Generalized covariation and extended Fukushima decompositions for Banach valued processes. Application to windows of Dirichlet processes.

Cristina Di Girolami, Francesco Russo

Generalized covariation and extended Fukushima decompositions for Banach valued processes. Application to windows of Dirichlet processes.

Cristina DI GIROLAMI*, and Francesco RUSSO ‡

May 21th, 2011

Abstract

This paper concerns a class of Banach valued processes which have finite quadratic variation. The notion introduced here generalizes the classical one, of Métivier and Pellaumail which is quite restrictive. We make use of the notion of χ-covariation which is a generalized notion of covariation for processes with values in two Banach spaces B_1 and B_2. χ refers to a suitable subspace of the dual of the projective tensor product of B_1 and B_2. We investigate some C^1 type transformations for various classes of stochastic processes admitting a χ-quadratic variation and related properties. If $\mathbb{X}^1$ and $\mathbb{X}^2$ admit a χ-covariation, $F^i : B_i \to \mathbb{R}$, $i = 1, 2$ are of class C^1 with some supplementary assumptions then the covariation of the real processes $F^1(\mathbb{X}^1)$ and $F^2(\mathbb{X}^2)$ exist.
A detailed analysis will be devoted to the so-called window processes. Let X be a real continuous process; the $C([-\tau, 0])$-valued process $X(\cdot)$ defined by $X_t(y) = X_{t+y}$, where $y \in [-\tau, 0]$, is called *window* process. Special attention is given to transformations of window processes associated with Dirichlet and weak Dirichlet processes. In fact we aim to generalize the following properties valid for $B = \mathbb{R}$. If $\mathbb{X} = X$ is a real valued Dirichlet process and $F : B \to \mathbb{R}$ of class $C^1(B)$ then $F(\mathbb{X})$ is still a Dirichlet process. If $\mathbb{X} = X$ is a weak Dirichlet process with finite quadratic variation, and $F : C^{0,1}([0, T] \times B)$ is of class $C^{0,1}$, then $(F(t, \mathbb{X}_t))$ is a weak Dirichlet process. We specify corresponding results when $B = C([-\tau, 0])$ and $\mathbb{X} = X(\cdot)$. This will consitute a significant Fukushima decomposition for functionals of windows of (weak) Dirichlet processes. As applications, we give a new technique for representing path-dependent random variables.

[**2010 Math Subject Classification**:] 60H05, 60H07, 60H10, 60H30, 91G80

Key words and phrases Covariation and Quadratic variation, Calculus via regularization, Infinite dimensional analysis, Tensor analysis, Dirichlet processes, Representation of path dependent random variables, Malliavin calculus, Generalized Fukushima decomposition.

*LUISS Guido Carli - Libera Università Internazionale degli Studi Sociali Guido Carli di Roma (Italy).
†ENSTA ParisTech, Unité de Mathématiques appliquées, 32, Boulevard Victor, F-75739 Paris Cedex 15 (France).
‡INRIA Rocquencourt and Cermics Ecole des Ponts, Projet MATHFI. Domaine de Voluceau, BP 105 F-78153 Le Chesnay Cedex (France) .

1 Introduction

The notion of covariation is historically defined for two real valued $(\mathcal{F}_t)$-semimartingales X and Y and it is denoted by $[X,Y]$. This notion was extended to the case of general processes by mean of discretization techniques, by [17], or via regularization, see for instance [23, 25]. In this paper we will follow the language of regularization; for simplicity we suppose that either X or Y are continuous. We propose here a slight different approach than [23].

Definition 1.1. Let X and Y be two real processes such that X is continuous. For $\epsilon > 0$, we denote

$$[X,Y]^\epsilon_t = \int_0^t \frac{(X_{s+\epsilon} - X_s)(Y_{s+\epsilon} - Y_s)}{\epsilon} ds\ , \quad t > 0\ . \tag{1.1}$$

We say that X and Y admit a **covariation** if

i) $\lim_{\epsilon\to 0}[X,Y]^\epsilon_t$ exists in probability for every $t > 0$ and

ii) the limiting process in **i)** admits a continuous modification that will be denoted by $[X,Y]$.

If $[X,X]$ exists, we say that X is a **finite quadratic variation process** (or it has finite quadratic variation) and it is also denoted by $[X]$. If $[X] = 0$, X is called **zero quadratic variation process**.
We say that (X,Y) admits its (or X and Y admit their) **mutual covariations** if $[X]$, $[Y]$ and $[X,Y]$ exist.

Remark 1.2.
1. Lemma 1.3 below allows to show that, whenever $[X,X]$ exists, then $[X,X]^\varepsilon$ also converges in the ucp sense as intended for instance in the [23, 25] sense. The basic results established there are still valid here, see the following items.
2. If X and Y are $(\mathcal{F}_t)$-local semimartingales, then $[X,Y]$ coincides with the classical covariation, see Corollaries 2 and 3 in [25].
3. If X (resp. A) is a finite (resp. zero) quadratic variation process, we have $[A,X] = 0$.

We recall two useful tools related to the notion of covariation for real valued processes, see Lemma 3.1 from [24] and Propostion 1.2 in [23].

Lemma 1.3. Let $(Z^\epsilon)_{\epsilon>0}$ be a family of continuous processes indexed by $[0,T]$. We suppose the following.

1) $\forall\epsilon > 0$, $t \longrightarrow Z^\epsilon_t$ is increasing.

2) There is a continuous process $(Z_t)_{t\in[0,T]}$ such that $Z^\epsilon_t \to Z_t$ in probability for any $t \in [0,T]$ when ϵ goes to zero.

Then Z^ε converges to Z ucp, where ucp stands for the uniform convergence in probability on each compact.

Proposition 1.4. Let X and Y be two continuous processes admitting their mutual covariations. Then for every cadlag process H on $\mathbb{R}^+$ we have

$$\int_0^\cdot H_s \frac{(X_{s+\epsilon} - X_s)(Y_{s+\epsilon} - Y_s)}{\epsilon} ds \xrightarrow[\epsilon\longrightarrow 0]{ucp} \int_0^\cdot H_s d[X,Y]_s\ .$$

Definition 1.5.
1. An $(\mathcal{F}_t)$-adapted real process A is called $(\mathcal{F}_t)$**-martingale orthogonal process** if $[A,N] = 0$ for any continuous $(\mathcal{F}_t)$-local martingale N.

2. A real process A is called $(\mathcal{F}_t)$**-strongly predictable** if there is $\rho > 0$ such that $X_{\cdot+\rho}$ is $(\mathcal{F}_t)$-adapted.

Previous notion was introduced in [6]; the proposition below was the object of Corollary 3.11 in [6].

Proposition 1.6. Let A be an $(\mathcal{F}_t)$-strongly predictable process and M be an $(\mathcal{F}_t)$-local martingale. Then $[A, M] = 0$. In particular an $(\mathcal{F}_t)$-strongly predictable process is an $(\mathcal{F}_t)$-martingale orthogonal process.

Important subclasses of finite quadratic variation processes are Dirichlet processes. Probably the best denomination should be Föllmer-Dirichlet processes, since a very similar notion was introduced by [16] in the discretization framework.

Definition 1.7. A real continuous process X is a called $(\mathcal{F}_t)$**-Dirichlet process** if X admits a decomposition $X = M + A$ where M is an $(\mathcal{F}_t)$-local martingale and A is a zero quadratic variation process. For convenience, we suppose $A_0 = 0$.

The decomposition is unique if for instance $A_0 = 0$, see Proposition 16 in [25]. An $(\mathcal{F}_t)$-Dirichlet process has in particular finite quadratic variation. An $(\mathcal{F}_t)$-semimartingale is also an $(\mathcal{F}_t)$-Dirichlet process, a locally bounded variation process is in fact a zero quadratic variation process.
The concept of $(\mathcal{F}_t)$-Dirichlet process can be weakened. An extension of such processes are the so-called $(\mathcal{F}_t)$-weak Dirichlet processes, which were first introduced and discussed in [12] and [19], but they appeared implicitly even in [13]. Recent developments concerning the subject appear in [4, 6, 28]. $(\mathcal{F}_t)$-weak Dirichlet processes are generally not $(\mathcal{F}_t)$-Dirichlet processes but they still maintain a decomposition property.

Definition 1.8. A real continuous process Y is called $(\mathcal{F}_t)$**-weak Dirichlet** if Y admits a decomposition $Y = M + A$ where M is an $(\mathcal{F}_t)$-local martingale and A is an $(\mathcal{F}_t)$-martingale orthogonal process. For convenience, we will always suppose $A_0 = 0$.

The decomposition is unique, see for instance Remark 3.5 in [19] or again Proposition 16 in [25]. Corollary 3.15 in [6] makes the following observation. If the underlying filtration $(\mathcal{F}_t)$ is the natural filtration associated with a Brownian motion W, then any $(\mathcal{F}_t)$-adapted process A is an $(\mathcal{F}_t)$-martingale orthogonal process if and only if $[A, W] = 0$. An $(\mathcal{F}_t)$-Dirichlet process is also an $(\mathcal{F}_t)$-weak Dirichlet process, a zero quadratic variation process is in fact also an $(\mathcal{F}_t)$-martingale orthogonal process. An $(\mathcal{F}_t)$-weak Dirichlet process is not necessarily a finite quadratic variation process; on the other hand, there are $(\mathcal{F}_t)$-weak Dirichlet processes with finite quadratic variation that are not Dirichlet, see for instance [13]. Let Y be an $(\mathcal{F}_t)$-weak Dirichlet process with decomposition $Y = W + A$, W being a $(\mathcal{F}_t)$-Brownian motion and the process A an $(\mathcal{F}_t)$-martingale orthogonal process; if A has with finite quadratic variation, then Y is also a finite quadratic variation process and $[Y] = [W] + [A]$. In Theorem 5.10 we will provide another class of examples of $(\mathcal{F}_t)$-weak Dirichlet processes with finite quadratic variation which are not $(\mathcal{F}_t)$-Dirichlet.
An important property in stochastic calculus concerns the conservation of the *semimartingale* or *Dirichlet process* features through some real transformations. Here are some classical results.

a) The class of real semimartingales with respect to a given filtration is known to be stable with respect to $C^2(\mathbb{R})$ transformations, i.e. if $f \in C^2(\mathbb{R})$ or difference of convex functions, X is an $(\mathcal{F}_t)$-semimartingale, then $f(X)$ is still an $(\mathcal{F}_t)$-semimartingale.

b) Finite quadratic variation processes are stable under $C^1(\mathbb{R})$ transformations.

c) Also Dirichlet processes are stable with respect to $C^1(\mathbb{R})$ transformations. If $f \in C^1(\mathbb{R})$ and $X = M + A$ is a real $(\mathcal{F}_t)$-Dirichlet process with M the $(\mathcal{F}_t)$-local martingale and A the zero quadratic variation process, then $f(X)$ is still an $(\mathcal{F}_t)$- Dirichlet process whose decomposition is $f(X) = \tilde{M}+\tilde{A}$, where $\tilde{M}_t = f(X_0) + \int_0^t f'(X_s)dM_s$ and $\tilde{A}_t = f(X_t) - \tilde{M}_t$; see [2] and [26] for details.

d) In some applications, in particular to control theory (as illustrated in [18]), one often needs to know the nature of process $(f(t, X_t))$ where $f \in C^{0,1}(\mathbb{R}^+ \times \mathbb{R})$ and X is a real continuous $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation. It was shown in [19], Proposition 3.10, that $\big(f(t, X_t)\big)$ is an $(\mathcal{F}_t)$-weak Dirichlet process. Obviously, $(f(t, X_t))$ does not need to be of finite quadratic variation. Consider, as an example, f only depending on time, deterministic, with infinite quadratic variation.

Let B_1, B_2 be two general Banach spaces. If $\mathbb{X}$ (resp. $\mathbb{Y}$) is a B_1 (resp. B_2) valued stochastic process it is not obvious to define an exploitable notion of covariation of $\mathbb{X}$ and $\mathbb{Y}$ even if they are H-valued martingales and $B_1 = B_2 = H$ is a separable Hilbert space. In Definition 3.4 we recall the notion of χ-covariation (resp. χ-quadratic variation) introduced in [7] in reference to a subspace χ of the dual of $B_1\hat{\otimes}_\pi B_2$, where $\mathbb{X}$ is B_1-valued and $\mathbb{Y}$ is B_2-valued. When χ equals the whole space $(B_1\hat{\otimes}_\pi B_2)^*$, we say that $\mathbb{X}$ and $\mathbb{Y}$ admit a global covariation. In [21, 10] one introduces two historical concepts of quadratic variations related to a Banach valued process $\mathbb{X}$, the **real and tensor quadratic variations**. In Definition 1.3, Propositions 1.5, 1.6 and Corollary 1.7 of [7] we recover in our regularization language those notions. In Proposition 3.16 of [7] we show that whenever $\mathbb{X}$ has a real and tensor quadratic variation then it has the global quadratic variation. Many Banach space valued processes do not admit a global quadratic variation, even though they admit a χ-quadratic variation for some suitable χ, see [7], Section 4 for several examples. In this paper, given different classes of stochastic processes $\mathbb{X}$ with values in some Banach space B and a functional $F : B \to \mathbb{R}$ with some Fréchet regularity, we are interested in finding natural sufficient conditions so that $F(\mathbb{X})$ is a real finite quadratic variation process, a Dirichlet or a weak Dirichlet process.

– In Theorem 4.6 we show that if $\mathbb{X}$ is a B-valued process with χ-quadratic variation and $F : B \to \mathbb{R}$ is of class C^1 Fréchet with some supplementary properties on DF, then $F(\mathbb{X})$ is a real finite quadratic variation process. This constitutes a natural generalization of previous item b) concerning real valued processes.

A typical Banach space which justifies the introduction of the notion of χ-quadratic variation is $B = C([-\tau, 0])$ for some $\tau > 0$. If X is a real continuous process, the $C([-\tau, 0])$-valued process $X(\cdot)$ defined by $X_t(y) = X_{t+y}$, where $y \in [-\tau, 0]$, is called **window process** (associated with X). If X is an $(\mathcal{F}_t)$-Dirichlet (resp. $(\mathcal{F}_t)$-weak Dirichlet), the process $X(\cdot)$ is called **window $(\mathcal{F}_t)$-Dirichlet** (resp. **$(\mathcal{F}_t)$-weak Dirichlet) process**. For window processes, we obtain more specific results. We introduce here a notation which will be re-defined in Section 2. Let a be the vector $(a_N, a_{N-1}, \dots, a_1, 0)$ which identifies $N + 1$ fixed points on $[-\tau, 0]$, $-\tau = a_N < a_{N-1} < \dots a_1 < a_0 = 0$. Space $\mathcal{D}_a([-\tau, 0])$ denotes the Hilbert space of measures μ on $[-\tau, 0]$ which can be written as a sum of Dirac's measures concentrated on points a_i, i.e. $\mu(dx) = \sum_{i=0}^N \lambda_i\delta_{a_i}(dx)$, $\lambda_i \in \mathbb{R}$. $\mathcal{D}_0([-\tau, 0])$ denotes the space $\mathcal{D}_a([-\tau, 0])$ when $a = (0)$, i.e. the linear space of multiples of Dirac's measure concentrated in 0. The following items are generalizations of properties c), d) valid for real processes. We set $B = C([-\tau, 0])$.

– Let X be an $(\mathcal{F}_t)$-Dirichlet process, with associated window process $\mathbb{X} = X(\cdot)$ and again $F : B \longrightarrow \mathbb{R}$ of class C^1 Fréchet. Theorem 5.10 gives conditions so that $F(\mathbb{X})$ is a real $(\mathcal{F}_t)$-weak Dirichlet process. Under a stronger condition, Theorem 5.8 shows that $F(\mathbb{X})$ is a real $(\mathcal{F}_t)$-Dirichlet process. More precisely Theorem 5.10 (resp. Theorem 5.8) states the following. Let $F : B \longrightarrow \mathbb{R}$ be of class $C^1\big(B\big)$ in the Fréchet sense such that the first derivative $DF(\eta)$ at each point $\eta \in B$, belongs to $\mathcal{D}_a([-\tau, 0]) \oplus L^2([-\tau, 0])$ (resp. $\mathcal{D}_0([-\tau, 0]) \oplus L^2([-\tau, 0])$). We suppose moreover that DF, with values in the mentioned space, is continuous. Then $F(\mathbb{X})$ is a real $(\mathcal{F}_t)$-weak Dirichlet process (resp. Dirichlet process).

- Previous item is extended to the case when X is an $(\mathcal{F}_t)$- weak Dirichlet process with finite quadratic variation in Theorem 5.12. Let $F : [0,T] \times B \longrightarrow \mathbb{R}$ as time dependent of class $C^{0,1}$. Similarly to the case when B is finite dimensional, [19], we cannot expect $\big(F(t, X_t(\cdot))\big)$ to be a Dirichlet process. In general it will not even be a finite quadratic variation process. In Theorem 5.12 we state the following. Suppose that the first derivative $DF(t,\eta)$, at each point $(t,\eta) \in [0,T] \times C([-\tau,0])$, belongs to $\mathcal{D}_a([-\tau,0]) \oplus L^2([-\tau,0])$. We suppose again that DF, with values in the mentioned space, is continuous. Then $\big(F(t, X_t(\cdot))\big)$ is at least a weak Dirichlet process.
- If DF does not necessarily live in $\mathcal{D}_a([-\tau,0]^2) \oplus L^2([-\tau,0])$, and in some cases even if $t \mapsto DF(t,\eta)$ for fixed η is only stepwise continuous but it fulfills a technical condition called the *support predictability condition* (see Definition 5.13), it is possible to recover the conclusion of previous statement, see Theorems 5.15 and 5.16.

One of the consequences of the paper is that, under some modest conditions on a functional $F : [0,T] \times B \to \mathbb{R}$ and on a B-valued process $\mathbb{X}$ which is a window of a semimartingale (with $B = C([-\tau,0])$), it is possible to characterize $F(t,\mathbb{X}_t)$ through a Fukushima type decomposition, which is unique, and it plays the role of Itô type formula under weak conditions. The Fukushima decomposition given in Theorems 5.12, 5.15 and 5.16 is innovating at the level of stochastic analysis. In fact it does not concern the decomposition of a functional of an infinite dimensional Dirichlet process (or maybe weak Dirichlet); in fact, even the window of a semimartingale is generally not a B-valued semimartingale, see Proposition 4.7 in [7].

In Section 6, we consider a diffusion X such that $X_t = X_0 + \int_0^t \sigma(r, X_r) dW_r + \int_0^t b(r, X_r) dr$ and $\sigma, b : [0,T] \times \mathbb{R} \to \mathbb{R}$ of class $C^{0,1}$ whose partial derivative in the second variable is bounded. Even if σ is possibly degenerate, we give representations of a class of path dependent random variables h depending on the whole history of X via a functional C^1 Fréchet. A first representation result is Proposition 6.11 which is based on Theorem 6.4. When the process X is a standard Brownian motion we allow the functional not be smooth, see Section 6.3. In Section 6.4 we consider h of the form $f(X_{t_1}, \ldots, X_{t_N})$ $0 < t_1 < \ldots < t_N = T$ and $f \in C^2(\mathbb{R}^N)$ with polynomial growth. In this case the representation can be associated with N PDEs, each-one stated when the time t varies in the subinterval (t_{i-1}, t_i), for $1 \leq i \leq N$.

The paper is organized as follows. After this introduction, Section 2 contains general notations and some preliminaries. Section 3 will be devoted to the definition of χ-covariation and χ-quadratic variation and some related results. In that section we will remind the evaluation of χ-covariation and χ-quadratic variation for different classes of processes. In Section 4, we discuss how a B-valued process having some χ-quadratic variation transforms. In Section 5 we concentrate on the case $B = C([-\tau,0]$ and on generalized Fukushima decomposition of windows of Dirichlet or weak Dirichlet processes. At Section 6 we provide an application to the problem of recovering quasi-explicit representation formulae for square integrable random variables.

2 Preliminaries

In this section we recall some definitions and notations concerning the whole paper. Let A and B be two general sets such that $A \subset B$; $1_A : B \to \{0,1\}$ will denote the indicator function of the set A, so $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$. We also write $1_A(x) = 1_{\{x \in A\}}$. Throughout this paper we will denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a fixed probability space, equipped with a given filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ fulfilling the usual conditions. Let E and F be Banach spaces over the scalar field $\mathbb{R}$. We shall denote by $L(E;F)$ the Banach space of F-valued bounded linear maps on E with the norm given by $\|\phi\| = \sup\{\|\phi(e)\|_F \,:\, \|e\|_E \leq 1\}$. When $F = \mathbb{R}$, the topological dual space of E will be denoted simply by E^*. If ϕ is a linear functional on E, we shall denote the value of ϕ at an element $e \in E$ either by $\phi(e)$ or $\langle \phi, e\rangle$ or even ${}_{E^*}\langle \phi, e\rangle_E$. Throughout the paper the symbols $\langle \cdot, \cdot\rangle$ will denote always some type of duality that will change depending on the context. We shall denote the space of $\mathbb{R}$-valued bounded bilinear forms on the product $E\times F$ by $\mathcal{B}(E\times F)$ with the norm given by $\|\phi\|_{\mathcal{B}} = \sup\{|\phi(e, f)| \,:\, \|e\|_E \leq 1; \|f\|_F \leq 1\}$. If $a < b$ are two real numbers, $C([a, b])$ will denote the Banach linear space of real continuous functions equipped with the uniform norm denoted by $\|\cdot\|_\infty$. If K is a compact subset of $\mathbb{R}^n$, $\mathcal{M}(K)$ will denote the dual space $C(K)^*$, i.e. the so-called set of finite signed measures on K. Our principal references about functional analysis and Banach spaces topologies are [11, 3].

