# New Semiconcavity Results for Optimal Control Problems

Piermarco Cannarsa

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University of Rome “Tor Vergata” (Italy)
http://www.mat.uniroma2.it

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# Outline

- Semiconcave functions
  - Definitions and general properties
  - Generalized gradients
  - Singularities of semiconcave functions
- Semiconcave functions and optimal control
  - Mayer problem
  - Differential inclusions
- Semiconcave functions and Hamilton-Jacobi equations
  - Smoothness of extremal trajectories
  - Semiconcavity of solutions
- Conclusions

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# definition

$\Omega \subset \mathbb{R}^N$ domain

## Definition

$u : \Omega \to \mathbb{R}$ *(linearly) semiconcave*

- *$u$ continuous in $\Omega$*
- *$\exists C \geqslant 0$ (semiconcavity constant) such that*

$$u(x+h) + u(x-h) - 2u(x) \leqslant C|h|^2$$

*for all $x, h \in \mathbb{R}^N$ such that $[x-h, x+h] \subset \Omega$*

more general notion

$$u(x+h) + u(x-h) - 2u(x) \leqslant |h|\omega(|h|)$$

where $\omega : [0,\infty) \to [0,\infty)$ (semiconcavity modulus) upper semicontinuous, nondecreasing, $\omega(r) \to 0 \quad (r \to 0)$

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# equivalent properties

$u : \Omega \to \mathbb{R}$ semiconcave with constant $C \geqslant 0$

**iff**

any of the following holds:

- for all $x, y$ such that $[x, y] \subset \Omega$ and all $\lambda \in [0, 1]$
$$\lambda u(x) + (1-\lambda)u(y) - u(\lambda x + (1-\lambda)y) \leqslant \frac{C}{2}\lambda(1-\lambda)|x-y|^2$$
- the function $x \to u(x) - \frac{C}{2}|x|^2$ is concave in $\Omega$
- for any $\nu \in \mathbb{R}^N$ such that $|\nu| = 1$
$$\frac{\partial^2 u}{\partial \nu^2} \leq C \qquad \text{in (distributional sense on)} \quad \Omega$$
- $u(x) = \inf_{i \in \mathcal{I}} u_i(x)$ where $\|D^2 u_i\|_\infty \leq C$ for all $i \in \mathcal{I}$

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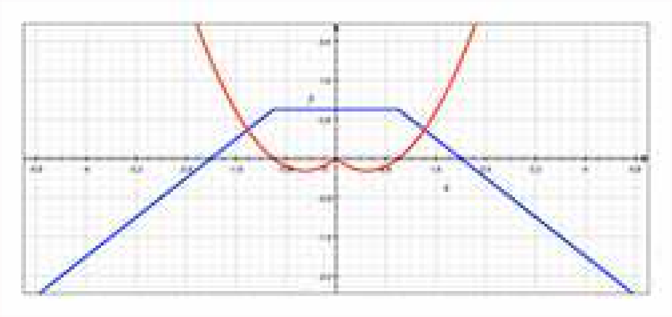
# why study semiconcave functions?

$$u(x) = \inf_{i \in \mathcal{I}} u_i(x) \qquad x \in \mathbb{R}^N$$

- $u_i$ affine $\Rightarrow$ $u$ concave
- $\|D^2 u_i\|_\infty \leq C$ $\Rightarrow$ $u$ semiconcave



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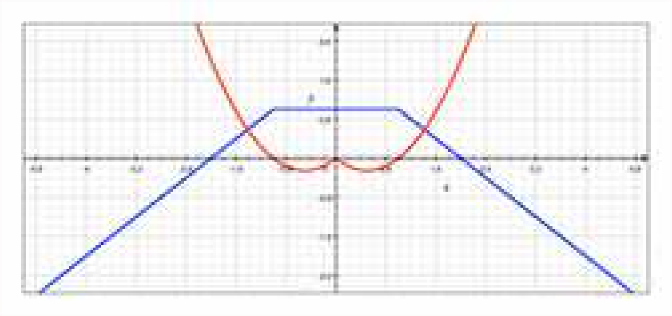

