The α-Arboricity of Complete Uniform Hypergraphs

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THE α-ARBORICITY OF COMPLETE UNIFORM HYPERGRAPHS

J.-C. BERMOND[†], Y. M. CHEE[‡], N. COHEN[†], AND X. ZHANG[‡]

Abstract. α-Acyclicity is an important notion in database theory. The α-arboricity of a hypergraph $\mathcal{H}$ is the minimum number of α-acyclic hypergraphs that partition the edge set of $\mathcal{H}$. The α-arboricity of the complete 3-uniform hypergraph is determined completely.

Key words. α-Arboricity, Acyclic hypergraph, Decomposition, Steiner system

AMS subject classifications. 05B07, 05C35

1. Introduction. There is a natural bijection between database schemas and hypergraphs, where each attribute of a database schema D corresponds to a vertex in a hypergraph $\mathcal{H}$, and each relation R of attributes in D corresponds to an edge in $\mathcal{H}$. Many properties of databases have therefore been studied in the context of hypergraphs. One such property of databases is the important notion of α-acyclicity. Besides being a desirable property in the design of databases [2, 3, 8, 9, 10], many NP-hard problems concerning databases can be solved in polynomial time when restricted to instances for which the corresponding hypergraphs are α-acyclic [3, 16, 19]. Examples of such problems include determining global consistency, evaluating conjunctive queries, and computing joins or projections of joins.

When faced with such computationally intractable problems on a general database schema, it is natural to decompose it into α-acyclic instances on which efficient algorithms can be applied. This has motivated some recent studies on the α-arboricity of hypergraphs, the minimum number of α-acyclic hypergraphs into which the edges of a given hypergraph can be partitioned [4, 14, 17].

In this paper, we give a general construction for partitioning complete uniform hypergraphs into α-acyclic hypergraphs based on Steiner systems, and completely determine the α-arboricity of complete 3-uniform hypergraphs.

2. Preliminaries. We assume familiarity with basic concepts and notions in graph theory.

Let n be a positive integer. The set $\{1, \ldots, n\}$ is denoted $[n]$. Disjoint union of sets is denoted by $\sqcup$. We use $\sqcup$ in place of $\cup$ when we want to emphasize the disjointness of the sets involved in a union.

For X a finite set and k a nonnegative integer, the set of all k-subsets of X is denoted $\binom{X}{k}$, that is $\binom{X}{k} = \{K \subseteq X : |K| = k\}$. A *hypergraph* is a pair $\mathcal{H} = (X, \mathcal{A})$, where X is a finite set, and $\mathcal{A} \subseteq 2^X$. The elements of X are called *vertices* and the elements of $\mathcal{A}$ are called *edges*. The *order* of $\mathcal{H}$ is the number of vertices in X, and the *size* of $\mathcal{H}$ is the number of edges in $\mathcal{A}$. If $\mathcal{A} \subseteq \binom{X}{k}$, then $\mathcal{H}$ is said to be *k-uniform*.

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A 2-uniform hypergraph is just the usual notion of a *graph*. The *complete k-uniform hypergraph* $(X, \binom{X}{k})$ of order n is denoted $K_n^{(k)}$. A hypergraph is *empty* if it has no edges. The degree of a vertex v is the number of edges containing v.

A *Steiner system* $S(t,k,n)$ is a k-uniform hypergraph $(X,\mathcal{A})$ such that every $T \in \binom{X}{t}$ is contained in exactly one edge in $\mathcal{A}$.

A *group divisible design* k-GDD is a triple $(X,\mathcal{G},\mathcal{A})$, where $(X,\mathcal{A})$ is a k-uniform hypergraph, $\mathcal{G} = \{G_1,\dots,G_t\}$ is a partition of X into parts G_i, $i \in [t]$, called *groups*, such that every $T \in \binom{X}{2}$ not contained in a group is contained in exactly one edge in $\mathcal{A}$, and every $T \in \binom{X}{2}$ contained in a group is not contained in any edge in $\mathcal{A}$. The *type* of a k-GDD $(X,\mathcal{G},\mathcal{A})$ is the multiset $[|G_1|,\dots,|G_t|]$. The exponential notation $g_1^{t_1}\dots g_s^{t_s}$ is used to denote the multiset where element g_i has multiplicity t_i, $i \in [s]$.

We require the following result from Colbourn *et al.* [5] on the existence of 3-GDDs.

Theorem 2.1 (Colbourn, Hoffman, and Rees [5]). *Let g, t, and u be nonnegative integers. There exists a* 3-GDD *of type $g^t u^1$ if and only if the following conditions are all satisfied:*

(i) *if $g > 0$ then $t \geq 3$, or $t = 2$ and $u = g$, or $t = 1$ and $u = 0$, or $t = 0$;*
(ii) *$u \leq g(t-1)$ or $gt = 0$;*
(iii) *$g(t-1) + u \equiv 0 \pmod 2$ or $gt = 0$;*
(iv) *$gt \equiv 0 \pmod 2$ or $u = 0$;*
(v) *$g^2\binom{t}{2} + gtu \equiv 0 \pmod 3$.*

2.1. Graphs of Hypergraphs. Given a hypergraph $\mathcal{H}$, we may define the following graphs on $\mathcal{H}$.

Definition 2.2. *Let $\mathcal{H} = (X,\mathcal{A})$ be a hypergraph. The* line graph *of $\mathcal{H}$ is the graph $L(\mathcal{H}) = (V,\mathcal{E})$, where $V = \mathcal{A}$ and $\mathcal{E} = \{\{A,B\} \subseteq \binom{V}{2} : A \cap B \neq \varnothing\}$.*

Definition 2.3. *Let $\mathcal{H} = (X,\mathcal{A})$ be a hypergraph. The* primal graph *or* 2-section *of $\mathcal{H}$ is the graph $G(\mathcal{H}) = (X,\mathcal{E})$ such that $\{x,y\} \in \mathcal{E}$ if and only if $\{x,y\} \subset A$ for some $A \in \mathcal{A}$.*

A hypergraph $\mathcal{H}$ is *conformal* if for every clique K in $G(\mathcal{H})$, there is an edge in $\mathcal{H}$ that contains K. A hypergraph $\mathcal{H}$ is *chordal* if $G(\mathcal{H})$ is chordal, that is, every cycle of length at least four in $G(\mathcal{H})$ contains two nonconsecutive vertices that are adjacent.