The capital letters $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$ (resp. X, Y, Z) will generally denote Banach valued (resp. real valued) processes indexed by the time variable $t \in [0, T]$ with $T > 0$. A stochastic process $\mathbb{X}$ will also be denoted by $(\mathbb{X}_t)_{t\in[0,T]}$. A B-valued (resp. $\mathbb{R}$-valued) stochastic process $\mathbb{X} : \Omega \times [0, T] \to B$ (resp. $\mathbb{X} : \Omega \times [0, T] \to \mathbb{R}$) is said to be measurable if $\mathbb{X} : \Omega \times [0, T] \longrightarrow B$ (resp. $\mathbb{X} : \Omega \times [0, T] \to \mathbb{R}$) is measurable with respect to the σ-algebras $\mathcal{F} \otimes \mathcal{B}or([0, T])$ and $\mathcal{B}or(B)$ (resp. $\mathcal{B}or(\mathbb{R})$), $\mathcal{B}or$ denoting the corresponding Borel σ-algebra. We recall that $\mathbb{X} : \Omega \times [0, T] \longrightarrow B$ (resp. $\mathbb{R}$) is said to be *strongly measurable* (or *measurable in the Bochner sense*) if it is the limit of measurable countable valued functions. If $\mathbb{X}$ is measurable and cadlag with B separable then $\mathbb{X}$ is strongly measurable. If B is finite dimensional then a measurable process $\mathbb{X}$ is also strongly measurable. If nothing else is mentioned, all the processes indexed by $[0, T]$ will be naturally prolonged by continuity setting $\mathbb{X}_t = \mathbb{X}_0$ for $t \leq 0$ and $\mathbb{X}_t = \mathbb{X}_T$ for $t \geq T$. A similar convention is done for deterministic functions. A sequence $(\mathbb{X}^n)_{n\in\mathbb{N}}$ of continuous B-valued processes indexed by $[0, T]$, will be said to converge *ucp* (*uniformly convergence in probability*) to a process $\mathbb{X}$ if $\sup_{0\leq t\leq T} \|\mathbb{X}^n_t - \mathbb{X}_t\|_B$ converges to zero in probability when $n \to \infty$. The Fréchet space $\mathscr{C}([0, T])$ will denote the linear space of continuous real processes equipped with the ucp topology and the metric $d(\mathbb{X}, \mathbb{Y}) = \mathbb{E}\left[\sup_{t\in[0,T]} |\mathbb{X}_t - \mathbb{Y}_t| \wedge 1\right]$. We go on with other notations.

The direct sum of two Banach spaces E_1 and E_2 will be denoted by $E := E_1 \oplus E_2$. E is still a Banach space under the 2-norm defined by $\|e_1 + e_2\|_E := (\|e_1\|^2_{E_1} + \|e_2\|^2_{E_2})^{1/2}$. If each of the spaces E_i is a Hilbert space then E coincides with the uniquely determined Hilbert space with scalar product $\langle e, f\rangle_E = \langle e_1 + e_2, f_1 + f_2\rangle_E = \sum_{i=1}^2 \langle e_i, f_i\rangle_i$, where $\langle \cdot, \cdot\rangle_i$ is the scalar product in E_i.

We recall now some basic concepts and results about tensor products of two Banach spaces E and F. For details and a more complete description of these arguments, the reader may refer to [27], the case with E and F Hilbert spaces being particularly exhaustive in [22]. If E and F are Banach spaces, the Banach space $E\hat{\otimes}_\pi F$ (resp. $E\hat{\otimes}_h F$) denotes the *projective (resp. Hilbert) tensor product* of the Banach spaces E and F. If E and F are Hilbert spaces the Hilbert tensor product $E\hat{\otimes}_h F$ is a Hilbert space. We recall that $E\hat{\otimes}_\pi F$ is obtained by a completion of the algebraic tensor product $E \otimes F$ equipped with the projective norm π. Let $\{x_i\}_{1\leq i\leq n} \subset E$ and $\{y_i\}_{1\leq i\leq n} \subset F$, for a general element $u = \sum_{i=1}^n x_i \otimes y_i$ in $E \otimes F$, $\pi(u) = \inf\{\sum_{i=1}^n \|x_i\| \|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i\}$. Let $e \in E$ and $f \in F$, symbol $e \otimes f$ (resp. $e\otimes^2$) will denote a basic element of the algebraic tensor product $E \otimes F$ (resp. $E \otimes E$). The space $(E\hat{\otimes}_\pi F)^*$ denotes, as usual, the topological dual of the projective tensor product. There is an isometric isomorphism between the dual space of the projective tensor product and the space of bounded bilinear forms equipped with the usual norm:

$$(E\hat{\otimes}_\pi F)^* \cong \mathcal{B}(E \times F) \cong L(E; F^*)\ . \tag{2.1}$$

Through relation

$${}_{(E\hat{\otimes}_\pi F)^*}\langle T, \sum_{i=1}^n x_i \otimes y_i\rangle_{E\hat{\otimes}_\pi F} = T\left(\sum_{i=1}^n x_i \otimes y_i\right) = \sum_{i=1}^n \tilde{T}(x_i, y_i) = \sum_{i=1}^n \bar{T}(x_i)(y_i) \tag{2.2}$$

we associate a bounded bilinear form $\tilde{T} \in \mathcal{B}(E \times F)$, a bounded linear functional T on $E\hat{\otimes}_\pi F$ and an element $\bar{T} \in L(E; F^*)$. In the sequel that identification will be often used without explicit mention.
The importance of tensor product spaces and their duals is justified first of all from identification (2.1). In fact the second order Fréchet derivative of a real function defined on a Banach space E belongs to $\mathcal{B}(E \times E)$. We recall another important property.

$$\mathcal{M}([-\tau,0]^2) \subset \left(C([-\tau,0])\hat{\otimes}_\pi C([-\tau,0])\right)^* . \tag{2.3}$$

Let η_1, η_2 be two elements in $C([-\tau,0])$. The element $\eta_1 \otimes \eta_2$ in the algebraic tensor product $C([-\tau,0])\otimes^2$ will be identified with the element η in $C([-\tau,0]^2)$ defined by $\eta(x,y) = \eta_1(x)\eta_2(y)$ for all x, y in $[-\tau,0]$. So if μ is a measure on $\mathcal{M}([-\tau,0]^2)$, the pairing duality ${}_{\mathcal{M}([-\tau,0]^2)}\langle \mu, \eta_1 \otimes \eta_2 \rangle_{C([-\tau,0]^2)}$ has to be understood as the following pairing duality:

$${}_{\mathcal{M}([-\tau,0]^2)}\langle \mu, \eta \rangle_{C([-\tau,0]^2)} = \int_{[-\tau,0]^2} \eta(x,y)\mu(dx,dy) = \int_{[-\tau,0]^2} \eta_1(x)\eta_2(y)\mu(dx,dy) . \tag{2.4}$$

Along the paper, spaces $\mathcal{M}([-\tau,0])$ and $\mathcal{M}([-\tau,0]^2)$ and their subsets will play a central role. We will introduce some other notations that will be used in the sequel. Let $-\tau = a_N < a_{N-1} < \dots a_1 < a_0 = 0$ be $N+1$ fixed points in $[-\tau,0]$. Symbol a will refer to the vector $(a_N, a_{N-1}, \dots, a_1, 0)$ which identifies $N+1$ points on $[-\tau,0]$.

- Symbol $\mathcal{D}_i([-\tau,0])$ (shortly $\mathcal{D}_i$), will denote the one dimensional Hilbert space of multiples of Dirac's measure concentrated at $a_i \in [-\tau,0]$, i.e. $\mathcal{D}_i([-\tau,0]) := \{\mu \in \mathcal{M}([-\tau,0]);\ s.t.\mu(dx) = \lambda\, \delta_{a_i}(dx) \text{ with } \lambda \in \mathbb{R}\}$; the space $\mathcal{D}_0$ will be the space of multiples of Dirac measure concentrated at 0.
- Symbol $\mathcal{D}_a([-\tau,0])$ (shortly $\mathcal{D}_a$), will denote the $(N+1)$-dimensional Hilbert space of multiples of Dirac's measure concentrated at $a_i \in [-\tau,0]$, $0 \le i \le N$, i.e.

$$\mathcal{D}_a([-\tau,0]) := \{\mu \in \mathcal{M}([-\tau,0]);\ s.t.\mu(dx) = \sum_{i=0}^N \lambda_i\, \delta_{a_i}(dx) \text{ with } \lambda_i \in \mathbb{R}\} = \bigoplus_{i=0}^N \mathcal{D}_i . \tag{2.5}$$

- Symbol $\mathcal{D}_{i,j}([-\tau,0]^2)$ (shortly $\mathcal{D}_{i,j}$), will denote the one dimensional Hilbert space of the multiples of Dirac measure concentrated at $(a_i,a_j) \in [-\tau,0]^2$, i.e. $\mathcal{D}_{i,j}([-\tau,0]^2) := \{\mu \in \mathcal{M}([-\tau,0]^2);\ s.t.\mu(dx,dy) = \lambda\, \delta_{a_i}(dx)\delta_{a_j}(dy) \text{ with } \lambda \in \mathbb{R}\} \cong \mathcal{D}_i\hat{\otimes}_h\mathcal{D}_j$. The space $\mathcal{D}_{0,0}$ will be the space of Dirac's measures concentrated at $(0,0)$.
- $L^2([-\tau,0])$ is a Hilbert subspace of $\mathcal{M}([-\tau,0])$, as well as $L^2([-\tau,0]^2) \cong L^2([-\tau,0])\hat{\otimes}_h^2$ is a Hilbert subspace of $\mathcal{M}([-\tau,0]^2)$, both equipped with the norm derived from the usual scalar product.
- $\mathcal{D}_i([-\tau,0]) \oplus L^2([-\tau,0])$ is a Hilbert subspace of $\mathcal{M}([-\tau,0])$. The particular case when $i=0$, the space $\mathcal{D}_0([-\tau,0]) \oplus L^2([-\tau,0])$, shortly $\mathcal{D}_0 \oplus L^2$, will be often recalled in the paper.
- $\mathcal{D}_i([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])$ is a Hilbert subspace of $\mathcal{M}([-\tau,0]^2)$.
- $\chi^2([-\tau,0]^2)$, χ^2 shortly, the Hilbert space defined as follows.

$$L^2([-\tau,0]^2) \oplus \bigoplus_{i=0}^N \left(L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_i([-\tau,0])\right) \oplus \bigoplus_{i=0}^N \left(\mathcal{D}_i([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])\right) \oplus \bigoplus_{i,j=0}^N \mathcal{D}_{i,j}([-\tau,0]^2) . \tag{2.6}$$

- As a particular case of $\chi^2([-\tau,0]^2)$ we will denote $\chi^0([-\tau,0]^2)$, χ^0 shortly, the subspace of measures defined as

$$\chi^0([-\tau,0]^2) = L^2([-\tau,0]^2) \oplus L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_0([-\tau,0]) \oplus \mathcal{D}_0([-\tau,0])\hat{\otimes}_h L^2([-\tau,0]) \oplus \mathcal{D}_{0,0}([-\tau,0]^2) . \tag{2.7}$$

Let B be a Banach space. A function $F : [0,T] \times B \longrightarrow \mathbb{R}$, is said to be $C^{1,2}([0,T] \times B)$ (Fréchet), or $C^{1,2}$ (Fréchet), if the following properties are fulfilled.

- F is once continuously differentiable; the partial derivative with respect to t will be denoted by $\partial_t F : [0,T] \times B \longrightarrow \mathbb{R}$;
- for any $t \in [0,T]$, $x \mapsto DF(t,x)$ is of class C^1 where $DF : [0,T] \times B \longrightarrow B^*$ denotes the derivative with respect to the second argument;
- the second order derivative with respect to the second argument $D^2F : [0,T] \times B \to (B\hat{\otimes}_\pi B)^*$ is continuous.

If $B = C([-\tau,0])$, we remark that DF defined on $[0,T] \times B$ takes values in $B^* \cong \mathcal{M}([-\tau,0])$. For all $(t,\eta) \in [0,T] \times C([-\tau,0])$, we will denote by $D_{dx}F(t,\eta)$ the measure such that

$$ {}_{\mathcal{M}([-\tau,0])}\langle DF(t,\eta), h\rangle_{C([-\tau,0])} = DF(t,\eta)(h) = \int_{[-\tau,0]} h(x) D_{dx}F(t,\eta) \qquad \forall\, h \in C([-\tau,0]). \quad (2.8)$$

Recalling (2.3), if $D^2F\,(t,\eta) \in \mathcal{M}([-\tau,0]^2)$ for all $(t,\eta) \in [0,T] \times C([-\tau,0])$ (which will happen in most of the treated cases) we will denote with $D^2_{dx\,dy}F(t,\eta)$, or $D_{dx}D_{dy}F(t,\eta)$, the measure on $[-\tau,0]^2$ such that

$$ {}_{\mathcal{M}([-\tau,0]^2)}\langle D^2F(t,\eta), g\rangle_{C([-\tau,0]^2)} = D^2F(t,\eta)(g) = \int_{[-\tau,0]^2} g(x,y)\, D^2_{dx\,dy}F(t,\eta) \qquad \forall\, g \in C([-\tau,0]^2). \quad (2.9)$$

A useful notation that will be used along all the paper is the following.

Notation 2.1. Let $F : [0,T] \times C([-\tau,0]) \longrightarrow \mathbb{R}$ be a Fréchet differentiable function, with Fréchet derivative $DF : [0,T] \times C([-\tau,0]) \longrightarrow \mathcal{M}([-\tau,0])$. For any given $(t,\eta) \in [0,T] \times C([-\tau,0])$ and $a \in [-\tau,0]$, we denote by $D^{ac}F\ (t,\eta)$ the absolutely continuous part of measure $DF\,(t,\eta)$, and by $D^{\delta_a}F\,(t,\eta) := DF\,(t,\eta)(\{a\})$. For every $\eta \in C([-\tau,0])$, we observe that $t \mapsto D^{\delta_a}F(t,\eta)$ is a real valued function.
We denote $D^{\perp}F\,(t,\eta) = DF\,(t,\eta) - DF\,(t,\eta)(\{0\})\delta_0$.

Example 2.2. If for example $DF(t,\eta) \in \mathcal{D}_0 \oplus L^2([-\tau,0])$ for every $(t,\eta) \in [0,T] \times C([-\tau,0])$, then we will often write

$$ D_{dx}F\,(t,\eta) = D^{\delta_0}F\,(t,\eta)\delta_0(dx) + D^{ac}_x F\,(t,\eta)dx\ . \quad (2.10)$$

3 Notions of χ-covariation between Banach valued processes

Let B_1, B_2 be two Banach spaces. Whenever $B_1 = B_2$ we will denote it simply by B.

Definition 3.1. A Banach subspace $(\chi, \|\cdot\|_\chi)$ continuously injected into $(B_1\hat{\otimes}_\pi B_2)^*$ will be called a **Chi-subspace** (of $(B_1\hat{\otimes}_\pi B_2)^*$).

Remark 3.2. Obviously the pairing between $(B_1\hat{\otimes}_\pi B_2)^*$ and $(B_1\hat{\otimes}_\pi B_2)^{**}$ is compatible with the paring between χ and χ^*.

Example 3.3. When $B = C([-\tau,0])$, typical examples of Chi-subspace of $(B\hat{\otimes}_\pi B)^*$ are $\mathcal{M}([-\tau,0]^2)$ equipped with the total variation norm and all Hilbert closed subspaces of $\mathcal{M}([-\tau,0]^2)$. For instance $L^2([-\tau,0]^2)$, $\mathcal{D}_i([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])$, $\mathcal{D}_{i,j}([-\tau,0]^2)$, for $1 \le i,j \le N$, $\chi^2([-\tau,0]^2)$ and $\chi^0([-\tau,0]^2)$.

We recall now the notion of χ-covariation between a B_1-valued stochastic process $\mathbb{X}$ and a B_2-valued stochastic process $\mathbb{Y}$. We suppose $\mathbb{X}$ to be a continuous B_1-valued stochastic process and $\mathbb{Y}$ to be a strongly measurable B_2-valued stochastic process such that $\int_0^T \|\mathbb{Y}_s\|_{B^*} ds < +\infty$ a.s. We remind that $\mathscr{C}([0,T])$ denotes the space of continuous processes equipped with the ucp topology.
Let χ be a Chi-subspace of $(B_1\hat{\otimes}_\pi B_2)^*$ and $\epsilon > 0$. We denote by $[\mathbb{X},\mathbb{Y}]^\epsilon$, the following application

$$[\mathbb{X},\mathbb{Y}]^\epsilon : \chi \longrightarrow \mathscr{C}([0,T]) \quad \text{defined by} \quad \phi \mapsto \left(\int_0^t {}_{\chi}\langle \phi, \frac{J\left((\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s)\right)}{\epsilon} \rangle_{\chi^*} \, ds \right)_{t\in[0,T]} \tag{3.1}$$

where $J : B_1\hat{\otimes}_\pi B_2 \longrightarrow (B_1\hat{\otimes}_\pi B_2)^{**}$ is the canonical injection between a space and its bidual. With application $[\mathbb{X},\mathbb{Y}]^\epsilon$ it is possible to associate another one, denoted by $\widetilde{[\mathbb{X},\mathbb{Y}]}^\epsilon$, defined by

$$\widetilde{[\mathbb{X},\mathbb{Y}]}^\epsilon(\omega,\cdot) : [0,T] \longrightarrow \chi^* \quad \text{given by} \quad t \mapsto \left(\phi \mapsto \int_0^t {}_{\chi}\langle \phi, \frac{J\left((\mathbb{X}_{s+\epsilon}(\omega) - \mathbb{X}_s(\omega)) \otimes (\mathbb{Y}_{s+\epsilon}(\omega) - \mathbb{Y}_s(\omega))\right)}{\epsilon} \rangle_{\chi^*} \, ds \right).$$

Definition 3.4. Let B_1, B_2 be two Banach spaces and χ be a Chi-subspace of $(B_1\hat{\otimes}_\pi B_2)^*$. Let $\mathbb{X}$ (resp. $\mathbb{Y}$) be a continuous B_1 (resp. strongly measurable B_2) valued stochastic process such that $\int_0^T \|\mathbb{Y}_s\|_{B^*} ds < +\infty$ a.s.. We say that $\mathbb{X}$ **and** $\mathbb{Y}$ **admit a χ-covariation** if

H1 For all (ϵ_n) there exists a subsequence (ϵ_{n_k}) such that

$$\begin{aligned}
&\sup_k \int_0^T \sup_{\|\phi\|_\chi \le 1} \left| \langle \phi, \frac{(\mathbb{X}_{s+\epsilon_{n_k}} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon_{n_k}} - \mathbb{Y}_s)}{\epsilon_{n_k}} \rangle \right| ds \\
&= \sup_k \int_0^T \frac{\left\| (\mathbb{X}_{s+\epsilon_{n_k}} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon_{n_k}} - \mathbb{Y}_s) \right\|_{\chi^*}}{\epsilon_{n_k}} ds \; < \infty \; a.s.
\end{aligned} \tag{3.2}$$

H2 (i) There exists an application $\chi \longrightarrow \mathscr{C}([0,T])$, denoted by $[\mathbb{X},\mathbb{Y}]$, such that

$$[\mathbb{X},\mathbb{Y}]^\epsilon(\phi) \xrightarrow[\epsilon \longrightarrow 0_+]{ucp} [\mathbb{X},\mathbb{Y}](\phi) \tag{3.3}$$

for every $\phi \in \chi \subset (B_1\hat{\otimes}_\pi B_2)^*$.

(ii) There is a measurable process $\widetilde{[\mathbb{X},\mathbb{Y}]} : \Omega \times [0,T] \longrightarrow \chi^*$, such that
- for almost all $\omega \in \Omega$, $\widetilde{[\mathbb{X},\mathbb{Y}]}(\omega,\cdot)$ is a (cadlag) bounded variation process,
- $\widetilde{[\mathbb{X},\mathbb{Y}]}(\cdot,t)(\phi) = [\mathbb{X},\mathbb{Y}](\phi)(\cdot,t)$ a.s. for all $\phi \in \chi$.

If $\mathbb{X}$ and $\mathbb{Y}$ admit a χ-covariation we will call χ**-covariation** of $\mathbb{X}$ and $\mathbb{Y}$ the χ^*-valued process $(\widetilde{[\mathbb{X},\mathbb{Y}]})_{0\le t\le T}$ defined for every $\omega \in \Omega$ and $t \in [0,T]$ by $\phi \mapsto \widetilde{[\mathbb{X},\mathbb{Y}]}(\omega,t)(\phi) = [\mathbb{X},\mathbb{Y}](\phi)(\omega,t)$. By abuse of notation, $[\mathbb{X},\mathbb{Y}]$ will also be often called χ-covariation and it will be confused with $\widetilde{[\mathbb{X},\mathbb{Y}]}$.

Definition 3.5. Let $\mathbb{X} = \mathbb{Y}$ be a B-valued stochastic process and χ be a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$. The χ-covariation $[\mathbb{X},\mathbb{X}]$ (or $\widetilde{[\mathbb{X},\mathbb{X}]}$) will also be denoted by $[\mathbb{X}]$ (or $\widetilde{[\mathbb{X}]}$), it will be called χ**-quadratic variation of** $\mathbb{X}$ and we will say that $\mathbb{X}$ has a χ-quadratic variation.

Definition 3.6. If the χ-covariation exists for $\chi = (B_1\hat{\otimes}_\pi B_2)^*$, we say that $\mathbb{X}$ and $\mathbb{Y}$ admit a **global covariation**. Analogously if $\mathbb{X}$ is B-valued and the χ-quadratic variation exists for $\chi = (B\hat{\otimes}_\pi B)^*$, we say that $\mathbb{X}$ admits a **global quadratic variation**.

We recall Corollary 3.2 from [7], which generalizes Proposition 1.4 in the Banach spaces framework.

Proposition 3.7. Let B_1, B_2 be two Banach spaces and χ be a Chi-subspace of $(B_1\hat{\otimes}_\pi B_2)^*$. Let $\mathbb{X}$ and $\mathbb{Y}$ be two stochastic processes with values in B_1 and B_2 admitting a χ-covariation; let $\mathbb{H}$ be a continuous measurable process $\mathbb{H} : \Omega \times [0,T] \longrightarrow \mathcal{V}$ where $\mathcal{V}$ is a closed separable subspace of χ. Then for every $t \in [0,T]$

$$\int_0^t {}_{\chi}\langle \mathbb{H}(\cdot,s), d\widetilde{[\mathbb{X},\mathbb{Y}]}^{\epsilon}(\cdot,s)\rangle_{\chi^*} \xrightarrow[\epsilon\longrightarrow 0]{} \int_0^t {}_{\chi}\langle \mathbb{H}(\cdot,s), d\widetilde{[\mathbb{X},\mathbb{Y}]}(\cdot,s)\rangle_{\chi^*} \tag{3.4}$$

in probability.

We recall some evaluations of χ-covariations and χ-quadratic variations for window processes given in Section 4 of [7], in particular we refer to Proposition 4.9 and Corollary 4.10.

Proposition 3.8. Let $0 < \tau \leq T$ and we make the same conventions about vector $a = (a_N = -\tau, \ldots, a_0 = 0)$ as those introduced after (2.4). Let X and Y be two real continuous processes with finite quadratic variation.

1) $X(\cdot)$ and $Y(\cdot)$ admit a zero χ-covariation, where $\chi = L^2([-\tau,0]^2)$.
2) $X(\cdot)$ and $Y(\cdot)$ admit zero χ-covariation for every given $i \in \{0,\ldots,N\}$, where $\chi = L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_i([-\tau,0])$ and $\mathcal{D}_i([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])$.

