

Viabale Trajectories and Optimal Synthesis in Optimal Control under State Constraints

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Viable Trajectories and Optimal Synthesis in Optimal Control under State Constraints

Hélène Frankowska

CNRS AND UNIVERSITÉ PIERRE ET MARIE CURIE

Kick Off Meeting SADCO

Paris, March 3-4, 2011



Control System under State Constraints

$$\begin{cases} x'(t) = f(t, x(t), u(t)), & u(t) \in U \quad \text{a.e. in } [0, 1] \\ x(0) = x_0 \\ x(t) \in K \quad \text{for all } t \in [0, 1] \end{cases}$$

U is a complete separable metric space, $K \subset \mathbb{R}^n$ is closed

$$f : [0, 1] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n, \quad x_0 \in \mathbb{R}^n$$

Controls are Lebesgue measurable functions $u(\cdot) : [0, 1] \rightarrow U$

A **viable trajectory** $x(\cdot)$ of control system is an absolutely continuous function satisfying $x(0) = x_0$,

$$x'(t) = f(t, x(t), u(t)) \text{ almost everywhere in } [0, 1]$$

for some control $u(\cdot)$ and $x(t) \in K$ for all $t \in [0, 1]$



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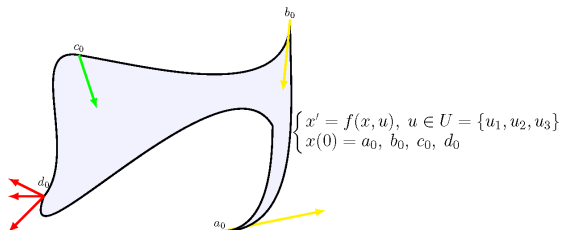
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Tangent Vectors



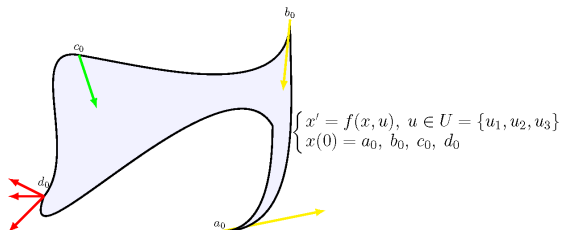
$f(a_0, u_1)$, $f(b_0, u_2)$, $f(c_0, u_2)$ are tangent to K at a_0 , b_0 and c_0 .
At d_0 there is no $u \in U$ such that $f(d_0, u)$ is tangent to K .

The **contingent** cone to K at $x \in K$ is defined by

$$T_K(x) := \left\{ v \in \mathbb{R}^n \mid \liminf_{h \rightarrow 0^+} \frac{\text{dist}(x + hv; K)}{h} = 0 \right\}$$



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Viability Theorem

$\mathcal{S}_K(x_0)$ – set of all viable trajectories starting at x_0 ,

$$W(t, x) := \{u \in U \mid f(t, x, u) \in T_K(x)\}$$

Then $x(\cdot) \in \mathcal{S}_K(x_0)$ **if and only if** $x(0) = x_0$ and $x(\cdot)$ is a trajectory of the following control system

$$x'(t) = f(t, x(t), u(t)), \quad u(t) \in W(t, x(t)) \quad \text{a.e. in } [0, 1]$$

for some control $u(\cdot)$

If $x \rightsquigarrow f(x, U) := \bigcup_{u \in U} \{f(x, u)\}$ is "Marchaud", f is continuous and K is closed, then the following statements are equivalent

- (i) $\mathcal{S}_K(x_0) \neq \emptyset \quad \forall x_0 \in K$
- (ii) $f(x, U) \cap T_K(x) \neq \emptyset \quad \forall x \in \partial K$



Viability Theorem

J. W. Bebernes & J. D. Schuur (1970)

For functional differential inclusions G. Haddad (1981)

For measurable time dependence of dynamics

HF & S. Plaskacz (1994); R. Vinter & A. Rapaport (1996)

There exist also extensions of the above theorem to stochastic control systems and to infinite dimensional control systems

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When K is not viable one considers its **Viability Kernel** - the largest viable subset of K J.-P. Aubin

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Example : Milyutin, 2000 (also for higher order)

In the presence of state constraints **optimal trajectories** may behave quite badly.

$$\begin{cases} \min \int_0^T \left(x(t) + \frac{|u(t)|^p}{p} \right) dt, \quad p > 1 \\ x'''(t) = u(t), \quad x(0) = \xi_0, \quad x'(0) = \xi_1, \quad x''(0) = \xi_2 \\ u(t) \in \mathbb{R}, \quad x(t) \geq 0 \end{cases}$$

If an initial condition (ξ_0, ξ_1, ξ_2) is admissible, then the optimal solution exists and is unique.

