Viable Trajectories and Optimal Synthesis in Optimal Control under State Constraints

Hélène Frankowska

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Control System under State Constraints

$$\begin{cases} x'(t) = f(t, x(t), u(t)), \quad u(t) \in U & \text{a.e. in } [0,1] \\ x(0) = x_0 \\ x(t) \in K \quad \text{for all } \; t \in [0,1] \end{cases}$$

U is a complete separable metric space, $K \subset \mathbb{R}^n$ is closed

$$f : [0,1] \times \mathbb{R}^n \times U \to \mathbb{R}^n, \quad x_0 \in \mathbb{R}^n$$

Controls are Lebesgue measurable functions $u(\cdot) : [0,1] \to U$

A **viable trajectory** $x(\cdot)$ of control system is an absolutely continuous function satisfying $x(0) = x_0$,

$$x'(t) = f(t, x(t), u(t)) \text{ almost everywhere in } [0,1]$$

for some control $u(\cdot)$ and $x(t) \in K$ for all $t \in [0,1]$

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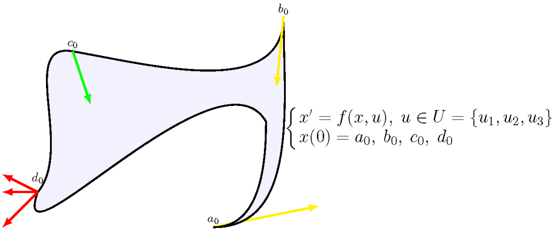
Tangent Vectors



$f(a_0, u_1)$, $f(b_0, u_2)$, $f(c_0, u_2)$ are tangent to K at a_0, b_0 and c_0.
At d_0 there is no $u \in U$ such that $f(d_0, u)$ is tangent to K.

The **contingent** cone to K at $x \in K$ is defined by

$$T_K(x) := \{v \in \mathbb{R}^n \mid \liminf_{h \to 0+} \frac{\mathrm{dist}(x + hv; K)}{h} = 0\}$$

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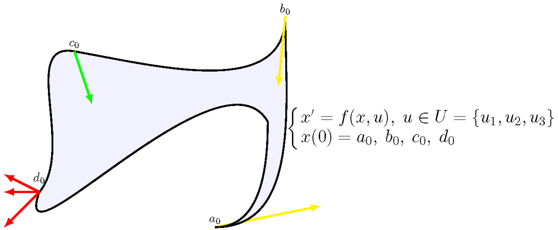

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Viability Theorem

$\mathcal{S}_K(x_0)$ – set of all viable trajectories starting at x_0,

$$W(t,x) := \{u \in U \,|\, f(t,x,u) \in T_K(x)\}$$

Then $x(\cdot) \in \mathcal{S}_K(x_0)$ if and only if $x(0) = x_0$ and $x(\cdot)$ is a trajectory of the following control system

$$x'(t) = f(t, x(t), u(t)), \quad u(t) \in W(t, x(t)) \quad \text{a.e. in } [0,1]$$

for some control $u(\cdot)$

If $x \rightsquigarrow f(x,U) := \bigcup_{u \in U}\{f(x,u)\}$ is "Marchaud", f is continuous and K is closed, then the following statements are equivalent

$$(i) \quad \mathcal{S}_K(x_0) \neq \emptyset \quad \forall\, x_0 \in K$$

$$(ii) \quad f(x,U) \cap T_K(x) \neq \emptyset \quad \forall\, x \in \partial K$$

Viability Theorem

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For functional differential inclusions G. Haddad (1981)
For measurable time dependence of dynamics
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There exist also extensions of the above theorem to stochastic control systems and to infinite dimensional control systems
J.-P. Aubin & G. Da Prato; S. Shuzhong; M.Bardi & R. Jensen; R. Buckhdahn, R. Peng, M. Quincampoix & C. Rainer, ...

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Example : Milyutin, 2000 (also for higher order)

In the presence of state constraints optimal trajectories may behave quite badly.

$$\begin{cases} \min \int_0^T \left(x(t) + \frac{|u(t)|^p}{p} \right) dt, \quad p > 1 \\ x'''(t) = u(t), \;\; x(0) = \xi_0, \; x'(0) = \xi_1, \; x''(0) = \xi_2 \\ u(t) \in \mathbb{R}, \;\; x(t) \geq 0 \end{cases}$$

If an initial condition (ξ_0, ξ_1, ξ_2) is admissible, then the optimal solution exists and is unique.