2.2. Acyclic Hypergraphs. Graham [11], and independently, Yu and Ozsoyoglu [20, 21], defined an acyclicity property (which has come to be known as α-acyclicity) for hypergraphs in the context of database theory, via a transformation now known as the *GYO reduction*. Given a hypergraph $\mathcal{H} = (X,\mathcal{A})$, the GYO reduction applies the following operations repeatedly to $\mathcal{H}$ until none can be applied:

(i) If a vertex $x \in X$ has degree one, then delete x from the edge containing it.
(ii) If $A, B \in \mathcal{A}$ are distinct edges such that $A \subseteq B$, then delete A from $\mathcal{A}$.
(iii) If $A \in \mathcal{A}$ is empty, that is $A = \varnothing$, then delete A from $\mathcal{A}$.

Definition 2.4. *A hypergraph $\mathcal{H}$ is* α-acyclic *if GYO reduction on $\mathcal{H}$ results in an empty hypergraph.*

The notion of α-acyclicity is closely related to conformality and chordality for hypergraphs. Beeri et al. [3] showed:

Theorem 2.5 (Beeri et al. [3]). *$\mathcal{H}$ is α-acyclic if and only if $\mathcal{H}$ is conformal and chordal.*

Let $\mathcal{H} = (X,\mathcal{A})$ be a hypergraph. Assign to every edge $\{A,B\}$ of $L(\mathcal{H})$ the weight $|A \cap B|$. We denote this weighted line graph of $\mathcal{H}$ by $L'(\mathcal{H})$. The maximum weight of a forest in $L'(\mathcal{H})$ is denoted $w(\mathcal{H})$. Acharya and Las Vergnas [1] introduced the hypergraph invariant

$$\mu(\mathcal{H}) = \sum_{A\in\mathcal{A}} |A| - \left|\bigcup_{A\in\mathcal{A}} A\right| - w(\mathcal{H}),$$

called the *cyclomatic number* of $\mathcal{H}$, and used it to characterize conformal and chordal hypergraphs.

THEOREM 2.6 (Acharya and Las Vergnas [1]). *A hypergraph $\mathcal{H}$ satisfies $\mu(\mathcal{H}) = 0$ if and only if $\mathcal{H}$ is conformal and chordal.*

Theorem 2.5 and Theorem 2.6 immediately imply the following.

COROLLARY 2.7. *A hypergraph $\mathcal{H}$ is α-acyclic if and only if $\mu(\mathcal{H}) = 0$.*

Li and Wang [15] were unaware of these connections and rediscovered Corollary 2.7 recently. An easy consequence is that a maximum α-acyclic k-uniform hypergraph of order n has size $n - k + 1$ [18]. Let $L_{k-1}(\mathcal{H})$ denote the spanning subgraph of $L'(\mathcal{H})$ containing only those edges of $L'(\mathcal{H})$ of weight $k - 1$. We derive the following characterizations of maximum α-acyclic k-uniform hypergraphs.

COROLLARY 2.8. *A k-uniform hypergraph $\mathcal{H} = (X, \mathcal{A})$ of order n and size $n-k+1$ is α-acyclic if and only if $L(\mathcal{H})$ contains a spanning tree, each edge of which has weight $k - 1$ (in other words, $L_{k-1}(\mathcal{H})$ is connected).*

Proof. By Corollary 2.7, we have

$$\begin{aligned} w(\mathcal{H}) &= \sum_{A\in\mathcal{A}} |A| - \left|\bigcup_{A\in\mathcal{A}} A\right| \\ &= (n - k + 1)k - n \\ &= (n - k)(k - 1). \end{aligned}$$

Since every edge in $L'(\mathcal{H})$ has weight at most $k - 1$, and a forest of $L'(\mathcal{H})$ contains at most $n - k$ edges (and contains exactly $n - k$ edges if and only if the forest is a spanning tree), the corollary follows. □

An *α-acyclic decomposition* of a hypergraph $\mathcal{H} = (X, \mathcal{A})$ is a set of α-acyclic hypergraphs $\{(X, \mathcal{A}_i)\}_{i=1}^{c}$ such that $\mathcal{A}_1, \ldots, \mathcal{A}_c$ partition $\mathcal{A}$, that is, $\mathcal{A} = \bigsqcup_{i=1}^{c} \mathcal{A}_i$. The *size* of the α-acyclic decomposition is c.

DEFINITION 2.9. *The* α-arboricity *of a hypergraph $\mathcal{H}$, denoted $\alpha\mathrm{arb}(\mathcal{H})$, is the minimum size of an α-acyclic decomposition of $\mathcal{H}$.*

3. Previous Work.

Trivially, $\alpha\mathrm{arb}(K_n^{(1)}) = \alpha\mathrm{arb}(K_n^{(n)}) = 1$, since both $K_n^{(1)}$ and $K_n^{(n)}$ are α-acyclic. It is also known that $\alpha\mathrm{arb}(K_n^{(2)}) = \alpha\mathrm{arb}(K_n^{(n-1)}) = \lceil n/2\rceil$ (see, for example, [4]). Li [14] also showed that $\alpha\mathrm{arb}(K_n^{(n-2)}) = \lceil n(n-1)/6\rceil$. In general, Li [14] showed that

$$\left\lceil \frac{1}{k}\binom{n}{k-1}\right\rceil \le \alpha\mathrm{arb}(K_n^{(k)}) \le \frac{1}{2}\binom{n+1}{k-1}. \tag{3.1}$$

The upper and lower bounds in (3.1) differ by approximately a factor of $k/2$. Wang [17] conjectured the lower bound to be the true value of $\alpha\mathrm{arb}(K_n^{(k)})$.