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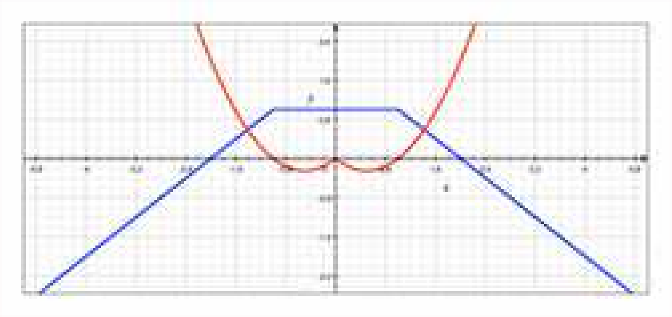

# references

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# Outline

# generalized differentials

$\Omega \subset \mathbb{R}^N$ domain $\quad u : \Omega \to \mathbb{R}$

$u$ semiconcave $\Longrightarrow$ $u$ Locally Lipschitz

$\Longrightarrow$ $u$ differentiable a.e.

$x \in \Omega$

- limiting gradients

$$D^* u(x) = \left\{ \lim_{i \to \infty} Du(x_i) \;\middle|\; x_i \to x \right\}$$

compact $\neq \emptyset$

- $D^+ u(x) = \mathrm{co}\big(D^* u(x)\big)$ superdifferential

$\Big( \Rightarrow \; D^+ u(x) = \partial u(x)$ Clarke's generalized differential $\Big)$

$$D^+ u(x) = \left\{ p \in \mathbb{R}^N \;\middle|\; \limsup_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \leqslant 0 \right\}$$

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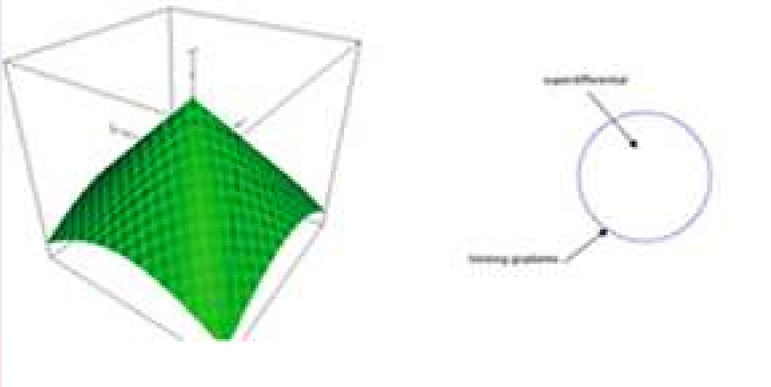

Figure: $u(x, y) = 3 - \sqrt{x^2 + y^2}$ and $D^+u(0, 0)$

# example 2

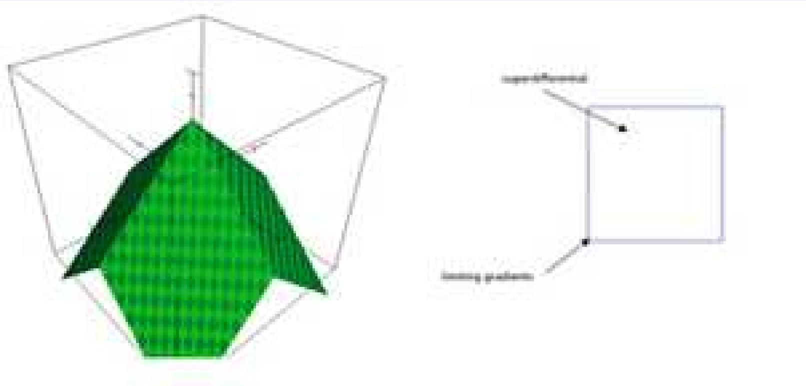

Figure: $u(x,y) = 3 - |x| - |y| \quad \text{and} \quad D^+u(0,0)$

# Outline

# singular point = nondifferentiability point

$u : \Omega \to \mathbb{R}$ semiconcave

## Definition (singular set)

$$\begin{aligned} \Sigma(u) &= \{x \in \Omega \mid \nexists Du(x)\} \\ &= \{x \in \Omega \mid \dim D^+u(x) \geqslant 1\} \end{aligned}$$

