Internal stabilization by noise of the Navier-Stokes equation and Internal stabilization of the Oseen-Stokes equations by Stratonovich noise

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1 Introduction and statement of the problem

This work is concerned with internal stabilization via Stratonovich noise feedback controller of Oseen–Stokes system

$$
\begin{aligned}
&\frac{\partial X}{\partial t}-\nu\Delta X+(f\cdot\nabla)X+(X\cdot\nabla)g=\nabla p\ \text{ in }\ (0,\infty)\times\mathcal{O},\\
&\nabla\cdot X=0\quad\text{ in }\ (0,\infty)\times\mathcal{O},\\
&X(0,\xi)=x(t),\ \xi\in\mathcal{O},\quad X=0\text{ on }(0,\infty)\times\partial\mathcal{O}.
\end{aligned}
\tag{1}
$$

Here, $\mathcal{O}$ is an open and bounded subset of $\mathbb{R}^d$, $d=2,3$, with smooth boundary $\partial\mathcal{O}$ and $f,g\in C^2(\overline{\mathcal{O}};\mathbb{R}^d)$ are given functions. In the special case $g\equiv 0$, system (1) describes the dynamic of a fluid Stokes flow with partial inclusion of convection acceleration $(f\cdot\nabla)X$ (X is the velocity field). The same equation describes the disturbance flow induced by a moving body with velocity f through the fluid. Should we mention also that in the special case $f\equiv g\equiv X_e$, where X_e is the equilibrium (steady-state) solution of the Navier–Stokes equation

$$
\begin{aligned}
&\frac{\partial X}{\partial t}-\nu\Delta X+(X\cdot\nabla)X=\nabla p+f_e,\\
&\nabla\cdot X=0,\quad X|_{\partial\mathcal{O}}=0,
\end{aligned}
\tag{2}
$$

and $f_e\in C(\overline{\mathcal{O}};\mathbb{R}^d)$, system (1) is the linearization of (2) around X_e. In this way, the stabilization of (1) can be interpreted as the first order stabilization procedure of steady-state Navier–Stokes flows.

Our aim here is to design a stochastic feedback controller of the form

$$u = \mathbf{1}_{\mathcal{O}_0} \sum_{k=1}^{M} R_k(X) \circ \dot{\beta}_k, \ \ R_k \in L((L^2(\mathcal{O}))^d), \tag{3}$$

which stabilizes in probability system (1) and has the support in an open subdomain $\mathcal{O}_0$ of $\mathcal{O}$.

Here, $\{\beta_k\}_{k=1}^M$ is a system of mutually independent Brownian motions on a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ with filtration $\{\mathcal{F}_t\}_{t>0}$ while the corresponding closed loop system

$$\begin{aligned} & dX - \nu\Delta X\,dt + (f\cdot\nabla)X\,dt + (X\cdot\nabla)g\,dt \\ & \qquad = \mathbf{1}_{\mathcal{O}_0} \sum_{k=1}^{M} R_k(X) \circ d\beta_k + \nabla p\,dt \\ & X(0) = x \end{aligned} \tag{4}$$

is taken in Stratonovich sense (see, e.g., [1]) and this is the significance of the symbol $R_k(X) \circ \dot{\beta}_k$ in the expression of the noise controller (3). We have denoted by $\mathbf{1}_{\mathcal{O}_0}$ the characteristic function of the open set $\mathcal{O}_0 \subset \mathcal{O}$.

In [5], [6], [7], [8], the author has designed similar stabilizable Ito noise controllers for equation (1) and related Navier–Stokes equations. However, it should be said that, with respect to Ito noise controllers, the Stratonovich feedback controller (3) has the advantage to be stable with respect to smooth changes $\dot{\beta}_k^\varepsilon$ of the noise $\dot{\beta}_k$ and this fact is crucial not only from the conceptual point of view, but also for numerical simulations and practical implementation into system (1) of the random stabilizable feedback controller

$$u_\varepsilon(t) = \mathbf{1}_{\mathcal{O}_0} \sum_{k=1}^{M} R_k(X(t)) \dot{\beta}_k^\varepsilon(t),$$

where $\dot{\beta}_k^\varepsilon$ is a smooth approximation of β_k^ε.