CONJECTURE 3.1. $\alpha\mathrm{arb}(K_n^{(k)}) = \left\lceil \frac{1}{k}\binom{n}{k-1}\right\rceil$.

Recently, Chee et al. [4] showed that Conjecture 3.1 holds when $k = n - 3$, so that Conjecture 3.1 is now known to hold for all n, when $k = 1, 2, n-3, n-2, n-1, n$.

Chee et al. [4] also showed that Conjecture 3.1 holds whenever there exists a Steiner system $S(n-k, n-k+1, n)$, and that Conjecture 3.1 holds in an asymptotic sense when k is large enough. More precisely, the following was obtained.

THEOREM 3.2 (Chee et al. [4]). *Let δ be a positive constant. Then for $k = n - O(\log^{1-\delta} n)$, we have*

$$\alpha\mathrm{arb}(K_n^{(k)}) = (1+o(1))\frac{1}{k}\binom{n}{k-1},$$

where the $o(1)$ is in n.

4. Decompositions based on Steiner Systems. First, note that the cardinality of the Steiner system $S(k-1,k,n)$ is precisely $\frac{1}{k}\binom{n}{k-1}$, i.e., when such a system exists, the lower bound given by equation 3.1. Therefore, the idea of our construction consists in using the blocks of a $S(k-1,k,n)$ as *centers* of our partitions of $K_n^{(k)}$ into α-acyclic hypergraphs. Each of these hypergraphs is based on a *center* C (in this case a block from the Steiner system) to which are added $n-3$ edges, each of which intersect the *center* on $k-1$ vertices (we name these hypergraphs *star-shaped*). The reader may find it helpful to consult Fig.4.1, which illustrates the following proof for $n = 7$ and $k = 3$, using the Steiner triple system $S(2,3,7)$ $(\mathbb{Z}_7, \mathcal{A})$, with $\mathcal{A} = \{\{i, i+1, i+3\} : i \in \mathbb{Z}_7\}$.

THEOREM 4.1. *If there exists an $S(k-1,k,n)$, then $\alpha\mathrm{arb}(K_n^{(k)}) = \frac{1}{k}\binom{n}{k-1}$.*

Proof. Let k and n be positive integers, $2 \le k \le n$. Let $(X, \mathcal{A})$ be an $S(k-1,k,n)$. Define a bipartite graph G with bipartition $V(G) = P \sqcup Q$, where $P = \{(A,x) : A \in \mathcal{A}$ and $x \in X \setminus A\}$ and $Q = \binom{X}{k} \setminus \mathcal{A}$, so that vertex $(A,x) \in P$ is adjacent to vertex $K \in Q$ if and only if $K \subset A \cup \{x\}$. Thus, the neighborhood of vertex $(A,x) \in P$ is the set $N(A,x) = \{(A \cup \{x\}) \setminus \{u\} : u \in A\}$, and the neighborhood of vertex $K \in Q$ is the set $N(K) = \{(A,x) : x \in K,\ A \in \mathcal{A}$, and $K \setminus \{x\} \subset A\}$. Evidently, $|N(A,x)| = k$ for all $(A,x) \in P$. To see that $|N(K)| = k$ for all $K \in Q$, note that each of the k $(k-1)$-subsets of K is contained in exactly one $A \in \mathcal{A}$, since $(X,\mathcal{A})$ is an $S(k-1,k,n)$. It follows that $|N(A,x)| = |N(K)| = k$, and G is k-regular. Hence, G has a perfect matching M.

Now, for each $A \in \mathcal{A}$, let us define the k-uniform hypergraph $\mathcal{H}_A = (X, \mathcal{B}_A)$, where $\mathcal{B}_A = \{A\} \cup \{K \in Q : \{(A,x), K\} \in M$ for some $x \in X \setminus A\}$. It is easy to check that $\binom{X}{k} = \bigsqcup_{A \in \mathcal{A}} \mathcal{B}_A$. We claim that, in fact, the set of hypergraphs $\{\mathcal{H}_A\}_{A \in \mathcal{A}}$ is an α-acyclic decomposition of $(X, \binom{X}{k})$. To see this, note that $\mathcal{H}_A$ has order n and size $n-k+1$, and observe that each edge in $\mathcal{B}_A \setminus \{A\}$ intersects A in exactly $k-1$ vertices. Hence, $L_{k-1}(\mathcal{H}_A)$ is connected. It follows from Corollary 2.8 that $\mathcal{H}_A$ is α-acyclic. The size of the α-acyclic decomposition $\{\mathcal{H}_A\}_{A \in \mathcal{A}}$ is the size of an $S(k-1,k,n)$, which is precisely $\frac{1}{k}\binom{n}{k-1}$. □

COROLLARY 4.2. *We have $\alpha\mathrm{arb}(K_n^{(k)}) = \frac{1}{k}\binom{n}{k-1}$ whenever any one of the following conditions hold:*

(i) $k = 2$ *and* $n \equiv 0 \pmod 2$, *or*
(ii) $k = 3$ *and* $n \equiv 1, 3 \pmod 6$, *or*
(iii) $k = 4$ *and* $n \equiv 2, 4 \pmod 6$, *or*
(iv) $k = 5$ *and* $n \in \{11, 23, 35, 47, 71, 83, 107, 131\}$, *or*
(v) $k = 6$ *and* $n \in \{12, 24, 36, 48, 72, 84, 108, 132\}$.

