The minimum time function around the origin

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The regularity of the minimum time function is a widely studied topic.

Which viewpoint?

We take the viewpoint started by Cannarsa and Sinestrari (Calc. Var. (1995) and book by Birkhäuser (2004)):

semiconcavity or semiconvexity of the minimum time function $\mathcal{T}$ under two assumptions:
(a) a first order small time controllability condition (Petrov's condition), which implies that $\mathcal{T}$ is locally Lipschitz;
(b) regularity conditions on the target and on the dynamics.

The starting question: can one prove similar regularity results by weakening the controllability assumptions (e.g., assuming $\mathcal{T}$ merely continuous)?

Yes.

Nonsmooth analysis and geometric measure theory results.

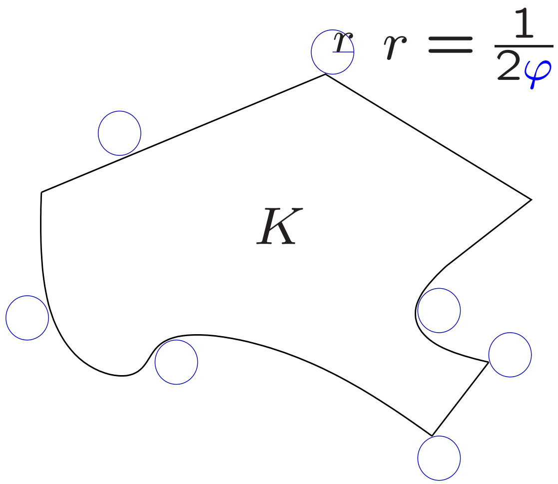
Let's focus on sublevels of $\mathcal{T}$, or, equivalently, to reachable sets from the target for the reversed dynamics at time T (denoted by $\mathcal{R}^T$).

Semiconvexity implies that sublevels of $\mathcal{T}$ satisfy a kind of external sphere condition, while semiconcavity implies they satisfy the same kind of internal sphere condition.

Definition. Let $K \subset \mathbb{R}^n$ be closed set. We say that
(a) K satisfies an external (uniform) sphere condition if $\exists r > 0$ such that $\forall x \in K$, there exists $v \in N(x; K)$, $v \neq 0$, for which

$$\langle v, y - x \rangle \leq \frac{1}{2r} \|v\| \, \|y - x\|^2 \qquad \forall y \in K. \tag{1}$$

(b) K has positive reach (Federer TAMS '59) (proximally smooth, prox-regular, φ-convex, ...) if (1) holds for all $v \in N(x; K)$.
(c) internal sphere condition $=$ external sphere condition for $\overline{K^c}$.



Of course, K is convex if and only if $r = \infty$. Inward corners are forbidden.

Well known facts:

– if $f : \mathbb{R}^n \to \mathbb{R}$ is semiconvex/semiconcave, then its epigraph/hypograph has positive reach;

– the epigraph of $f(x) = \sqrt{|x|}$ has positive reach, but f is not locally Lipschitz;

– K has positive reach iff the metric projection onto K is a singleton in a neighborhood of K.

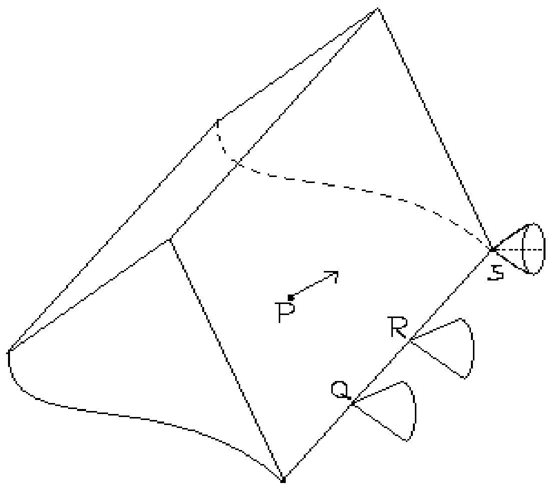
In general, (a) and (b) are different. However (Nour-Stern-Takche, JCA (2009)) if K is epi-Lipschitz (= the normal cone is pointed), then (a) $\Leftrightarrow$ (b).

