Cosparse Analysis Modeling

Sangnam Nam, Mike E. Davies, Michael Elad, Rémi Gribonval

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Sangnam Nam[1] *Michael E. Davies*[2] *Michael Elad*[3] *Rémi Gribonval*[1]

[1] IRISA, INRIA Rennes – Bretagne Atlantique, France
[2] IDCOM & Joint Research Institute for Signal and Image Processing, Edinburgh University, UK
[3] Department of Computer Science, The Technion – Israel Institute of Technology, Israel

ABSTRACT

A ubiquitous problem that has found many applications, from signal processing to machine learning, is to estimate a high-dimensional vector $\mathbf{x}_0 \in \mathbf{R}^d$ from a set of incomplete linear observations $y = \mathbf{M}\mathbf{x}_0 \in \mathbf{R}^m$. This is an ill-posed problem which admits infinitely many solutions, hence solving it is hopeless unless we can use additional prior knowledge on $\mathbf{x}_0$.

The assumption that $\mathbf{x}_0$ admits a *sparse* representation $\mathbf{z}_0$ in some *synthesis dictionary* $\mathbf{D}$ is known to be of significant help, and it is now well understood that under incoherence assumptions on the matrix $\mathbf{MD}$, one can recover vectors $\mathbf{x}_0$ with sufficiently sparse representations by solving the optimization problem:

$$\hat{\mathbf{x}}_S := \mathbf{D}\hat{\mathbf{z}}; \quad \hat{\mathbf{z}} := \arg\min_{\mathbf{z}} \|\mathbf{z}\|_\tau \text{ subject to } \mathbf{y} = \mathbf{MDz} \quad (1)$$

for $0 \leq \tau \leq 1$.

An alternative to (1) which has been used successfully in practice is to consider the *analysis* ℓ_τ-optimization:

$$\hat{\mathbf{x}}_A := \arg\min_{\mathbf{x}} \|\Omega\mathbf{x}\|_\tau \text{ subject to } \mathbf{y} = \mathbf{Mx}, \quad (2)$$

where $\Omega : \mathbf{R}^d \to \mathbf{R}^p$ is an *analysis operator*. Typically the dimensions are $m \leq d \leq p, n$.

The focus of our work is the study of a data model that makes possible to identify a collection of signals $\mathbf{x}_0$ that can be recovered via the optimization (2). Roughly speaking, in the case of (1), the signals $\mathbf{x}_0$ that are sparse, in other words, satisfy the *sparse synthesis model*, can be recovered via (1). In the sparse synthesis model, we consider the number of the non-zeros $\|\mathbf{z}\|_0$ of the representation $\mathbf{z}$ of $\mathbf{x}_0$ (meaning that $\mathbf{x}_0 = \mathbf{Dz}$), and we say that $\mathbf{x}_0$ is sparse if there is a representation $\mathbf{z}_0$ of $\mathbf{x}_0$ with small $\|\mathbf{z}_0\|_0$. To the contrary, in the case of (2), we are more interested in the number of the zeros $p - \|\Omega\mathbf{x}_0\|_0$ of the representation $\Omega\mathbf{x}_0$ of $\mathbf{x}_0$. We call the quantity $\ell = p - \|\Omega\mathbf{x}_0\|_0$ the *cosparsity* of $\mathbf{x}_0$ and say that $\mathbf{x}_0$ is cosparse, or it satisfies *cosparse analysis model*, if ℓ is large.

For the cosparse analysis model, we have the following uniqueness result:

Proposition 1. *Let Ω be an analysis operator in general position. Then, for* almost all $\mathbf{M}$ *(with respect to the Lebesgue measure), a necessary and sufficient condition for* (2) *with $\tau = 0$ to have a unique minimum is*

$$m \geq 2(d - \ell). \quad (3)$$

The proposition characterizes the collection of signals that can be recovered from (2) for the generic case. We also observe that by comparing with the uniqueness result for the sparse synthesis model, again in generic cases, $d - \ell$ is playing the role of the sparsity $k = \|\mathbf{z}_0\|_0$.

In the sparse synthesis model, armed with the fact that $\mathbf{x}_0$ has a sparse representation, one may try to recover/approximate $\mathbf{x}_0$ by greedily selecting atoms from $\mathbf{MD}$ and matching the observation $\mathbf{y}$. The Orthogonal Matching Pursuit (OMP) is an example of such approachs. Similarly, we can consider a greedy algorithm as an alternative to (2) in the cosparse analysis model. As such an algorithm, we propose the Greedy Analysis Pursuit (GAP). In this algorithm, contrary to the greedy algorithms for the synthesis model, we try to identify the rows of Ω that correspond to the *zeros* of $\Omega\mathbf{x}_0$. Moreover, again contrary to the typical greedy approach, GAP greedily—perhaps, generously—removes a row that is likely not to correspond the zeros from a collection of candidate rows that may do. This way, instead of building the support of $\mathbf{z}_0$ by growing the size of the support as is done in, for example, OMP, we are carving out unwanted rows in order to obtain the correct cosupport of $\mathbf{x}_0$, i.e., the set of the indices that correspond to the zeros of $\Omega\mathbf{x}_0$.

Finally, we run a synthetic experiment to demonstrate the effectiveness of the proposed algorith, GAP. Interestingly, we observe from the phase transition diagrams obtained from the experiment that GAP performs better than the analysis ℓ_1-minimization.

Keywords— Sparse Representation, Inverse Problems, Algorithms, Cosparse Analysis Model

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