- $u \in \mathrm{Lip}_{loc}(\Omega) \implies |\Sigma(u)| = 0$ (Lebesgue)
- $Du \in BV_{loc}(\Omega; \mathbb{R}^N) \implies \Sigma(u)$ countably $\mathcal{H}^{N-1}$-rectifiable
- magnitude of a singular point $x$

$$\kappa(x) = \dim D^+u(x)$$

# singular point = nondifferentiability point

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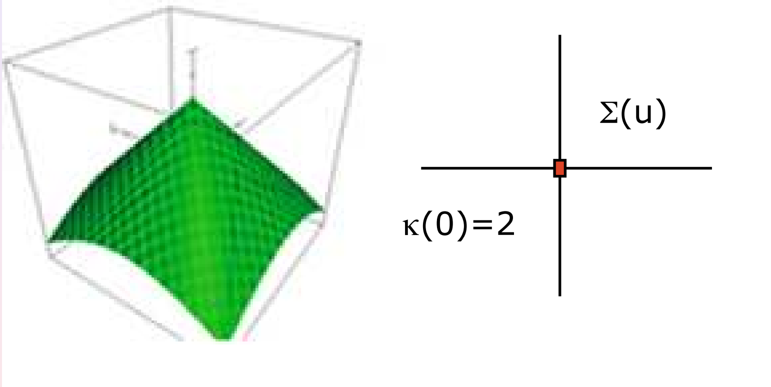


Figure: singular set of $u(x,y) = 3 - \sqrt{x^2 + y^2}$

# example 2

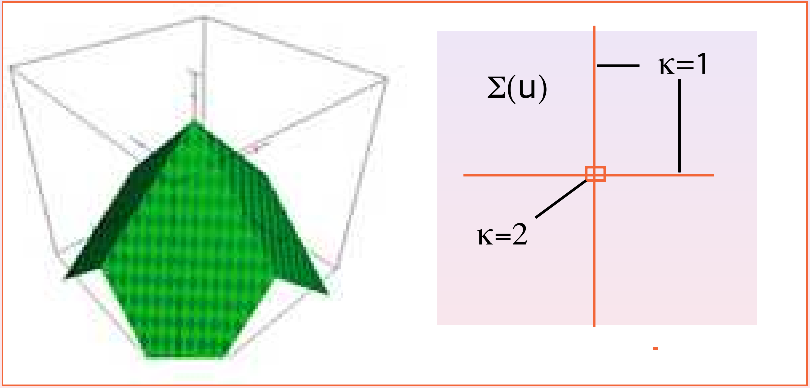


Figure: singular set of $u(x, y) = 3 - |x| - |y|$

# sharp rectifiability

$u : \Omega \to \mathbb{R}$ semiconcave

singular sets of prescribed magnitude $k = 1, \ldots, N$

$$\Sigma^k(u) := \{x \in \Omega \mid \dim D^+u(x) = k\}$$

**Theorem**

$\Sigma^k(u)$ *countably* $(N-k)$*-rectifiable*

- $\Sigma(u) = \cup_{k=1}^{N} \Sigma^k(u)$ countably $(N-1)$-rectifiable
- references
  - Zajíček (1978), Veselý (1979)
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# propagation of singularities

