Viable Trajectories and Optimal Synthesis in Optimal Control under State Constraints

Hélène Frankowska

Viable Trajectories and Optimal Synthesis in Optimal Control under State Constraints

Hélène Frankowska

CNRS AND UNIVERSITÉ PIERRE ET MARIE CURIE

Kick Off Meeting SADCO

Paris, March 3-4, 2011

Control System under State Constraints

$$\begin{cases} x'(t) & = & f(t,x(t),u(t)), \quad u(t) \in U \quad \text{a.e. in } [0,1] \\ x(0) & = & x_0 \\ x(t) & \in & K \quad \text{for all} \quad t \in [0,1] \end{cases}$$

U is a complete separable metric space, $K \subset \mathbb{R}^n$ is closed

$$f : [0,1] \times \mathbb{R}^n \times U \to \mathbb{R}^n, \quad x_0 \in \mathbb{R}^n$$

Controls are Lebesgue measurable functions $u(\cdot) : [0,1] \to U$

A **viable trajectory** $x(\cdot)$ of control system is an absolutely continuous function satisfying $x(0) = x_0$,

$$x'(t) = f(t,x(t),u(t)) \text{ almost everywhere in } [0,1]$$

for some control $u(\cdot)$ and $x(t) \in K$ for all $t \in [0,1]$

Control System under State Constraints

$$\begin{cases} x'(t) & = & f(t, x(t), u(t)), \quad u(t) \in U \quad \text{a.e. in } [0,1] \\ x(0) & = & x_0 \\ x(t) & \in & K \quad \text{for all } \ t \in [0,1] \end{cases}$$

U is a complete separable metric space, $K \subset \mathbb{R}^n$ is closed

$$f : [0,1] \times \mathbb{R}^n \times U \to \mathbb{R}^n, \quad x_0 \in \mathbb{R}^n$$

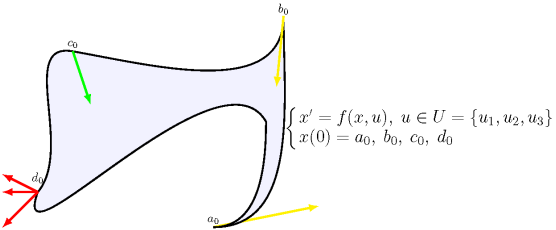
Controls are Lebesgue measurable functions $u(\cdot) : [0,1] \to U$

A **viable trajectory** $x(\cdot)$ of control system is an absolutely continuous function satisfying $x(0) = x_0$,

$$x'(t) = f(t, x(t), u(t)) \text{ almost everywhere in } [0,1]$$

for some control $u(\cdot)$ and $x(t) \in K$ for all $t \in [0,1]$.

Tangent Vectors

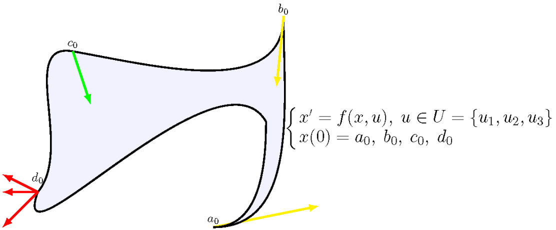


$f(a_0, u_1)$, $f(b_0, u_2)$, $f(c_0, u_2)$ are tangent to K at a_0, b_0 and c_0.
At d_0 there is no $u \in U$ such that $f(d_0, u)$ is tangent to K.

The **contingent** cone to K at $x \in K$ is defined by

$$T_K(x) := \{v \in \mathbb{R}^n \mid \liminf_{h \to 0+} \frac{\mathrm{dist}(x + hv; K)}{h} = 0\}$$

Tangent Vectors



$f(a_0, u_1)$, $f(b_0, u_2)$, $f(c_0, u_2)$ are tangent to K at a_0, b_0 and c_0.
At d_0 there is no $u \in U$ such that $f(d_0, u)$ is tangent to K.