If moreover the covariation $[X_{\cdot+a_i}, Y_{\cdot+a_j}]$ exists for a given $i,j \in \{0,\ldots,N\}$, the following statements hold.

3) $X(\cdot)$ and $Y(\cdot)$ admit a χ-covariation, where $\chi = \mathcal{D}_{i,j}([-\tau,0]^2)$ and it equals

$$[X(\cdot),Y(\cdot)](\mu) = \mu(\{a_i,a_j\})[X_{\cdot+a_i}, Y_{\cdot+a_j}], \quad \forall \mu \in \mathcal{D}_{i,j}([-\tau,0]^2). \tag{3.5}$$

4) In the case $i = j = 0$, i.e. X and Y admit a covariation $[X,Y]$, then $X(\cdot)$ and $Y(\cdot)$ admit $\chi^0([-\tau,0]^2)$-covariation which equals

$$[X(\cdot),Y(\cdot)](\mu) = \mu(\{0,0\})[X,Y], \quad \forall \mu \in \chi^0. \tag{3.6}$$

If $[X_{\cdot+a_i}, Y_{\cdot+a_j}]$ exists for all $i,j = 0,\ldots,N$, then

5) $X(\cdot)$ and $Y(\cdot)$ admit a $\chi^2([-\tau,0]^2)$-covariation which equals

$$[X(\cdot),Y(\cdot)](\mu) = \sum_{i,j=0}^{N} \mu(\{a_i,a_j\})[X_{\cdot+a_i}, Y_{\cdot+a_j}], \quad \forall \mu \in \chi^2([-\tau,0]^2)\,. \tag{3.7}$$

As application of Proposition 3.8 we obtain the following.

Corollary 3.9. Let X be a real $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation and decomposition $X = M + A$, M being its $(\mathcal{F}_t)$-local martingale component. Let N be a real $(\mathcal{F}_t)$-martingale. We set $\chi = \mathcal{D}_{0,0} \oplus \chi_2$ with

$$\chi_2 = \oplus_{i=1}^{N}\mathcal{D}_{i,0} \oplus \left(L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_0\right). \tag{3.8}$$

We have the following.

1. $X(\cdot)$ and $N(\cdot)$ admit a $\mathcal{D}_{0,0}$-covariation given, for $\mu \in \mathcal{D}_{0,0}$, by

$$[X(\cdot), N(\cdot)](\mu) = \mu(\{0,0\})[M,N] \ . \tag{3.9}$$

2. $X(\cdot)$ and $N(\cdot)$ admit a zero χ_2-covariation.
3. $X(\cdot)$ and $N(\cdot)$ admit a χ-covariation where for any $\mu \in \chi$, (3.9) holds.
4. $X(\cdot)$ and $N(\cdot)$ admit a χ^0-covariation given, for $\mu \in \chi^0$, by (3.9).

Corollary 3.10. Let X be a real $(\mathcal{F}_t)$-Dirichlet process with decomposition $X = M + A$, M being its $(\mathcal{F}_t)$-local martingale component. Let N be a real $(\mathcal{F}_t)$-martingale. Then we have the following.

1. $X(\cdot)$ admits a χ^2-quadratic variation given by

$$[X(\cdot)](\mu) = \sum_{i=0}^{N} \mu(\{a_i, a_i\})[M]_{\cdot + a_i} \ .$$

2. $X(\cdot)$ and $N(\cdot)$ admit a χ^2-covariation given by

$$[X(\cdot), N(\cdot)](\mu) = \sum_{i=0}^{N} \mu(\{a_i, a_i\})[M,N]_{\cdot + a_i} \ .$$

Remark 3.11. More details about Dirichlet processes and their properties will be given in section 5. Examples of finite quadratic variation weak Dirichlet processes are provided in Section 2 of [13]. For an $(\mathcal{F}_t)$-weak Dirichlet process X the covariations $[X_{\cdot + a_i}, X_{\cdot + a_j}]$ are not a priori determined.

Proof of Corollary 3.9.

1. Using Proposition 1.6 $[X,Y] = [M,N]$. So this point follows by item 3) of Proposition 3.8.
2. We keep in mind the direct sum decomposition of χ_2 given in (3.8). We compute the $\oplus_{i=1}^{N} \mathcal{D}_{0,i}$-covariation. Covariations $[X_{\cdot + a_i}, N] = 0$ because it is the sum of $[M_{\cdot + a_i}, N]$ and $[A_{\cdot + a_i}, N]$ which are zero by Proposition 1.6 for $i = 1, \ldots, N$. Using item 3) of Proposition 3.8, $X(\cdot)$ and $N(\cdot)$ have zero $\oplus_{i=1}^{N} \mathcal{D}_{i,0}$-covariation. By item 2) of Proposition 3.8, $X(\cdot)$ and $N(\cdot)$ have zero $L^2([-\tau, 0]\hat{\otimes}_h \mathcal{D}_i$-covariation for every i. Proposition 3.18 in [7] concludes the proof of item 2, since it allows to express the χ-covariation in a sum of χ-covariation whenever χ is a direct sum of Chi-subspaces.
3. It follows by 1., 2. and again by Proposition 3.18 in [7].
4. We know that that $[X,N] = [M,N]$. So this point follows by item 4) of Proposition 3.8.

□

Proof of Corollary 3.10.

1. If $i \neq j$, by Proposition 1.6 and Remark 1.2, it follows that $[X_{\cdot + a_i}, X_{\cdot + a_j}] = 0$. If $i = j$, by Remark 1.2 and by definition of quadratic variation we get $[X_{\cdot + a_i}] = [M]_{\cdot + a_i}$. The result follows by item 5) of Proposition 3.8.
2. Similarly as in the proof of item 1. we have that $[X_{\cdot + a_i}, N_{\cdot + a_j}] = 0$ if $i \neq j$ and $[X_{\cdot + a_i}, N_{\cdot + a_i}] = [M,N]_{\cdot + a_i}$. The result follows as a consequence of item 5) of Proposition 3.8.

□

Other interesting results about χ-covariation and χ-quadratic variation for a window of a finite quadratic variation process are given in Proposition 6.4 in [9] and with more details in [7], Propositions 4.16 and 4.18.

4 Transformation of χ-quadratic variation and of χ-covariation

Let X be a real finite quadratic variation process and $f \in C^1(\mathbb{R})$. We recall that $f(X)$ is again a finite quadratic variation process. We will illustrate some natural generalizations to the infinite dimensional framework. In this section, we analyze how transform Banach valued processes having a χ-covariation through C^1 Fréchet differentiable functions. We first recall the finite dimensional case framework, see [15] Remark 3.

Proposition 4.1. Let $X = (X^1, \dots, X^n)$ be a $\mathbb{R}^n$-valued process having all its mutual covariations $[X^i, X^j]_t$ and F, $G \in C^1(\mathbb{R}^n)$. Then the covariation $[F(X), G(X)]$ exists and is given by

$$[F(X), G(X)]_{\cdot} = \sum_{i,j=1}^{n} \int_0^{\cdot} \partial_i F(X) \partial_j G(X) d[X^i, X^j] \tag{4.1}$$

This includes the case of Proposition 2.1 in [23], setting $n = 2$, $F(x,y) = f(x)$, $G(x,y) = g(y)$, $f, g \in C^1(\mathbb{R})$.

When the value space is a general Banach space, we need to recall some other preliminary results.

Proposition 4.2. Let E be a Banach space, $S, T : E \longrightarrow \mathbb{R}$ be linear continuous forms. There is a unique linear continuous forms from $E \hat{\otimes}_\pi E$ to $\mathbb{R} \hat{\otimes}_\pi \mathbb{R} \cong \mathbb{R}$, denoted by $S \otimes T$, such that $S \otimes T(e_1 \otimes e_2) = S(e_1) \cdot T(e_2)$ and $\|S \otimes T\| = \|S\| \, \|T\|$.

Proof. See Proposition 2.3 in [27]. $\square$

Remark 4.3.

1. If $T = S$, we will denote $S \otimes S = S \otimes^2$.
2. Let B be a Banach space and $F, G : E \longrightarrow \mathbb{R}$ of class $C^1(E)$ in the Fréchet sense. If x and y are fixed, $DF(x)$ and $DF(y)$ are linear continuous form from E to $\mathbb{R}$. We remark that the symbol $DF(x) \otimes DF(y)$ is defined according to Proposition 4.2, we insist on the fact that "a priori" $DF(x) \otimes DF(y)$ does not denote an element of some tensor product $E^* \otimes E^*$.

When E is a Hilbert space, the application $S \otimes T$ of Proposition 4.2 can be further specified.

Proposition 4.4. Let E be a Hilbert space, S, $T \in E^*$ and $\mathcal{S}$, $\mathcal{T}$ the associated elements in E via Riesz identification. $S \otimes T$ can be characterized as the continuous bilinear form

$$S \otimes T(x \otimes y) = \langle \mathcal{S}, \mathcal{T} \rangle_E \cdot \langle x, y \rangle_E = \langle \mathcal{S} \otimes \mathcal{T}, x \otimes y \rangle_{E \hat{\otimes}_h E}, \quad \forall x, y \in E. \tag{4.2}$$

In particular the linear form $S \otimes T$ belongs to $(E \hat{\otimes}_h E)^*$ and via Riesz it is identified with the tensor product $\mathcal{S} \otimes \mathcal{T}$. That Riesz identification will be omitted in the sequel.

Proof. The application ϕ defined in the right-side of (4.2) belongs to $(E \hat{\otimes}_h E)^*$ by construction. Since $(E \hat{\otimes}_h E)^* \subset (E \hat{\otimes}_\pi E)^*$, it also belongs to $(E \hat{\otimes}_\pi E)^*$. Moreover we have

$$\|\phi\|_{\mathcal{B}} = \sup_{\|f\|_E \le 1, \|g\|_E \le 1} |\phi(f,g)| = \sup_{\|f\|_E \le 1} |\langle \mathcal{S}, f \rangle| \sup_{\|g\|_E \le 1} |\langle \mathcal{T}, g \rangle| = \|S\|_{E^*} \, \|T\|_{E^*} .$$

By uniqueness in Proposition 4.2, ϕ must coincide with $S \otimes T$. $\square$

As application of Proposition 4.4, setting the Hilbert space $E = \mathcal{D}_a \oplus L^2([-\tau, 0])$, we state the following useful result that will be often used in Section 5 devoted to $C([-\tau, 0])$-valued window processes.

Example 4.5. Let F^1 and F^2 be two functions from $C([-\tau, 0])$ to $\mathcal{D}_a \oplus L^2([-\tau, 0])$ such that $\eta \mapsto F^j(\eta) = \sum_{i=0,\ldots N} \lambda_i^j(\eta)\delta_{a_i} + g^j(\eta)$ with $\eta \in C([-T, 0])$, $\lambda_i^j : C([-\tau, 0]) \longrightarrow \mathbb{R}$ and $g^j : C([-\tau, 0]) \longrightarrow L^2([-T, 0])$ continuous for $j = 1, 2$. Then for any $\eta_1, \eta_2 \in C([-\tau, 0])$, $(F^1 \otimes F^2)(\eta_1, \eta_2)$ will be identified with the true tensor product $F^1(\eta_1) \otimes F^2(\eta_2)$ which belongs to $\chi^2([-\tau, 0]^2)$. In fact we have

$$F^1(\eta_1) \otimes F^2(\eta_2) = \sum_{i,j=0,\ldots,N} \lambda_i^1(\eta_1)\lambda_j^2(\eta_2)\delta_{a_i} \otimes \delta_{a_j} + g^1(\eta_1) \otimes \sum_{i=0,\ldots,N} \lambda_i^2(\eta_2)\delta_{a_i} +$$

$$+ \sum_{i=0,\ldots,N} \lambda_i^1(\eta_1)\delta_{a_i} \otimes g^2(\eta_2) + g^1(\eta_1) \otimes g^2(\eta_2) \tag{4.3}$$

We now state a result related to the generalization of Proposition 4.1 to functions of processes admitting a χ-covariation.

Theorem 4.6. Let B be a separable Banach space, χ a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$ and $\mathbb{X}^1$, $\mathbb{X}^2$ two B-valued continuous stochastic processes admitting a χ-covariation. Let $F^1, F^2 : B \longrightarrow \mathbb{R}$ be two functions of class C^1 in the Fréchet sense. We suppose moreover that the applications

$$\begin{aligned} DF^i(\cdot) \otimes DF^j(\cdot) : B \times B &\longrightarrow \chi \subset (B\hat{\otimes}_\pi B)^* \\ (x, y) &\mapsto DF^i(x) \otimes DF^j(y) \end{aligned}$$

are continuous for $i, j = 1, 2$.
Then, for every $i, j \in \{1, 2\}$, the covariation between $F^i(\mathbb{X}^i)$ and $F^j(\mathbb{X}^j)$ exists and is given by

$$[F^i(\mathbb{X}^i), F^j(\mathbb{X}^j)] = \int_0^{\cdot} \langle DF^i(\mathbb{X}^i_s) \otimes DF^j(\mathbb{X}^j_s), d\widetilde{[\mathbb{X}^i, \mathbb{X}^j]}_s \rangle. \tag{4.4}$$

Remark 4.7. In view of an application of Proposition 3.7 in the proof of Theorem 4.6, we observe the following. Since B is separable and $DF^i(\cdot) \otimes DF^j(\cdot) : B \times B \longrightarrow \chi$ is continuous, the process $H_t = DF^i(\mathbb{X}^i_t) \otimes DF^j(\mathbb{X}^j_t)$ takes values in a separable closed subspace $\mathcal{V}$ of χ.

Corollary 4.8. Let us formulate the same assumptions as in Theorem 4.6. If there is a χ^*-valued stochastic process $\mathbb{H}^{i,j}$ such that $\widetilde{[\mathbb{X}^i, \mathbb{X}^j]}_s = \int_0^s \mathbb{H}^{i,j}_u \, du$ in the Bochner sense then

$$[F^i(\mathbb{X}^i), F^j(\mathbb{X}^j)]_t = \int_0^t \langle DF^i(\mathbb{X}^i_s) \otimes DF^j(\mathbb{X}^j_s), \mathbb{H}^{i,j}_s \rangle \, ds, \quad t \in [0, T]. \tag{4.5}$$

Proof of Theorem 4.6. We make use in an essential manner of Proposition 3.7. Without restriction of generality we only consider the case $F^1 = F^2 = F$ and $\mathbb{X}^1 = \mathbb{X}^2 = \mathbb{X}$.
Let $t \in [0, T]$. By definition of the quadratic variation of a real process in Definition 1.1, it will be enough to show that the quantity

$$\int_0^t \frac{(F(\mathbb{X}_{s+\epsilon}) - F(\mathbb{X}_s))^2}{\epsilon} ds \, .$$

converges in probability to the right-hand side of (4.5). Using Taylor's expansion we have

$$\begin{aligned}\frac{1}{\epsilon}\int_0^t (F(\mathbb{X}_{s+\epsilon}) - F(\mathbb{X}_s))^2\, ds = \frac{1}{\epsilon}\int_0^t \Big(&\langle DF(\mathbb{X}_s), \mathbb{X}_{s+\epsilon} - \mathbb{X}_s\rangle + \\ &+ \int_0^1 \langle DF\left((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}\right) - DF(\mathbb{X}_s), \mathbb{X}_{s+\epsilon} - \mathbb{X}_s\rangle\, d\alpha\Big)^2 ds = \\ = A_1(\epsilon) + A_2(\epsilon) + A_3(\epsilon),&\end{aligned}$$

where

$$\begin{aligned}A_1(\epsilon) &= \frac{1}{\epsilon}\int_0^t \langle DF(\mathbb{X}_s), \mathbb{X}_{s+\epsilon} - \mathbb{X}_s\rangle^2 ds = \\ &= \int_0^t \langle DF(\mathbb{X}_s)\otimes DF(\mathbb{X}_s), \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)^{\otimes 2}}{\epsilon}\rangle ds \\ A_2(\epsilon) &= \frac{2}{\epsilon}\int_0^t \langle DF(\mathbb{X}_s), \mathbb{X}_{s+\epsilon} - \mathbb{X}_s\rangle \cdot \\ &\qquad \cdot \int_0^1 \langle DF\left((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}\right) - DF(\mathbb{X}_s), \mathbb{X}_{s+\epsilon} - \mathbb{X}_s\rangle d\alpha\, ds = \\ &= 2\int_0^t\int_0^1 \langle DF(\mathbb{X}_s)\otimes\left(DF\left((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}\right) - DF(\mathbb{X}_s)\right), \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)^{\otimes 2}}{\epsilon}\rangle d\alpha\, ds \\ A_3(\epsilon) &= \frac{1}{\epsilon}\int_0^t \Big(\int_0^1 \langle DF\left((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}\right) - DF(\mathbb{X}_s), \mathbb{X}_{s+\epsilon} - \mathbb{X}_s\rangle d\alpha\Big)^2 ds \le \\ &\le \frac{1}{\epsilon}\int_0^t\int_0^1 \langle DF\left((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}\right) - DF(\mathbb{X}_s), \mathbb{X}_{s+\epsilon} - \mathbb{X}_s\rangle^2 d\alpha\, ds = \\ &= \int_0^t\int_0^1 \langle\left(DF\left((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}\right) - DF(\mathbb{X}_s)\right)^{\otimes 2}, \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)^{\otimes 2}}{\epsilon}\rangle d\alpha\, ds\ .\end{aligned}$$

According to Remark 4.7 and Proposition 3.7 with $X = Y$, it follows

$$A_1(\epsilon) \xrightarrow{\mathbb{P}} \int_0^t \langle DF(\mathbb{X}_s)\otimes DF(\mathbb{X}_s), d\widetilde{[\mathbb{X}]}_s\rangle\ .$$

It remains to show the convergence in probability of $A_2(\epsilon)$ and $A_3(\epsilon)$ to zero.
About $A_2(\epsilon)$ the following decomposition holds:

$$DF(\mathbb{X}_s)\otimes\left(DF\left((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}\right) - DF(\mathbb{X}_s)\right) = DF(\mathbb{X}_s)\otimes DF\left((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}\right) - DF(\mathbb{X}_s)\otimes DF(\mathbb{X}_s); \tag{4.6}$$

concerning $A_3(\epsilon)$ we get

$$\begin{aligned}\left(DF\left((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}\right) - DF(\mathbb{X}_s)\right)^{\otimes 2} = &\ DF\left((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}\right)^{\otimes 2} + \\ &- DF\left((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}\right)\otimes DF(\mathbb{X}_s) + \\ &+ DF(\mathbb{X}_s)\otimes DF(\mathbb{X}_s) + \\ &- DF(\mathbb{X}_s)\otimes DF\left((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}\right)\ .\end{aligned} \tag{4.7}$$

Using (4.6), we obtain

$$|A_2(\epsilon)| \le 2\int_0^t \int_0^1 \left| \langle DF(\mathbb{X}_s) \otimes \left(DF\left((1-\alpha)\mathbb{X}_s + \alpha \mathbb{X}_{s+\epsilon}\right) - DF(\mathbb{X}_s)\right), \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)\otimes^2}{\epsilon} \rangle \right| d\alpha\, ds \le$$
$$\le \int_0^t \int_0^1 \left\| DF(\mathbb{X}_s) \otimes DF\left((1-\alpha)\mathbb{X}_s + \alpha \mathbb{X}_{s+\epsilon}\right) - DF(\mathbb{X}_s) \otimes DF(\mathbb{X}_s) \right\|_\chi \left\| \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)\otimes^2}{\epsilon} \right\|_{\chi^*} d\alpha\, ds\ . \tag{4.8}$$

For fixed $\omega \in \Omega$ we denote by $\mathcal{V}(\omega) := \{\mathbb{X}_t(\omega);\ t \in [0,T]\}$ and

$$\mathcal{U} = \mathcal{U}(\omega) = \overline{conv(\mathcal{V}(\omega))}, \tag{4.9}$$

i.e. the set $\mathcal{U}$ is the closed convex hull of the compact subset $\mathcal{V}(\omega)$ of B. From (4.8) we deduce

$$|A_2(\epsilon)| \le \varpi^{\mathcal{U}\times\mathcal{U}}_{DF\otimes DF}\left(\varpi_{\mathbb{X}}(\epsilon)\right) \int_0^t \left\| \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)\otimes^2}{\epsilon} \right\|_{\chi^*} ds,$$

where $\varpi^{\mathcal{U}\times\mathcal{U}}_{DF\otimes DF}$ is the continuity modulus of the application $DF(\cdot)\otimes DF(\cdot) : B \times B \longrightarrow \chi$ restricted to $\mathcal{U}\times\mathcal{U}$ and $\varpi_{\mathbb{X}}$ is the continuity modulus of the continuous process $\mathbb{X}$. We recall that

$$\varpi^{\mathcal{U}\times\mathcal{U}}_{DF\otimes DF}(\delta) = \sup_{\|(x_1,y_1)-(x_2,y_2)\|_{B\times B} \le \delta} \| DF(x_1)\otimes DF(y_1) - DF(x_2) \otimes DF(y_2)\|_\chi$$

where the space $B \times B$ is equipped with the norm obtained summing the norms of the two components. According to Theorem 5.35 in [1], $\mathcal{U}(\omega)$ is compact, so the function $DF(\cdot)\otimes DF(\cdot)$ on $\mathcal{U}(\omega)\times\mathcal{U}(\omega)$ is uniformly continuous and $\varpi^{\mathcal{U}\times\mathcal{U}}_{DF\otimes DF}$ is a positive, increasing function on $\mathbb{R}^+$ converging to 0 when the argument converges to zero.
Let (ϵ_n) converging to zero; Condition **H1** in the definition of χ-quadratic variation, implies the existence of a subsequence (ϵ_{n_k}) such that $A_2(\epsilon_{n_k})$ converges to zero a.s. This implies that $A_2(\epsilon) \to 0$ in probability. With similar arguments, using (4.7), we can show that $A_3(\epsilon) \to 0$ in probability. We observe in fact

$$|A_3(\epsilon)| \le \int_0^t \int_0^1 \left\| DF\left((1-\alpha)\mathbb{X}_s + \alpha \mathbb{X}_{s+\epsilon}\right)\otimes^2 - DF(\mathbb{X}_s) \otimes DF\left((1-\alpha)\mathbb{X}_s + \alpha \mathbb{X}_{s+\epsilon}\right)\right\|_\chi \cdot$$
$$\cdot \left\| \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)\otimes^2}{\epsilon} \right\|_{\chi^*} d\alpha\, ds +$$
$$+ \int_0^t \int_0^1 \left\| DF\left((1-\alpha)\mathbb{X}_s + \alpha \mathbb{X}_{s+\epsilon}\right) \otimes DF(\mathbb{X}_s) - DF(\mathbb{X}_s)\otimes^2 \right\|_\chi \left\| \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)\otimes^2}{\epsilon} \right\|_{\chi^*} d\alpha\, ds \le$$
$$\le 2\varpi^{\mathcal{U}\times\mathcal{U}}_{DF\otimes DF}\left(\varpi_{\mathbb{X}}(\epsilon)\right) \int_0^t \left\| \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)\otimes^2}{\epsilon} \right\|_{\chi^*} ds\ .$$

The result is now established. □

Corollary 4.9. Let B be a separable Banach space and B_0 be a Banach space such that $B_0 \supset B$ continuously. Let $\chi = (B_0 \hat{\otimes}_\pi B_0)^*$ and $\mathbb{X}$ a continuous B-valued stochastic process admitting a χ-quadratic variation. Let $F^1, F^2 : B \longrightarrow \mathbb{R}$ be functions of class C^1 Fréchet such that DF^i, $i = 1, 2$ are continuous as applications from B to B_0^*.
Then the covariation of $F^i(\mathbb{X})$ and $F^j(\mathbb{X})$ exists and it is given by

$$[F^i(\mathbb{X}), F^j(\mathbb{X})]_\cdot = \int_0^\cdot \langle DF^i(\mathbb{X}_s) \otimes DF^j(\mathbb{X}_s), d\widetilde{[\mathbb{X}]}_s \rangle\ . \tag{4.10}$$

Proof. It is clear that χ is a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$. For any given $x, y \in B$, $i, j = 1, 2$, by the characterization of $DF^i(x) \otimes DF^j(y)$ given in Proposition 4.2 and Remark 4.3, the following applications

$$DF^i(x) \otimes DF^j(y) : B_0 \hat{\otimes}_\pi B_0 \longrightarrow \mathbb{R}$$

are continuous for $i, j \in \{1, 2\}$. The result follows by Theorem 4.6. $\square$

Remark 4.10. Under the same assumptions as Corollary 4.9 we suppose moreover that B_0 is a Hilbert space. For any $x, y \in B$, $DF(x) \otimes DG(y)$ belongs to $\left(B_0 \hat{\otimes}_h B_0\right)^*$ because of Proposition 4.4 and it will be associated to a true tensor product in the sense explained in the same proposition.