Properties of sets (and epigraphs) with positive reach.

1. Proximal, Fréchet, Clarke normal cones to K do coincide.
Moreover (Federer, TAMS (1959)), the set

$$K^{(k)} = \{x \in K \, : \, \mathcal{H} - \mathrm{dim}(N(x;K)) \geq n - k\}$$

is $\mathcal{H}^k$-rectifiable (analogous to a k-face of a convex set). In particular, the set of points where the normal cone has dimension $\geq$ 2 (i.e., where K has a “corner”) is $\mathcal{H}^{n-2}$-rectifiable.



2. (GC-Marigonda, Calc. Var. (2006)) Let $K \subset \mathbb{R}^n$ be compact with positive reach. Then K has finite perimeter in $\mathbb{R}^n$ and the De Giorgi outer normal coincides with the unit proximal normal . Under a natural nondegeneracy condition, the reduced boundary coincides $\mathcal{H}^{n-1}$-a.e. with the topological boundary.

3. Let $\Omega \subseteq \mathbb{R}^n$ be open and let $f : \overline{\Omega} \to \mathbb{R} \cup \{+\infty\}$ be continuous and assume that its epigraph has positive reach.
Then (GC-Marigonda, Calc. Var. (2006)) for a.e. $x \in \Omega$ there esists $\delta_x > 0$ such that f is semiconvex in $B(x, \delta_x)$. In particular, f is a.e. differentiable in Ω, is locally BV in Ω and admits a second order Taylor expansion around a.e. $x \in \Omega$.
Moreover, (GC-Marigonda-Wolenski (2011)) a representation formula for the generalized gradient of f using limits of convex combinations of gradients, similar to the classical formula for Lipschitz functions, holds.

Therefore epi/hypo$f$ with positive reach is a reasonable substitute of semiconvexity/concavity.

Properties of sets (and epigraphs) with uniform external sphere condition.

Theorem. (Khai T. Nguyen, JMAA 2010) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and such that its epigraph satisfies a (locally) uniform external sphere condition. Then the set of points x where $N(\text{epi} f; (x, f(x))$ is not pointed (= contains lines) is a closed set with Lebesgue n-dimensional measure zero.

Therefore, such functions enjoy essentially the same properties of functions whose epigraph has positive reach.

Results for the minimum time function.

Consider the controlled dynamics

$$\dot{x} = f(x) + g(x)u, \qquad x(0) = x$$

$x \in \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$, U compact and convex.

We are given a closed target S.
Let $\mathcal{T}(x)$ be the minimum time to reach the target S from x.

Assumptions.

1) f and g are $\mathcal{C}^{1,1}$.

2) S satisfies an internal sphere condition with uniform radius $\rho > 0$.

3) $\mathcal{T}$ is continuous on $\overline{S^c}$ (controllability + small time controllability).

Theorem. (Khai T. Nguyen, JMAA 2010). Under the above assumptions, the hypograph of $\mathcal{T}$ satisfies a locally uniform internal sphere condition.

Theorem. (GC- Khai T. Nguyen, SICON 2010) Assume furthermore that the normal cone to the hypograph of $\mathcal{T}$ is pointed. Then the hypograph of $\mathcal{T}$ has positive reach and we can compute the generalized gradient of T.

The result is perhaps natural: if the target satisfies an internal sphere condition and the dynamics is reasonably regular, than one can expect the sublevels of $\mathcal{T}$ to satisfy the same condition. In fact they are backwards reachable sets from the target, therefore they can be seen (essentially) as union of sets with internal spheres.

There is a lot of work in progress (Khai T. Nguyen, A. Marigonda, D. Vittone) for the case where the target satisfies a "uniform internal Fréchet normal condition". Similar results are being obtained, but with entirely different methods. This work will have applications to the regularity of $\mathcal{T}$ in the case of a $\mathcal{C}^{1,\alpha}$-dynamics, $0 < \alpha < 1$.

The case where the target is “fat” seems to be well understood.

We switch now to the “opposite” case: we take the target to be the origin.