$u : \Omega \to \mathbb{R}$ semiconcave $x_0 \in \Sigma(u)$

## Problem

$$??? \implies \Sigma(u) \neq \{x_0\}$$

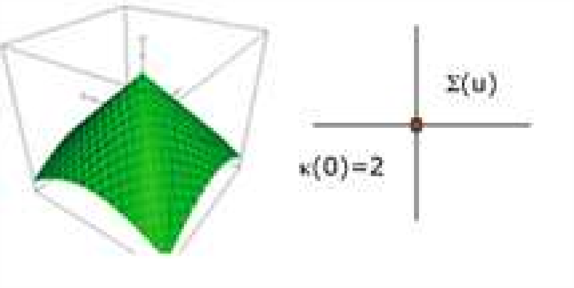


Figure: no propagation for $u(x,y) = 3 - \sqrt{x^2+y^2}$

# do lower magnitude singularities propagate?

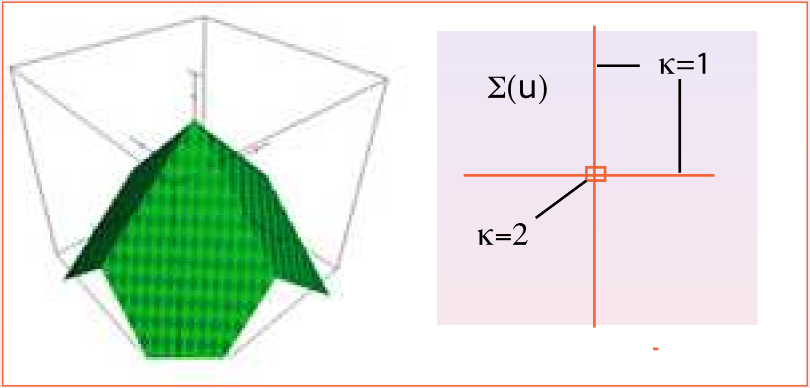


Figure: magnitude 1 singularities of $u(x, y) = 3 - |x| - |y|$ do propagate along straight lines

# NO

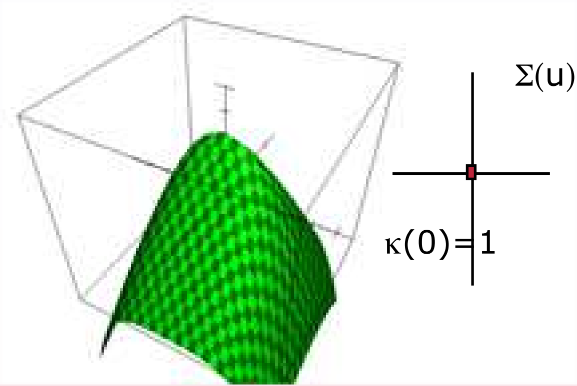


Figure: an isolated singularity of magnitude 1 at the origin

$$u(x,y) = 3 - \sqrt{\left(\frac{3x}{2}\right)^2 + \left(\frac{2y}{3}\right)^4}$$