We discuss rapidly the finite dimensional framework. A detailed analysis was performed in Paragraph 1, Chapter 6 of [8]. We also recall that $\mathbb{R}^n \hat{\otimes}_\pi \mathbb{R}^n$ can be identified with the space of matrices $\mathbb{M}_{n\times n}(\mathbb{R})$. Since $\mathbb{R}^n \hat{\otimes}_\pi \mathbb{R}^n$ is finite dimensional and all the topologies are equivalent, it is enough to show the identification on $\mathbb{R}^n \otimes \mathbb{R}^n$. In fact let $u \in \mathbb{R}^n \otimes \mathbb{R}^n$ in the form $u = \sum_{1\leq i,j\leq n} u_{i,j}\; e_i \otimes e_j$ where $(e_i)_{1\leq i\leq n}$ is the canonical basis for $\mathbb{R}^n$. To u is possible to associate a unique matrix $U = (u_{i,j})_{1\leq i,j\leq n}$, $U \in \mathbb{M}_{n\times n}(\mathbb{R})$. Conversely given a matrix $U \in \mathbb{M}_{n\times n}(\mathbb{R})$ of the form $U = (U_{i,j})_{1\leq i,j\leq n}$, we ssociate the unique element $u \in \mathbb{R}^n \otimes \mathbb{R}^n$ in the form $u = \sum_{1\leq i,j\leq n} U_{i,j}\; e_i \otimes e_j$. Concerning the dual space we have $(\mathbb{R}^n \otimes \mathbb{R}^n)^* \cong L(\mathbb{R}^n; L(\mathbb{R}^n))$ which is naturally identified with $\mathbb{M}_{n\times n}(\mathbb{R})$. So a matrix $T \in \mathbb{M}_{n\times n}(\mathbb{R})$ of the form $T = (T_{i,j})_{1\leq i\leq m, 1\leq j\leq n}$ is associated with the linear form $t : \mathbb{R}^n \otimes \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $t(x \otimes y) = {}_{\mathbb{R}^n}\langle Tx\,, y\rangle_{\mathbb{R}^n}$.
Moreover the duality pairing between an element $t \in (\mathbb{R}^n \hat{\otimes}_\pi \mathbb{R}^n)^*$ and an element $u \in (\mathbb{R}^n \hat{\otimes}_\pi \mathbb{R}^n)$ (or simply $(\mathbb{R}^n \otimes \mathbb{R}^m)$, denoted by $\langle t, u\rangle$, coincides with the trace $Tr(TU)$, whenever U (resp. T) is the $\mathbb{M}_{n\times n}(\mathbb{R})$ matrix associated with u (resp. t).

Example 4.11. Let $\mathbb{X} = (X^1, \cdots, X^n)$ be a $\mathbb{R}^n$-valued stochastic process admitting all its mutual covariations, and $F, G : \mathbb{R}^n \longrightarrow \mathbb{R} \in C^1(\mathbb{R}^n)$. We recall that $\mathbb{X}$ admits a global quadratic variation $\widetilde{[\mathbb{X}]}$ which coincides with the tensor element associated with the matrix $([\mathbb{X}^*, \mathbb{X}])_{1\leq i,j\leq n} := ([X^i, X^j])_{i,j}$, see Proposition 6.2 in [8].
The application of Theorem 4.6 to this context provides a new proof of Proposition 4.1.
According to Proposition 6.2 item 2.(b) in [8], the right-hand side of (4.4) equals

$$\int_0^{\cdot} Tr\left(DF(\mathbb{X}_s) \otimes DG(\mathbb{X}_s) \cdot d[\mathbb{X}^*, \mathbb{X}]_s\right)$$

which coincides with the right-hand side of (4.1).

5 Transformation of window Dirichlet processes and window weak Dirichlet processes

5.1 Some preliminary result on measure theory

We set now $B = C([-\tau, 0])$ and we formulate now some related Fukushima type decomposition involving B-valued window Dirichlet and window weak Dirichlet processes. First we need a preliminary result on measure theory.
We start with some notations appearing for instance in [10], Chapter 1, Section D, Definition 18. Let E be a Banach space and $g : [0, T] \longrightarrow E^*$ be a bounded variation function. Then the real function

$\|g\|_{E^*} : [0,T] \longrightarrow \mathbb{R}$ has also bounded variation. If $f : [0,T] \longrightarrow E$ is a Bochner measurable, then the Bochner integral $\int_0^T {}_E\langle f(s), dg(s)\rangle_{E^*}$ is well-defined provided that

$$\int_0^T \|f(s)\|_E d\|g\|_{E^*}(s) < +\infty \; . \tag{5.1}$$

We denote by $L^1_E(g)$ the linear space of functions f verifying (5.1).

Lemma 5.1. Let E be a topological direct sum $E_1 \oplus E_2$ where E_1, E_2 are Banach spaces equipped with norms $\|\cdot\|_{E_i}$. We denote by P_i the projectors $P_i : E \to E_i$, $i \in 1, 2$.
Let $\tilde{g} : [0,T] \to E^*$ and we define $\tilde{g}_i : [0,T] \to E_i^*$ setting $\tilde{g}_i(t)(\eta) := \tilde{g}(t)(\eta)$ for all $\eta \in E_i$, i.e. the restriction of $\tilde{g}(t)$ to E_i^*. We suppose $\tilde{g}_i$ continuous with bounded variation, $i = 1, 2$.
Let $f : [0,T] \to E$ measurable with projections $f_i := P_i(f)$ defined from $[0,T]$ to E_i.
Then the following statements hold.

1. f in $L^1_E(\tilde{g})$ if and only if f_i in $L^1_{E_i}(\tilde{g}_i), i = 1, 2$ and yields

$$\int_0^t {}_E\langle f(s), d\tilde{g}(s)\rangle_{E^*} = \int_0^t {}_{E_1}\langle f_1(s), d\tilde{g}_1(s)\rangle_{E_1^*} + \int_0^t {}_{E_2}\langle f_2(s), d\tilde{g}_2(s)\rangle_{E_2^*} \; . \tag{5.2}$$

2. If $\tilde{g}_2(t) \equiv 0$ and f_1 in $L^1_{E_i}(\tilde{g}_1)$ then

$$\int_0^t {}_E\langle f(s), d\tilde{g}(s)\rangle_{E^*} = \int_0^t {}_{E_1}\langle f_1(s), d\tilde{g}_1(s)\rangle_{E_1^*} \; . \tag{5.3}$$

Proof.

1. By the hypothesis on $\tilde{g}_i$ we deduce that $\tilde{g} : [0,T] \to E^*$ has bounded variation. If $f : [0,T] \to E$ belongs to L^1_E, then $f_i = P_i(f) : [0,T] \to E_i$, $i = 1, 2$ belong to $L^1_{E_i}$ by the property $\|P_i f\|_{E_i} \le \|f\|_E$. We prove (5.2) for a step function $f : [0,T] \to E$ defined by $f(s) = \sum_{j=1}^N \phi_{A_j}(s) f_j$ with ϕ_{A_j} indicator functions of the subsets A_j of $[0,T]$ and $f_j \in E$. We have $f_j = f_{1j} + f_{2j}$ with $f_{ij} = P_i f_j$, $i = 1, 2$, so

$$\begin{aligned}\int_0^T {}_E\langle f(s), d\tilde{g}(s)\rangle_{E^*} &= \sum_{j=1}^N \int_{A_j} {}_E\langle f_j, d\tilde{g}(s)\rangle_{E^*} = \sum_{j=1}^N {}_E\langle f_j, \int_{A_j} d\tilde{g}(s)\rangle_{E^*} = \sum_{j=1}^N {}_E\langle f_j, d\tilde{g}(A_j)\rangle_{E^*} = \\ &= \sum_{j=1}^N {}_{E_1}\langle f_{1j}, d\tilde{g}_1(A_j)\rangle_{E_1^*} + \sum_{j=1}^N {}_{E_2}\langle f_{2j}, d\tilde{g}_2(A_j)\rangle_{E_2^*} = \\ &= \int_0^T {}_{E_1}\langle f_1(s), d\tilde{g}_1(s)\rangle_{E_1^*} + \int_0^T {}_{E_2}\langle f_2(s), d\tilde{g}_2(s)\rangle_{E_2^*} \; .\end{aligned}$$

   A general function f in $L^1_E(\tilde{g})$ is a sum of $f_1 + f_2$, $f_i \in L^1_{E_i}(\tilde{g}_i)$ for $i = 1, 2$. Both f_1 and f_2 can be approximated by step functions. Vector integration $L^1_E(\tilde{g})$, as well as on $L^1_{E_i}(\tilde{g}_i)$, is defined by density on step functions. The result follows by an approximation argument.

2. It follows directly by 1.

□

A useful consequence of Lemma 5.1 is the following.

Proposition 5.2. Let $E_1 = \mathcal{D}_{i,j}([-\tau,0]^2)$ and E_2 be a Banach subspace of $\mathcal{M}([-\tau,0]^2)$ such that $E_1 \cap E_2 = \{0\}$.
– Let $\tilde{g} : [0,T] \to E^*$ such that $\tilde{g}(t)_{|E_2} \equiv 0$.
– We set $g_1 : [0,T] \to \mathbb{R}$ by $g_1(t) = {}_{E_1}\langle \delta_{(a_i,a_j)}, \tilde{g}_1(t)\rangle_{E_1^*}$, supposed continuous with bounded variation.
– Let $f : [0,T] \to E$ such that $t \to f(t)(\{(a_i,a_j\}) \in L^1(d|g_1|)$.
Then

$$\int_0^t {}_E\langle f(s), d\tilde{g}(s)\rangle_{E^*} = \int_0^t f(s)(\{a_i,a_j\}) dg_1(s) \ . \tag{5.4}$$

Remark 5.3. Let g_1 be the real function defined in the second item of the hypotheses.
Defining $\tilde{g}_1 : [0,T] \to E_1^*$ by $\tilde{g}_1(t) = g_1(t)\ \delta_{(a_i,a_j)}$, by construction it follows $\tilde{g}_1(t)(f) = \tilde{g}(t)(f)$ for every $f \in E_1, t \in [0,T]$. Since for $a, b \in [0,T]$, with $a < b$, we have

$$\|\tilde{g}(b) - \tilde{g}(a)\|_{E^*} = \|\tilde{g}_1(b) - \tilde{g}_1(a)\|_{E_1^*} = |g_1(b) - g_1(a)| \ ;$$

then g_1 is continuous with bounded variation if and only if $\tilde{g}$ is continuous with bounded variation.

Proof of Proposition 5.2. We apply Lemma 5.1.2. Clearly we have $P_1(f) = f(\{a_i,a_j\})\delta_{(a_i,a_j)}$. It follows that

$$\int_0^t {}_E\langle f(s), d\tilde{g}(s)\rangle_{E^*} = \int_0^t {}_{E_1}\langle f(s)(\{a_i,a_j\})\delta_{(a_i,a_j)}, d\tilde{g}_1(s)\rangle_{E_1^*}.$$

Since $g_1(t) = {}_{E_1}\langle \delta_{(a_i,a_j)}, \tilde{g}_1(t)\rangle_{E_1^*}$ and because of Theorem 30 in Chapter 1, paragraph 2 of [10], previous expression equals the right-hand side of (5.4). ☐

Remark 5.4. Let E be a Banach subspace of $\mathcal{M}([-\tau,0]^2)$ containing $\mathcal{D}_{i,j}([-\tau,0]^2)$. A typical example of application of Proposition 5.2 is given by $E_1 = \mathcal{D}_{i,j}([-\tau,0]^2)$ and $E_2 = \{\mu \in E \mid \mu(\{a_i,a_j\}) = 0\}$. Any $\mu \in E$ can be decomposed into $\mu_1 + \mu_2$, where $\mu_1 = \mu(\{a_i,a_j\})\delta_{(a_i,a_j)}$, which belongs to E_1, and $\mu_2 \in E_2$.

In the proof of item 3. in proposition below we will use Proposition 5.2 considering $\tilde{g}$ as the χ-covariation of two processes $X(\cdot)$ and $Y(\cdot)$.

Proposition 5.5. Let $i, j \in \{0, \ldots, N\}$ and let χ_2 be a Banach subspace of $\mathcal{M}([-\tau,0]^2)$ such that $\mu(\{a_i,a_j\}) = 0$ for every $\mu \in \chi_2$. We set $\chi = \mathcal{D}_{i,j}([-\tau,0]^2) \oplus \chi_2$.
Let X, Y be two real continuous processes such that $(X_{\cdot+a_i}, Y_{\cdot+a_j})$ admits their mutual covariations and such that $X(\cdot)$ and $Y(\cdot)$ admit a zero χ_2-covariation. Then following properties hold.

1. χ is a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$, with $B = C([-\tau,0])$.
2. $X(\cdot)$ and $Y(\cdot)$ admit a χ-covariation of the type

$$[X(\cdot), Y(\cdot)] : \chi \longrightarrow \mathscr{C}([0,T]) \ , \qquad\qquad [X(\cdot), Y(\cdot)](\mu) = \mu(\{a_i,a_j\})[X_{\cdot+a_i}, Y_{\cdot+a_j}] \ .$$

3. For every χ-valued process $\mathbb{Z}$ with locally bounded paths (for instance cadlag) we have

$$\int_0^{\cdot} \langle \mathbb{Z}_s, d\widetilde{[X(\cdot), Y(\cdot)]}_s\rangle = \int_0^{\cdot} \mathbb{Z}_s(\{a_i,a_j\}) d[X_{\cdot+a_i}, Y_{\cdot+a_j}]_s \ . \tag{5.5}$$

Proof.

1. By Proposition 3.4 in [7], χ is a closed subspace of $\mathcal{M}([-\tau,0]^2)$. The claim follows by Proposition 3.3 in [7].
2. We denote here $\chi_1 = \mathcal{D}_{i,j}([-\tau,0]^2)$; χ_1 and χ_2 are closed subspaces of $\mathcal{M}([-\tau,0]^2)$. By Proposition 3.8, item 3) $X(\cdot)$ and $Y(\cdot)$ admit a χ_1-covariation. Proposition 3.18 in [7] implies that $X(\cdot)$ and $Y(\cdot)$ admit a χ-covariation which can be determined from the χ_1-covariation and the χ_2-covariation. More precisely, for μ in χ with decomposition $\mu_1 + \mu_2$, $\mu_1 \in \chi_1$ and $\mu_2 \in \chi_2$, with a slight abuse of notations, we have

$$\begin{aligned} [X(\cdot), Y(\cdot)](\mu) &= [X(\cdot), Y(\cdot)](\mu_1) + [X(\cdot), Y(\cdot)](\mu_2) = [X(\cdot), Y(\cdot)](\mu_1) \\ &= \mu_1(\{a_i, a_j\})[X_{\cdot + a_i}, Y_{\cdot + a_j}] = \mu(\{a_i, a_j\})[X_{\cdot + a_i}, Y_{\cdot + a_j}] \,. \end{aligned}$$

3. Since both sides of (5.5) are continuous processes, it is enough to show that they are equal a.s. for every fixed $t \in [0,T]$. This follows for almost all $\omega \in \Omega$ using Proposition 5.2 where $f = \mathbb{Z}(\omega)$ and $\tilde{g} = [\widetilde{X(\cdot), Y(\cdot)}](\omega)$. We remark that here $\tilde{g}_1 = [\widetilde{X_{\cdot + a_i}(\cdot), Y_{\cdot + a_j}(\cdot)}](\omega)$ and $g_1 = [X_{\cdot + a_i}, Y_{\cdot + a_j}](\omega)$.

□

Remark 5.6. Proposition 5.5 will be used in the sequel especially in the case $a_i = a_j = 0$.

Remark 5.7. Under the same assumptions as Proposition 5.5, if $\mathbb{Z}$ takes values in $\mathcal{D}_{i,j}$, then

$$\int_0^{\cdot} \langle \mathbb{Z}_s, d[\widetilde{X(\cdot), Y(\cdot)}]_s \rangle = \int_0^{\cdot} \mathbb{Z}_s(\{a_i, a_j\}) d[X_{\cdot + a_i}, Y_{\cdot + a_j}]_s \,. \tag{5.6}$$

In fact, the left-hand side equals

$$\int_0^{\cdot} \mathbb{Z}_s(a_i, a_j) \langle \delta_{a_i, a_j}, d[\widetilde{X(\cdot), Y(\cdot)}]_s \rangle.$$

That expression equals the right-hand side of (5.6) because of item 3) in Proposition 3.8 and Theorem 30, Chapter 1, par 2 of [10].

5.2 On some generalized Fukushima decomposition

We are ready now to show some decomposition results.

Theorem 5.8. Let X be a real continuous $(\mathcal{F}_t)$-Dirichlet process with decomposition $X = M + A$, where M is the $(\mathcal{F}_t)$-local martingale and A is a zero quadratic variation process with $A_0 = 0$. Let $F : C([-\tau,0]) \longrightarrow \mathbb{R}$ be a Fréchet differentiable function such that the range of DF is $\mathcal{D}_0([-\tau,0]) \oplus L^2([-\tau,0])$. Moreover we suppose that $DF : C([-\tau,0]) \longrightarrow \mathcal{D}_0([-\tau,0]) \oplus L^2([-\tau,0])$ is continuous.
Then $F(X(\cdot))$ is an $(\mathcal{F}_t)$-Dirichlet process with local martingale component equal to

$$\bar{M}_{\cdot} = F\big(X_0(\cdot)\big) + \int_0^{\cdot} D^{\delta_0} F\big(X_s(\cdot)\big) dM_s \,, \tag{5.7}$$

where from Notation 2.1 we recall that $D^{\delta_0} F(\eta) = DF(\eta)(\{0\})$.

Remark 5.9. The Itô integral in (5.7) makes sense because $\big(D^{\delta_0} F\left(X_t(\cdot)\right)\big)$ is $(\mathcal{F}_t)$-adapted.

Proof. We need to show that $[\bar{A}] = 0$ where $\bar{A} := F(X(\cdot)) - \bar{M}$. For simplicity of notations, in this proof we will denote $\alpha_0(\eta) = D^{\delta_0}F(\eta)$. By the linearity of the covariation of real processes, we have $[\bar{A}] = A_1 + A_2 - 2A_3$ where

$$
\begin{aligned}
A_1 &= [F(X.(\cdot))] \\
A_2 &= \left[\int_0^{\cdot} \alpha_0\big(X_s(\cdot)\big)dM_s\right] \\
A_3 &= \left[F(X(\cdot)), \int_0^{\cdot} \alpha_0\big(X_s(\cdot)\big)dM_s\right] .
\end{aligned}
$$

Since X is a finite quadratic variation process, by Proposition 3.8 4), its window process $X(\cdot)$ admits a $\chi^0([-\tau,0]^2)$-quadratic variation. Moreover by Example 4.5 and Remark 4.10 the map $DF \otimes DF$: $C([-\tau,0]) \times C([-\tau,0]) \longrightarrow \chi^0([-\tau,0]^2)$ is a continuous application. Applying Theorem 4.6 and (5.5) of Proposition 5.5 we obtain

$$
\begin{aligned}
A_1 &= \int_0^{\cdot} \langle DF(X_s(\cdot)) \otimes DF(X_s(\cdot)), d\widetilde{[X.(\cdot)]}_s\rangle = \\
&= \int_0^{\cdot} \alpha_0^2(X_s(\cdot))d[X]_s = \int_0^{\cdot} \alpha_0^2(X_s(\cdot))d[M]_s .
\end{aligned}
$$

The term A_2 is the quadratic variation of a local martingale; by Remark 1.2 item 2. we get

$$
A_2 = \int_0^{\cdot} \alpha_0^2(X_s(\cdot))d[M]_s .
$$

It remains to prove that $A_3 = \int_0^{\cdot} \alpha_0^2(X_s(\cdot))d[M]_s$. We define $G : C([-\tau,0]) \longrightarrow \mathbb{R}$ by $G(\eta) = \eta(0)$. We observe that $\bar{M} = G(\bar{M}(\cdot))$ where $\bar{M}(\cdot)$ denotes as usual the window process associated to $\bar{M}$. G is Fréchet differentiable and $DG(\eta) = \delta_0$, therefore DG is continuous from $C([-\tau,0])$ to $\mathcal{D}_0([-\tau,0]) \oplus L^2([-\tau,0])$. Moreover by Example 4.5 we know that $DF \otimes DG : C([-\tau,0]) \times C([-\tau,0]) \longrightarrow \chi^0([-\tau,0]^2)$ is continuous. Corollary 3.9 item 2. says that the $\chi^0([-\tau,0]^2)$-covariation between $X(\cdot)$ and $\bar{M}(\cdot)$ exists and it is given by

$$
[X(\cdot), \bar{M}(\cdot)](\mu) = \mu(\{0,0\})[X, \bar{M}] . \tag{5.8}
$$

We have $[X, \bar{M}] = [M, \bar{M}] + [A, \bar{M}] = [M, \bar{M}]$. By Remark 1.2 item 2. and the usual properties of stochastic calculus we have

$$
[X, \bar{M}] = \left[M, \int_0^{\cdot} \alpha_0\big(X_s(\cdot)\big)dM_s\right] = \int_0^{\cdot} \alpha_0\big(X_s(\cdot)\big)d[M]_s . \tag{5.9}
$$

Finally, applying again Theorem 4.6, relation (5.5) in Proposition 5.5 and (5.9) we obtain

$$
\begin{aligned}
A_3 &= [F(X(\cdot)), G(\tilde{M}(\cdot))] = \int_0^{\cdot} \langle DF(X_s(\cdot)) \otimes DG(\bar{M}_s(\cdot)), d\widetilde{[X(\cdot), \bar{M}(\cdot)]}_s\rangle \\
&= \int_0^{\cdot} \alpha_0(X_s(\cdot))d[X, \bar{M}]_s = \int_0^{\cdot} \alpha_0^2(X_s(\cdot))d[M]_s .
\end{aligned}
$$

The result is now established. ∎

Theorem 5.8 admits a slight generalization, in which will intervene the space $\mathcal{D}_a$ as defined at equation (2.5) but the final process is no longer a Dirichlet process, only a weak Dirichlet.

