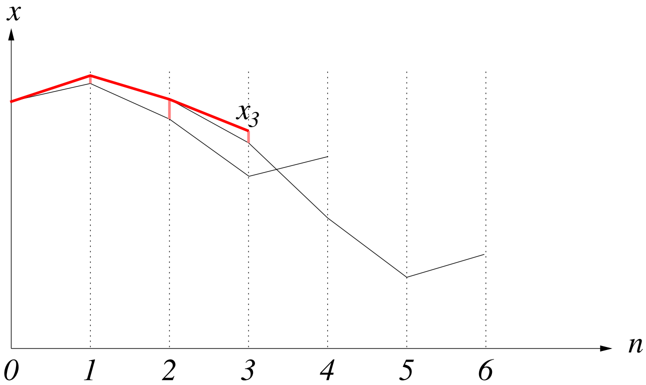

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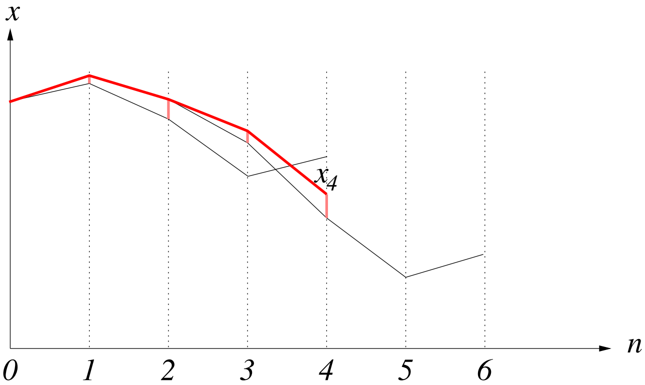

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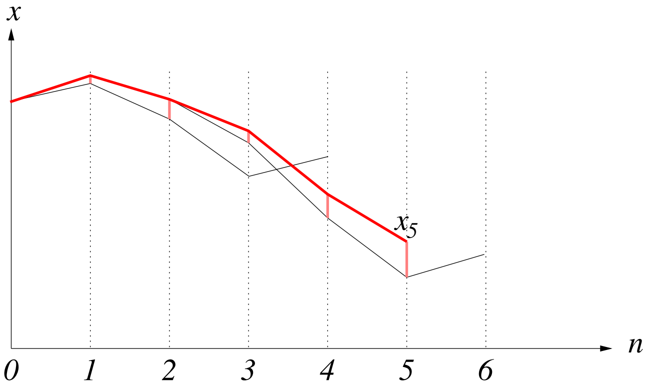

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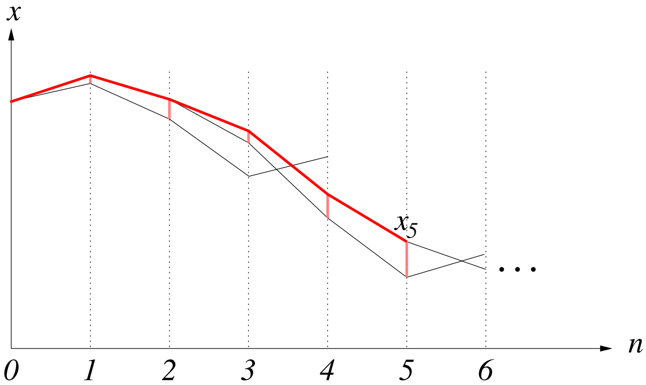

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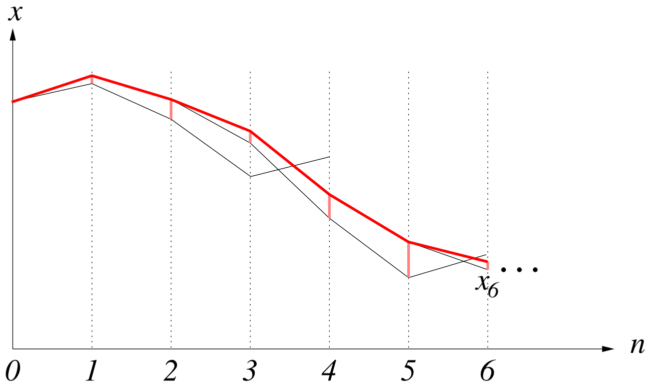

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Remedy:

- use sensitivity based techniques to update the “tails” of the optimal control sequences
- perform an integrated robustness and stability analysis

This will be the starting point for SADCO Task 3.3

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