# “pattern recognition”

a closer look at $D^+u$ and $D^*u$

## Example

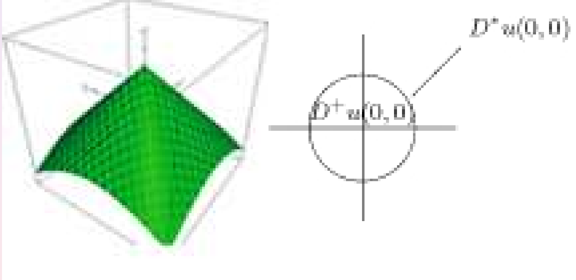


Figure: in example 1 $\boxed{D^*u(0,0) = \partial D^+u(0,0)}$

# example 2

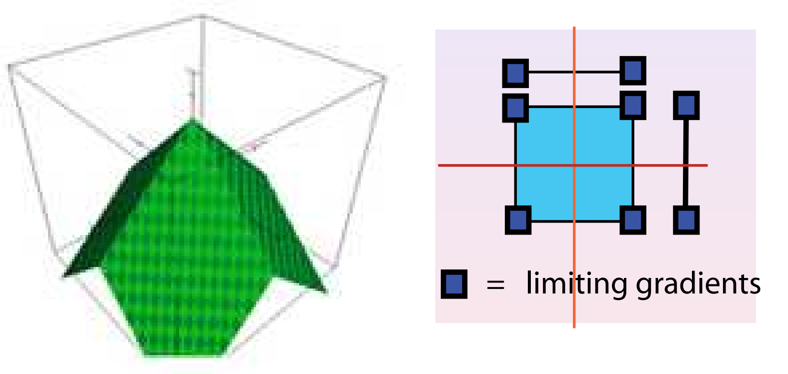


Figure: here $\boxed{D^* u(0,0) \subsetneq \partial D^+ u(0,0)}$

# example 3

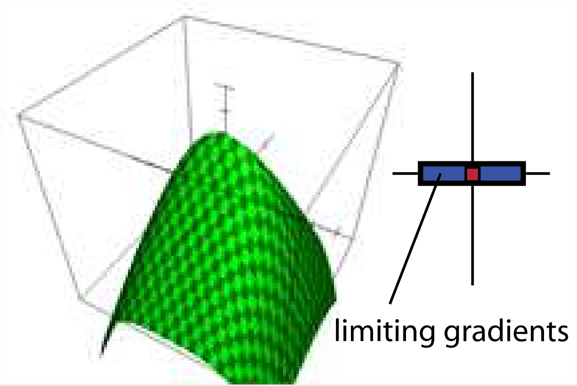


Figure: here $\boxed{D^* u(0,0) = D^+ u(0,0)}$

# propagation principle

$u : \Omega \to \mathbb{R}$ semiconcave $x_0 \in \Sigma(u)$

Theorem (Albano – C 1999; C – Yu 2007)

*Let*

- $\emptyset \neq \partial D^+u(x_0) \setminus D^*u(x_0) \ni p_0$
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*Then* $\exists \tau > 0$ *and* $x(\cdot) : [0, \tau] \to \Sigma(u)$ *Lipschitz*

1. $\dot{x}(s) \in q - p_0 + D^+u(x(s))$ *a.e.* $s \in [0, \tau]$
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$u : \Omega \to \mathbb{R}$ semiconcave $x_0 \in \Sigma(u)$

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# back to example 2

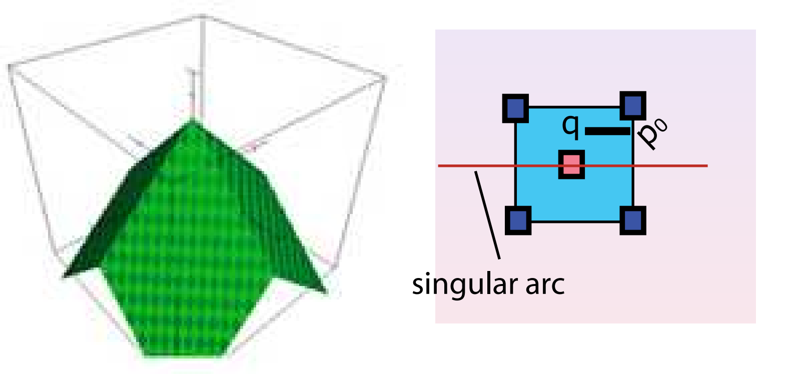


Figure: the propagation principle at work

# Outline

# value function of a Mayer problem

$t \leq T \quad x \in \mathbb{R}^N \quad y_{t,x,\alpha}(\cdot)$ solution

$$\begin{cases} \dot{y}(s) = f(y(s), \alpha(s)) & s \in (t, T) \\ y(t) = x \end{cases}$$

where

- $A$ compact
- $\alpha : [t, T] \to A$ measurable
- $f \in C(\mathbb{R}^N \times A; \mathbb{R}^N)$ continuous
  - $|f(x, a)| \leq C_0(1 + |x|)$
  - $|f(x, a) - f(y, a)| \leq C_1|x - y|$

given $\phi \in C(\mathbb{R}^N)$ define value function

$$V(t, x) = \inf_{\alpha} \phi(y_{t,x,\alpha}(T))$$

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# an ‘old’ semiconcavity result

## Theorem

*Assume*

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- $\|f_x(x,a) - f_x(y,a)\| \le C_2|x-y|$