As regards the literature on stabilization of linear differential systems by Stratonovich noise, the pioneering works [2], [3] should be primarily cited. For linear PDEs, this procedure was developed in [9], [10] which are related to this work. For general results on internal stabilization of Navier–Stokes systems with deterministic feedback controllers, we refer to [4].

2 The noise stabilizing feedback controller

Consider the standard space of free divergence vectors $H = \{y \in (L^2(\mathcal{O}))^d;\ \nabla \cdot y = 0 \text{ in } \mathcal{O},\ y \cdot n = 0 \text{ on } \partial\mathcal{O}\}$ and denote by $\mathcal{A}_0 : D(\mathcal{A}_0) \subset H \to H$ the realization of the Oseen–Stokes operator in this space, that is,

$$\mathcal{A}_0 y = P(-\nu\Delta y + (f \cdot \nabla)y + (y \cdot \nabla)g), \quad y \in D(\mathcal{A}_0), \tag{5}$$

where $D(\mathcal{A}_0) = H \cap (H^2(\mathcal{O}))^d \cap (H_0^1(\mathcal{O}))^d$. Here, P is the Leray projector on H and $H^2(\mathcal{O})$, $H_0^1(\mathcal{O})$ are standard Sobolev spaces on $\mathcal{O}$. In the following, it will be more convenient to represent equation (1) in the complex Hilbert space $\mathcal{H} = H + iH$ by extending $\mathcal{A}_0$ to $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ via standard procedure, $\mathcal{A}(x + iy) = \mathcal{A}_0 x + i\mathcal{A}_0 y$. The operator $\mathcal{A}$ has a countable set of eigenvalues $\{\lambda_j\}_{j=1}^\infty$ (eventually complex) with the corresponding eigenvectors φ_j. Denote by $\mathcal{A}^*$ the adjoint of $\mathcal{A}$ with eigenvalues $\overline{\lambda}_j$ and eigenvectors φ_j^*. Each eigenvalue is repeated in the following according to its algebraic multiplicity m_j. We denote by N the number of eigenvalues λ_j for which

$$\operatorname{Re} \lambda_j > 0 \text{ for } j \le N,\ \ \lambda_1 + \lambda_2 + \cdots + \lambda_N > 0. \tag{6}$$

(In the above sequence, each λ_j is taken together its conjugate $\overline{\lambda}_j$ and, clearly, there is such a natural number N.)

Set $\mathcal{X}_u$=lin span$\{\varphi_j\}_{j=1}^N$ and denote by $\mathcal{X}_s$ the algebraic complement of $\mathcal{X}_u$ in $\mathcal{X}$. It is well known that $\mathcal{X}_u$ and $\mathcal{X}_s$ are both invariant for $\mathcal{A}$ and, if we set

$$\mathcal{A}_u = \mathcal{A}_0|_{\mathcal{X}_u}, \quad \mathcal{A}_s = \mathcal{A}_0|_{\mathcal{X}_s},$$

we have for their spectra $\sigma(\mathcal{A}_u) = \{\lambda_j\}_{j=1}^N$, $\sigma(\mathcal{A}_s) = \{\lambda_j\}_{j=N+1}^\infty$ and, since $-\mathcal{A}_s$ is the generator of an analytic C-semigroup $e^{-\mathcal{A}_s t}$ in $\mathcal{X}_s$, we have

$$\|e^{-\mathcal{A}_s t}\|_{L(\mathcal{H})} \le C \exp(-\mathrm{Re}\,\lambda_{N+1} t), \quad t \ge 0, \tag{7}$$