For “most” of the admissible initial conditions (ξ_0, ξ_1, ξ_2) there exist $T = T(\xi_0, \xi_1, \xi_2) > 0$ such that the optimal trajectory reaches the boundary of constraints an **infinite** (countable) number of times with T being their **only** accumulation point.



Lipschitz Dependence on Initial Conditions

$$C_K(x) := \{v \in \mathbb{R}^n \mid \lim_{h \rightarrow 0+, K \ni y \rightarrow x} \frac{\text{dist}(y + hv; K)}{h} = 0\}$$

Inward Pointing Condition $f(t, x, U) \cap \text{Int } C_K(x) \neq \emptyset \quad \forall x \in \partial K$

Assume $f(t, x, U)$ are closed, $f(t, x, \cdot)$ is continuous, $f(\cdot, \cdot, u)$ is locally Lipschitz (uniformly in u), $\sup_{u \in U} |f(t, x, u)| \leq \gamma(1 + |x|)$

Theorem (HF & Rampazzo, 2000; FP4 1994-1998)

If the *inward pointing condition* holds true, then the set-valued map $S_K(\cdot)$ is locally \mathcal{C} -Lipschitz on K .

$S_K(\cdot)$ is \mathcal{C} -Lipschitz at $x_0 \in K$ if it is Lipschitz as a set-valued map from a neighborhood of x_0 in K into $\mathcal{C} = C([0, 1]; \mathbb{R}^n)$:

$\exists L \geq 0, \forall x_1, x_2 \in K$ close to $x_0, \forall \bar{x}_1(\cdot) \in S_K(x_1),$

$\exists \bar{x}_2(\cdot) \in S_K(x_2)$ satisfying $\|\bar{x}_1(\cdot) - \bar{x}_2(\cdot)\|_{\mathcal{C}} \leq L \|x_1 - x_2\|$



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State Constraints with Smooth Boundaries

When the boundary of K is **smooth** and f depends on time in a measurable way a Neighboring Feasible Trajectory Theorem (NFT) proved via a constructive argument implies Lipschitz dependence of viable trajectories on initial conditions.

If K is an intersection of closed sets $K_j \subset \mathbb{R}^n$ with smooth boundaries

$$K = \bigcap_{j=1}^m K_j$$

a counterexample to (NFT) was recently given by
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Below d_j denotes the oriented distance to K_j and $J(x)$ the set of all active indices at x .



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Assumptions

Assume $K_j \subset \mathbb{R}^n$ is closed and has $C_{loc}^{1,1}$ -boundary for every j .

- (i) $f(\cdot, x, u)$ is measurable $\forall x, u$;
- (ii) $f(t, x, \cdot)$ is continuous $\forall t, x$;
- (iii) $\forall r > 0, \exists \lambda_r > 0$ such that $\forall t, u$
 $f(t, \cdot, u)$ is λ_r -Lipschitz on rB ;
- (iv) $\exists \gamma > 0$ such that $\forall t, x, \max_{u \in U} |f(t, x, u)| \leq \gamma(1 + |x|)$;
- (v) $f(t, x, U)$ is closed $\forall t, x$;
- (vi) $0 \notin \text{conv} \{ \nabla d_j(x) \mid j \in J(x) \}$ **transversality condition**.

Then

$$N_K(x) := C_K(x)^- = \mathbb{R}_+ \text{conv} \{ \nabla d_j(x) \mid j \in J(x) \}$$



Multiple State Constraints $K = \bigcap_{j=1}^m K_j$

Relaxed Inward Pointing Condition : $\exists \rho > 0 \forall x \in \partial K, \forall t$

$$\exists v_{t,x} \in \text{co} f(t, x, U), \quad \langle n_x, v_{t,x} \rangle \leq -\rho \quad \forall n_x \in N_K(x), \quad |n_x| = 1$$

Theorem (HF & Bettiol, to appear, DCDS-A, FP6 2002-06)

Assume the **relaxed inward pointing condition** and that either f is differentiable with respect to x or the sets $f(t, x, U)$ are convex. Then the set-valued map $\mathcal{S}_K(\cdot)$ is locally \mathcal{C} -Lipschitz on $\text{Int } K$.