The **contingent** cone to K at $x \in K$ is defined by

$$T_K(x) := \{v \in \mathbb{R}^n \mid \liminf_{h \to 0+} \frac{\mathrm{dist}(x + hv; K)}{h} = 0\}$$

Viability Theorem

$\mathcal{S}_K(x_0)$ – set of all viable trajectories starting at x_0,

$$W(t,x) := \{u \in U \mid f(t,x,u) \in T_K(x)\}$$

Then $x(\cdot) \in \mathcal{S}_K(x_0)$ if and only if $x(0) = x_0$ and $x(\cdot)$ is a trajectory of the following control system

$$x'(t) = f(t, x(t), u(t)), \quad u(t) \in W(t, x(t)) \quad \text{a.e. in } [0,1]$$

for some control $u(\cdot)$

If $x \rightsquigarrow f(x,U) := \bigcup_{u \in U}\{f(x,u)\}$ is "Marchaud", f is continuous and K is closed, then the following statements are equivalent

$$(i) \quad \mathcal{S}_K(x_0) \neq \emptyset \quad \forall\, x_0 \in K$$

$$(ii) \quad f(x,U) \cap T_K(x) \neq \emptyset \quad \forall\, x \in \partial K$$

Viability Theorem

J. W. Bebernes & J. D. Schuur (1970)
For functional differential inclusions G. Haddad (1981)
For measurable time dependence of dynamics
HF & S. Plaskacz (1994); R. Vinter & A. Rapaport (1996)

There exist also extensions of the above theorem to stochastic control systems and to infinite dimensional control systems
J.-P. Aubin & G. Da Prato; S. Shuzhong; M.Bardi & R. Jensen; R. Buckhdahn, R. Peng, M. Quincampoix & C. Rainer, ...

When K is not viable one considers its **Viability Kernel** - the largest viable subset of K J.-P. Aubin

In general however there is no Lipschitz dependence of viable trajectories on the initial conditions.

Viability Theorem

J. W. Bebernes & J. D. Schuur (1970)
For functional differential inclusions G. Haddad (1981)
For measurable time dependence of dynamics
HF & S. Plaskacz (1994); R. Vinter & A. Rapaport (1996)

There exist also extensions of the above theorem to stochastic control systems and to infinite dimensional control systems
J.-P. Aubin & G. Da Prato; S. Shuzhong; M.Bardi & R. Jensen; R. Buckhdahn, R. Peng, M. Quincampoix & C. Rainer, ...

When K is not viable one considers its **Viability Kernel** - the largest viable subset of K J.-P. Aubin

In general however there is no Lipschitz dependence of viable trajectories on the initial conditions.

Example : Milyutin, 2000 (also for higher order)

In the presence of state constraints optimal trajectories may behave quite badly.

$$\begin{cases} \min \int_0^T \left(x(t) + \frac{|u(t)|^p}{p} \right) dt, \quad p > 1 \\ x'''(t) = u(t), \quad x(0) = \xi_0, \ x'(0) = \xi_1, \ x''(0) = \xi_2 \\ u(t) \in \mathbb{R}, \quad x(t) \geq 0 \end{cases}$$

If an initial condition (ξ_0, ξ_1, ξ_2) is admissible, then the optimal solution exists and is unique.

For "most" of the admissible initial conditions (ξ_0, ξ_1, ξ_2) there exist $T = T(\xi_0, \xi_1, \xi_2) > 0$ such that the optimal trajectory reaches the boundary of constraints an infinite (countable) number of times with T being their only accumulation point.

Lipschitz Dependence on Initial Conditions

$$C_K(x) := \{v \in \mathbb{R}^n \mid \lim_{h\to 0+,\, K\ni y\to x} \frac{\mathrm{dist}(y+hv;K)}{h} = 0\}$$

Inward Pointing Condition $f(t,x,U) \cap \mathrm{Int}\, C_K(x) \neq \emptyset \quad \forall\, x \in \partial K$

Assume $f(t,x,U)$ are closed, $f(t,x,\cdot)$ is continuous, $f(\cdot,\cdot,u)$ is locally Lipschitz (uniformly in u), $\sup_{u\in U}|f(t,x,u)| \leq \gamma(1+|x|)$

Theorem (HF & Rampazzo, 2000; FP4 1994-1998)

*If the **inward pointing condition** holds true, then the set-valued map $\mathcal{S}_K(\cdot)$ is locally $\mathcal{C}-$Lipschitz on K.*

$\mathcal{S}_K(\cdot)$ is $\mathcal{C}-$Lipschitz at $x_0 \in K$ if it is Lipschitz as a set-valued map from a neighborhood of x_0 in K into $\mathcal{C} = C([0,1];\mathbb{R}^n)$:
$\exists\, L \geq 0,\ \forall\, x_1, x_2 \in K$ close to x_0, $\forall\, \bar{x}_1(\cdot) \in \mathcal{S}_K(x_1)$,
$\exists\, \bar{x}_2(\cdot) \in \mathcal{S}_K(x_2)$ satisfying $\|\bar{x}_1(\cdot) - \bar{x}_2(\cdot)\|_{\mathcal{C}} \leq L|x_1 - x_2|$