Theorem 5.10. Let X be a real continuous $(\mathcal{F}_t)$-Dirichlet process with decomposition $X = M + A$, M being a local martingale and A a zero quadratic variation process with $A_0 = 0$. Let $F : C([-\tau, 0]) \longrightarrow \mathbb{R}$ be a Fréchet differentiable function such that $DF : C([-\tau, 0]) \longrightarrow \mathcal{D}_a([-\tau, 0]) \oplus L^2([-\tau, 0])$ is continuous. We have the following.

1. $F(X(\cdot))$ is an $(\mathcal{F}_t)$-weak Dirichlet process with decomposition $F(X(\cdot)) = \bar{M} + \bar{A}$, where $\bar{M}$ is the local martingale defined by

$$\bar{M}_\cdot := F\left(X_0(\cdot)\right) + \int_0^\cdot D^{\delta_0} F(X_s(\cdot)) dM_s$$

   and $\bar{A}$ is the $(\mathcal{F}_t)$-martingale orthogonal process, corresponding to Definition 1.8.

2. $F(X(\cdot))$ is a finite quadratic variation process and

$$[F(X(\cdot))] = \sum_{i=0,\dots,N} \int_0^t \left(D^{\delta_{a_i}} F(X_s(\cdot))\right)^2 d[M]_{s+a_i} \tag{5.10}$$

3. Process $\bar{A}$ is a finite quadratic variation process and

$$[\bar{A}]_t = \sum_{i=1,\dots,N} \int_0^t \left(D^{\delta_{a_i}} F(X_s(\cdot))\right)^2 d[M]_{s+a_i} \tag{5.11}$$

4. In particular $\{F(X_t(\cdot)); t \in [0, -a_1]\}$ is a Dirichlet process with local martingale component $\bar{M}$.

Proof. In this proof $\alpha_i(\eta)$ will denote $D^{\delta_{a_i}} F(\eta) = DF(\eta)(\{a_i\})$ if $\eta \in C([-\tau, 0])$.

1. To show that $F(X(\cdot))$ is an $(\mathcal{F}_t)$-weak Dirichlet process we need to show that $[F(X(\cdot)) - \int_0^\cdot \alpha_0\left(X_s(\cdot)\right) dM_s, N]$ is zero for every $(\mathcal{F}_t)$-continuous local martingale N. Again we set $G : C([-\tau, 0]) \longrightarrow \mathbb{R}$ by $G(\eta) = \eta(0)$. It holds $N_t = G(N_t(\cdot))$. We remark that function G is Fréchet differentiable with $DG : C([-\tau, 0]) \longrightarrow \mathcal{D}_0([-\tau, 0])$ continuous and $DG(\eta) \equiv \delta_0$.
   In view of applying Corollary 3.9, we set $\chi := \mathcal{D}_{0,0} \oplus \chi_2$ where $\chi_2 = \oplus_{i=1}^N \mathcal{D}_{i,0} \oplus \left(L^2([-\tau, 0]) \hat{\otimes}_h \mathcal{D}_0\right)$.
   In particular for every $\mu \in \chi_2$ we have $\mu(\{0, 0\}) = 0$. $X(\cdot)$ and $N(\cdot)$ admit a χ-covariation by Corollary 3.9 3. On the other hand $DF \otimes DG : C([-\tau, 0]) \times C([-\tau, 0]) \longrightarrow \chi$ and it is a continuous map. By Theorem 4.6 we have

$$[F(X(\cdot)), N]_t = [F(X(\cdot)), G(N(\cdot))]_t = \int_0^t \langle DF(X_s(\cdot)) \otimes \delta_0, d\widetilde{[X(\cdot), N(\cdot)]}_s \rangle \ . \tag{5.12}$$

   By (5.5) in Proposition 5.5 it follows that

$$\begin{aligned} [F(X(\cdot)), N]_t &= \int_0^t (D^{\delta_0} F(s, X_s(\cdot)) \otimes \delta_0)(\{0, 0\}) d[X, N]_s \\ &= \int_0^t D^{\delta_0} F(s, X_s(\cdot)) d[M, N]_s \ . \end{aligned} \tag{5.13}$$

By Remark 1.2 item 2. and usual properties of stochastic calculus, it yields

$$\left[\int_0^{\cdot} \alpha_0\big(X_s(\cdot)\big)dM_s, N\right]_t = \int_0^t \alpha_0\big(X_s(\cdot)\big)d[M,N]_s$$

and the result follows.

2. By Example 4.5 we know that $DF \otimes DF : C([-\tau,0]) \times C([-\tau,0]) \longrightarrow \chi^2([-\tau,0]^2)$ and it is a linear continuous map. We decompose $DF(\eta) = \sum_{i=1}^N \alpha_i(\eta)\delta_{a_i} + g(\eta)$ where $g : C([-\tau,0] \to L^2([-\tau,0])$ so that

$$\begin{aligned} DF(\eta) \otimes DF(\eta) = &\sum_{i,j=0}^N \alpha_i(\eta)\alpha_j(\eta)\delta_{a_i} \otimes \delta_{a_j} + \sum_{i=0}^N \alpha_i(\eta)\delta_{a_i} \otimes g(\eta) + \sum_{j=0}^N \alpha_j(\eta) g(\eta) \otimes \delta_{a_j} \\ &+ g(\eta) \otimes g(\eta). \end{aligned}$$

Applying Theorem 4.6, relation (5.5) in Proposition 5.5 and obvious bilinearity arguments, we obtain

$$\begin{aligned} [F(X(\cdot))]_t &= \int_0^t \langle DF(X_s(\cdot)) \otimes DF(X_s(\cdot)), d\widetilde{[X_s(\cdot)]}\rangle \\ &= \sum_{i,j=0}^N \int_0^t \langle \mathbb{Z}_s^{i,j}, d\widetilde{[X_s(\cdot)]}\rangle + \int_0^t \langle \mathbb{Z}_s, d\widetilde{[X_s(\cdot)]}\rangle \end{aligned} \tag{5.14}$$

where

$$\begin{aligned} \mathbb{Z}_s^{i,j} &= \alpha_i(X_s(\cdot))\alpha_j(X_s(\cdot))\ \delta_{a_i} \otimes \delta_{a_j} \\ \mathbb{Z}_s &= DF(X_s(\cdot)) \otimes DF(X_s(\cdot)) - \sum_{i,j=0}^N \mathbb{Z}_s^{i,j}. \end{aligned} \tag{5.15}$$

A wise application of Proposition 5.5 and Remark 5.7 show that (5.14) equals

$$\sum_{i,j=0\ldots,N} \int_0^t \alpha_i(X_s(\cdot))\alpha_j(X_s(\cdot))d[X_{\cdot+a_i}, X_{\cdot+a_j}]_s = \sum_{i=0,\ldots,N} \int_0^t \alpha_i^2(X_s(\cdot))d[M]_{s+a_i}.$$

The last equality is a consequence of Proposition 1.6 and of the definition of weak Dirichlet process. Finally (5.10) is proved.

3. By bilinearity of the covariation of real processes we have $[\bar{A}] = [F(X(\cdot))] + [\bar{M}] - 2[F(X(\cdot)), \bar{M}]$. The first bracket is equal to (5.10) and the second term gives

$$\left[\int_0^{\cdot} \alpha_0\big(X_s(\cdot)\big)dM_s\right] = \int_0^t \alpha_0^2(X_s(\cdot))d[M]_s\ .$$

Setting $N_t = \int_0^t \alpha_0(X_s(\cdot))dM_s$, (5.13) gives

$$\left[F(X(\cdot)), \int_0^{\cdot} \alpha_0\big(X_s(\cdot)\big)dM_s\right] = \int_0^t \alpha_0^2(X_s(\cdot))d[M]_s$$

and (5.11) follows.

4. It is an easy consequence of (5.11) since $(\bar{A}_t)_{t\in[0,-a_1[}$ is a zero quadratic variation process.

□

Remark 5.11.
1. Theorem 5.10 gives a class of examples of $(\mathcal{F}_t)$-weak Dirichlet processes with finite quadratic variation which are not necessarily $(\mathcal{F}_t)$-Dirichlet processes.
2. An example of $F : C([-\tau, 0]) \longrightarrow \mathbb{R}$ Fréchet differentiable such that $DF : C([-\tau, 0]) \longrightarrow \mathcal{D}_a([-\tau, 0]) \oplus L^2([-\tau, 0])$ continuously is, for instance, $F(\eta) = f(\eta(a_0), \ldots, \eta(a_N))$, with $f \in C^1(\mathbb{R}^N)$. We have $DF(\eta) = \sum_{i=0}^{N} \partial_i f(\eta(a_0), \ldots, \eta(a_N))\, \delta_{a_i}$.
3. Let $a \in [-\tau, 0[$ and W be a classical $(\mathcal{F}_t)$-Brownian motion, process X defined as $X_t := W_{t+a}$ is an $(\mathcal{F}_t)$-weak Dirichlet process that is not $(\mathcal{F}_t)$-Dirichlet.
This follows from Theorem 5.10, point 2. and 3. taking $F(\eta) = \eta(a)$. In particular point 3. implies that the quadratic variation of the martingale orthogonal process is $[\bar{A}]_t = (t + a)^+$. This result was also proved directly in Proposition 4.11 in [5].

We now go on with a C^1 transformation of window of weak Dirichlet processes.

Theorem 5.12. Let X be an $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation where M is the local martingale part. Let $F : [0, T] \times C([-\tau, 0]) \longrightarrow \mathbb{R}$ continuous. We suppose moreover that $(t, \eta) \mapsto DF(t, \eta)$ exists with values in $\mathcal{D}_a([-\tau, 0]) \oplus L^2([-\tau, 0])$ and $DF : [0, T] \times C([-\tau, 0]) \longrightarrow \mathcal{D}_a([-\tau, 0]) \oplus L^2([-\tau, 0])$ is continuous.
Then $(F(t, X_t(\cdot)))_{t\in[0,T]}$ is an $(\mathcal{F}_t)$-weak Dirichlet process with martingale part

$$\bar{M}^F_t := F(0, X_0(\cdot)) + \int_0^t D^{\delta_0} F(s, X_s(\cdot)) dM_s \; . \tag{5.16}$$

Proof. In this proof we will denote real processes $\bar{M}^F$ simply by $\bar{M}$ and χ will denote the following Chi-subspace $\chi := \big(\mathcal{D}_a([-\tau, 0]) \oplus L^2([-\tau, 0])\big) \hat{\otimes}_h \mathcal{D}_0([-\tau, 0])$. We need to show that for any $(\mathcal{F}_t)$-continuous local martingale N

$$\big[F(\cdot, X(\cdot)) - \bar{M}, N\big] \equiv 0. \tag{5.17}$$

Since the covariation of semimartingales coincides with the classical covariation, see Remark 1.2 item 2., it follows

$$\big[\bar{M}, N\big]_t = \int_0^t D^{\delta_0} F(s, X_s(\cdot)) d[M, N]_s \; . \tag{5.18}$$

It remains to check that, for every $t \in [0, T]$,

$$[F(\cdot, X(\cdot)), N]_t = \int_0^t D^{\delta_0} F(s, X_s(\cdot)) d[M, N]_s \; .$$

For this, for fixed $t \in [0, T]$, we will evaluate the limit in probability of

$$\int_0^t \Big(F(s + \epsilon, X_{s+\epsilon}(\cdot)) - F(s, X_s(\cdot))\Big) \frac{N_{s+\epsilon} - N_s}{\epsilon} ds \tag{5.19}$$

if it exists. (5.19) can be written as the sum of the two terms

$$I_1(t,\epsilon) = \int_0^t \Big(F(s+\epsilon, X_{s+\epsilon}(\cdot)) - F(s+\epsilon, X_s(\cdot))\Big)\frac{N_{s+\epsilon}-N_s}{\epsilon}ds\,,$$
$$I_2(t,\epsilon) = \int_0^t \Big(F(s+\epsilon, X_s(\cdot)) - F(s, X_s(\cdot))\Big)\frac{N_{s+\epsilon}-N_s}{\epsilon}ds\,.$$

First we prove that $I_1(t,\epsilon)$ converges to $\int_0^t D^{\delta_0}F(s,X_s(\cdot))d[M,N]_s$.
If $G : C([-\tau,0]) \to \mathbb{R}$ is the function $G(\eta) = \eta(0)$, then G is of class C^1 and $DG(\eta) = \delta_0$ for all $\eta \in C([-\tau,0])$ so that $DG : C([-\tau,0]) \longrightarrow \mathcal{D}_0([-\tau,0])$ is continuous. In particular it holds the equality $\eta(0) = G(\eta(\cdot)) = \langle \delta_0, \eta\rangle$. We express

$$\begin{aligned} I_1(t,\epsilon) &= \int_0^t \langle DF(s+\epsilon, X_s(\cdot)), (X_{s+\epsilon}(\cdot)-X_s(\cdot))\rangle \frac{N_{s+\epsilon}-N_s}{\epsilon}ds + R_1(t,\epsilon) \\ &= \int_0^t \langle DF(s+\epsilon, X_s(\cdot)), (X_{s+\epsilon}(\cdot)-X_s(\cdot))\rangle \frac{\langle \delta_0, N_{s+\epsilon}(\cdot)-N_s(\cdot)\rangle}{\epsilon}ds + R_1(t,\epsilon), \end{aligned} \tag{5.20}$$

and

$$\begin{aligned} R_1(t,\epsilon) &= \int_0^t \left[\int_0^1 \langle DF\big(s+\epsilon,(1-\alpha)X_s(\cdot)+\alpha X_{s+\epsilon}(\cdot)\big) - DF\big(s+\epsilon, X_s(\cdot)\big), \big(X_{s+\epsilon}(\cdot)-X_s(\cdot)\big)\rangle d\alpha\right] \times \\ &\qquad \times \frac{\langle \delta_0, N_{s+\epsilon}(\cdot)-N_s(\cdot)\rangle}{\epsilon}ds = \\ &= \int_0^t\int_0^1 \langle DF\big(s+\epsilon,(1-\alpha)X_s(\cdot)+\alpha X_{s+\epsilon}(\cdot)\big)\otimes\delta_0 - DF\big(s+\epsilon,X_s(\cdot)\big)\otimes\delta_0\,, \\ &\qquad \frac{\big(X_{s+\epsilon}(\cdot)-X_s(\cdot)\big)\otimes\big(N_{s+\epsilon}(\cdot)-N_s(\cdot)\big)}{\epsilon}\rangle d\alpha\, ds\,. \end{aligned} \tag{5.21}$$

We will show that $R_1(\cdot,\varepsilon)$ converges ucp to zero, when $\varepsilon \to 0$. Since χ is a Hilbert space, making the proper Riesz identification for $t \in [0,T], \eta_1,\eta_2 \in C([-\tau,0])$ the map $DF(t,\eta_1)\otimes DG(\eta_2)$ coincides with the tensor product $DF(t,\eta_1)\otimes\delta_0$, see Proposition 4.4. As in Example 4.5 the map $DF\otimes\delta_0 : [0,T]\times C([-\tau,0])$ takes values in the separable space $\chi^2([-\tau,0]^2)$ and it is a continuous map. In particular it takes values in χ which is a Hilbert subspace of $\chi^2([-\tau,0]^2)$.
We denote by $\mathcal{U} = \mathcal{U}(\omega)$ the closed convex hull of the compact subset $\mathcal{V}$ of $C([-\tau,0])$ defined, for every ω, by

$$\mathcal{V} = \mathcal{V}(\omega) := \{X_t(\omega);\, t\in[0,T]\}\,. \tag{5.22}$$

According to Theorem 5.35 from [1], $\mathcal{U}(\omega) = \overline{conv(\mathcal{V})(\omega)}$ is compact, so the function $DF(\cdot,\cdot)\otimes\delta_0$ on $[0,T]\times\mathcal{U}$ is uniformly continuous and we denote by $\varpi^{[0,T]\times\mathcal{U}}_{DF(\cdot,\cdot)\otimes\delta_0}$ the continuity modulus of the application $DF(\cdot,\cdot)\otimes\delta_0$ restricted to $[0,T]\times\mathcal{U}$ and by ϖ_X the continuity modulus of the continous process X. $\varpi^{[0,T]\times\mathcal{U}}_{DF(\cdot,\cdot)\otimes\delta_0}$ is, as usual, a positive, increasing function on $\mathbb{R}^+$ converging to zero when the argument converges to zero. So we have

$$\sup_{t\in[0,T]}|R_1(t,\epsilon)| \le \int_0^T \varpi^{[0,T]\times\mathcal{U}}_{DF(\cdot,\cdot)\otimes\delta_0}(\varpi_X(\epsilon))\left\|\frac{\big(X_{s+\epsilon}(\cdot)-X_s(\cdot)\big)\otimes\big(N_{s+\epsilon}(\cdot)-N_s(\cdot)\big)}{\epsilon}\right\|_\chi ds\,. \tag{5.23}$$

We recall by Corollary 3.9, item 3. that $X(\cdot)$ and $N(\cdot)$ admit a χ-covariation. In particular using condition **H1** and (5.23) the claim $R_1(\cdot,\epsilon) \xrightarrow[\epsilon\to 0]{ucp} 0$ follows.
On the other hand, the first addend in (5.20) can be rewritten as

$$\int_0^t \langle DF\big(s, X_s(\cdot)\big) \otimes \delta_0, \frac{\big(X_{s+\epsilon}(\cdot) - X_s(\cdot)\big) \otimes \big(N_{s+\epsilon}(\cdot) - N_s(\cdot)\big)}{\epsilon} \rangle ds + R_2(t,\epsilon) \tag{5.24}$$

where $R_2(\cdot,\epsilon) \xrightarrow[\epsilon\to 0]{ucp} 0$ arguing similarly as for $R_1(t,\epsilon)$.
In view of the application of Proposition 3.7, since $DF \otimes \delta_0 : [0,T] \times C([-\tau,0]) \longrightarrow \chi$ is continuous, we observe that the process $H_s = DF\big(s, X_s(\cdot)\big) \otimes \delta_0$ takes obviously values in the separable space χ which is a closed subspace of $\chi^2([-\tau,0]^2)$. Using bilinearity and Proposition 3.7, the integral in (5.24) converges then in probability to

$$\int_0^t \langle DF\big(s, X_s(\cdot)\big) \otimes \delta_0, d\widetilde{[X(\cdot), N(\cdot)]}_s \rangle \ . \tag{5.25}$$

As in Theorem 5.8, item 1., we decompose χ in the following direct sum $\mathcal{D}_{0,0} \oplus \chi_2$ where we recall that $\chi_2 = \oplus_{i=1}^N \mathcal{D}_{i,0} \oplus \big(L^2([-\tau,0])\hat{\otimes}_h \mathcal{D}_0\big)$. By Corollary 3.9 2., $X(\cdot)$ and $N(\cdot)$ admit a zero χ_2-covariation. By (5.5) in Proposition 5.5 it follows that (5.25) equals

$$\int_0^t (D^{\delta_0} F(s, X_s(\cdot)) \otimes \delta_0)(\{0,0\}) d[X,N]_s = \int_0^t D^{\delta_0} F(s, X_s(\cdot)) d[M,N]_s \ . \tag{5.26}$$

We will show now that $I_2(\cdot,\epsilon) \xrightarrow[\epsilon\to 0]{ucp} 0$.
By stochastic Fubini's theorem we obtain

$$I_2(t,\epsilon) = \int_0^t \xi(\epsilon, r) dN_r$$

where

$$\xi(\epsilon, r) = \frac{1}{\epsilon} \int_{0\vee(r-\epsilon)}^r \left[F(s+\epsilon, X_s(\cdot)) - F(s, X_s(\cdot))\right] ds \ .$$

Proposition 2.26, chapter 3 of [20] says that $I_2(\cdot,\epsilon) \xrightarrow[\epsilon\to 0]{ucp} 0$ if

$$\int_0^T \xi^2(\epsilon, r) d[N]_r \xrightarrow[\epsilon\to 0]{} 0 \tag{5.27}$$

in probability. We fix $\omega \in \Omega$ and we show that the convergence in (5.27) holds in particular a.s. We denote by $\varpi_F^{[0,T]\times\mathcal{U}}$ the continuity modulus of the application F restricted to the compact set $[0,T]\times\mathcal{U}$. For every $r \in [0,T]$ we have

$$|\xi(\epsilon, r)| \le \sup_{r\in[0,T]} |F(r+\epsilon, X_r(\cdot)) - F(r, X_r(\cdot))| \le \varpi_F^{[0,T]\times\mathcal{U}}(\epsilon)$$

which converges to zero for ϵ going to zero since function F on $[0,T]\times\mathcal{U}$ is uniformly continuous on the compact set and $\varpi_F^{[0,T]\times\mathcal{U}}$ is, as usual, a positive, increasing function on $\mathbb{R}^+$ converging to zero when the argument converges to zero. By Lebesgue's dominated convergence theorem we finally obtain (5.27). $\square$

If DF does not necessarily leave in some $\mathcal{D}_a([-T,0]) \oplus L^2([-T,0])$ space, it is still possible to express a variant of Theorem 5.12. The price to pay is a new property required for DF which will be called *support predictability property.* It is described below.