(see, e.g., [5], p. 14). In the following, we shall assume that

(i) *All the eigenvalues* $\{\lambda_j\}_{j=1}^N$ *are semisimple.*

This means that the algebraic multiplicity of each λ_j, $j = 1, ..., N$, coincides with its geometric multiplicity or, in other words, the finite-dimensional operator (matrix) $\mathcal{A}_u$ is diagonalizable. As we will see later on, this assumption is not essentially necessary but it simplifies however the construction of the stabilizing controller because it reduces the unstable part of the system to a diagonal finite-dimensional differential system. In particular, it follows by (i) that we can choose the dual systems $\{\varphi_j\}$ and $\{\varphi_j^*\}$ in such a way that

$$\left\langle \varphi_i, \varphi_j^* \right\rangle = \delta_{ij}, \quad i, j = 1, ..., N. \tag{8}$$

(Here, and everywhere in the following, $\langle \cdot, \cdot \rangle$ stands for the scalar product in$\mathcal{H}$ and H. By $|\cdot|_{\mathcal{H}}$ and $|\cdot|_H$ we denote the corresponding norms.)

We note that the uncontrolled Oseen–Stokes system (1) can be rewritten in the space $\mathcal{H}$ as

$$\frac{dX}{dt} + \mathcal{A}X = 0,\ t \geq 0, \quad X(0) = x, \tag{9}$$

and setting $X_u = \sum_{j=1}^{N} y_j \varphi_j$, $X_s = (I - P_N)X$, where P_N is the algebraic projector on $\mathcal{X}_u$, we have

$$\frac{dX_u}{dt} + \mathcal{A}_u X_u = 0, \quad X_u(0) = P_N x, \tag{10}$$

$$\frac{dX_s}{dt} + \mathcal{A}_s X_s = 0, \quad X_s(0) = (I - P_N)x. \tag{11}$$

We set $A_u = \{\langle \mathcal{A}\varphi_j, \varphi_k^* \rangle\}_{j,k=1}^{N} = \mathrm{diag}\|\lambda_j\|_{j=1}^{N}$ and so, by (8), we may rewrite (10) in terms of $y = \left\{ y_j = \left\langle X_u, \varphi_j^* \right\rangle \right\}_{j=1}^{N}$ as

$$\frac{dy}{dt} + A_u y = 0, \quad y(0) = \left\{ \left\langle P_N x, \varphi_j^* \right\rangle \right\}_{j=1}^{N}. \tag{12}$$

Since $\mathrm{Tr}(-A_u) = -\lambda_1 - \lambda_2 - \cdots - \lambda_N < 0$, it follows by Theorem 2 in [3] that there is a sequence of skew-symmetric matrices $\{C^k\}_{k=1}^{M}$, where $M = N - 1$ such that the solution y to the Stratonovich stochastic system

$$dy + A_u y\, dt = \sum_{k=1}^{M} C^k y \circ d\beta_k, \quad t \geq 0, \tag{13}$$

has the property

$$|y(t)| \leq C|y(0)|e^{-\gamma_0 t},\ \mathbb{P}\text{-a.s.},\ \forall t > 0, \tag{14}$$

where $\gamma_0 > 0$. The matrix C^k is explicitly constructed in [3] and it will be used below to construct a stabilizable feedback controller of the form (3). Namely, we set in (3)

$$R_k(X) = \sum_{i,j=1}^{N} C_{ij}^k \left\langle X, \varphi_j^* \right\rangle \phi_i, \quad k = 1, ..., M, \tag{15}$$

where $\|C^k_{ij}\|^N_{i,j=1} = C^k$,

$$\phi_i = \sum_{\ell=1}^{N} \alpha_{i\ell}\varphi^*_\ell, \quad i = 1, ..., N, \tag{16}$$

and $\alpha_{i\ell}$ are chosen in such a way that

$$\sum_{\ell=1}^{N} \alpha_{i\ell}\gamma_{\ell j} = \delta_{ij}, \quad i, j = 1, ..., N. \tag{17}$$