Theorem. (G. C, A. Marigonda, P. Wolenski, SICON (2006)) Let the target S be convex and the dynamics be linear:

$$\begin{cases} \dot{x} &= Ax + Bu \\ x(0) &= \xi. \end{cases}$$

If $\mathcal{T}$ is continuous on $\overline{S^c}$, then $\mathrm{epi}(\mathcal{T})$ has positive reach.

The proof uses a computation of the proximal subdifferential of $\mathcal{T}$ using normals to the target (transported using the adjoint equation), which give the direction, and the Hamiltonian, which gives the norm (see Wolenski-Yu, 1998).

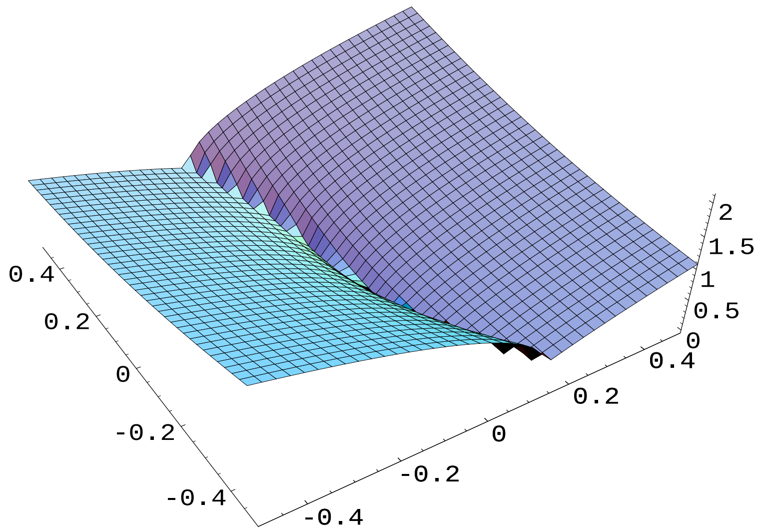
This is a generalization of the first result of this type, due to Cannarsa and Sinestrari (Calc. Var. '95), where semiconvexity of $\mathcal{T}$ is proved, provided T is locally Lipschtz.

Example 1 (rocket car). Consider the problem of reaching in minimum time the origin subject to the dynamics

$$\ddot{x} = u \in [-1, 1].$$

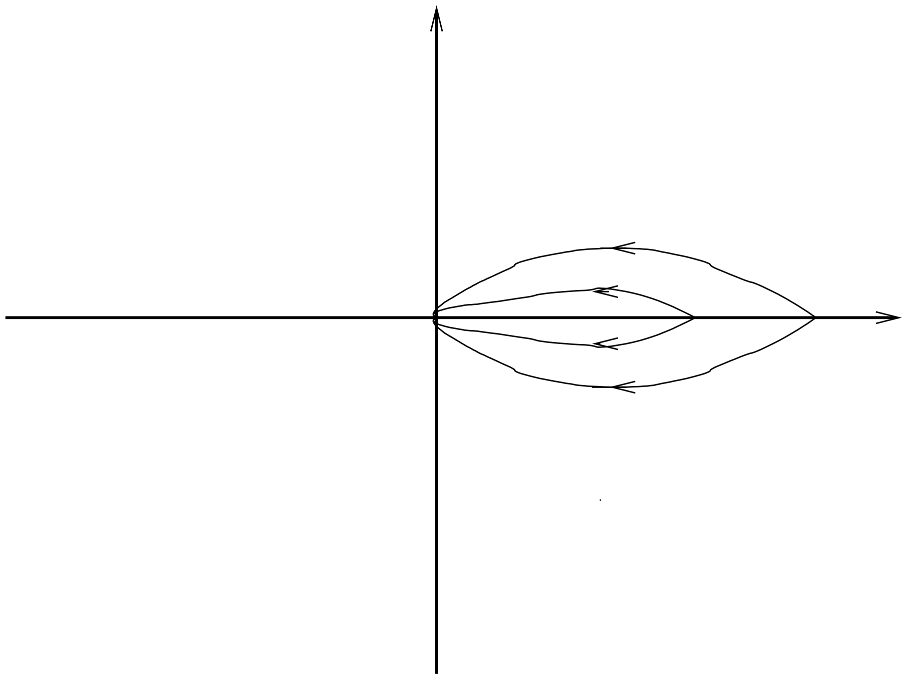
The minimum time function is:

$$\mathcal{T}(x,y) = \begin{cases} y + 2\sqrt{y^2/2 + x} & \text{for} \quad x \geq -y|y|/2, \\ -y + 2\sqrt{y^2/2 - x} & \text{for} \quad x < -y|y|/2 \end{cases}$$