Definition 5.13. Let $0 \le a < b \le T$. A function $F : [a,b] \times C([-\tau,0]) \longrightarrow \mathbb{R}$ such that $F(t,\cdot)$ is differentiable for any $t \in [a,b]$ is said to fulfill the **support predictability property** if the following holds. For every compact K of $C([-\tau,0])$, we have

$$\int_a^b \left[\sup_{\eta \in K} \frac{1}{\epsilon} \int_{(-\epsilon)\vee(-\tau)}^0 \left| D_{dr}^{\perp} F \right| (t,\eta) \right] dt = O(\epsilon) \ , \tag{5.28}$$

where we recall that $D_{dr}^{\perp} F = D_{dr} F - DF(\{0\})\delta_0(dr)$.

Remark 5.14. Suppose that $F(t,\cdot)$ is differentiable for any $t \in [a,b]$.

1. Suppose the existence of $\rho > 0$ such that $D^{\perp} F(t,\eta)$ has support in $[-\tau,-\rho]$ for any $t \in [a,b]$, $\eta \in C([-\tau,0])$. Then F fulfills the support predictability property; in fact quantity (5.28) vanishes for ϵ small.
2. Suppose that $D^{\perp} F(t,\eta)$ is absolutely continuous for every $t \in [a,b]$. We denote $\big(D_r^{\perp} F(t,\eta), r \in [-\tau,0]\big)$, the corresponding density. If for any compact K of $C([-\tau,0])$ there is $\rho_1 > 0$ such that $t \mapsto \sup_{r\in[-\rho_1,0],\eta\in K} \left| D_r^{\perp} F(t,\eta) \right|$ belongs to $L^1([a,b])$, then F fulfills the support predictability property. This is for instance verified if $(r,t,\eta) \mapsto D_r^{\perp} F(t,\eta)$ is continuous.

As announced a variant of Theorem 5.12 is given below.

Theorem 5.15. Let $0 \le a < b \le T$ and X be an $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation and decomposition $X = M + A$, M local martingale. Let $F : [a,b] \times C([-\tau,0]) \longrightarrow \mathbb{R}$ continuous such that
i) $F(t,\cdot)$ is differentiable for every $t \in [a,b]$,
ii) $(t,\eta) \mapsto D^{\perp} F(t,\eta)$ is bounded on each compact of $[a,b] \times C([-\tau,0])$,
iii) F fulfills the support predictability property,
iv) $(t,\eta) \mapsto D^{\delta_0} F(t,\eta)$ is continuous on $]a,b] \times C([-\tau,0])$ and it admits a continuous extension on $[a,b] \times C([-\tau,0])$.
Then $F(\cdot, X_\cdot(\cdot))$ is an $(\mathcal{F}_t)$-weak Dirichlet process with martingale part

$$\tilde{M}_t^F = F(a, X_a(\cdot)) + \int_a^t D^{\delta_0} F(s, X_s(\cdot))\, dM_s \ , \quad t \in [a,b] \ . \tag{5.29}$$

Proof. Without restriction of generality we will suppose $a = 0$ and $b = T$. The proof follows from a modification of the one of Theorem 5.12. (5.19) was expressed as the sum of $I_1(t,\epsilon)$ and $I_2(t,\epsilon)$. $I_1(t,\epsilon)$ is the sum of $I_{11}(t,\epsilon)$ and $I_{12}(t,\epsilon)$ where

$$I_{11}(t,\epsilon) = \int_0^t D^{\delta_0} F(s+\epsilon, X_s(\cdot)) \frac{(X_{s+\epsilon} - X_s)(N_{s+\epsilon} - N_s)}{\epsilon} ds \, ,$$

$$I_{12}(t,\epsilon) = \int_0^t \int_0^1 \left[D^{\delta_0} F(s+\epsilon, (1-\alpha)X_s(\cdot) + \alpha X_{s+\epsilon}(\cdot)) - D^{\delta_0} F(s+\epsilon, X_s(\cdot)) \right] d\alpha \ \frac{(X_{s+\epsilon} - X_s)(N_{s+\epsilon} - N_s)}{\epsilon} ds \, ,$$

$$I_{13}(t,\epsilon) = \int_0^t \int_0^1 {}_{\mathcal{M}([-\tau,0])}\langle D^{\perp} F(s+\epsilon, (1-\alpha)X_s(\cdot) + \alpha X_{s+\epsilon}(\cdot)) \ , (X_{s+\epsilon}(\cdot) - X_s(\cdot)) \rangle_{C([-\tau,0])} d\alpha \ \frac{(N_{s+\epsilon} - N_s)}{\epsilon} ds \ .$$

We have

$$I_{11}(t,\epsilon) = J_{11}(t,\epsilon) + R_{11}(t,\epsilon)$$

where

$$J_{11}(t,\epsilon) = \int_0^t D^{\delta_0} F\left(s, X_s(\cdot)\right) \frac{\left(X_{s+\epsilon} - X_s\right)\left(N_{s+\epsilon} - N_s\right)}{\epsilon} ds$$

and $\sup_{t \le T} |R_{11}(t,\epsilon)| \xrightarrow[\epsilon \longrightarrow 0]{} 0$ in probability because $D^{\delta_0} F$ is continuous by item iv) and there uniformly continuous on each compact. In fact (X, N) have all their mutual covariations, then by Proposition 1.4 and the fact that $[X, N] = [M, N]$, clearly,

$$J_{11}(t,\epsilon) \xrightarrow[\epsilon \longrightarrow 0]{ucp} \int_0^t D^{\delta_0} F\left(s, X_s(\cdot)\right) d[M,N]_s \ .$$

$I_{12}(t,\epsilon)$ behaves similarly to $R_1(t,\epsilon)$ in (5.21), so it converges ucp to zero. Term $I_{13}(t,\epsilon)$ can be rewritten as

$$I_{13}(t,\epsilon) = \int_0^t \int_0^1 \int_{-\tau}^0 D_{dr}^{\perp} F\left(s+\epsilon, (1-\alpha)X_s(\cdot) + \alpha X_{s+\epsilon}(\cdot)\right)\left(X_{s+r+\epsilon} - X_{s+r}\right) d\alpha \ \frac{N_{s+\epsilon} - N_s}{\epsilon} ds$$

and it decomposes into $J_{13}(t,\epsilon) + R_{13}(t,\epsilon)$ where

$$J_{13}(t,\epsilon) = \int_0^t Z_s(\epsilon) \frac{N_{s+\epsilon} - N_s}{\epsilon} ds$$

with

$$Z_s(\epsilon) = \int_0^1 \int_{-\tau}^{-\epsilon} D_{dr}^{\perp} F\left(s+\epsilon, (1-\alpha)X_s(\cdot) + \alpha X_{s+\epsilon}(\cdot)\right)\left(X_{s+r+\epsilon} - X_{s+r}\right) d\alpha$$

and

$$\begin{aligned} R_{13}(t,\epsilon) &= \int_0^t \int_0^1 \int_{-\epsilon}^0 D_{dr}^{\perp} F\left(s+\epsilon, (1-\alpha)X_s(\cdot) + \alpha X_{s+\epsilon}(\cdot)\right)\left(X_{s+r+\epsilon} - X_{s+r}\right) d\alpha \ \frac{N_{s+\epsilon} - N_s}{\epsilon} ds \\ &= \int_{\epsilon}^{t+\epsilon} \int_0^1 \int_{-\epsilon}^0 D_{dr}^{\perp} F\left(s, (1-\alpha)X_{s-\epsilon}(\cdot) + \alpha X_s(\cdot)\right)\left(X_{s+r} - X_{s+r-\epsilon}\right) d\alpha \ \frac{N_s - N_{s-\epsilon}}{\epsilon} ds \end{aligned}$$

By stochastic Fubini's theorem we obtain

$$J_{13}(t,\epsilon) = \int_0^t \xi(u,\epsilon) dN_u \ , \quad \text{with} \quad \xi(u,\epsilon) = \frac{1}{\epsilon} \int_{(u-\epsilon)^+}^u Z_s(\epsilon) ds \ .$$

Proposition 2.26, chapter 3 of [20] says that $J_{13}(\cdot,\epsilon) \xrightarrow[\epsilon \to 0]{ucp} 0$ if

$$\int_0^T \xi^2(u,\epsilon) d[N]_u \xrightarrow[\epsilon \to 0]{} 0 \tag{5.30}$$

in probability. We have

$$
\begin{aligned}
|Z_s(\epsilon)| &\leq \int_{(u-\epsilon)^+}^{u} \frac{1}{\epsilon} \int_0^1 \int_{-\tau}^{-\epsilon} \left\| D_{dr}^{\perp} F\left(s+\epsilon, (1-\alpha)X_s(\cdot) + \alpha X_{s+\epsilon}(\cdot)\right) \right\| |X_{s+r+\epsilon} - X_{s+r}| \, d\alpha \; ds \\
&\leq \varpi_X(\epsilon) \int_0^1 \left\| D_{dr}^{\perp} F\left(s+\epsilon, (1-\alpha)X_s(\cdot) + \alpha X_{s+\epsilon}(\cdot)\right) \right\|_{Var([-\tau,0])} d\alpha \\
&\leq \varpi_X(\epsilon) \sup_{(t,\eta)\in[0,T]\times\mathcal{U}} \left\| D_{dr}^{\perp} F \right\|_{Var([-\tau,0])} (t,\eta)
\end{aligned}
$$

where $\mathcal{U} = \mathcal{U}(\omega) = \overline{conv(\mathcal{V})(\omega)}$ where $\mathcal{V}(\omega)$ was defined in (5.22). Previous expression is bounded because of item ii) in the assumptions and since $\mathcal{U}$ is a compact set in the infinite dimensional space $C([-\tau,0])$. So $\xi^2(u,\epsilon) \leq \varpi_X^2(\epsilon) \sup_{(t,\eta)\in[0,T]\times\mathcal{U}} \left\| D_{dr}^{\perp} F \right\|_{Var([-\tau,0])}^2$ since

$$
|\xi(u,\epsilon)| = \left| \frac{1}{\epsilon} \int_{(u-\epsilon)^+}^{u} Z_s(\epsilon) ds \right| \leq \varpi_X(\epsilon) \sup_{(t,\eta)\in[0,T]\times\mathcal{U}} \left\| D_{dr}^{\perp} F \right\|_{Var([-\tau,0])} (t,\eta) \; .
$$

Finally the left-hand side of (5.30) is bounded by

$$
\varpi_X^2(\epsilon) \sup_{(t,\eta)\in[0,T]\times\mathcal{U}} \left\| D_{dr}^{\perp} F \right\|_{Var([-\tau,0])}^2 \int_0^T d[N]_u = \varpi_X^2(\epsilon) \sup_{(t,\eta)\in[0,T]\times\mathcal{U}} \left\| D_{dr}^{\perp} F \right\|_{Var([-\tau,0])}^2 [N]_T
$$

which converges to zero a.s. for $\epsilon \longrightarrow 0$.
It remains to control $R_{13}(t,\epsilon)$. This term is bounded by

$$
\varpi_N(\epsilon) \; \varpi_X(\epsilon) \int_0^{T+\epsilon} \frac{1}{\epsilon} \left[\sup_{\eta\in\mathcal{U}(\omega)} \int_{-\epsilon}^{0} |D_{dr} F| (t,\eta) \right] dt
$$

The result follows since F fulfills the support predictability property. □

We present a slight generalization of Theorem 5.15.

Theorem 5.16. Let X be an $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation with decomposition $X = M + A$, M local martingale. Let $F : [0,T] \times C([-\tau,0]) \longrightarrow \mathbb{R}$ continuous, fulfilling assumptions i), ii) and iii) of Theorem 5.15 for $a = 0$ and $b = T$. Suppose the existence of $0 = a_0 < a_1 < \ldots < a_N = T$ such that $(t,\eta) \mapsto D^{\delta_0} F(t,\eta)$ is continuous on $]a_i, a_{i+1}] \times C([-\tau,0])$ admitting a continuous extension on $[a_i, a_{i+1}] \times C([-\tau,0])$, for $0 \leq i \leq (N-1)$.
Then $\left(F\left(t, X_t(\cdot)\right)\right)_{t\in[0,T]}$ is an $(\mathcal{F}_t)$-weak Dirichlet process with local martingale part

$$
\tilde{M}_t^F = F\left(0, X_0(\cdot)\right) + \int_0^t D^{\delta_0} F\left(s, X_s(\cdot)\right) dM_s \; . \tag{5.31}
$$

Proof. Let N be an $(\mathcal{F}_t)$-local martingale. Since

$$
\left[\tilde{M}^F, N\right]_t = \int_0^t D^{\delta_0} F\left(s, X_s(\cdot)\right) d[M,N]_s \; , \quad t \in [0,T] \; , \tag{5.32}
$$

it will be enough to show that

$$[(F(t, X_t(\cdot))), N]_t = \int_0^t D^{\delta_0} F(s, X_s(\cdot))\, d[M,N]_s \ , \quad t \in [0,T] \ . \tag{5.33}$$

We observe that $F|_{[a_i,a_{i+1}]\times C([-\tau,0])}$ verifies the assumptions of Theorem 5.15 with $a = a_i$, $b = a_{i+1}$. Consequently for $t \in]a_i, a_{i+1}]$, $i \in \{0, \ldots, (N-1)\}$, $(F(t, X_t(\cdot)))_{t\in[a_i,a_{i+1}]}$ is an $(\mathcal{F}_t)$-weak Dirichlet process with local martingale part

$$\tilde{M}_t^i = F(a_i, X_{a_i}(\cdot)) + \int_{a_i}^t D^{\delta_0} F(s, X_s(\cdot))\, dM_s \ , \quad t \in [a_i, a_{i+1}[\ .$$

(5.33) follows by summation. ☐

We discuss now some consequences related to the martingale representation.

5.3 About some martingale representation

Suppose that X is an $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation with decomposition $X = M + A$, M local martingale. Let $h \in L^1(\Omega)$. We are interested in sufficient conditions so that

$$h = h_0 + \int_0^T \xi_s dM_s \tag{5.34}$$

where (ξ_s) is an explicit predictable process, $h_0 \in \mathbb{R}$.

The two results below are a consequence respectively of Theorems 5.12 and 5.16. They settle the basis for a representation of integrable random variables. $\mathcal{D}_a \oplus L^2$ will denote here $\mathcal{D}_a([-\tau,0]) \oplus L^2([-\tau,0])$.

Proposition 5.17. Let $F : [0,T] \times C([-\tau,0]) \longrightarrow \mathbb{R}$ continuous such that $(s,\eta) \mapsto DF(s,\eta)$ exists with values in $\mathcal{D}_a \oplus L^2$ and $DF : [0,T] \times C([-\tau,0]) \longrightarrow \mathcal{D}_a \oplus L^2$ is continuous. If moreover

$$\mathbb{E}[h|\mathcal{F}_t] = F(t, X_t(\cdot)) \quad \text{a.s.} \quad \forall\, t \in [0,T[\tag{5.35}$$

then

$$h = F(0, X_0(\cdot)) + \int_0^T D^{\delta_0} F(s, X_s(\cdot)) dM_s \ . \tag{5.36}$$

Remark 5.18. We observe that $F(0, X_0(\cdot)) = \mathbb{E}[h|\mathcal{F}_0]$.

Proof. Since F verifies the assumptions of Theorem 5.12, then $F(\cdot, X_\cdot(\cdot))$ is an $(\mathcal{F}_t)$-weak Dirichlet process with martingale part given by

$$M_t^F = F(0, X_0(\cdot)) + \int_0^t D^{\delta_0} F(s, X_s(\cdot)) dM_s \ , \tag{5.37}$$

according to (5.16). By (5.35), $F(\cdot, X_\cdot(\cdot))$ is obviously an $(\mathcal{F}_t)$-martingale being a conditional expectation with respect to filtration $(\mathcal{F}_t)$. By the uniqueness of the decomposition of $(\mathcal{F}_t)$-weak Dirichlet processes, it follows

$$F(t, X_t(\cdot)) = F(0, X_0(\cdot)) + \int_0^t D^{\delta_0} F(s, X_s(\cdot)) dM_s \; .$$

In particular the $(\mathcal{F}_t)$-martingale orthogonal component is zero. Since $h = F(T, X_T(\cdot))$ and $F(0, X_0(\cdot)) = \mathbb{E}[h|\mathcal{F}_0]$ the result follows. $\square$

Corollary 5.19. Let X be an $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation with decomposition $X = M + A$, M local martingale.
Let $0 < a_1 < \ldots < a_N = T$. Let $h \in L^1(\Omega)$. Let $F : [0, T] \times C([-\tau, 0]) \longrightarrow \mathbb{R}$ verifying the following properties.
a) $F(s, \cdot)$ is differentiable $\forall \; s \in [0, T]$.
b) F fulfills the support predictability property.
c) $(s, \eta) \mapsto D^{\delta_0} F(s, \eta)$ is continuous on $]a_i, a_{i+1}]$, $0 \leq i \leq N-1$ and it admits a continuous extension on $[a_i, a_{i+1}]$.
d) $(s, \eta) \mapsto D^{\perp} F(s, \eta)$ is bounded for each compact, with respect to the total variation norm.
e) $F(s, X_s(\cdot)) = \mathbb{E}[h|\mathcal{F}_s]$.
Then

$$h = F(0, X_0(\cdot)) + \int_0^T D^{\delta_0} F(s, X_s(\cdot)) \, dM_s \; . \tag{5.38}$$

Proof. By Theorem 5.16 $(F((t, X_t(\cdot)))_{t \in [0,T]}$ is an $(\mathcal{F}_t)$-weak Dirichlet process. On the other hand by construction $(F((t, X_t(\cdot)))_{t \in [0,T]}$ is an $(\mathcal{F}_t)$-martingale. The result follows again by uniqueness of the decomposition of an $(\mathcal{F}_t)$-weak Dirichlet process. $\square$

Remark 5.20. Suppose $h \in L^2(\Omega)$. Indeed previous corollary can be stated requiring the same assumptions a), b), c), d), e) but on F restricted to $[0, b] \times C([-\tau, 0])$ for every $b \in]a_{N-1}, T[$. In fact, the square integrable martingale $\mathbb{E}[h|\mathcal{F}_t]$ can be decomposed according to Kunita-Watanabe's theorem, it can be decomposed into the sum

$$F(0, X_0(\cdot)) + \int_0^t \xi_s dM_s + N_t \; , \quad t \in [0, T]$$

where N is an $(\mathcal{F}_t)$-local martingale strongly orthogonal to M, i.e. $[N, M] = 0$; moreover we know that $\int_0^T \xi_s^2 d[M]_s < +\infty$ a.s. Since $(F(t, X_t(\cdot)))_{t \in [0,b]}$ is also an $(\mathcal{F}_t)$-weak Dirichlet process by Theorem 5.16, the uniqueness of the decomposition implies that

$$\int_0^t D^{\delta_0} F(s, X_s(\cdot)) \, dM_s = \int_0^t \xi_s dM_s + N_t \; , \quad t \in [0, b]$$

and so $N_s = 0, s \in [0, b]$ and $\xi_s \equiv D^{\delta_0} F(s, X_s(\cdot))$. This implies (5.38).

6 Consequences on quasi explicit representations of path dependent random variables

6.1 General considerations

This section has illustrative features. We test our method on the representation of a random variable which depends on the path of a diffusion process. It will be a toy model for future investigations with applications in verification theorems in control theory on functional dependent equation. Let $(W_t)_{t\in[0,T]}$ be a standard Brownian motion with respect to some usual filtration $(\mathcal{F}_t)$. In this section η (resp. γ) will denote an element in $C([-T,0])$ (resp. $C([0,T])$). Let $(X_t)_{t\in[0,T]}$ be a real continuous process; with X we will also denote the whole trajectory of X in $C([0,T])$. a will stand for a grid $0 = a_0 < a_1 < \ldots < a_N = T$, not anymore in $[-T,0]$ as before. In this section τ will be equal to T.

We aim at implementing Corollary 5.19 when X is solution of a SDE of the type

$$X_t = X_0 + \int_0^t \sigma(r, X_r)dW_r + \int_0^t b(r, X_r)dr \ , \tag{6.1}$$

where $\sigma, b : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous, $\partial_x \sigma$ and $\partial_x b$ exist and they are bounded. We remark that σ is possibly degenerate. With $\partial_x \sigma$ (resp. $\partial_x b$) we will denote the derivative of σ (resp. b) with respect to the second argument. In the present context X is an $(\mathcal{F}_t)$-semimartingale, therefore an $(\mathcal{F}_t)$-weak Dirichlet, with decomposition $X = M + A$, M the local martingale given by $M_t = X_0 + \int_0^t \sigma(r, X_r)dW_r$.
The idea is to evaluate

$$h := \Phi(X)$$

with $\Phi : C([0,T]) \longrightarrow \mathbb{R}$ such that $h \in L^1(\Omega)$.
It is well known that the following flow property holds:

$$X_t = Y_t^{s, X_s(\cdot)} \tag{6.2}$$

where for $(s,\eta) \in [0,T] \times C([-\tau,0])$, where $Y^{s,\eta}$ is an element in $C([0,T])$, in fact $Y_t = Y_t^{s,\eta}$, $t \in [0,T]$ is defined by

$$Y_t = \begin{cases} \eta(t-s) & t \in [0,s] \\ \eta(0) + \int_s^t \sigma(r, Y_r)dW_r + \int_s^t b\,(r, Y_r)\,dr & t \in [s,T] \ . \end{cases} \tag{6.3}$$

$Y^{s,\eta}$ also symbolizes a function in $C([0,T])$. We start with some basic estimates.

Proposition 6.1. For every $q \geq 1$, there is a constant $C(q)$ such that

$$\sup_{s\in[0,T]} \mathbb{E}\left[\|Y^{s,\eta}\|_\infty^q\right] \leq C(q)\,\|\eta\|_\infty^q$$

Proof. Since σ, b have linear growth, by Burkholder-Davis-Gundy inequality, for every $q \geq 1$, and by Gronwall lemma, there is a constant $C(q)$ such that

$$\sup_{0\leq s\leq t\leq T} \mathbb{E}\left[|Y_t^{s,\eta}|^q\right] \leq C(q) \ \ |\eta(0)|^q \ .$$

This implies the result. □

Process $Y = Y^{s,\eta}$ given in (6.3) also solves

$$Y_t = \eta(0) + \int_s^t \sigma\left(r, Y_r^{s,\eta}\right) d\bar{W}_r + \int_s^t b\left(r, Y_r^{s,\eta}\right) dr$$

where $\bar{W} = W_{s+\cdot} - W_s$ is independent of $\mathcal{F}_t = \sigma\left(Y_r, r \le t\right)$. Suppose now Φ being continuous. Taking into account (6.2), for $s \le t$, we get

$$M_s := \mathbb{E}\left[\Phi(X) | \mathcal{F}_s\right] = u\left(s, X_s(\cdot)\right) \tag{6.4}$$

where $u : [0,T] \times C([-T,0]) \longrightarrow \mathbb{R}$

$$u\left(s, \eta\right) = \mathbb{E}\left[\Phi(Y^{s,\eta})\right] \tag{6.5}$$

and $Y^{s,\eta}$ is given by (6.3)

Remark 6.2.
1. We remark that $Y^{s,\eta} : \Omega \longrightarrow C([0,T])$.
2. It is possible to show that $u \in C^0\left([0,T] \times C([-T,0])\right)$.