Proof. For (i), note that an $S(1,2,n)$ is a perfect matching in the complete graph K_n, and hence exists if and only if n is even. For (ii), an $S(2,3,n)$ is a Steiner triple system and exists if and only if $n \equiv 1$ or $3 \pmod 6$ (see, for example, [7]). For (iii),

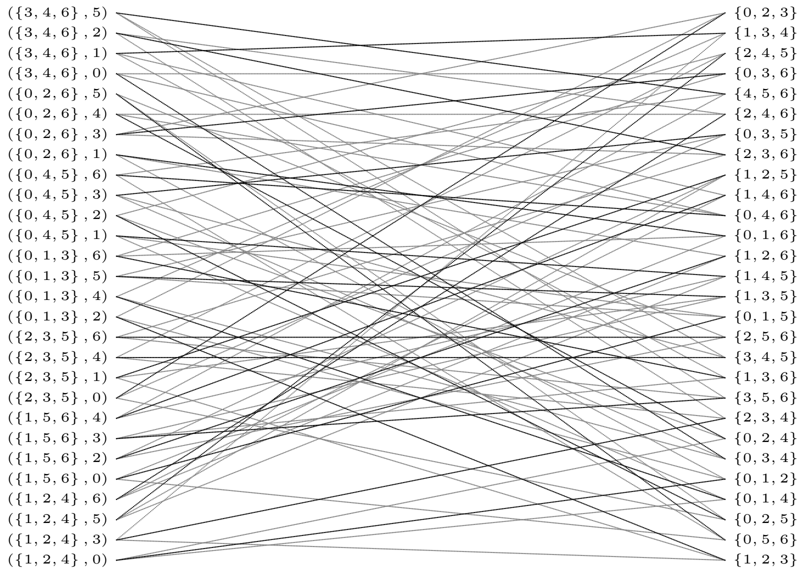

FIG. 4.1. *Case* $n = 7$, $k = 3$

an $S(3, 4, n)$ is a Steiner quadruple system, existence for which was settled by Hanani [13], who showed that $n \equiv 2$ or $4 \pmod 6$ is necessary and sufficient. For (iv)–(v), see [12, 6] for existence results. ☐

5. α-Arboricity of $K_n^{(3)}$. We determine $\alpha\mathrm{arb}(K_n^{(3)})$ completely in this section. Corollary 4.2 determined $\alpha\mathrm{arb}(K_n^{(3)})$ for all $n \equiv 1, 3 \pmod 6$, so we focus on the remaining cases of $n \equiv 0, 2, 4, 5 \pmod 6$ here.

5.1. The Case $n \equiv 0, 4 \pmod 6$. In this subsection, $n \equiv 0, 4 \pmod 6$, $n \geq 4$.

Let $X = Y \sqcup Z$, where $|Y| = n - 3$ and $Z = \{\infty_1, \infty_2, \infty_3\}$. Let $(Y, \mathcal{A})$ be an $S(2, 3, n - 3)$.

Our proof here is similar to the one given previously. Our classes, however, are now of two different kinds: not only do we need our former *star-shaped* hypergraphs whose *centers* belong to a Steiner triple system on Y, but also classes whose *centers* are two triples $\{y, \infty_1, \infty_2\}$ and $\{y, \infty_1, \infty_3\}$ (intersecting on y, ∞_1), for all $y \in Y$. As previously, any edge of our α-acyclic hypergraphs intersects at least one edge of its center on exactly two vertices. The decomposition is completed by another *star-shaped* class containing the triples $\{y, \infty_2, \infty_3\}$ where $y \in X \backslash \{\infty_2, \infty_3\}$.

We define the bipartite graph Γ with bipartition $V(\Gamma) = P \sqcup Q$, where

$$P = \left(\bigcup_{A \in \mathcal{A}} \{(A, x) : x \in X \setminus A\} \right) \cup \left(\bigcup_{y \in Y} \{(\{y, \infty_1, \infty_2\}, \{y, \infty_1, \infty_3\}, z) : z \in Y \setminus \{y\}\} \right),$$

$$Q = \binom{X}{3} \setminus (\mathcal{A} \cup \{\{y, \infty_1, \infty_2\}, \{y, \infty_1, \infty_3\}, \{y, \infty_2, \infty_3\} : y \in Y\} \cup \{Z\}),$$

with adjacency of vertices in Γ as follows:

(i) Vertex $(\{a,b,c\},x) \in P$ is adjacent to vertices $\{a,b,x\},\{a,c,x\},\{b,c,\ x\} \in Q$.

(ii) Vertex $(\{y,\infty_1,\infty_2\},\{y,\infty_1,\infty_3\},z) \in P$ is adjacent to vertices $\{y,z,\infty_h\} \in Q$, $h \in [3]$.

Every vertex in P being of degree 3, let us prove the same holds for the vertices of Q. For any pair of vertices $u,v \in Y$, we name A_{uv} the unique triple of $\mathcal{A}$ containing both u and v.

(i) $\{a,b,c\} \subseteq Y$ is adjacent to (A_{ab},c), (A_{bc},a), and (A_{ac},b)

(ii) $\{a,b,\infty_h\} \in Q$ is adjacent to (A_{ab},∞_h), $(\{b,\infty_1,\infty_2\},\{b,\infty_1,\infty_3\},a)$, and $(\{a,\infty_1,\ \infty_2\},\ \{a,\infty_1,\infty_3\},b)$.