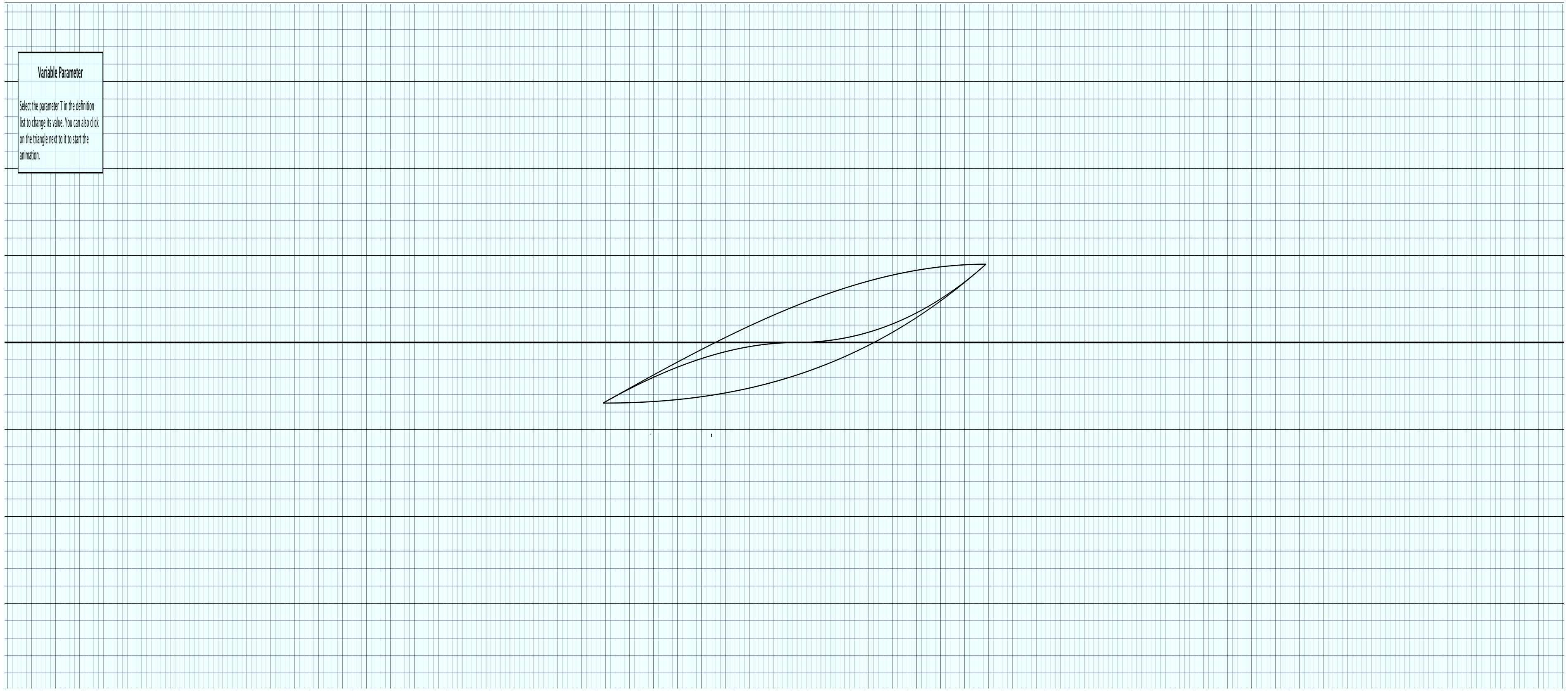
Example 2 (Cannarsa and Sinestrari). Consider the problem of reaching in minimum time the origin subject to the dynamics

$$\begin{cases} \dot{x} & = -y^2 + u_1, \quad u_1 \in [-1, 1] \\ \dot{y} & = u_2, \quad u_2 \in [-1, 1]. \end{cases}$$



Example 3.

$$\begin{cases} \dot{x} &= y^2 + y \\ \dot{y} &= -y^2 + u, \quad u \in [-1, 1]. \end{cases}$$



The linearization around the origin is:

Example 1.