Here, $\gamma_{\ell j} = \int_{\mathcal{O}_0} \varphi^*_\ell \overline{\varphi}^*_j d\xi$ and since, by the unique continuation property (see [5], p. 157), the eigenfunction system $\{\varphi^*_j\}$ is linearly independent on $\mathcal{O}_0$, we infer that the matrix $\|\gamma_{\ell j}\|^N_{\ell,j=1}$ is not singular and, therefore, there is a unique system $\{\alpha_{i\ell}\}$ which satisfies (17). Then, by (16), we see that

$$\left\langle \mathbf{1}_{\mathcal{O}_0}\phi_i, \varphi^*_j \right\rangle = \delta_{ij}, \quad i, j = 1, ..., N. \tag{18}$$

As mentioned earlier, $\mathcal{O}_0$ is an arbitrary open subset of $\mathcal{O}$.

Theorem 1 is the main result.

Theorem 1 *The solution X to the closed loop system* (4)*, where R_k are defined by* (15)*, is exponentially stable in probability, that is,*

$$|X(t)|_{\widetilde{H}} \le Ce^{-\gamma t}|x|_H, \quad \forall t \ge 0, \ \mathbb{P}\text{-a.s.}, \tag{19}$$

where $\gamma > 0$.

2.1 The proof of Theorem 1

The idea of the proof, already used in stabilization theory of infinite-dimensional systems with finite-dimensional unstable subspaces, is to stabilize the finite-dimensional system (10) by a feedback controller of the form (3) and to reconstruct consequently the system via the infinite-dimensional stable complement (11).

Namely, taking into account (10), (11), we write the closed loop system (4), that is,

$$dX + \mathcal{A}X\,dt = P\left[\mathbf{1}_{\mathcal{O}_0}\sum_{k=1}^{M} R_k(X)\circ d\beta_k\right] \tag{20}$$

as

$$dX_u + \mathcal{A}_u X_u dt = P_N \sum_{k=1}^{M}\sum_{i,j=1}^{N} C_{ij}^k \left\langle X_u, \varphi_j^*\right\rangle P(,\phi_i)\circ d\beta_k, \tag{21}$$

$$dX_s + \mathcal{A}_s X_s dt = (I-P_N)\sum_{k=1}^{M}\sum_{i,j=1}^{N} C_{ij}^k \left\langle X_u, \varphi_j^*\right\rangle P(\phi_i)\circ d\beta_k, \tag{22}$$

where $X_u + X_s = X$.

Taking into account (18) and that $X_u = \sum_{j=1}^N y_j\varphi_j$, we may rewrite (21) as

$$dy_\ell + \lambda_\ell y_\ell dt = \sum_{k=1}^{M} C_{\ell j}^k y_j \circ d\beta_k, \quad \ell = 1,...,N, \tag{23}$$

and so, by (19), we have that

$$|y_\ell(t)| \le Ce^{-\gamma t}|y_\ell(0)|,\ \mathbb{P}\text{-a.s.},\ \forall t>0,\ \ell=1,...,N. \tag{24}$$

(We have denoted by C several positive constants independent of t.)

As regards the existence of a solution X_s to (22), this is standard and follows from the general theory of linear infinite-dimensional stochastic equations.

Now, in order to estimate X_s, it is convenient to replace (22) by its Ito formulation (see, e.g., [10])

$$dX_s+\mathcal{A}_sX_s\,dt=\frac{1}{2}\sum_{k=1}^{M}[P(\mathbf{1}_{\mathcal{O}_0}R_k]^2X_u dt+\sum_{k=1}^{M}P(\mathbf{1}_{\mathcal{O}_0}R_k(X_u))d\beta_k. \tag{25}$$