We would like to examine situations where $F := u$ fulfills the conditions for the application of Corollary 5.19. In particular Theorem 6.4 below verifies condition a) of that corollary. The other assumptions will be the object of Proposition 6.11. In this section the natural candidate for function u is given in (6.5). First we introduce a notation and we recall that $D^{\delta_0} u(t,\eta) = Du(t,\eta)\{0\}$ and $D_{dr}^{\perp} u(t,\eta)$ is singular with respect to δ_0.

Notation 6.3. If X is a continuous real $(\mathcal{F}_t)$-semimartingale, symbol $\mathcal{E}\left(X\right)$ denotes the Doléans exponential of X, in particular $\mathcal{E}\left(X\right)_t = \exp\left\{X_t - X_0 - \frac{1}{2}[X]_t\right\}$.

Theorem 6.4. Let X be a diffusion process of type (6.1) and $\Phi : C([0,T]) \longrightarrow \mathbb{R}$ of class C^1 Fréchet such that $D\Phi$ has polynomial growth. For $s \in [0,T]$ and $\eta \in C([-T,0])$, we set $u : [0,T] \times C([-T,0]) \longrightarrow \mathbb{R}$ by $u(s,\eta) = \mathbb{E}\left[\Phi\left(Y^{s,\eta}\right)\right]$ where $Y^{s,\eta}$ belongs to $C([0,T])$ and it is defined by (6.3).
Then $u \in C^0([0,T] \times C([-T,0]))$. Moreover for every $s \in [0,T]$, $u(s,\cdot)$ belongs to $C^1\left(C([-T,0])\right)$ and $Du(s,\eta)$, with $s \in [0,T]$ and $\eta \in C([-T,0])$, is characterized by

$$D_{dr} u(s,\eta) = D_{dr}^{\perp} u(s,\eta) + D^{\delta_0} u(s,\eta) \delta_0(dr) \tag{6.6}$$

with

$$\begin{aligned} D_{dr}^{\perp} u(s,\eta) &= \mathbb{E}\left[D_{s+dr}\Phi\left(Y^{s,\eta}\right) 1_{[-s,0[}(r)\right] \quad \text{and} \\ D^{\delta_0} u(s,\eta) &= \mathbb{E}\left[\int_{[s,T]} D_{d\rho}\Phi\left(Y^{s,\eta}\right)\ \mathcal{E}\left\{\int_s^\rho \partial_x \sigma\left(\xi, Y_\xi^{s,\eta}\right) dW_\xi + \int_s^\rho \partial_x b\left(\xi, Y_\xi^{s,\eta}\right) d\xi\right\}\right]. \end{aligned} \tag{6.7}$$

In particular item a) of Corollary 5.19 is verified.

Proof. We recall that $Y : [0,T] \times C([-T,0]) \times \Omega \longrightarrow C([0,T])$ is a.s. continuous. We suppose for a moment that $\partial_x \sigma$ and $\partial_x b$ are Hölder continuous. In this case it is possible to show that $\eta \mapsto Y^{s,\eta}$ is of class $C^1\left(C([-T,0])\,; C([0,T])\right)$ a.s. Let $Y : [0,T] \times C([-T,0]) \times \Omega \longrightarrow C([0,T])$, $(s,\eta,\omega) \mapsto \left(Y_t^{s,\eta}(\omega)\right)_{t\in[0,T]}$; then the first order Fréchet derivative with respect to the second argument η will be $DY : [0,T] \times C([-T,0]) \times \Omega \longrightarrow \mathcal{L}\left(C([-T,0]), C([0,T])\right)$, i.e. $DY^{s,\eta} : C([-T,0]) \longrightarrow C([0,T])$ is a linear functional.

Remark 6.5. If we fix $t \in [0,T]$ ω-a.s. then it holds $Y_t : [0,T] \times C([-T,0]) \times \Omega \longrightarrow \mathbb{R}$; in this case the first order Fréchet derivative with respect to the second argument η will be $DY_t : [0,T] \times C([-T,0]) \times \Omega \longrightarrow (C([-T,0]))^* = \mathcal{M}([-T,0])$. In particular if $f \in C([-T,0])$,

$$_{\mathcal{M}([-T,0])}\langle DY_t^{s,\eta}(\omega)\ ,\ f\rangle_{C([-T,0])} = \int_{[-T,0]} f(r) D_{dr} Y_t^{s,\eta}(\omega) \qquad \square$$

We go on with the proof of Theorem 6.4. Computing the derivative $DY_t^{s,\eta}$ it gives

$$D_{dr}Y_t^{s,\eta} = \begin{cases} \delta_{t-s}(dr) & t \le s \\ \delta_0(dr) + \int_s^t \partial_x \sigma\left(\xi, Y_\xi^{s,\eta}\right) D_{dr}Y_\xi^{s,\eta} dW_\xi + \int_s^t \partial_x b\left(\xi, Y_\xi^{s,\eta}\right) D_{dr} Y_\xi^{s,\eta} d\xi & t \ge s\ . \end{cases} \tag{6.8}$$

Consequently, for $t \ge s$, it follows

$$D_{dr}Y_t^{s,\eta} = \begin{cases} \delta_{t-s}(dr) = \delta_{dr}(t-s) & t \le s \\ \\ \delta_0(dr)\mathcal{E}\left\{\int_s^t \partial_x\sigma\left(\xi, Y_\xi^{s,\eta}\right) dW_\xi + \int_s^t \partial_x b\left(\xi, Y_\xi^{s,\eta}\right) d\xi\right\} & t \ge s\ . \end{cases} \tag{6.9}$$

We remind from Notation 6.3 that

$$\mathcal{E}\left\{\int_s^t \partial_x\sigma\left(\xi, Y_\xi^{s,\eta}\right) dW_\xi + \int_s^t \partial_x b\left(\xi, Y_\xi^{s,\eta}\right) d\xi\right\}$$

equals

$$\exp\left\{\int_s^t \partial_x\sigma\left(\xi, Y_\xi^{s,\eta}\right) dW_\xi - \frac{1}{2}\int_s^t (\partial_x\sigma)^2\left(\xi, Y_\xi^{s,\eta}\right) d\xi + \int_s^t \partial_x b\left(\xi, Y_\xi^{s,\eta}\right) d\xi\right\}\ .$$

Moreover, by usual integration theory for every $t \in [0,T]$, $u(s,\cdot)$ is of class $C^1\left(C([-T,0])\right)$ and

$$D_{dr}u\left(s,\eta\right) = \mathbb{E}\left[\int_{[0,T]} D_{d\rho}\Phi\left(Y^{s,\eta}\right) D_{dr}Y_\rho\right] \tag{6.10}$$

Taking into account (6.9), by composition, one obtains a precise evaluation which can be done again via the usual integration results. Omitting the details we have

$$\begin{aligned} D_{dr}u(s,\eta) &= \mathbb{E}\left[\int_{[0,s[} D_{d\rho}\Phi\left(Y^{s,\eta}\right)\ D_{dr}Y_\rho^{s,\eta}\right] + \mathbb{E}\left[\int_{[s,T]} D_{d\rho}\Phi\left(Y^{s,\eta}\right)\ D_{dr}Y_\rho^{s,\eta}\right] \\ &= \mathbb{E}\left[\int_{[0,s[} D_{d\rho}\Phi\left(Y^{s,\eta}\right)\ \delta_{\rho-s}(dr)\right] \\ &+ \mathbb{E}\left[\int_{[s,T]} D_{d\rho}\Phi\left(Y^{s,\eta}\right)\ \delta_0(dr)\mathcal{E}\left\{\int_s^\rho \partial_x\sigma\left(\xi, Y_\xi^{s,\eta}\right) dW_\xi + \int_s^\rho \partial_x b\left(\xi, Y_\xi^{s,\eta}\right) d\xi\right\}\right] \\ &= \mathbb{E}\left[D_{s+dr}\Phi\left(Y^{s,\eta}\right)\right] 1_{[-s,0[}(r) + \\ &+ \delta_0(dr)\mathbb{E}\left[\int_{[s,T]} D_{d\rho}\Phi\left(Y^{s,\eta}\right)\ \mathcal{E}\left\{\int_s^\rho \partial_x\sigma\left(\xi, Y_\xi^{s,\eta}\right) dW_\xi + \int_s^\rho \partial_x b\left(\xi, Y_\xi^{s,\eta}\right) d\xi\right\}\right]\ . \end{aligned} \tag{6.11}$$

Finally we obtain (6.6) and (6.7).
If $\partial_x\sigma$ and $\partial_x b$ are not Hölder continuous, Y will not be a.s. differentiable in η, but only in $L^2(\Omega)$, i.e. in quadratic mean. However the two expressions in (6.7) still remain valid. We omit the details. $\square$

Remark 6.6. In fact, the regularity assumption on σ and $b : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ can be partly reduced, see for instance using the technique developed in [14]. The expression $DY^{s,\eta}$ is the same as in the one of (6.9) under Hypotheses given in Section 1.2 and 6 of [14].

6.2 Some representations

In this section we discuss a basis example with some particular cases of Φ and we express the consequences on u appearing in Theorem 6.4.

∇ will denote the derivative with respect to the argument in $L^2([-T,0])$. Symbol $D^{ac}_\rho \Phi(\gamma)$ denotes the absolute continuous component of the first order Fréchet derivative of Φ and in all cases it coincides with the partial derivative of f in L^2 with respect to the last argument which will be denoted in the sequel by $\nabla_\rho f(\gamma(a_1), \ldots, \gamma(a_N), \gamma)$, $\rho \in [0,T]$.

Example 6.7. Let $a_0 = 0 < a_1 < \ldots < a_N = T$.

Let $\Phi(\gamma) = f(\gamma(a_1), \ldots, \gamma(a_N), \gamma)$ with $f : \mathbb{R}^N \times L^2([0,T]) \longrightarrow \mathbb{R}$, $f \in C^1\left(\mathbb{R}^N \times L^2([0,T]) \longrightarrow \mathbb{R}\right)$ such that all the derivatives have polynomial growth. In this case $D\Phi$ has the following particular form

$$D_{dr}\Phi(\gamma) = \sum_{i=1}^N D^{\delta_{a_i}}\Phi(\gamma)\,\delta_{a_i}(dr) + (D^{ac}_r\Phi)(\gamma)dr\ , \text{ with } \begin{cases} D^{\delta_{a_i}}\Phi(\gamma) = \partial_i f(\gamma(a_1), \ldots, \gamma(a_N), \gamma) \\ (D^{ac}_r\Phi)(\gamma)dr = (\nabla_r f)(\gamma(a_1), \ldots, \gamma(a_N), \gamma)\, dr \end{cases} \tag{6.12}$$

where the $D^{ac}_r u(t,\eta)$ denotes the absolute continuous part of the measure $D\Phi$. In particular $D\Phi \in \mathcal{D}_a([0,T]) \oplus L^2([0,T])$.

By (6.6) and (6.7) in Theorem 6.4, it yields

$$\begin{aligned}
D^{\delta_0}u(s,\eta) &= \mathbb{E}\left[\sum_{a_i \geq s} D^{\delta_{a_i}}\Phi\left(Y^{s,\eta}\right)\mathcal{E}\left\{\int_s^{a_i} \partial_x\sigma\left(\xi, Y^{s,\eta}_\xi\right) dW_\xi + \int_s^{a_i} \partial_x b\left(\xi, Y^{s,\eta}_\xi\right) d\xi\right\}\right] \\
&\quad + \mathbb{E}\left[\int_s^T \nabla_\rho\Phi\left(Y^{s,\eta}\right)\mathcal{E}\left\{\int_s^\rho \partial_x\sigma\left(\xi, Y^{s,\eta}_\xi\right) dW_\xi + \int_s^\rho \partial_x b\left(\xi, Y^{s,\eta}_\xi\right) d\xi\right\}\ d\rho\right] \\
&= \mathbb{E}\Bigg[\sum_{a_i \geq s} \partial_i f\left(\eta(a_1 - s), \ldots, \eta(a_{i-1} - s), Y^{s,\eta}_{a_i}, \ldots, Y^{s,\eta}_{a_N}, Y^{s,\eta}\right) \\
&\qquad\qquad \mathcal{E}\left\{\int_s^{a_i} \partial_x\sigma\left(\xi, Y^{s,\eta}_\xi\right) dW_\xi + \int_s^{a_i} \partial_x b\left(\xi, Y^{s,\eta}_\xi\right) d\xi\right\}\Bigg] \\
&\quad + \mathbb{E}\Bigg[\int_s^T \nabla_\rho f\left(\eta(a_1 - s), \ldots, \eta(a_{i-1} - s), Y^{s,\eta}_{a_i}, \ldots, Y^{s,\eta}_{a_N}, Y^{s,\eta}\right) \\
&\qquad\qquad \mathcal{E}\left\{\int_s^\rho \partial_x\sigma\left(\xi, Y^{s,\eta}_\xi\right) dW_\xi + \int_s^\rho \partial_x b\left(\xi, Y^{s,\eta}_\xi\right) d\xi\right\} d\rho\Bigg]\ .
\end{aligned} \tag{6.13}$$

and

$$D^{\perp}_{dr}u(s,\eta) = \mathbb{E}\left[\sum_{a_i<s} D^{\delta_{a_i}}\Phi\left(Y^{s,\eta}\right)\delta_{a_i-s}(dr)\right] + \mathbb{E}\left[\int_0^s \nabla_\rho\Phi\left(Y^{s,\eta}\right)\delta_{\rho-s}(dr)\ d\rho\right]$$

$$= \mathbb{E}\left[\sum_{a_i<s} \partial_i f\left(\eta(a_1-s),\ldots,\eta(a_{i-1}-s),Y^{s,\eta}_{a_i},\ldots,Y^{s,\eta}_{a_N},Y^{s,\eta}\right)\delta_{a_i-s}(dr)\right]$$

$$+ \mathbb{E}\left[1_{[-s,0]}(r)\nabla_{r+s}f\left(\eta(a_1-s),\ldots,\eta(a_{i-1}-s),Y^{s,\eta}_{a_i},\ldots,Y^{s,\eta}_{a_N},Y^{s,\eta}\right)dr\right] \tag{6.14}$$

where ∇f denotes the Fréchet derivative of f with respect to the last argument which is absolutely continuous. In fact $\nabla_\cdot f$ is a function in $L^2([0,T])$ (for any argument of f) defined by $r \mapsto \nabla_r f(\cdot)$.

Remark 6.8. Let $j \in \{1,\ldots,N\}$.

1. If $s \in]a_{j-1},a_j]$ we have

$$D^{\delta_0}u(s,\eta) = \mathbb{E}\left[\sum_{i=j}^N D^{\delta_{a_i}}\Phi\left(Y^{s,\eta}\right)\mathcal{E}\left\{\int_s^{a_i}\partial_x\sigma\left(\xi,Y^{s,\eta}_\xi\right)dW_\xi + \int_s^{a_i}\partial_x b\left(\xi,Y^{s,\eta}_\xi\right)d\xi\right\}\right]$$

$$+ \mathbb{E}\left[\int_s^T \nabla_\rho\Phi\left(Y^{s,\eta}\right)\mathcal{E}\left\{\int_s^\rho\partial_x\sigma\left(\xi,Y^{s,\eta}_\xi\right)dW_\xi + \int_s^\rho\partial_x b\left(\xi,Y^{s,\eta}_\xi\right)d\xi\right\}\ d\rho\right] \tag{6.15}$$

and

$$D^{\perp}_{dr}u(s,\eta) = \mathbb{E}\left[\sum_{i=1}^{j-1} D^{\delta_{a_i}}\Phi\left(Y^{s,\eta}\right)\delta_{a_i-s}(dr)\right]$$

$$+ \mathbb{E}\left[\int_0^s \nabla_\rho\Phi\left(Y^{s,\eta}\right)\delta_{\rho-s}(dr)\ d\rho\right]\ . \tag{6.16}$$

In particular for fixed $\eta \in C([-T,0])$, $s \mapsto D^{\delta_0}u(s,\eta)$ is continuous on $]a_{j-1},a_j[$, is left-continuous and it admits a continuous extension to $[a_{j-1},a_j]$.

2. If $s \in [0,a_1]$, the (6.15) holds with $j=1$ and $s \mapsto D^{\delta_0}u(s,\eta)$ is continuous on $[0,a_1]$.
3. We remark that u is not necessarily of class $C^{0,1}\left([0,T]\times C([-T,0])\right)$ excepted if $N=1$.

The following is a particular case of Example 6.7 where Φ only depends on the maturity and on the whole trajectory in L^2.

Example 6.9. Suppose that $\Phi(\gamma) = f(\gamma(T),\gamma)$ for $f : \mathbb{R}\times L^2([-T,0]) \longrightarrow \mathbb{R}$ of class C^1 with polynomial growth derivatives. (6.13) and (6.14) give

$$D^{\delta_0}u(s,\eta) = \mathbb{E}\left[\partial_1 f\left(Y^{s,\eta}_T,Y^{s,\eta}\right)\mathcal{E}\left\{\int_s^T\partial_x\sigma\left(\xi,Y^{s,\eta}_\xi\right)dW_\xi + \int_s^T\partial_x b\left(\xi,Y^{s,\eta}_\xi\right)d\xi\right\}\right]$$

$$+ \mathbb{E}\left[\int_s^T \nabla_\rho f\left(Y^{s,\eta}_T,Y^{s,\eta}\right)\mathcal{E}\left\{\int_s^\rho\partial_x\sigma\left(\xi,Y^{s,\eta}_\xi\right)dW_\xi + \int_s^\rho\partial_x b\left(\xi,Y^{s,\eta}_\xi\right)d\xi\right\}d\rho\right]\ .$$

$$D^{\perp}_{dr}u(s,\eta) = \mathbb{E}\left[1_{[-s,0]}(r)\nabla_{r+s}f\left(Y^{s,\eta}_T,Y^{s,\eta}\right)dr\right]\ .$$

In this case u is of class C^1 for every $(s,\eta) \in [0,T]\times C([-T,0])$, $Du(s,\eta) \in \mathcal{D}_0 \oplus L^2([-T,0])$ and $Du : [0,T]\times C([-T,0]) \longrightarrow \mathcal{D}_0 \oplus L^2([-T,0])$ is continuous.

The following is a particular case of Example 6.7 where Φ only depends pointwise on a finite number of points; in this case $\nabla_r f \equiv 0$.

Example 6.10. Let $\Phi(\gamma) = f(\gamma(a_1), \ldots, \gamma(a_N))$ with $f : \mathbb{R}^N \longrightarrow \mathbb{R}$. In this case (6.12) reduces to $D_{dr}\Phi(\gamma) = \sum_{i=1}^N \partial_i f(\gamma(a_1), \ldots, \gamma(a_N))\, \delta_{a_i}(dr)$. (6.13) reduces to

$$D^{\delta_0} u(s,\eta) = \mathbb{E}\left[\sum_{a_i \geq s (i \geq 1)} \partial_i f\left(Y^{s,\eta}_{a_1}, \ldots, Y^{s,\eta}_{a_N}\right) \mathcal{E}\left\{\int_s^{a_i} \partial_x \sigma\left(\xi, Y^{s,\eta}_\xi\right) dW_\xi + \int_s^{a_i} \partial_x b\left(\xi, Y^{s,\eta}_\xi\right) d\xi\right\}\right] . \tag{6.17}$$

We recall that, if $s \in [a_{N-1}, a_N[$ then $s \geq a_i$ for all $i = 1, \ldots, (N-1)$ and $Y^{s,\eta}_{a_i} = \eta(a_i - s)$, by definition (6.3).

The result below is a fundamental step for obtaining a quasi-explicit representation of integrable random variables. In fact we verify the validity of Corollary 5.19.

Proposition 6.11. Let $\Phi : C([0,T]) \longrightarrow \mathbb{R}$ of the type

$$\Phi(\gamma) = f(\gamma(a_1), \ldots, \gamma(a_N), \gamma)$$

with $f : \mathbb{R}^N \times L^2([0,T]) \longrightarrow \mathbb{R}$ of class C^1 with partial derivatives having polynomial growth. Let $\Phi : C([0,T]) \longrightarrow \mathbb{R}$ defined by $\Phi(\gamma) = f(\gamma(a_1), \ldots, \gamma(a_N), \gamma)$. We set $u(s,\eta) = \mathbb{E}\left[\Phi\left(Y^{s,\eta}\right)\right]$ for $s \in [0,T]$ and $\eta \in C([-T,0])$. Then

$$h = u(0, X_0(\cdot)) + \int_0^T D^{\delta_0} u(s, X_s(\cdot))\, \sigma(s, X_s)\, dW_s\ , \tag{6.18}$$

where $D^{\delta_0} u(s,\eta)$ is given in (6.13).