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

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The linear part in Example 2 is not strong enough.

Definition. The linear system

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n, \quad u \in \mathcal{U} \subset \mathbb{R}^m, \tag{2}$$

where $B = (b_1, \ldots, b_m) \in \mathbb{M}_{n\times m}$ is normal if for all $i = 1, \ldots, m$

$$\text{rank}\big[b_i, Ab_i, \ldots, A^{n-1}b_i\big] = n$$

(i.e., the system is controllable at the origin by using each control separately). Of course the linearized dynamics of Example 2 is not normal.

Classical (Hermes-LaSalle): the system (2) is normal iff the reachable set from the origin at time T is strictly convex (for all times T).

A refinement of this classical result.

Theorem. (GC, Khai T. Nguyen). Assume that (2) is normal. Then for all $T > 0$ there exists $\gamma > 0$ (depending only on A, B, n, m, T) such that for all $x \in \mathcal{R}^T$, for all $v \in N(\mathcal{R}^T; x)$ one has

$$\langle v, y - x \rangle \leq -\gamma \|v\| \, \|y - x\|^n \qquad \forall y \in \mathcal{R}^T.$$

Remark. The power n is optimal: $x^{(n)} = u \in [-1, 1]$.

The proof is rather technical and has two main ingredients:

- an analysis of the zeros of the switching functions

$$g_i(s) = \langle e^{As} b_i, \zeta \rangle, \qquad \zeta \in \mathbb{R}^n$$

  and of its derivatives (essentially the interval $[0, T]$ is split into n disjoint sets where one of $g_i, g_i', \ldots, g_i^{(n-1)}$ is bounded away from zero; those sets are disjoint union of a finite [and computable] number of intervals);

- an integral inequality

$$\int_a^b (t-a)^k K(t)\,dt \geq \frac{1}{k+1} \left(\int_a^b K(t)\,dt \right)^{k+1},$$

  where $K \in L^\infty$, $0 \leq K \leq 1$, $k \in \mathbb{N}$.

It is natural conjecturing that for a (smooth enough) nonlinear system

$$\dot{x} = F(x) + G(x)u, \qquad u \in \mathcal{U} \subset \mathbb{R}^m, x \in \mathbb{R}^n$$

such that

- $F(0) = 0$

- the linearization at the origin is normal

- $\|G(x) - G(0)\| = o(\|x\|^n)$ for $x \to 0$

then the reachable set from the origin $\mathcal{R}^T$ at any time T sufficiently small is strictly convex (or at least satisfies a uniform external sphere condition).

However, for the moment we can prove the following

Theorem. (GC, Khai T. Nguyen) consider the dynamics

$$\dot{x} = F(x) + G(x)u, \qquad x \in \mathbb{R}^2, \qquad |u| \leq 1.$$

Assume

$$F(0) = 0, \qquad DG(0) = 0, \qquad \text{rank}[G(0), DF(0)G(0)] = 2.$$

Then, for any $T > 0$ sufficiently small,

- $\mathcal{R}^T$ is strictly convex;
- every $x \in \mathcal{R}^T$ is optimal (i.e., there exists a time optimal trajectory passing through x; this requires extending the optimal control/adjoint vector after x);
- the epigraph of $\mathcal{T}$ has positive reach.

This result is obtained by linearization. We show that the switching function of the nonlinear problem is essentially the same as the switching function of the linearized (at the origin) problem.

The obstruction for obtaining a result in higher dimension is the approximation of the optimal trajectory with a suitable trajectory of the linearized system: we need a power n with respect to the distance between two final points, but we are able to obtain only a power 2. So $n = 2$ is OK.

Trying to obtain a similar result in higher dimension is the first open problem I would like to mention.

The second open problem is studying cases where both type of singularities may occur (upwards/downwards corners). This happens, e.g., for larger times, i.e., not too close to the origin.

This is wide open, both from the viewpoint of nonsmooth analysis and of control theory.