*Then* $V$ **(linearly) semiconcave** $(-\infty, T] \times \mathbb{R}^N$

references

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# proof of semiconcavity of $V(t,\cdot)$

$x, z \in \mathbb{R}^N$

- $\alpha$ optimal at $x$
- $y(\cdot) = y_{t,x,\alpha}(\cdot)\,, \quad y_{\pm}(\cdot) = y_{t,x\pm z,\alpha}(\cdot)$

by semiconcavity of $\phi$ and smoothness of flow

$$
\begin{aligned}
& V(t,x+z) + V(t,x-z) - 2V(t,x) \\
\leq \quad & \phi(y_+(T)) + \phi(y_-(T)) - 2\phi(y(T)) \\
= \quad & \underbrace{\phi(y_+(T)) + \phi(y_-(T)) - 2\phi\Big(\frac{y_+(T)+y_-(T)}{2}\Big)}_{\leq c|y_+(T)-y_-(T)|^2 \leq c|z|^2} \\
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# Outline

# multifunctions and trajectories

$F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ multifunction

$$\begin{cases} \mathbf{1)}\ F(x) \text{ nonempty, convex, and compact } \forall x \in \mathbb{R}^N \\ \mathbf{2)}\ F \text{ Lipschitz: } F(y) \subset F(x) + c|y-x|B\ \forall x, y \in \mathbb{R}^N \\ \mathbf{3)}\ \exists r > 0 \text{ so that } \max\{|v| : v \in F(x)\} \leq r(1+|x|) \end{cases}$$

given

- $F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$
- $T \in \mathbb{R}$, let $t \leq T$, $x \in \mathbb{R}^N$

denote by $\boxed{Y_T(t,x)}$ all absolutely continuous arcs

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# parameterizations

- control system

$$\begin{cases} \dot{y}(s) = f(y(s), \alpha(s)) & s \in (t, T) \\ y(t) = x \end{cases}$$

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$$F(x) = \{ f(x, a) \ : \ a \in A \} \qquad (*)$$

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## difficulty: smoothness of $f(\cdot, a)$

Example

$$F(x) = [-|x|, |x|] \qquad (N = 1)$$

*has the $C^{1+}$ parametrization*

$$F(x) = \{xu : |u| \leq 1\}$$

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trajectories of the two systems coincide but known theorems asserting semiconcavity only apply to former parametrization

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# Hamiltonian

$H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ associated to $F$ is defined by

$$H(x,p) = \sup_{v \in F(x)} v \cdot p$$

one-to-one correspondence between $H$ and $F$

$$v \in F(x) \iff v \cdot p \leq H(x,p) \quad \forall p \in \mathbb{R}^N$$

**Assumptions**

(H1) for all compact convex $K \subseteq \mathbb{R}^N$ there is a constant $c_K \geq 0$ so that $x \mapsto H(x,p)$ semiconvex on $K$ with constant $c_K|p|$

(H2) $\nabla_p H(x,p)$ locally Lipschitz in $x$ uniformly for $|p| \neq 0$

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given terminal cost $\varphi \in C(\mathbb{R}^N)$
define value function

$$V(t,x) = \inf\left\{\varphi(y(T)) \;:\; y \in Y_T(t,x)\right\}$$

Theorem (C – Wolenski, 2011)

*Assume*

- *$F$ and $H$ as above*
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*Then $V$ locally semiconcave on $(-\infty, T] \times \mathbb{R}^N$*

time optimal control

- C—, Marino and Wolenski
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# proof of semiconcavity of $V(t,\cdot)$

- fix $x, z \in \mathbb{R}^N$ and let $y(\cdot)$ be **optimal** at $x$: want to show $V(t,x+z)+V(t,x-z)-2V(t,x) \le c|z|^2$
- invoke **maximum principle** to obtain adjoint arc

$$\begin{cases} -\dot{p}(s) \in \partial_x H\big(y(s),p(s)\big)\,, \quad \dot{y}(s) \in \partial_p H\big(y(s),p(s)\big) \\ -p(T) \in \partial^P \varphi\big(y(T)\big) \end{cases}$$