Lipschitz Dependence on Initial Conditions

$$C_K(x) := \{v \in \mathbb{R}^n \mid \lim_{h\to 0+,\, K\ni y\to x} \frac{\mathrm{dist}(y+hv;K)}{h} = 0\}$$

Inward Pointing Condition $f(t,x,U)\cap \mathrm{Int}\, C_K(x) \neq \emptyset \;\; \forall\, x \in \partial K$

Assume $f(t,x,U)$ are closed, $f(t,x,\cdot)$ is continuous, $f(\cdot,\cdot,u)$ is locally Lipschitz (uniformly in u), $\sup_{u\in U}|f(t,x,u)| \leq \gamma(1+|x|)$

Theorem (HF & Rampazzo, 2000; FP4 1994-1998)

*If the **inward pointing condition** holds true, then the set-valued map $\mathcal{S}_K(\cdot)$ is locally $\mathcal{C}-$Lipschitz on K.*

$\mathcal{S}_K(\cdot)$ is $\mathcal{C}-$Lipschitz at $x_0 \in K$ if it is Lipschitz as a set-valued map from a neighborhood of x_0 in K into $\mathcal{C} = C([0,1];\mathbb{R}^n)$:
$\exists\, L \geq 0,\;\; \forall\, x_1, x_2 \in K$ close to x_0, $\forall\, \bar{x}_1(\cdot) \in \mathcal{S}_K(x_1)$,
$\exists\, \bar{x}_2(\cdot) \in \mathcal{S}_K(x_2)$ satisfying $\|\bar{x}_1(\cdot) - \bar{x}_2(\cdot)\|_{\mathcal{C}} \leq L|x_1 - x_2|$

Lipschitz Dependence on Initial Conditions

$$C_K(x) := \{v \in \mathbb{R}^n \mid \lim_{h \to 0+,\, K \ni y \to x} \frac{\mathrm{dist}(y + hv; K)}{h} = 0\}$$

Inward Pointing Condition $f(t,x,U) \cap \mathrm{Int}\, C_K(x) \neq \emptyset \quad \forall x \in \partial K$

Assume $f(t,x,U)$ are closed, $f(t,x,\cdot)$ is continuous, $f(\cdot,\cdot,u)$ is locally Lipschitz (uniformly in u), $\sup_{u \in U} |f(t,x,u)| \leq \gamma(1 + |x|)$

Theorem (HF & Rampazzo, 2000; FP4 1994-1998)

*If the **inward pointing condition** holds true, then the set-valued map $\mathcal{S}_K(\cdot)$ is locally $\mathcal{C}-$Lipschitz on K.*

$\mathcal{S}_K(\cdot)$ is $\mathcal{C}-$Lipschitz at $x_0 \in K$ if it is Lipschitz as a set-valued map from a neighborhood of x_0 in K into $\mathcal{C} = C([0,1]; \mathbb{R}^n)$:
$\exists\, L \geq 0,\ \forall\, x_1, x_2 \in K$ close to x_0, $\forall\, \bar{x}_1(\cdot) \in \mathcal{S}_K(x_1)$,
$\exists\, \bar{x}_2(\cdot) \in \mathcal{S}_K(x_2)$ satisfying $\|\bar{x}_1(\cdot) - \bar{x}_2(\cdot)\|_{\mathcal{C}} \leq L|x_1 - x_2|$

State Constraints with Smooth Boundaries

When the boundary of K is smooth and f depends on time in a measurable way a Neighboring Feasible Trajectory Theorem (NFT) proved via a constructive argument implies Lipschitz dependence of viable trajectories on initial conditions.

If K is an intersection of closed sets $K_j \subset \mathbb{R}^n$ with smooth boundaries

$$K = \cap_{j=1}^{m} K_j$$

a counterexample to (NFT) was recently given by
P. Bettiol, A. Bressan & R. Vinter (2010)

Below d_j denotes the oriented distance to K_j and $J(x)$ the set of all active indices at x.

State Constraints with Smooth Boundaries

When the boundary of K is smooth and f depends on time in a measurable way a Neighboring Feasible Trajectory Theorem (NFT) proved via a constructive argument implies Lipschitz dependence of viable trajectories on initial conditions.

If K is an intersection of closed sets $K_j \subset \mathbb{R}^n$ with smooth boundaries

$$K = \cap_{j=1}^{m} K_j$$

a counterexample to (NFT) was recently given by
P. Bettiol, A. Bressan & R. Vinter (2010)

Below d_j denotes the oriented distance to K_j and $J(x)$ the set of all active indices at x.