Taking into account that, by (7), $e^{-\mathcal{A}_st}$ is an exponentially stable semigroup on $\mathcal{X}_s$, without loss of generality we may assume that Re $\langle\mathcal{A}_sx,x\rangle \geq \gamma|x|^2_H$. (Otherwise, proceeding as in [6], we replace the scalar product $\langle x,y\rangle$ by $\langle Qx,y\rangle$ where Q is the solution to the Lyapunov equation $\mathcal{A}_sQ+Q\mathcal{A}^*_s=\gamma I$.) Then, applying Ito's formula in (25), we see that

$$\begin{aligned}d|X_s(t)|^2_{\mathcal{H}}+2\gamma|X_s(t)|^2_{\mathcal{H}}dt=&\left\langle X_s(t),\sum_{k=1}^{M}[P(\mathbf{1}_{\mathcal{O}_0}R_k]^2X_u(t)\right\rangle dt\\ &+\sum_{k=1}^{M}|P(\mathbf{1}_{\mathcal{O}}R_k(X_u))|^2dt+2\sum_{k=1}^{M}P(\mathbf{1}_{\mathcal{O}}R_k(X_u))X_sd\beta_k.\end{aligned} \tag{26}$$

Taking into account that, by (24), $|X_u(t)|_{\mathcal{H}}\leq e^{-\gamma t}|X_u(0)|$, we infer by (26) that

$$E|X_s(t)|^2_{\mathcal{H}}\leq C|X_s(0)|^2e^{-\gamma t},\quad \forall t\geq 0.$$

Since $M(t)=\int_0^t\sum_{k=1}^{M}P(\mathbf{1}_{\mathcal{O}_0}R_k(X_u)X_s\,d\beta_k$ is a local martingale, it follows by (26) and Lemma 3.1 in [4] that

$$|X_s(t)|_{\mathcal{H}}\leq C|X_s(0)|e^{-\gamma t},\quad \forall t\geq 0,\ \mathbb{P}\text{-a.s.},$$

which completes the proof.

3 The design of a real valued noise controller

A nice feature of the feedback controller given by Theorem 1 is its simple structure. Moreover, its computation does not rise any special problem because the stabilizing matrices C^k for the diagonal system (23) can be explicitly expressed.

If all λ_j, $j = 1, ..., N$, are real, then the feedback controller $\{R_k\}$ given by Theorem 1 as well as the closed loop system (4) are real, too. However, if some λ_j are complex, then the above feedback controller does not stabilize system (1), but its complex realization in the space $\mathcal{H} = H + iH$. In this case, though the stabilizing feedback controller (3) is quite simple, its implementation into real systems rises some delicate problems because the feedback controller is in implicit form as function of $\mathrm{Re}\, X$, $\mathrm{Im}\, X$. In order to circumvent these inconvenience, we will derive below from the above construction a real valued noise feedback controller which has a stabilizing effect on the Oseen–Stokes system (1).

To this end, we denote by $\{\psi_j\}_{j=1}^N$ the orthogonalized system (via Schmidt procedure) $\{\mathrm{Re}\, \varphi_j, \mathrm{Im}\, \varphi_j,\ j = 1, ..., N\}$. Taking into account that each φ_j in the above system arises together with its conjugate, the dimension of the system $\{\psi_j\}$ is again N. We also set

$$\mathcal{X}_u^{\mathrm{re}} = \mathrm{lin\ span}\{\psi_j\}_{j=1}^N$$

and note that $\mathcal{X}_u^{\mathrm{re}} = \{\mathrm{Re}\, y;\ y \in \mathcal{X}_u\}$ and $H = \mathcal{X}_u^{\mathrm{re}} \oplus \mathcal{X}_s^{\mathrm{re}}$, where $X_s^{\mathrm{re}} = \{\mathrm{Re}\, y;\ y \in \mathcal{X}_s\}$. Moreover, the operator $\mathcal{A}_0$ leaves invariant spaces $\mathcal{X}_u^{\mathrm{re}}$, $\mathcal{X}_s^{\mathrm{re}}$ and if we set

$$\mathcal{A}_u^{\mathrm{re}} = \mathcal{A}_0|_{\mathcal{X}_u^{\mathrm{re}}}, \quad \mathcal{A}_s^{\mathrm{re}} = \mathcal{A}_0|_{\mathcal{X}_s^{\mathrm{re}}},$$

we have, of course,

$$\|e^{-\mathcal{A}_s^{\mathrm{re}} t}\|_{L(H)} \le C e^{-\gamma_0 t}, \quad \forall t > 0, \tag{27}$$