- supposing $p(s) \neq 0$ define $y_\pm(\cdot)$ by

$$\begin{cases} \dot{y}_\pm(s) = \nabla_p H\big(y_\pm(s),p(s)\big) \\ y_\pm(t) = x \pm z \end{cases}$$

observe

- $|y_+(s)-y_-(s)| \le c|z|$ and $|y_\pm(s)-y(s)| \le c|z|$
- $p \cdot \dot{y} = H(y,p)$ and $p \cdot \dot{y}_\pm = H(y_\pm,p)$

- $V(t,x+z)+V(t,x-z)-2V(t,x)$

$$\leq \varphi(y_+(T))+\varphi(y_-(T))-2\varphi(y(T))$$
$$=\underbrace{\varphi(y_+(T))+\varphi(y_-(T))-2\varphi\left(\frac{y_+(T)+y_-(T)}{2}\right)}_{\leq c|y_+(T)-y_-(T)|^2\leq c|z|^2}$$
$$+2\left[\varphi\left(\frac{y_+(T)+y_-(T)}{2}\right)-\varphi(y(T))\right]$$

- since $-p(T)\in\partial^P\varphi(y(T))$

$$\varphi\left(\frac{y_+(T)+y_-(T)}{2}\right)-\varphi(y(T))$$
$$\leq -p(T)\cdot\left[\frac{y_+(T)+y_-(T)}{2}-y(T)\right]+c\underbrace{\left|\frac{y_+(T)+y_-(T)}{2}-y(T)\right|^2}_{\leq c|z|^2}$$

- $$-p(T)\cdot\left[\frac{y_+(T)+y_-(T)}{2}-y(T)\right]$$
$$=\frac{1}{2}\int_t^T\left[-\dot p\cdot(y_+ + y_- - 2y) + p\cdot(2\dot y-\dot y_+-\dot y_-)\right]ds$$
- $$\frac{1}{2}\int_t^T p\cdot(2\dot y-\dot y_+-\dot y_-)\,ds$$
$$=\int_t^T\left[H(y,p)-\frac{H\big(y_+,p\big)+H\big(y_-,p\big)}{2}\right]ds$$
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$$+\int_t^T\Bigg[\underbrace{H\Big(\frac{y_++y_-}{2},p\Big)-\frac{H(y_+,p)+H(y_-,p)}{2}}_{\le c|p||y_+-y_-|^2\le c|z|^2}\Bigg]ds$$
- $$\int_t^T\left[H(y,p)-H\Big(\frac{y_++y_-}{2},p\Big)\right]ds \qquad (-\dot p\in\partial_x H(y,p)!)$$
$$\le\int_t^T\left[\dot p\cdot\Big(\frac{y_++y_-}{2}-y\Big)+\underbrace{c|p|\left|\frac{y_++y_-}{2}-y\right|^2}_{\le c|z|^2}\right]ds$$

# nonsmooth HJ equations

$$\begin{cases} H(x, -Du(x)) = 1 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}$$

## Example

$$|A(x)Du(x)| = 1 \qquad \text{in } \Omega$$

where

- $A : \mathbb{R}^N \to \mathbb{R}^{N\times N}$ is $\alpha$-Hölder continuous on $\mathbb{R}^N$
- $A(x)$ is invertible for all $x \in \mathbb{R}^N$, and

$$|A(x)| \leq C \quad \text{and} \quad |A(x)^{-1}| \leq C \qquad \forall x \in \mathbb{R}^N$$

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- $A : \mathbb{R}^N \to \mathbb{R}^{N\times N}$ is $\alpha$-Hölder continuous on $\mathbb{R}^N$
- $A(x)$ is invertible for all $x \in \mathbb{R}^N$, and

$$|A(x)| \le C \quad \text{and} \quad |A(x)^{-1}| \le C \qquad \forall x \in \mathbb{R}^N$$