State Constraints with Smooth Boundaries

When the boundary of K is smooth and f depends on time in a measurable way a Neighboring Feasible Trajectory Theorem (NFT) proved via a constructive argument implies Lipschitz dependence of viable trajectories on initial conditions.

If K is an intersection of closed sets $K_j \subset \mathbb{R}^n$ with smooth boundaries

$$K = \cap_{j=1}^{m} K_j$$

a counterexample to (NFT) was recently given by P. Bettiol, A. Bressan & R. Vinter (2010)

Below d_j denotes the oriented distance to K_j and $J(x)$ the set of all active indices at x.

Assumptions

Assume $K_j \subset \mathbb{R}^n$ is closed and has $\mathcal{C}^{1,1}_{loc}$-boundary for every j.

(i) $f(\cdot, x, u)$ is measurable $\forall\, x,\ u$;

(ii) $f(t, x, \cdot)$ is continuous $\forall\, t,\ x$;

(iii) $\forall\, r > 0, \exists\, \lambda_r > 0$ such that $\forall\, t,\ u$

$f(t, \cdot, u)$ is λ_r-Lipschitz on rB;

(iv) $\exists\, \gamma > 0$ such that $\forall\, t,\ x,\ \ \max_{u \in U} |f(t, x, u)| \leq \gamma(1 + |x|)$;

(v) $f(t, x, U)$ is closed $\forall\, t,\ x$;

(vi) $0 \notin conv\, \{\nabla d_j(x) \mid j \in J(x)\}$ transversality condition.

Then

$$N_K(x) := C_K(x)^- = \mathbb{R}_+ conv\, \{\nabla d_j(x) \mid j \in J(x)\}$$

Multiple State Constraints $K = \cap_{j=1}^{m} K_j$

Relaxed Inward Pointing Condition : $\exists\, \rho > 0\ \forall\, x \in \partial K,\ \forall\, t$

$$\exists\, v_{t,x} \in co\, f(t,x,U), \ \ \langle n_x, v_{t,x} \rangle \leq -\rho \quad \forall\, n_x \in N_K(x), \ \ |n_x| = 1$$

Theorem (HF & Bettiol, to appear, DCDS-A, FP6 2002-06)

*Assume the **relaxed inward pointing condition** and that either f is differentiable with respect to x or the sets $f(t,x,U)$ are convex. Then the set-valued map $\mathcal{S}_K(\cdot)$ is locally $\mathcal{C}-$Lipschitz on $\mathrm{Int}\, K$.*

Corollary (Approximation from the interior)

*Assume that the **relaxed inward pointing condition** holds true, that f is differentiable with respect to x and let $x_0 \in Int\, K$. Then for every $\bar{x} \in \mathcal{S}_K(x_0)$ and every $\varepsilon > 0$ there exists $x_\varepsilon \in \mathcal{S}_K(x_0)$ such that $\|x_\varepsilon - \bar{x}\|_{\mathcal{C}} < \varepsilon$ and $x_\varepsilon(t) \in Int\, K$ for all $t \in [0,1]$.*

Value Function of the Mayer Problem

$$\text{minimize}\,\{\varphi(x(1)) \mid x(\cdot) \in \mathcal{S}_K(x_0)\}$$

$\varphi : K \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, $\mathrm{Dom}\,\varphi \neq \emptyset$.

$$x'(t) = f(t, x(t), u(t)),\; u(t) \in U \text{ a.e. in } [t_0, 1],\; x(t_0) = y_0 \qquad (1)$$

$$V(t_0, y_0) = \inf\{\varphi(x(1)) \mid x(\cdot) \text{ satisfies } (1),\; x(t) \in K \;\; \forall\, t \in [t_0, 1]\}$$

Even for smooth data and in the absence of state constraints the value function is, in general, nonsmooth. For this reason derivatives of V are replaced by "generalized" derivatives.

Value Function of the Mayer Problem

$$\text{minimize}\,\{\varphi(x(1)) \mid x(\cdot) \in \mathcal{S}_K(x_0)\}$$

$\varphi : K \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, $\mathrm{Dom}\,\varphi \neq \emptyset$.

$$x'(t) = f(t, x(t), u(t)),\ u(t) \in U \text{ a.e. in } [t_0, 1],\ x(t_0) = y_0 \quad (1)$$

$$V(t_0, y_0) = \inf\{\varphi(x(1)) \mid x(\cdot) \text{ satisfies } (1),\ x(t) \in K\ \ \forall\, t \in [t_0, 1]\}$$

Even for smooth data and in the absence of state constraints the value function is, in general, nonsmooth. For this reason derivatives of V are replaced by "generalized" derivatives.