Corollary (Approximation from the interior)

Assume that the **relaxed inward pointing condition** holds true, that f is differentiable with respect to x and let $x_0 \in \text{Int } K$. Then for every $\bar{x} \in \mathcal{S}_K(x_0)$ and every $\varepsilon > 0$ there exists $x_\varepsilon \in \mathcal{S}_K(x_0)$ such that $\|x_\varepsilon - \bar{x}\|_{\mathcal{C}} < \varepsilon$ and $x_\varepsilon(t) \in \text{Int } K$ for all $t \in [0, 1]$.



Value Function of the Mayer Problem

$$\text{minimize } \{\varphi(x(1)) \mid x(\cdot) \in \mathcal{S}_K(x_0)\}$$

$\varphi : K \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, $\text{Dom } \varphi \neq \emptyset$.

$$x'(t) = f(t, x(t), u(t)), \quad u(t) \in U \text{ a.e. in } [t_0, 1], \quad x(t_0) = y_0 \quad (1)$$

$$V(t_0, y_0) = \inf\{\varphi(x(1)) \mid x(\cdot) \text{ satisfies (1), } x(t) \in K \quad \forall t \in [t_0, 1]\}$$

Even for **smooth** data and in the absence of state constraints the value function is, in general, **nonsmooth**. For this reason derivatives of V are replaced by “generalized” derivatives.



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Value Function and Optimal Synthesis

If $K = \mathbb{R}^n$ and $V \in C^1$, then optimal trajectories are those satisfying

$$x'(t) \in f(t, x(t), \Lambda(t, x(t))) \quad \text{a.e.}, \quad x(0) = x_0,$$

$$\Lambda(t, y) := \{u \in U \mid H(t, y, -V'_x(t, y)) = \langle -V'_x(t, y), f(t, y, u) \rangle\}$$

and a trajectory-control pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ is optimal **if and only if**

$$\bar{u}(t) \in \Lambda(t, \bar{x}(t)) \quad \text{a.e.}$$

Generalizations to more natural **nonsmooth** value functions lead to **highly irregular** $\Lambda(\cdot, \cdot)$ and $f(\cdot, \cdot, \Lambda(\cdot, \cdot))$.

There is **no** analogous characterization for the state constrained case, but some sufficient conditions are available.



Local Lipschitz Continuity of Value Function

Assume that $K = \bigcap_{j=1}^m K_j$, $K_j \subset \mathbb{R}^n$ is closed with $C_{loc}^{1,1}$ -boundary and satisfy the transversality condition.

Theorem ($m=1$)

If the **(relaxed) inward pointing condition** holds true and φ is locally Lipschitz, then V is locally Lipschitz on $[0, 1] \times K$.

Theorem ($m > 1$)

Assume that the **relaxed inward pointing condition** holds true, that φ is locally Lipschitz and that either f is differentiable with respect to x , or $f(t, x, U)$ is convex for every $(t, x) \in [0, T] \times \mathbb{R}^n$. Then V is locally Lipschitz on $[0, 1] \times \text{Int } K$.



Optimal Synthesis - necessity

$\forall t \in [0, T)$, $x \in K$ such that $V(t, x) \neq +\infty$ and all $\bar{v} \in \mathbb{R}^n$ define the directional derivative in the direction $(1, \bar{v}) \in \mathbb{R}^{n+1}$ by

$$D_{\uparrow} V(t, x)(1, \bar{v}) := \liminf_{h \rightarrow 0+, v \rightarrow \bar{v}} \frac{V(t+h, x+hv) - V(t, x)}{h}$$

and consider the sets

$$F(t, x) = \{u \in U \mid D_{\uparrow} V(t, x)(1, f(t, x, u)) \leq 0\}.$$

Then the dynamic programming principle implies that if $x(\cdot) \in \mathcal{S}_K(x_0)$ is optimal for the Mayer problem and $u(\cdot)$ is a corresponding optimal control, then

$$u(t) \in F(t, x(t)) \text{ a.e. in } [0, T].$$

In other words, $x(\cdot)$ satisfies

$$x'(t) \in f(t, x(t), F(t, x(t))) \text{ a.e., } x(0) = x_0.$$



Optimal Synthesis - sufficiency

The above condition is also sufficient when V is locally Lipschitz on K .