For "most" of the admissible initial conditions (ξ_0, ξ_1, ξ_2) there exist $T = T(\xi_0, \xi_1, \xi_2) > 0$ such that the optimal trajectory reaches the boundary of constraints an infinite (countable) number of times with T being their only accumulation point.

Lipschitz Dependence on Initial Conditions

$$C_K(x) := \{v \in \mathbb{R}^n \mid \lim_{h \to 0+,\, K \ni y \to x} \frac{\mathrm{dist}(y + hv; K)}{h} = 0\}$$

Inward Pointing Condition $f(t,x,U) \cap \mathrm{Int}\, C_K(x) \neq \emptyset \;\; \forall x \in \partial K$

Assume $f(t,x,U)$ are closed, $f(t,x,\cdot)$ is continuous, $f(\cdot,\cdot,u)$ is locally Lipschitz (uniformly in u), $\sup_{u \in U} |f(t,x,u)| \leq \gamma(1+|x|)$

Theorem (HF & Rampazzo, 2000; FP4 1994-1998)

*If the **inward pointing condition** holds true, then the set-valued map $\mathcal{S}_K(\cdot)$ is locally $\mathcal{C}-$Lipschitz on K.*

$\mathcal{S}_K(\cdot)$ is $\mathcal{C}-$Lipschitz at $x_0 \in K$ if it is Lipschitz as a set-valued map from a neighborhood of x_0 in K into $\mathcal{C} = C([0,1]; \mathbb{R}^n)$:
$\exists\, L \geq 0, \;\; \forall\, x_1, x_2 \in K$ close to x_0, $\forall\, \bar{x}_1(\cdot) \in \mathcal{S}_K(x_1)$,
$\exists\, \bar{x}_2(\cdot) \in \mathcal{S}_K(x_2)$ satisfying $\|\bar{x}_1(\cdot) - \bar{x}_2(\cdot)\|_{\mathcal{C}} \leq L|x_1 - x_2|$

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State Constraints with Smooth Boundaries

When the boundary of K is smooth and f depends on time in a measurable way a Neighboring Feasible Trajectory Theorem (NFT) proved via a constructive argument implies Lipschitz dependence of viable trajectories on initial conditions.

If K is an intersection of closed sets $K_j \subset \mathbb{R}^n$ with smooth boundaries

$$K = \cap_{j=1}^{m} K_j$$

a counterexample to (NFT) was recently given by
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Assumptions

Assume $K_j \subset \mathbb{R}^n$ is closed and has $\mathcal{C}^{1,1}_{loc}$-boundary for every j.

(i) $f(\cdot, x, u)$ is measurable $\forall\, x,\ u$;

(ii) $f(t, x, \cdot)$ is continuous $\forall\, t,\ x$;

(iii) $\forall\, r > 0, \exists\, \lambda_r > 0$ such that $\forall\, t,\ u$

$f(t, \cdot, u)$ is λ_r-Lipschitz on rB;

(iv) $\exists\, \gamma > 0$ such that $\forall\, t,\ x,\ \max_{u \in U} |f(t, x, u)| \leq \gamma(1 + |x|)$;

(v) $f(t, x, U)$ is closed $\forall\, t,\ x$;

(vi) $0 \notin conv\,\{\nabla d_j(x) \mid j \in J(x)\}$ transversality condition.

Then

$$N_K(x) := C_K(x)^- = \mathbb{R}_+ conv\,\{\nabla d_j(x) \mid j \in J(x)\}$$

Multiple State Constraints $K = \cap_{j=1}^{m} K_j$

Relaxed Inward Pointing Condition : $\exists\, \rho > 0\ \forall\, x \in \partial K,\ \forall\, t$

$$\exists\, v_{t,x} \in co\, f(t,x,U),\ \ \langle n_x, v_{t,x} \rangle \leq -\rho \quad \forall\, n_x \in N_K(x),\ \ |n_x| = 1$$

Theorem (HF & Bettiol, to appear, DCDS-A, FP6 2002-06)