Value Function of the Mayer Problem

$$\text{minimize}\,\{\varphi(x(1)) \mid x(\cdot) \in \mathcal{S}_K(x_0)\}$$

$\varphi : K \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, $\mathrm{Dom}\,\varphi \neq \emptyset$.

$$x'(t) = f(t, x(t), u(t)),\ u(t) \in U \text{ a.e. in } [t_0, 1],\ x(t_0) = y_0 \quad (1)$$

$$V(t_0, y_0) = \inf\{\varphi(x(1)) \mid x(\cdot) \text{ satisfies } (1),\ x(t) \in K\ \ \forall\, t \in [t_0, 1]\}$$

Even for smooth data and in the absence of state constraints the value function is, in general, nonsmooth. For this reason derivatives of V are replaced by "generalized" derivatives.

Value Function and Optimal Synthesis

If $K = \mathbb{R}^n$ and $V \in C^1$, then optimal trajectories are those satisfying

$$x'(t) \in f(t, x(t), \Lambda(t, x(t))) \quad \text{a.e.,} \quad x(0) = x_0,$$

$$\Lambda(t, y) := \{u \in U \mid H(t, y, -V'_x(t, y)) = \langle -V'_x(t, y), f(t, y, u)\rangle\}$$

and a trajectory-control pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ is optimal if and only if

$$\bar{u}(t) \in \Lambda(t, \bar{x}(t)) \quad \text{a.e.}$$

Generalizations to more natural nonsmooth value functions lead to highly irregular $\Lambda(\cdot, \cdot)$ and $f(\cdot, \cdot, \Lambda(\cdot, \cdot))$.

There is no analogous characterization for the state constrained case, but some sufficient conditions are available.

Local Lipschitz Continuity of Value Function

Assume that $K = \cap_{j=1}^{m} K_j$, $K_j \subset \mathbb{R}^n$ is closed with $\mathcal{C}^{1,1}_{loc}$-boundary and satisfy the transversality condition.

Theorem (m=1)

If the ***(relaxed) inward pointing condition*** *holds true and φ is locally Lipschitz, then V is locally Lipschitz on $[0,1] \times K$.*

Theorem ($m > 1$)

Assume that the ***relaxed inward pointing condition*** *holds true, that φ is locally Lipschitz and that either f is differentiable with respect to x, or $f(t,x,U)$ is convex for every $(t,x) \in [0,T] \times \mathbb{R}^n$. Then V is locally Lipschitz on $[0,1] \times \mathrm{Int}\, K$.*

Optimal Synthesis - necessity

$\forall\ t \in [0, T)$, $x \in K$ such that $V(t,x) \neq +\infty$ and all $\bar{v} \in \mathbb{R}^n$ define the directional derivative in the direction $(1, \bar{v}) \in \mathbb{R}^{n+1}$ by

$$D_{\uparrow}V(t,x)(1,\bar{v}) := \liminf_{h\to 0+,\ v\to\bar{v}} \frac{V(t+h, x+hv) - V(t,x)}{h}$$

and consider the sets

$$F(t,x) = \{u \in U \mid D_{\uparrow}V(t,x)(1, f(t,x,u)) \leq 0\}\,.$$

Then the dynamic programming principle implies that if $x(\cdot) \in \mathcal{S}_K(x_0)$ is optimal for the Mayer problem and $u(\cdot)$ is a corresponding optimal control, then

$$u(t) \in F(t, x(t)) \ \text{ a.e. in } \ [0, T].$$

In other words, $x(\cdot)$ satisfies

$$x'(t) \in f(t, x(t), F(t, x(t))) \ \text{ a.e., } \ x(0) = x_0.$$

Optimal Synthesis - sufficiency

The above condition is also sufficient when V is locally Lipschitz on K.

Assume that $K = \cap_{j=1}^{m} K_j$, that for every j, $K_j \subset \mathbb{R}^n$ is closed with $\mathcal{C}^{1,1}_{loc}$-boundary and that the transversality condition holds true.