Hence Γ is 3-regular, and consequently has a perfect matching M.

For each $A \in \mathcal{A}$, let us define the 3-uniform hypergraph $\mathcal{H}_A = (X,\mathcal{B}_A)$, where

$$\mathcal{B}_A = \{A\} \cup \{T \in Q : \{(A,x),T\} \in M \text{ for some } x \in X \setminus A\}.$$

Then $\mathcal{H}_A$ is of order n and size $n-2$. Each edge in $\mathcal{B}_A \setminus \{A\}$ intersects A in exactly two vertices. Hence, $L_2(\mathcal{H}_A)$ is connected. It follows from Corollary 2.8 that $\mathcal{H}_A$ is α-acyclic.

In addition, for each $y \in Y$, define the 3-uniform hypergraph $\mathcal{H}_y = (X,\mathcal{B}_y)$, where

$$\begin{aligned}\mathcal{B}_y =& \{\{y,\infty_1,\infty_2\},\{y,\infty_1,\infty_3\}\} \cup \\ & \{T \in Q:\ \{(\{y,\infty_1,\infty_2\},\{y,\infty_1,\infty_3\},z),T\} \in M \text{ for some } z \in Y \setminus \{y\}\}.\end{aligned}$$

Then $\mathcal{H}_y$ is of order n and size $n-2$. In $L_2(\mathcal{H}_y)$, the vertex $\{y,\infty_1,\infty_2\}$ is adjacent to $\{y,\infty_1,\infty_3\}$, and each vertex in $\mathcal{B}_y \setminus \{\{y,\infty_1,\infty_2\},\{y,\infty_1,\infty_3\}\}$ is adjacent to one of the vertices $\{y,\infty_1,\infty_2\}$ or $\{y,\infty_1,\infty_3\}$. Hence, $L_2(\mathcal{H}_y)$ is connected. It follows from Corollary 2.8 that $\mathcal{H}_y$ is α-acyclic.

Finally, define the 3-uniform hypergraph $\mathcal{H} = (X,\mathcal{B})$, where $\mathcal{B} = \{\{y,\infty_2,\ \infty_3\} : y \in X \backslash \{\infty_2,\infty_3\}\}$. Note that $\mathcal{H}$ is α-acyclic, since it GYO-reduces to an empty hypergraph.

Now, we have

$$\binom{X}{3} = \left(\bigsqcup_{A \in \mathcal{A}} \mathcal{B}_A\right) \sqcup \left(\bigsqcup_{y \in Y} \mathcal{B}_y\right) \sqcup \mathcal{B},$$

so that $\{\mathcal{H}_A\}_{A\in\mathcal{A}} \cup \{\mathcal{H}_y\}_{y\in Y} \cup \{\mathcal{H}\}$ is an α-acyclic decomposition of $K_n^{(3)}$. The size of this decomposition is

$$\frac{(n-3)(n-4)}{6} + (n-3) + 1 = \frac{n(n-1)}{6},$$

which matches the lower bound in (3.1). This gives the following result.

Proposition 5.1. $\alpha\mathrm{arb}(K_n^{(3)}) = n(n-1)/6$ *for all* $n \equiv 0,4 \pmod 6$.

5.2. The Case $n \equiv 5 \pmod 6$. In this subsection, $n \equiv 5 \pmod 6$, $n \geq 5$. Write $n = 6k+5$. Let $X = Y \sqcup \{\infty_1,\infty_2\}$, where $|Y| = 6k+3$, and let $(Y,\mathcal{G},\mathcal{A})$ be a 3-GDD of type 3^{2k+1}, which exists by Theorem 2.1. Our construction is still based on *star-shaped* hypergraphs *centered* on the triples of the 3-GDD, but this time we will need to define *centers* consisting of 3 triples, pairwise intersecting on two elements. Also, for numerical reasons, $2k+1$ of our classes are of order only $n-2$ and size $n-4$.

Suppose $\mathcal{G} = \{G_1, \ldots, G_{2k+1}\}$, where $G_i = \{g_{i,1}, g_{i,2}, g_{i,3}\}$, $i \in [2k+1]$. To keep our expressions succinct, we let

$$T^h_{i,j,j'} = \{g_{i,j}, g_{i,j'}, \infty_h\}$$

for $i \in [2k+1]$, $1 \le j < j' \le 3$ and $h \in [2]$ and

$$G_{i,j} = \{g_{i,j}, \infty_1, \infty_2\}$$

for $i \in [2k+1]$ and $j \in [3]$.

Define the bipartite graph Γ with bipartition $V(\Gamma) = P \sqcup Q$, where

$$\begin{aligned} P = &\left(\bigcup_{A \in \mathcal{A}} \{(A, x) : x \in X \setminus A\} \right) \cup \left(\bigcup_{G \in \mathcal{G}} \{(G, x) : x \in Y \setminus G\} \right) \cup \\ &\left(\bigcup_{i=1}^{2k+1} \{(T^1_{i,1,2}, T^1_{i,1,3}, G_{i,1}, x) : x \in Y \setminus G_i\} \right) \cup \\ &\left(\bigcup_{i=1}^{2k+1} \{(T^2_{i,1,2}, T^2_{i,2,3}, G_{i,2}, x) : x \in Y \setminus G_i\} \right), \\ Q = &\binom{X}{3} \setminus \Big(\mathcal{A} \cup \mathcal{G} \cup \bigcup_{\substack{i,h \\ j<j'}} \{T^h_{i,j,j'}, T^h_{i,j,j'}, T^h_{i,j,j'}\} \cup \bigcup_{i,j} G_{i,j} \Big), \end{aligned}$$

with adjacency of vertices in Γ as follows:

(i) Vertex $(\{a, b, c\}, x) \in P$ is adjacent to vertices $\{a, b, x\}, \{a, c, x\}, \{b, c, x\} \in Q$.