*Assume the **relaxed inward pointing condition** and that either f is differentiable with respect to x or the sets $f(t,x,U)$ are convex. Then the set-valued map $\mathcal{S}_K(\cdot)$ is locally $\mathcal{C}-$Lipschitz on $\mathrm{Int}\, K$.*

Corollary (Approximation from the interior)

*Assume that the **relaxed inward pointing condition** holds true, that f is differentiable with respect to x and let $x_0 \in Int\, K$. Then for every $\bar{x} \in \mathcal{S}_K(x_0)$ and every $\varepsilon > 0$ there exists $x_\varepsilon \in \mathcal{S}_K(x_0)$ such that $\|x_\varepsilon - \bar{x}\|_{\mathcal{C}} < \varepsilon$ and $x_\varepsilon(t) \in Int\, K$ for all $t \in [0,1]$.*

Value Function of the Mayer Problem

$$\text{minimize}\,\{\varphi(x(1)) \mid x(\cdot) \in \mathcal{S}_K(x_0)\}$$

$\varphi : K \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, $\mathrm{Dom}\,\varphi \neq \emptyset$.

$$x'(t) = f(t, x(t), u(t)),\ u(t) \in U \text{ a.e. in } [t_0, 1],\ x(t_0) = y_0 \quad (1)$$

$$V(t_0, y_0) = \inf\{\varphi(x(1)) \mid x(\cdot) \text{ satisfies } (1),\ x(t) \in K\ \ \forall\, t \in [t_0, 1]\}$$

Even for smooth data and in the absence of state constraints the value function is, in general, nonsmooth. For this reason derivatives of V are replaced by "generalized" derivatives.

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Even for smooth data and in the absence of state constraints the value function is, in general, nonsmooth. For this reason derivatives of V are replaced by "generalized" derivatives.

Value Function and Optimal Synthesis

If $K = \mathbb{R}^n$ and $V \in C^1$, then optimal trajectories are those satisfying

$$x'(t) \in f(t, x(t), \Lambda(t, x(t))) \quad \text{a.e.}, \quad x(0) = x_0,$$

$$\Lambda(t, y) := \{u \in U \mid H(t, y, -V'_x(t, y)) = \langle -V'_x(t, y), f(t, y, u)\rangle\}$$

and a trajectory-control pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ is optimal if and only if

$$\bar{u}(t) \in \Lambda(t, \bar{x}(t)) \quad \text{a.e.}$$

Generalizations to more natural nonsmooth value functions lead to highly irregular $\Lambda(\cdot, \cdot)$ and $f(\cdot, \cdot, \Lambda(\cdot, \cdot))$.

There is no analogous characterization for the state constrained case, but some sufficient conditions are available.

Local Lipschitz Continuity of Value Function

Assume that $K = \cap_{j=1}^{m} K_j$, $K_j \subset \mathbb{R}^n$ is closed with $\mathcal{C}^{1,1}_{loc}$-boundary and satisfy the transversality condition.

Theorem (m=1)

*If the **(relaxed) inward pointing condition** holds true and φ is locally Lipschitz, then V is locally Lipschitz on $[0,1] \times K$.*

Theorem ($m > 1$)

*Assume that the **relaxed inward pointing condition** holds true, that φ is locally Lipschitz and that either f is differentiable with respect to x, or $f(t,x,U)$ is convex for every $(t,x) \in [0,T] \times \mathbb{R}^n$. Then V is locally Lipschitz on $[0,1] \times \mathrm{Int}\ K$.*

Optimal Synthesis - necessity

$\forall\ t \in [0, T),\ x \in K$ such that $V(t,x) \neq +\infty$ and all $\bar{v} \in \mathbb{R}^n$ define the directional derivative in the direction $(1, \bar{v}) \in \mathbb{R}^{n+1}$ by

$$D_{\uparrow}V(t,x)(1,\bar{v}) := \liminf_{h \to 0+,\ v \to \bar{v}} \frac{V(t+h, x+hv) - V(t,x)}{h}$$

and consider the sets

$$F(t,x) = \{u \in U \mid D_{\uparrow}V(t,x)(1, f(t,x,u)) \leq 0\}.$$

Then the dynamic programming principle implies that if $x(\cdot) \in \mathcal{S}_K(x_0)$ is optimal for the Mayer problem and $u(\cdot)$ is a corresponding optimal control, then

$$u(t) \in F(t, x(t)) \ \text{ a.e. in } \ [0, T].$$

In other words, $x(\cdot)$ satisfies

$$x'(t) \in f(t, x(t), F(t, x(t))) \ \text{ a.e.}, \ \ x(0) = x_0.$$

Optimal Synthesis - sufficiency

The above condition is also sufficient when V is locally Lipschitz on K.