for some $\gamma_0 > 0$. It is also easily seen that

$$\mathrm{Tr}[-\mathcal{A}_u^{\mathrm{re}}] = \mathrm{Tr}[-\mathcal{A}_u] < 0.$$

Then, we define as in (15) the operators

$$\widetilde{R}_k(x) = \sum_{i,j=1}^{N} \widetilde{C}_{ij}^k \left\langle X, \varphi_j \right\rangle \widetilde{\phi}_i, \quad k = 1, ..., M, \tag{28}$$

where $\widetilde{C}^k = \|\widetilde{C}_{ij}^k\|_{i,j=1}^N$ is the matrix system which stabilizes in probability, via Theorem 2 in [3], the finite-dimensional system

$$dy + A_u^{\mathrm{re}} y\, dt = \sum_{k=1}^{M} \widetilde{C}^k y \circ d\beta_k.$$

Here, $A_u^{\mathrm{re}} = \|\left\langle \mathcal{A}_u^{\mathrm{re}} \psi_i, \psi_j \right\rangle \|_{i,j=1}^N$ and $\widetilde{\phi}_i$ is given by

$$\widetilde{\phi}_i = \sum_{\ell=1}^{N} \widetilde{\alpha}_{i\ell} \psi_\ell, \quad i = 1, ..., N,$$

where $\widetilde{\alpha}_{i\ell}$ are chosen in such a way that

$$\sum_{\ell=1}^{N} \widetilde{\alpha}_{i\ell} \widetilde{\gamma}_{\ell j} = \delta_{ij}, \quad \widetilde{\gamma}_{\ell j} = \int_{\mathcal{O}_0} \psi_\ell \psi_j d\xi.$$

As in the previous case, the system $\{\psi_j\}_{j=1}^N$ is still linearly independent on $\mathcal{O}_0$ which implies that $\det \|\widetilde{\gamma}_{\ell j}\| \neq 0$ and the above system has a unique solution $\widetilde{\alpha}_{i\ell}$.

Then, arrived at this point, the proof of Theorem 1 applies neatly to conclude that the feedback noise controller (3) with $\widetilde{R}_k$ instead of R_k is stabilizing in the probability system (9).

Namely, we have the following stabilization result.

Theorem 2 *The solution X to the closed loop system* (4) *with* $R_k = \widetilde{R}_k$ *defined by* (28) *satisfies* (19) *in the real norm* $|X(t)|_H$.

Remark 3 It should be emphasized that there is a close connection between the unique continuation property of eigenfunctions φ_j (or φ_j^*) of the Oseen–Stokes operator and the above construction of a stabilizing finite-dimensional feedback controller (3). In fact, as seen above, the design of u in the form (15) or (28) is essentially based on this sharp property through existence in the algebraic system (17).

Remark 4 One might speculate that the noise feedback controller

$$u = \mathbf{1}_{\mathcal{O}_0} \sum_{k=1}^{M} R_k(X - X_e) \circ d\beta_k \tag{29}$$

inserted in the right hand side of Navier–Stokes system (2) stabilizes exponentially in probability the equilibrium solution X_e to (2). In general, this might not be true, but it happens for Ito noise of the form (29) for any x in a sufficiently small neighborhood of X_e (see [8]) and one might expect that the fixed point argument used there for the equivalent random system is still applicable in the present case. We expect to give details in a later work.

4 Conclusions

We have designed in this paper a Stratonovich stochastic feedback controller which exponentially stabilizes in probability a general Oseen–Stokes system from fluid dynamics. The controller has the support in an arbitrary open subset $\mathcal{O}_0$ of the velocity field domain $\mathcal{O} \subset \mathbb{R}^d$, $d = 2, 3$, and has a finite-dimensional linear structure which involves the dual eigenfunctions corresponding to unstable eigenvalues of the systems. The stabilization effect is independent of the Reynold number $1/\nu$ through the dimension N of the stabilizing controller (3) might depend on ν.

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