# assumptions on $H$

$\exists\ C_0, r, R$, and $\alpha \in (0, 1/2)$ such that

- $|H(x,p) - H(y,p)| \le C_0|x-y|^{2\alpha}|p| \qquad \forall x, y, p \in \mathbb{R}^N$
- $H(x,\cdot)$ is convex, 1-homogeneous, and

$$r|p| \ \le\ H(x,p)\ \le R|p| \qquad \forall x, p \in \mathbb{R}^N$$

- $H(x,\cdot)$ is continuously differentiable on $\mathbb{R}^N\backslash\{0\}$, and

$$\begin{aligned} -\tfrac{1}{2r}\,&|D_pH(x,q) - D_pH(x,p)|^2 \\ &\le\ \langle D_pH(x,q) - D_pH(x,p), \tfrac{p}{|p|}\rangle \\ &\qquad \le -\tfrac{1}{2R}\,|D_pH(x,q) - D_pH(x,p)|^2 \end{aligned}$$

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# associated control problem

define

$$F(x) = \text{co}\left\{ D_p H(x,p) \ : \ p \in \mathbb{R}^N \setminus \{0\} \right\} \qquad \forall x \in \mathbb{R}^N$$

then

$$\begin{cases} H(x, -Du(x)) = 1 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}$$

if and only if

$$u(x) = \inf \left\{ t \geq 0 \ : \ \exists \begin{cases} y'(s) \in F(y(s)) & s \geq 0 \ a.e. \\ y(0) = x\,, & y(t) \in \partial\Omega \end{cases} \right\}$$

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# properties of $F(\cdot)$

- $F(x) \subset F(y) + C_0|x-y|^{2\alpha}B \qquad \forall x, y \in \mathbb{R}^N$
- *curvature* estimates

$$B\left(f_p(x) - r\frac{p}{|p|}, r\right) \subset F(x) \quad \text{and} \quad F(x) \subset B\left(f_p(x) - R\frac{p}{|p|}, R\right)$$

- *controllability* condition

$$B(0,r) \subset F(x) \subset B(0,R)$$

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# Outline

# $C^{1,\alpha/2}$ regularity of extremal trajectories

$K \subset \mathbb{R}^N$ closed

$$\begin{cases} x'(t) \in F(x(t)) & t \geq 0 \text{ a.e.} \\ x(0) \in K \end{cases} \qquad (DI)$$

*reachable set* (from $K$) in time $t$

$$\mathcal{R}(t) = \{x(t) \ : \ x(\cdot) \text{ is a trajectory of } (DI) \ \}$$

$x(\cdot)$ extremal trajectory on $[0, t]$ if $x(t) \in \partial\mathcal{R}(t)$

Theorem (C–Cardaliaguet)

$x(\cdot)$ *extremal on* $[0, T] \quad \Rightarrow \quad |x'(t) - x'(s)| \leq C|t-s|^{\alpha/2} \ \forall t, s$

$F(x) = c(x)B$

- Su and Berger (2006, 2007) $\quad N = 2$
- Cardaliaguet, Ley and Monteillet

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# Outline

# 'new' semiconcavity for HJ

Theorem (C–Cardaliaguet)

- *u solution*

$$\begin{cases} H(x, -Du(x)) = 1 & in\ \Omega \\ u(x) = 0 & on\ \partial\Omega \end{cases}$$

- $\theta \in (0, \frac{\alpha}{4+\alpha})$

$\Rightarrow$ *u locally semiconcave*

$$u(x+h) + u(x-h) - 2u(x) \leqslant C|h|^{1+\theta}$$

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# conclusions

- **semiconcavity** is a useful property to study nonsmooth solutions of
  - optimal control problems
  - partial differential equations

  not only for smooth data
- **extensions** of such a property ($\varphi$-concavity, exterior ball property of hypograph. . . ) have been developed
- modelling control systems by **differential inclusions** provides intrinsic description which is independent of smoothness of parameterization
- **maximum principle** and **geometric properties of dynamics** (rather than smoothness of flow) should play an essential role in such generalized contexts

# Thank you for your attention