Assume that $K = \bigcap_{j=1}^m K_j$, that for every j , $K_j \subset \mathbb{R}^n$ is closed with $\mathcal{C}_{loc}^{1,1}$ -boundary and that the transversality condition holds true.

Theorem (Sufficient optimality condition)

Assume that φ is locally Lipschitz, that the **relaxed inward pointing condition** holds true and that V is **continuous** on $[0, 1] \times K$. Let $\bar{x} \in W^{1,1}$ satisfy

$$x'(t) \in f(t, x(t), F(t, x(t))) \text{ a.e., } x(0) = x_0.$$

If the set $\{t \in [0, T] \mid \bar{x}(t) \in \partial K\}$ is finite, then $\bar{x}(\cdot)$ is an optimal solution to the Mayer problem.



Optimal Trajectories

$$\text{epi}(V) = \{(t, x, r) \in [0, 1] \times K \times \mathbb{R} \mid r \geq V(t, x)\}$$

Assume that $V(0, x_0) < +\infty$ and consider the viability problem

$$\left\{ \begin{array}{l} s'(t) = 1, \quad s(0) = 0 \\ x'(t) = f(t, x(t), u(t)), \quad u(t) \in U \text{ a.e., } \quad x(0) = x_0 \\ z'(t) = 0, \quad z(0) = V(0, x_0) \\ (s(t), x(t), z(t)) \in \text{epi}(V) \quad \text{for all } t \in [0, 1]. \end{array} \right. \quad (2)$$

Then a trajectory $\bar{x}(\cdot)$ is optimal **if and only if** the mapping $[0, 1] \ni t \mapsto (t, \bar{x}(t), V(0, x_0))$ satisfies (2) on $[0, 1]$.

Some **algorithms** were developed to get $\text{epi}(V)$ as the **viability kernel** of an auxiliary control system under state constraints.



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Maximum Principle + Sensitivity Relation

Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be optimal and $V(0, \cdot)$ be locally Lipschitz at x_0 . Assume that transversality condition on K_j holds true, that φ is differentiable and f is differentiable with respect to x . Then $\exists \lambda \in \{0, 1\}$, $\psi(\cdot) \in NBV([0, 1]; \mathbb{R}^n)$ and $p(\cdot) \in W^{1,1}([0, 1]; \mathbb{R}^n)$, $(\lambda, p, \psi) \neq 0$ satisfying the **sensitivity relation** $-p(0) \in \lambda \partial_x V(0, x_0)$ (**generalized gradient** of $V(0, \cdot)$ at x_0) and the **maximum principle**

$$-p'(t) = \frac{\partial f}{\partial x}(t, \bar{x}(t), \bar{u}(t))^*(p(t) + \psi(t)) \quad \text{a.e. in } [0, 1],$$

$$\langle p(t) + \psi(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle = \max_{u \in U} \langle p(t) + \psi(t), f(t, \bar{x}(t), u) \rangle \quad \text{a.e.}$$

$$-p(1) - \psi(1) = \lambda \nabla \varphi(\bar{x}(1)), \quad \psi(t) = \int_{[0,t]} \nu(s) d\mu(s) \quad \forall t \in (0, 1].$$

for a positive Radon measure μ on $[0, 1]$ and a Borel measurable $\nu(s) \in N_K(\bar{x}(s)) \cap B$ μ -a.e.



Normality of the Maximum principle

Assume that $K = \bigcap_{j=1}^m K_j$, that for every j , $K_j \subset \mathbb{R}^n$ is closed with $\mathcal{C}_{loc}^{1,1}$ -boundary and that the transversality condition holds true.

Theorem

Assume that $x_0 \in \text{Int } K$ and the **relaxed inward pointing condition** holds true. Then $V(0, \cdot)$ is locally Lipschitz at x_0 and in the maximum principle $\lambda = 1$.

Sensitivity relations link the value function and the maximum principle and imply some sufficient optimality conditions.



Thank you for your attention!