(ii) Vertex $(T^1_{i,1,2}, T^1_{i,1,3}, G_{i,1}, x) \in P$ is adjacent to vertices $\{g_{i,\ell}, \infty_1, x\} \in Q$, $\ell \in [3]$.

(iii) Vertex $(T^2_{i,1,2}, T^2_{i,2,3}, G_{i,2}, x) \in P$ is adjacent to vertices $\{g_{i,\ell}, \infty_2, x\} \in Q$, $\ell \in [3]$.

Every vertex in P being of degree 3, let us prove the same holds for the vertices of Q. $\forall\, u, v \in Y$, we name A_{uv} the unique triple of $\mathcal{A} \cup \mathcal{G}$ containing both u and v.

(i) $\{a, b, c\} \subseteq Y$ is adjacent to (A_{ab}, c), (A_{bc}, a), and (A_{ac}, b).

(ii) $\{a, b, \infty_1\} \in Q$, where $a \in G_i$ and $b \in G_{i'}$ with $i \ne i'$, is adjacent to (A_{ab}, ∞_1), $(T^1_{i,1,2}, T^1_{i,1,2}, G_{i,1}, b)$ and $(T^1_{i',1,2}, T^1_{i',1,3}, G_{i',1}, a)$.

(iii) $\{a, b, \infty_2\} \in Q$, where $a \in G_i$ and $b \in G_{i'}$ with $i \ne i'$, is adjacent to (A_{ab}, ∞_2), $(T^2_{i,1,2}, T^2_{i,2,3}, G_{i,2}, b)$ and $(T^2_{i',1,2}, T^2_{i',2,3}, G_{i',2}, a)$.

Hence Γ is 3-regular, and consequently has a perfect matching M.

For each $A \in \mathcal{A}$, let us define the 3-uniform hypergraph $\mathcal{H}_A = (X, \mathcal{B}_A)$, where

$$\mathcal{B}_A = \{A\} \cup \{T \in Q : \{(A, x), T\} \in M \text{ for some } x \in X \setminus A\}.$$

Then $\mathcal{H}_A$ is of order n and size $n - 2$. Each edge in $\mathcal{B}_A \setminus \{A\}$ intersects A in exactly two vertices. Hence, $L_2(\mathcal{H}_A)$ is connected. It follows from Corollary 2.8 that $\mathcal{H}_A$ is α-acyclic.

In addition, for each $G \in \mathcal{G}$, define the 3-uniform hypergraph $\mathcal{H}_G = (Y, \mathcal{B}_G)$, where $\mathcal{B}_G = \{G\} \cup \{T \in Q : \{(G, x), T\} \in M \text{ for some } x \in Y \setminus G\}$. Then $\mathcal{H}_G$ is of order $n - 2$ and size $n - 4$. By the same reason as for $\mathcal{H}_A$, $\mathcal{H}_G$ is α-acyclic.

Furthermore, for each $i \in [2k+1]$, define the 3-uniform hypergraphs $\mathcal{H}_i = (X, \mathcal{B}_i)$ and $\mathcal{H}'_i = (X, \mathcal{B}'_i)$, where

$$\begin{aligned}\mathcal{B}_i &= \{T^1_{i,1,2}, T^1_{i,1,3}, G_{i,1}\} \cup \{T \in Q : \{(T^1_{i,1,2}, T^1_{i,1,3}, G_{i,1}, x), T\} \in M \text{ for some } \\ &\quad x \in Y \setminus G_i\}, \\ \mathcal{B}'_i &= \{T^2_{i,1,2}, T^2_{i,2,3}, G_{i,2}\} \cup \{T \in Q : \{(T^2_{i,1,2}, T^2_{i,2,3}, G_{i,2}, x), T\} \in M \text{ for some } \\ &\quad x \in Y \setminus G_i\}.\end{aligned}$$

Then $\mathcal{H}_i$ and $\mathcal{H}'_i$ are each of order n and size $n-2$. In $L_2(\mathcal{H}_i)$ (respectively, $L_2(\mathcal{H}'_i)$), the vertex $T^1_{i,1,2}$ (respectively, $T^2_{i,1,2}$) is adjacent to vertices $T^1_{i,1,3}$ and $G_{i,1}$ (respectively, $T^2_{i,2,3}$ and $G_{i,2}$), and each vertex in $\mathcal{B}_i \setminus \{T^1_{i,1,2}, T^1_{i,1,3}, G_{i,1}\}$ (respectively, $\mathcal{B}'_i \setminus \{T^2_{i,1,2}, T^2_{i,2,3}, G_{i,2}\}$) is adjacent to at least one of the vertices $T^1_{i,1,2}, T^1_{i,1,3}, G_{i,1}$ (respectively, $T^2_{i,1,2}, T^2_{i,2,3}, G_{i,2}$). Hence $L_2(\mathcal{H}_i)$ (respectively, $L_2(\mathcal{H}'_i)$) is connected. It follows from Corollary 2.8 that $\mathcal{H}_i$ (respectively, $\mathcal{H}'_i$) is α-acyclic.

Finally, define the 3-uniform hypergraph $\mathcal{H} = (X, \mathcal{B})$, where

$$\mathcal{B} = \bigcup_{i=1}^{2k+1} \{T^2_{i,1,3}, T^1_{i,2,3}, G_{i,3}\}.$$

It is easy to see that $\mathcal{H}$ is α-acyclic.