Assume that $K = \cap_{j=1}^{m} K_j$, that for every j, $K_j \subset \mathbb{R}^n$ is closed with $\mathcal{C}^{1,1}_{loc}$-boundary and that the transversality condition holds true.

Theorem (Sufficient optimality condition)

Assume that φ is locally Lipschitz, that the ***relaxed inward pointing condition*** *holds true and that V is continuous on $[0,1] \times K$. Let $\bar{x} \in W^{1,1}$ satisfy*

$$x'(t) \in f(t, x(t), F(t, x(t))) \ \text{a.e.}, \quad x(0) = x_0.$$

If the set $\{t \in [0, T] \,|\, \bar{x}(t) \in \partial K\}$ is finite, then $\bar{x}(\cdot)$ is an optimal solution to the Mayer problem.

Optimal Trajectories

$$epi(V) = \{(t,x,r) \in [0,1] \times K \times \mathbb{R} \mid r \geq V(t,x)\}$$

Assume that $V(0,x_0) < +\infty$ and consider the viability problem

$$\begin{cases} s'(t) = 1, \quad s(0) = 0 \\ x'(t) = f(t,x(t),u(t)), \quad u(t) \in U \text{ a.e.}, \quad x(0) = x_0 \\ z'(t) = 0, \quad z(0) = V(0,x_0) \\ (s(t),x(t),z(t)) \in epi(V) \quad \text{for all} \quad t \in [0,1]. \end{cases} \tag{2}$$

Then a trajectory $\bar{x}(\cdot)$ is optimal if and only if the mapping $[0,1] \ni t \mapsto (t, \bar{x}(t), V(0,x_0))$ satisfies (2) on $[0,1]$.

Some algorithms were developed to get $epi(V)$ as the **viability kernel** of an auxiliary control system under state constraints.

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Maximum Principle + Sensitivity Relation

Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be optimal and $V(0,\cdot)$ be locally Lipschitz at x_0. Assume that transversality condition on K_j holds true, that φ is differentiable and f is differentiable with respect to x. Then $\exists\ \lambda \in \{0,1\}$, $\psi(\cdot) \in NBV([0,1];\mathbb{R}^n)$ and $p(\cdot) \in W^{1,1}([0,1];\mathbb{R}^n)$, $(\lambda, p, \psi) \neq 0$ satisfying the sensitivity relation $-p(0) \in \lambda \partial_x V(0, x_0)$ (**generalized gradient** of $V(0,\cdot)$ at x_0) and the maximum principle

$$-p'(t) = \frac{\partial f}{\partial x}(t, \bar{x}(t), \bar{u}(t))^*(p(t) + \psi(t)) \ \text{ a.e. in } [0,1],$$

$$\langle p(t)+\psi(t), f(t, \bar{x}(t), \bar{u}(t))\rangle = \max_{u \in U}(\langle p(t)+\psi(t), f(t, \bar{x}(t), u)\rangle \ \text{ a.e.}$$

$$-p(1) - \psi(1) = \lambda \nabla\varphi(\bar{x}(1)), \ \ \psi(t) = \int_{[0,t]} \nu(s) d\mu(s) \quad \forall\ t \in (0,1].$$

for a positive Radon measure μ on $[0,1]$ and a Borel measurable $\nu(s) \in N_K(\bar{x}(s)) \cap B \quad \mu$-a.e.

Normality of the Maximum principle

Assume that $K = \cap_{j=1}^{m} K_j$, that for every j, $K_j \subset \mathbb{R}^n$ is closed with $\mathcal{C}^{1,1}_{loc}$-boundary and that the transversality condition holds true.

Theorem

*Assume that $x_0 \in \mathrm{Int}\, K$ and the **relaxed inward pointing condition** holds true. Then $V(0, \cdot)$ is locally Lipschitz at x_0 and in the maximum principle $\lambda = 1$.*

Sensitivity relations link the value function and the maximum principle and imply some sufficient optimality conditions.

Thank you for your attention!