Theorem (Sufficient optimality condition)

*Assume that φ is locally Lipschitz, that the **relaxed inward pointing condition** holds true and that V is continuous on $[0,1] \times K$. Let $\bar{x} \in W^{1,1}$ satisfy*

$$x'(t) \in f(t, x(t), F(t, x(t))) \text{ a.e.}, \quad x(0) = x_0.$$

If the set $\{t \in [0, T] \,|\, \bar{x}(t) \in \partial K\}$ is finite, then $\bar{x}(\cdot)$ is an optimal solution to the Mayer problem.

Optimal Trajectories

$$epi(V) = \{(t, x, r) \in [0, 1] \times K \times \mathbb{R} \mid r \geq V(t, x)\}$$

Assume that $V(0, x_0) < +\infty$ and consider the viability problem

$$\begin{cases} s'(t) = 1, \quad s(0) = 0 \\ x'(t) = f(t, x(t), u(t)), \quad u(t) \in U \text{ a.e.}, \quad x(0) = x_0 \\ z'(t) = 0, \quad z(0) = V(0, x_0) \\ (s(t), x(t), z(t)) \in epi(V) \quad \text{for all} \quad t \in [0, 1]. \end{cases} \tag{2}$$

Then a trajectory $\bar{x}(\cdot)$ is optimal if and only if the mapping $[0, 1] \ni t \mapsto (t, \bar{x}(t), V(0, x_0))$ satisfies (2) on $[0, 1]$.

Some algorithms were developed to get $epi(V)$ as the **viability kernel** of an auxiliary control system under state constraints.

Optimal Trajectories

$$epi(V) = \{(t,x,r) \in [0,1] \times K \times \mathbb{R} \mid r \geq V(t,x)\}$$

Assume that $V(0,x_0) < +\infty$ and consider the viability problem

$$\begin{cases} s'(t) = 1, \quad s(0) = 0 \\ x'(t) = f(t,x(t),u(t)), \quad u(t) \in U \text{ a.e.}, \quad x(0) = x_0 \\ z'(t) = 0, \quad z(0) = V(0,x_0) \\ (s(t),x(t),z(t)) \in epi(V) \quad \text{for all} \quad t \in [0,1]. \end{cases} \tag{2}$$

Then a trajectory $\bar{x}(\cdot)$ is optimal if and only if the mapping $[0,1] \ni t \mapsto (t, \bar{x}(t), V(0,x_0))$ satisfies (2) on $[0,1]$.

Some algorithms were developed to get $epi(V)$ as the **viability kernel** of an auxiliary control system under state constraints.

Maximum Principle + Sensitivity Relation

Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be optimal and $V(0,\cdot)$ be locally Lipschitz at x_0. Assume that transversality condition on K_j holds true, that φ is differentiable and f is differentiable with respect to x. Then $\exists\, \lambda \in \{0,1\}$, $\psi(\cdot) \in NBV([0,1];\mathbb{R}^n)$ and $p(\cdot) \in W^{1,1}([0,1];\mathbb{R}^n)$, $(\lambda, p, \psi) \neq 0$ satisfying the sensitivity relation $-p(0) \in \lambda \partial_x V(0, x_0)$ (**generalized gradient** of $V(0,\cdot)$ at x_0) and the maximum principle

$$-p'(t) = \frac{\partial f}{\partial x}(t, \bar{x}(t), \bar{u}(t))^* (p(t) + \psi(t)) \ \text{ a.e. in } [0,1],$$

$$\langle p(t)+\psi(t), f(t,\bar{x}(t),\bar{u}(t))\rangle = \max_{u \in U}(\langle p(t)+\psi(t), f(t,\bar{x}(t),u)\rangle \ \text{ a.e.}$$

$$-p(1)-\psi(1) = \lambda \nabla \varphi(\bar{x}(1)), \ \ \psi(t) = \int_{[0,t]} \nu(s) d\mu(s) \quad \forall\, t \in (0,1].$$

for a positive Radon measure μ on $[0,1]$ and a Borel measurable $\nu(s) \in N_K(\bar{x}(s)) \cap B \quad \mu$-a.e.

Normality of the Maximum principle

Assume that $K = \cap_{j=1}^{m} K_j$, that for every j, $K_j \subset \mathbb{R}^n$ is closed with $\mathcal{C}^{1,1}_{loc}$-boundary and that the transversality condition holds true.

Theorem

*Assume that $x_0 \in \mathrm{Int}\, K$ and the **relaxed inward pointing condition** holds true. Then $V(0,\cdot)$ is locally Lipschitz at x_0 and in the maximum principle $\lambda = 1$.*

Sensitivity relations link the value function and the maximum principle and imply some sufficient optimality conditions.

Thank you for your attention!