Now, we have

$$\binom{X}{3} = \left(\bigsqcup_{A \in \mathcal{A}} \mathcal{B}_A\right) \sqcup \left(\bigsqcup_{G \in \mathcal{G}} \mathcal{B}_G\right) \sqcup \left(\bigsqcup_{i=1}^{2k+1} \mathcal{B}_i\right) \sqcup \left(\bigsqcup_{i=1}^{2k+1} \mathcal{B}'_i\right) \sqcup \mathcal{B},$$

so that $\{\mathcal{H}_A\}_{A \in \mathcal{A}} \cup \{\mathcal{H}_G\}_{G \in \mathcal{G}} \cup \{H_i\}_{i \in [2k+1]} \cup \{H'_i\}_{i \in [2k+1]} \cup \{\mathcal{H}\}$ is an α-acyclic decomposition of $K_n^{(3)}$. The size of this decomposition is

$$\begin{aligned}3\binom{2k+1}{2} + (2k+1) + (2k+1) + (2k+1) + 1 &= 6k^2 + 9k + 4 \\ &= \left\lceil \frac{n(n-1)}{6} \right\rceil,\end{aligned}$$

which matches the lower bound in (3.1). This gives the following result.

Proposition 5.2. $\alpha\text{arb}(K_n^{(3)}) = \lceil n(n-1)/6 \rceil$ *for all* $n \equiv 5 \pmod 6$.

5.3. The Case $n \equiv 2 \pmod 6$. We treat the remaining case of $n \equiv 2 \pmod 6$.

Lemma 5.3. $\alpha\text{arb}(K_8^{(3)}) = 10$.

Proof. The lower bound in (3.1) showed that $\alpha\text{arb}(K_8^{(3)}) \geq 10$. We construct an α-acyclic decomposition meeting this lower bound.

Consider the $S(2,3,7)$, $(\mathbb{Z}_7, \mathcal{A})$, with $\mathcal{A} = \{\{i, i+1, i+3\} : i \in \mathbb{Z}_7\}$. Let $\{\mathcal{H}_A\}_{A \in \mathcal{A}}$ be the α-acyclic decomposition of $(\mathbb{Z}_7, \binom{\mathbb{Z}_7}{3})$ produced by the construction of Section 4. We use this to construct an α-acyclic decomposition of $K_8^{(3)}$ as follows. Let

$X = \mathbb{Z}_7 \sqcup \{\infty\}$, and let

$$\begin{aligned}
\mathcal{B}_1 &= \{\{i, i+1, \infty\} : i \in \mathbb{Z}_7 \setminus \{0\}\},\\
\mathcal{B}_2 &= \{\{i, i+3, \infty\} : i \in \mathbb{Z}_7 \setminus \{1\}\},\\
\mathcal{B}_3 &= \{\{i+1, i+3, \infty\} : i \in \mathbb{Z}_7 \setminus \{2\}\},\\
\mathcal{B}_4 &= E(\mathcal{H}_{\{0,1,3\}}) \cup \{\{0, 1, \infty\}\},\\
\mathcal{B}_5 &= E(\mathcal{H}_{\{1,2,4\}}) \cup \{\{1, 4, \infty\}\},\\
\mathcal{B}_6 &= E(\mathcal{H}_{\{2,3,5\}}) \cup \{\{3, 5, \infty\}\}.
\end{aligned}$$

Then $\{(X, \mathcal{B}_i)\}_{i\in[6]} \cup \{\mathcal{H}_{\{3,4,6\}}, \mathcal{H}_{\{0,4,5\}}, \mathcal{H}_{\{1,5,6\}}, \mathcal{H}_{\{0,2,6\}}\}$ is an α-acyclic decomposition of $(X, \binom{X}{3})$ of size 10. $\square$

Henceforth, in what follows, let $n \equiv 2 \pmod 6$, $n \geq 14$. Write $n = 6k + 2$. Let $X = Y \sqcup \{\infty\}$, where $|Y| = 6k + 1$, and let $(Y, \mathcal{G}, \mathcal{A})$ be a 3-GDD of type $3^{2k}1^1$, which exists by Theorem 2.1. Here again, we use *star-shaped* hypergraphs *centered* on the triples of $\mathcal{A}$, but also classes whose *centers* consist of two triples intersecting in 2 elements. They will be completed with a last *star-shaped* class of order $2k + 2$ and size $2k$ (in order to reach the bound).

Suppose $\mathcal{G} = \{G_1, \ldots, G_{2k}, \{g\}\}$, where $G_i = \{g_{i,1}, g_{i,2}, g_{i,3}\}$, $i \in [2k]$. To keep our expressions succinct, we let

$$\begin{aligned}
G_i' &= \{g_{i,1}, g_{i,2}, \infty\},\\
G_i'' &= \{g_{i,1}, g_{i,3}, \infty\},\\
G_i''' &= \{g_{i,2}, g_{i,3}, \infty\},
\end{aligned}$$

for $i \in [2k]$. Define the bipartite graph Γ with bipartition $V(\Gamma) = P \sqcup Q$, where

$$\begin{aligned}
P = {} & \left(\bigcup_{A\in\mathcal{A}} \{(A, x) : x \in X \setminus A\}\right) \cup \left(\bigcup_{i=1}^{2k} \{(G_i, G_i'', x) : x \in Y \setminus G_i\}\right) \cup \\
& \left(\bigcup_{i=1}^{2k} \{(G_i', G_i''', x) : x \in Y \setminus G_i\}\right) \cup \{G_i : i \in [2k]\},\\
Q = {} & \binom{X}{3} \setminus \left(\mathcal{A} \cup \mathcal{G} \cup \{G_i', G_i'', G_i''' : i \in [2k]\}\right),
\end{aligned}$$

with adjacency of vertices in Γ as follows:

(i) Vertex $(\{a, b, c\}, x) \in P$ is adjacent to vertices $\{a, b, x\}, \{a, c, x\}, \{b, c, x\} \in Q$.

(ii) Vertex $(G_i, G_i'', x) \in P$ is adjacent to vertices $\{g_{i,1}, g_{i,2}, x\}, \{g_{i,1}, g_{i,3}, x\}, \{g_{i,2}, g_{i,3}, x\} \in Q$.

(iii) Vertex $(G_i', G_i''', x) \in P$ is adjacent to vertices $\{g_{i,1}, \infty, x\}, \{g_{i,2}, \infty, x\}, \{g_{i,3}, \infty, x\} \in Q$.

(iv) Vertex $G_i \in P$ is adjacent to vertices $\{g_{i,j}, g, \infty\} \in Q$, $j \in [3]$.

Every vertex in P being of degree 3, let us prove the same holds for the vertices of Q. $\forall\, u, v \in Y$, we name A_{uv} the unique triple of $\mathcal{A}$ containing both u and v.

(i) $\{a, b, c\} \subseteq Y$, where a, b, and c belong to 3 different groups, is adjacent to (A_{ab}, c), (A_{ac}, b), and (A_{bc}, a).

(ii) $\{a, b, c\} \subseteq Y$, where a and b belong to the same group G_i and $c \notin G_i$, is adjacent to (A_{ac}, b), (A_{bc}, a) and (G_i, G_i'', c).

(iii) $\{g_{i,j}, g_{i',j'}, \infty\} \in Q$ (hence $i \neq i'$) is adjacent to $(A_{g_{i,j}g_{i',j'}}, \infty)$, $(G'_i, G'''_i, g_{i',j'})$ and $(G'_{i'}, G'''_{i'}, g_{i,j})$.

(iv) $\{g_{i,j}, g, \infty\}$ is adjacent to $(A_{g_{i,j}g}, \infty)$, G_i, and (G'_i, G'''_i, g).

Hence Γ is 3-regular, and consequently has a perfect matching M.

For each $A \in \mathcal{A}$, let us define the 3-uniform hypergraph $\mathcal{H}_A = (X, \mathcal{B}_A)$, where

$$\mathcal{B}_A = \{A\} \cup \{T \in Q : \{(A, x), T\} \in M \text{ for some } x \in X \setminus A\}.$$

Then $\mathcal{H}_A$ is of order n and size $n-2$. Each edge in $\mathcal{B}_A \setminus \{A\}$ intersects A in exactly two vertices. Hence, $L_2(\mathcal{H}_A)$ is connected. It follows from Corollary 2.8 that $\mathcal{H}_A$ is α-acyclic.

In addition, for each $i \in [2k]$, define the 3-uniform hypergraphs $\mathcal{H}_i = (X, \mathcal{B}_i)$ and $\mathcal{H}'_i = (X, \mathcal{B}'_i)$, where

$$\mathcal{B}_i = \{G_i, G''_i\} \cup \{T \in Q : \{(G_i, G''_i, x), T\} \in M \text{ for some } x \in Y \setminus G_i\},$$
$$\mathcal{B}'_i = \{G'_i, G'''_i\} \cup \{T \in Q : \{(G'_i, G'''_i, x), T\} \in M \text{ for some } x \in Y \setminus G_i\}.$$

Then $\mathcal{H}_i$ and $\mathcal{H}'_i$ are each of order n and size $n-2$. By the same reason as for $\mathcal{H}_A$, $\mathcal{H}_i$ and $\mathcal{H}'_i$ are α-acyclic.

Finally, define the 3-uniform hypergraph $\mathcal{H} = (X, \mathcal{B})$, where

$$\mathcal{B} = \bigcup_{i=1}^{2k} \{T \in Q : \{G_i, T\} \in M\}.$$

It is easy to see that $\mathcal{H}$ is α-acyclic, has order $2k+2$ and size $2k$.

Now, we have

$$\binom{X}{3} = \left(\bigsqcup_{A \in \mathcal{A}} \mathcal{B}_A \right) \sqcup \left(\bigsqcup_{i=1}^{2k} \mathcal{B}_i \right) \sqcup \left(\bigsqcup_{i=1}^{2k} \mathcal{B}'_i \right) \sqcup \mathcal{B},$$

so that $\{\mathcal{H}_A\}_{A \in \mathcal{A}} \cup \{\mathcal{H}_i\}_{i \in [2k]} \cup \{\mathcal{H}'_i\}_{i \in [2k]} \cup \{\mathcal{H}\}$ is an α-acyclic decomposition of $K_n^{(3)}$. The size of this decomposition is

$$\begin{aligned} \left(3\binom{2k}{2} + 2k \right) + 2k + 2k + 1 &= 6k^2 + 3k + 1 \\ &= \left\lceil \frac{n(n-1)}{6} \right\rceil, \end{aligned}$$

which matches the lower bound in (3.1). Together with Lemma 5.3, this gives the following result.

Proposition 5.4. *$\alpha\mathrm{arb}(K_n^{(3)}) = \lceil n(n-1)/6 \rceil$ for all $n \equiv 2 \pmod 6$, $n \geq 8$.*

5.4. Summary. Corollary 4.2(i), and Propositions 5.1, 5.2, 5.4, combine to give:

Theorem 5.5. *$\alpha\mathrm{arb}(K_n^{(3)}) = \lceil n(n-1)/6 \rceil$ for all $n \geq 3$.*

6. Conclusion. The problem of determining the α-arboricity of hypergraphs is a problem motivated by database theory. In this paper, we continue the study of the α-arboricity of complete uniform hypergraphs. We give a general construction based on Steiner systems and determine completely the value of $\alpha\mathrm{arb}(K_n^{(3)})$. Previously, $\alpha\mathrm{arb}(K_n^{(k)})$ was only known for $k = 1, 2, n-3, n-2, n-1, n$.

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