Proof. We verify the assumptions of Corollary 5.19. u is of course continuous. e) is fulfilled by construction. b) was the object of Theorem 6.4. In Example 6.7 we have given the expression of $Du(s,\cdot)$ for any $s \in [0,T]$. (6.13) and (6.14) give the explicit expression respectively for $D^{\delta_0} u(s,\eta)$ and $D^{\perp}_{dr} u(s,\eta)$. c) follows from those explicit expressions. d) follows by the fact that $(t,\eta) \mapsto D^{\perp} u(t,\eta)$ is bounded on each compact of $[a_i, a_{i+1}] \times C([-\tau, 0])$, $1 \leq i \leq N-1$. It remains to check a), i.e. if u fulfills the support predictability property. Since

$$(r,s,\eta) \mapsto \mathbb{E}\left[1_{[-s,0]}(r) \nabla_{r+s} f\left(Y^{s,\eta}_{a_1}, \ldots, \ldots, Y^{s,\eta}_{a_N}, Y^{s,\eta}\right)\right]$$

is bounded on each compact, in order to verify the support predictability condition we only need to show that for each compact $K \subset C([-T,0])$,

$$\int_0^T \left[\sup_{\eta \in K} \frac{1}{\epsilon} \int_{-\epsilon}^0 g(dr, s, \eta)\right] ds = O(\epsilon) \tag{6.19}$$

where

$$g(dr, s, \eta) = \sum_{a_j < s} \mathbb{E}\left[\partial_j f\left(Y^{s,\eta}_{a_1}, \ldots, \ldots, Y^{s,\eta}_{a_N}, Y^{s,\eta}\right)\right] \delta_{a_j - s}(dr)\ .$$

For $\epsilon < \min_{j\in\{1,\dots,N\}}\{a_j - a_{j-1}\}$, the left-hand side of (6.19) gives

$$\sum_{i=0}^{N-1}\int_{a_i}^{a_{i+1}}\left[\sup_{\eta\in K}\frac{1}{\epsilon}\int_{-\epsilon}^{0}\sum_{a_j<s}\mathbb{E}\left[\partial_j f\left(Y^{s,\eta}_{a_1},\dots,\dots,Y^{s,\eta}_{a_N},Y^{s,\eta}\right)\right]\delta_{a_j-s}(dr)\right]ds$$

$$= \sum_{i=0}^{N-1}\int_{a_i}^{a_{i+1}}\left[\sup_{\eta\in K}\frac{1}{\epsilon}\sum_{a_j<s}\mathbb{E}\left[\partial_j f\left(Y^{s,\eta}_{a_1},\dots,\dots,Y^{s,\eta}_{a_N},Y^{s,\eta}\right)\right]\int_{-\epsilon}^{0}\delta_{a_j-s}(dr)\right]ds. \tag{6.20}$$

We remark that

$$\int_{-\epsilon}^{0}\delta_{a_j-s}(dr) = \begin{cases} 1 & \text{if } (a_j - s)\in[-\epsilon,0] \Leftrightarrow s\in[a_j,a_j+\epsilon] \\ 0 & \text{otherwise}\end{cases} . \tag{6.21}$$

So in the second sum of (6.20), for $a_j < s$ only remains the term $a_j = a_i$ and (6.20) gives

$$\sum_{i=0}^{N-1}\int_{a_i}^{a_i+\epsilon}\left[\sup_{\eta\in K}\frac{1}{\epsilon}\mathbb{E}\left[\partial_i f\left(Y^{s,\eta}_{a_1},\dots,\dots,Y^{s,\eta}_{a_N},Y^{s,\eta}\right)\right]\right]ds\ .$$

Since all the derivatives of f have polynomial growth, previous expression is bounded by

$$C(N,f,T)\sup_{\eta\in K,s\in[0,T]}\{1+\mathbb{E}\left[\|Y^{s,\eta}\|^q_\infty\right]\}$$

where $C(N,f,T)$ is some constant and q is some positive real. The conclusion follows by Proposition 6.1. So u fulfills the support predictability condition. □

6.3 About the representation of non-smooth random variables

The next example is essentially illustrative. It will be developed and treated in a more general context in a paper in preparation. It will be possible to represent, still using Corollary 5.19, random variables of the type

$$h := \Phi(X)\ , \quad \text{where} \quad \Phi(\gamma) = f\left(\int_0^T\varphi_1(r)d\gamma(r),\dots,\int_0^T\varphi_N(r)d\gamma(r)\right)\ ,$$

f continuous with polynomial growth and $\varphi_1,\dots,\varphi_N$ of class $C^1([0,T])$. $\int_0^T\varphi_i(r)d\gamma(r)$ are naturally defined by an integration by parts as $\varphi(T)\gamma(T) - \varphi(0)\gamma(0) - \int_0^T\gamma(r)d\varphi(r) = \varphi(T)\gamma(T) - \int_0^T\gamma(r)d\varphi(r)$. More general formulations can be performed even with less regularity using specific approximation techniques. Here we only aim at obtaining a representation directly, without approximations. For simplicity we consider X to be a diffusion process of type (6.1) with $\sigma\equiv 1$ and $b\equiv 0$, so that X is a Brownian motion W and $Y^{s,\eta}$ introduced in (6.3) will be a Brownian flow.
In this illustrative subsection we only suppose $N=1$. Let $h = f\left(\int_0^T\varphi(r)dW_r\right)$, $\varphi\in C^1([0,T])$, $\varphi(T)\neq 0$, $f:\mathbb{R}\longrightarrow\mathbb{R}$ measurable with polynomial growth. We define $u(s,\eta) = \mathbb{E}\left[\Phi(Y^{s,\eta})\right]$ as in (6.5). It is possible to show the assumptions of Theorem 5.15 setting $a=0$ and $b<T$. We suppose for a moment that $f\in C^1_b(\mathbb{R})$. We come back to Theorem 6.4 which says that that for every $s\in[0,T]$, $u(s,\cdot)$ is of class C^1 Fréchet and

$$D_{dr}u(s,\eta) = D^{\delta_0}u(s,\eta)\delta_0(dr) + D^{\perp}_{dr}u(s,\eta)\ .$$

Since

$$D_{dr}\Phi(\gamma) = f'\left(\int_0^T \varphi(r)d\gamma(r)\right)\left(\varphi(T)\delta_T(dr) - \varphi(0)\delta_0(dr) - \dot{\varphi}(r)1_{[0,T]}(r)dr\right) \ . \tag{6.22}$$

Taking into account (6.7) we get

$$\begin{aligned} D^{\delta_0}u(s,\eta) &= \mathbb{E}\left[Z_\varphi\right]\left[\varphi(s) - \varphi(0)1_{\{0\}}(s)\right] \\ D^{\perp}_{dr}u(s,\eta) &= \mathbb{E}\left[Z_\varphi\right]\left[\left(-\varphi(0)\delta_{-s}(dr) - \dot{\varphi}(s+y)\right)1_{[-s,0[}(y)dy\right] \ , \end{aligned} \tag{6.23}$$

where

$$Z_\varphi = f'\left(\int_0^T \varphi(r)dY^{s,\eta}(r)\right) \ . \tag{6.24}$$

Using Malliavin calculus and the fact that

$$Z_\varphi = f'\left(\eta(0)\varphi(s) - \varphi(0)\eta(-s) - \int_0^s \eta(r-s)d\varphi(r) + \int_s^T \varphi(r)dW_r\right) \ .$$

We set the random variable $h = \Phi(Y^{s,\eta})$. Denoting D^m the Malliavin derivative we obtain

$$D^m_r(h) = D^m_r\left(\Phi(Y^{s,\eta})\right) = Z_\varphi \varphi(r)1_{]s,T]}(r) \ , \tag{6.25}$$

then

$$\langle D^m(h), \varphi\rangle_{L^2([0,T])} = \int_0^T D^m_r(h)\varphi(r)dr = Z_\varphi \int_s^T \varphi^2(r)dr \tag{6.26}$$

and so

$$\mathbb{E}\left[Z_\varphi\right] = \frac{1}{\int_s^T \varphi^2(r)dr}\mathbb{E}\left[\langle D^m(h), \varphi\rangle\right]. \tag{6.27}$$

Consequently, using (6.27) in (6.23) we obtain

$$D^{\delta_0}u(s,\eta) = \frac{\varphi(s) - \varphi(0)1_{\{0\}}(s)}{\int_s^T \varphi^2(r)dr}\mathbb{E}\left[\langle D^m(h), \varphi\rangle\right] \tag{6.28}$$

and

$$D^{\perp}_{dy}u(s,\eta) = \frac{\left(-\varphi(0)\delta_{-s}(dr) - \dot{\varphi}(s+y)\right)1_{[-s,0[}(y)dy}{\int_s^T \varphi^2(r)dr}\mathbb{E}\left[\langle D^m(h), \varphi\rangle\right] \ . \tag{6.29}$$

By the well-known integration by parts on the Wiener space, it follows

$$\mathbb{E}\left[\langle D^m(h), \varphi\rangle\right] = \mathbb{E}\left[h \cdot \delta(\varphi)\right] = \mathbb{E}\left[\Phi(Y^{s,\eta}) \cdot \int_0^T \varphi(r)dW_r\right] \ . \tag{6.30}$$

Consequently, for $s \in [0, T[$

$$D^{\delta_0} u(s, \eta) = \frac{\varphi(s) - \varphi(0)\mathbb{1}_{\{0\}}(s)}{\int_s^T \varphi^2(r)dr} \mathbb{E}\left[\Phi(Y^{s,\eta}) \cdot \int_0^T \varphi(r)dW_r\right] \tag{6.31}$$

$$D^{\perp}_{dy} u(s, \eta) = \frac{(-\varphi(0)\delta_{-s}(dr) - \dot{\varphi}(s+y))\, 1_{[-s,0[}(y)dy}{\int_s^T \varphi^2(r)dr} \mathbb{E}\left[\Phi(Y^{s,\eta}) \cdot \int_0^T \varphi(r)dW_r\right] . \tag{6.32}$$

Remark 6.12. We remark that in (6.31) and (6.32) does not appear the derivative of f. At this point we admit a technical point not to overcharge the proof. Even if f is not of class C^1, $u(t, \cdot)$ is still of class C^1 for every $s \in [0, T[$ and (6.31) and (6.32) still hold.

As promised, we verify the assumptions of Theorem 5.15. We first observe that

$$(s, \eta) \mapsto \mathbb{E}\left[\Phi(Y^{s,\eta}) \cdot \int_0^T \varphi(r)dW_r\right] \tag{6.33}$$

is continuous, therefore bounded on each compact of $[0, T] \times C([-T, 0])$. Given a compact subset K of $C([-T, 0])$, we denote

$$C(K) := \sup_{\eta \in K} \mathbb{E}\left[\Phi(Y^{s,\eta}) \cdot \int_0^T \varphi(r)dW_r\right] .$$

Assumptions i), ii) and iv) are clearly verified taking into account (6.33), (6.28) and (6.29). It remains to check the support predictability property. Let K be a compact of $C([0, T])$. Since

$$\sup_{y \in [0,b], \eta \in K} \left| \frac{\dot{\varphi}(s+y)1_{[-s,0[}(y)}{\int_s^T \varphi^2(r)dr} \mathbb{E}\left[\Phi(Y^{s,\eta}) \cdot \int_0^T \varphi(r)dW_r\right] \right|$$

is finite, Remark 5.14, item 2. implies that it will be enough to show that

$$\int_0^b \sup_{\eta \in K} \frac{1}{\epsilon} \int_{(-\epsilon)\vee(-T)}^0 \left| \frac{\varphi(0)\delta_{-s}(dr)1_{[-s,0[}(y)dy}{\int_s^T \varphi^2(r)dr} \mathbb{E}\left[\Phi(Y^{s,\eta}) \cdot \int_0^T \varphi(r)dW_r\right] \right| ds = O(\epsilon) . \tag{6.34}$$

Let $\epsilon < b$. The left-hand side of (6.34) is bounded by

$$\frac{C(K)}{\epsilon} \int_0^\epsilon \frac{\varphi(0)}{\int_s^T \varphi^2(r)dr} ds \leq \frac{C(K)}{\int_b^T \varphi^2(r)dr} \|\varphi\|_\infty < +\infty . \tag{6.35}$$

So also assumption iii) of Theorem 5.15, i.e. the support predictability property is verified. The conclusion follows by Remark 5.20.

6.4 Link to a finite dimensional PDE

We come back to the assumptions on coefficients σ and b of Section 6. They characterize again a diffusion process X as solution of (6.1). We link now Corollary 5.19 and Proposition 6.11 with a well-known result of representation related to hedging theory in mathematical finance. We consider a *contingent claim* defined by $h = \Phi(\gamma) = f(\gamma(a_1), \ldots, \gamma(a_N))$, i.e.

$$h = f(X_{a_1}, \ldots, X_{a_N}) \ , \quad 0 < a_1 < \ldots < a_N = T \tag{6.36}$$

with the usual convention $a_0 = 0$. We consider here the case f of class C^2 with polynomial growth but σ may become degenerate.

Proposition 6.13. Let X be a diffusion process of type (6.1). Let $N \geq 2$ and $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ of class C^2 with polynomial growth. We suppose $\sigma : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ of class $C^{0,2}$, such that $\partial_x \sigma$, $\partial^2_{xx}\sigma$ are bounded. There exist functions

$$\nu^i : \mathbb{R}^{i-1} \times [a_{i-1}, a_i] \times \mathbb{R} \longrightarrow \mathbb{R} \ , \quad 1 \leq i \leq N \ ,$$

such that for $y_1, \ldots, y_{i-1} \in \mathbb{R}$,

$$\nu^i(s,y) := \nu^i(y_1, \ldots, y_{i-1}; s, y)$$

solves

$$\begin{cases} \partial_s \nu^i(s,y) + \frac{1}{2}\sigma^2(s,y)\ \partial^2_{yy}\nu^i(s,y) + b(s,y)\partial_y \nu^i(s,y) = 0 & s \in]a_{i-1}, a_i[\\ \nu^i(a_i, y) = \nu^{i+1}(y_1, \ldots, y_{i-1}, y; a_i, y) & i < N \\ \nu^N(T,y) = f(y_1, \ldots, y_{N-1}, y) & i = N \end{cases} \tag{6.37}$$

such that

$$f(X_{a_1}, \ldots, X_{a_N}) = H_0 + \int_0^T \xi_s dX_s \tag{6.38}$$

and, for $1 \leq i \leq N$,

$$\begin{aligned} &\xi_s = \partial_y \nu^i\left(X_{a_1}, \ldots, X_{a_{i-1}}; s, X_s\right) \ , \qquad s \in]a_{i-1}, a_i[\\ &H_0 = \nu^1(0, X_0) \end{aligned} \ . \tag{6.39}$$

In particular $(s,y) \mapsto \nu^i(s,y) \in C^{1,2}(]a_{i-1}, a_i[\times\mathbb{R}) \cap C^0([a_{i-1}, a_i] \times \mathbb{R})$.

Remark 6.14. Let X be a finite quadratic variation process such that $[X]_t = \int_0^t \sigma^2(s, X_s)\, ds$. An example is of course our basic process X introduced in (6.1) but there are plenty of other examples. Let $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ be continuous with linear growth. If there are functions $\nu^1, \ldots, \nu^N$ as in Proposition 6.13, then representation (6.38), with (6.39), holds replacing dX_s with the forward integral d^-X_s. If X is a martingale we recall that $d^-X_s = dX_s$. This was proved in [5], Proposition 4.30. The aim of Proposition 6.13 is to construct effectively such functions $\nu^1, \ldots, \nu^N$.

Proof of Proposition 6.13. According to Corollary 5.19, representation (6.38) holds with $H_0 = u(0, X_0(\cdot))$ and $\xi_s = D^{\delta_0}u(s, X_s(\cdot))$ where $u(s,\eta) := \mathbb{E}\left[f\left(Y^{s,\eta}_{a_1}, \ldots, Y^{s,\eta}_{a_N}\right)\right]$. In fact $\Phi(\gamma) = f(\gamma(a_1), \ldots, \gamma(a_N))$ is C^1 Fréchet differentiable with polynomial growth.
We denote by $(Z^{s,y}_t)$ the flow such that $Z = Z^{s,y}$ verifies

$$Z_t = y + \int_s^t \sigma(r, Z_r)\, dW_r + \int_s^t b(r, Z_r)\, dr \ , \quad t \geq s \ .$$

In particular we have $Y^{s,\eta} = Z^{s,\eta(0)}$.
Let $1 \le i \le N$. Clearly, if $s \in [a_{i-1}, a_i]$,

$$
\begin{aligned}
u(s,\eta) &= \mathbb{E}\left[f\left(\eta(a_1 - s), \ldots, \eta(a_{i-1} - s), Z^{s,\eta(0)}_{a_i}, \ldots, Z^{s,\eta(0)}_{a_N} \right) \right] \\
&= \nu^i \left(\eta(a_1 - s), \ldots, \eta(a_{i-1} - s); s, \eta(0) \right)
\end{aligned}
\tag{6.40}
$$

where

$$
\nu^i(s,y) = \mathbb{E}\left[f\left(y_1, \ldots, y_{i-1}, Z^{s,y}_{a_i}, \ldots, Z^{s,y}_{a_N} \right) \right] \ . \tag{6.41}
$$

Now ν^i is continuous. We keep in mind the representation (6.18) in Proposition 6.11. In particular (6.38) holds with $H_0 = u(0, X_0(\cdot))$ and $\xi_s = D^{\delta_0} u(s, X_s(\cdot))$. We recall the expression of $D^{\delta_0} u$ calculated in Example 6.7 in a more general situation. We had

$$
\begin{aligned}
D^{\delta_0} u(s,\eta)\ 1_{]a_{i-1},a_i[}(s) = \mathbb{E}\Bigg[\sum_{a_j \ge s} & f\left(\eta(a_1 - s), \ldots, \eta(a_{i-1} - s), Z^{s,\eta(0)}_{a_i}, \ldots, Z^{s,\eta(0)}_{a_N} \right) \\
& \mathcal{E}\left\{ \int_s^{a_j} \partial_x \sigma\left(r, Z^{s,\eta(0)}_r \right) dW_r + \int_s^{a_j} \partial_x b\left(r, Z^{s,\eta(0)}_r \right) dr \right\} \Bigg]
\end{aligned}
$$

where $\mathcal{E}$ denotes the Doléans exponential operator as usual.
In fact by usual integration theory it is not difficult to show that for any $s \in]a_{i-1}, a_i[$, $y_1, \ldots, y_{i-1} \in \mathbb{R}$ fixed, $y \mapsto \nu^i(s,y)$ is of class C^2. We observe that $u(0, X_0(\cdot)) = \nu^1(0, X_0)$. Moreover $s \mapsto \partial_y \nu^i(s,y)$ and $s \mapsto \partial^2_{yy} \nu^i(s,y)$ are continuous on $]a_{i-1}, a_i[$. In particular if $s \in]a_{i-1}, a_i[$

$$
\partial_y \nu^i(s,y) = \mathbb{E}\left[\sum_{j \ge i} \partial_j f\left(y_1, \ldots, y_{i-1}, Z^{s,y}_{a_i}, \ldots, Z^{s,y}_{a_N} \right) \partial_y Z^{s,y}_{a_j} \right] \ .
$$

Since, at least in $L^2(\Omega)$,

$$
\partial_y Z^{s,y}_{a_j} = \mathcal{E}\left(\int_s^{a_j} \partial_x \sigma\left(r, Z^{s,y}_r \right) dW_r \right) \ ,
$$

it follows that, for $s \in]a_{i-1}, a_i[$,

$$
D^{\delta_0} u(s,\eta) = \partial_y \nu^i \left(\eta(a_1 - s), \ldots, \eta(a_{i-1} - s); s, \eta(0) \right)
$$

and so (6.39) is established.
It remains to prove (6.37).
We remark that we can evaluate the second order derivative with respect to y. It gives

$$
\begin{aligned}
\partial^2_{yy} \nu^i(s,y) &= \mathbb{E}\left[\sum_{k,j \ge i} \partial^2_{kj} f\left(y_1, \ldots, y_{i-1}, Z^{s,y}_{a_i}, \ldots, Z^{s,y}_{a_N} \right) \partial_y Z^{s,y}_{a_j} \partial_y Z^{s,y}_{a_k} \right] \\
&\quad + \mathbb{E}\left[\sum_{j \ge i} \partial_j f\left(y_1, \ldots, y_{i-1}, Z^{s,y}_{a_i}, \ldots, Z^{s,y}_{a_N} \right) \partial^2_{yy} Z^{s,y}_{a_j} \right]
\end{aligned}
$$

where $\partial^2_{yy} Z^{s,y}_{a_j}$ could be calculated explicitly.
It remains to provide the partial derivative with respect to s. Let $s, s+h \in]a_{i-1}, a_i[$ and suppose $h > 0$. Since $Z^{s,y}_{a_i} = Z^{s+h, Z^{s,y}_{s+h}}_{a_i}$, we easily obtain that

$$u(s,y) = \mathbb{E}\left[u\left(s+h, Z^{s,y}_{s+h}\right)\right] \ .$$

So, by Itô formula,

$$\begin{aligned}
\frac{\nu^i(s+h,y) - \nu^i(s,h)}{h} &= \frac{u(s+h,y) - u(s,y)}{h} \\
&= \frac{1}{h}\mathbb{E}\left[u(s+h,y) - u(s+h, Z^{s,y}_{s+h}\right] \\
&= \frac{1}{h}\mathbb{E}\left[-\int_s^{s+h} \partial_y u\left(s+h, Z^{s,y}_r\right) dZ^{s,y}_r - \frac{1}{2}\int_s^{s+h} \sigma^2\left(r, Z^{s,y}_r\right) \partial^2_{yy} u\left(s+h, Z^{s,y}_r\right) dr\right]
\end{aligned}$$

where $dZ^{s,y}_r = \sigma\left(r, Z^{s,y}_r\right) dW_r + b\left(r, Z^{s,y}_r\right) dr$.
Similar arguments allow to discuss the limit when $h \to 0$. Letting h go to zero we get

$$\begin{aligned}
\partial_s \nu^i(s,y) = \lim_{h\to 0} \frac{u(s+h,y) - u(s,y)}{h} &= -\frac{1}{2}\sigma^2\left(s,y\right) \partial^2_{yy} u\left(s,y\right) - b(s,y)\partial_y u(s,y) \\
&= -\frac{1}{2}\sigma^2(s,y)\partial^2_{yy}\nu^i(s,y) - b(s,y)\partial_y \nu^i(s,y) \ .
\end{aligned}$$

We have finally established the first line of (6.37). The second and third conditions in (6.37) are verified by inspection using (6.41). □

ACKNOWLEDGEMENTS:
The work of the second named author was partially supported by the ANR Project MASTERIE 2010 BLAN 0121 01.

References

[1] Charalambos D. Aliprantis and Kim C. Border. *Infinite dimensional analysis.* Springer, Berlin, third edition, 2006. A hitchhiker's guide.

[2] Jean Bertoin. Les processus de Dirichlet en tant qu'espace de Banach. *Stochastics*, 18(2):155–168, 1986.

[3] Haïm Brezis. *Analyse fonctionnelle.* Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[4] François Coquet, Adam Jakubowski, Jean Mémin, and Leszek Słomiński. Natural decomposition of processes and weak Dirichlet processes. In *In memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX*, volume 1874 of *Lecture Notes in Math.*, pages 81–116. Springer, Berlin, 2006.

[5] Rosanna Coviello, Cristina Di Girolami, and Francesco Russo. On stochastic calculus related to financial assets without semimartingales. *Preprint*, http://hal.archives-ouvertes.fr/inria-00564756/en/, 2011.

[6] Rosanna Coviello and Francesco Russo. Nonsemimartingales: stochastic differential equations and weak Dirichlet processes. *Ann. Probab.*, 35(1):255–308, 2007.

[7] Cristina Di Girolami and Francesco Russo. Generalized covariation for Banach valued processes and Itô formula. *HAL-INRIA, Preprint*, http://hal.archives-ouvertes.fr/inria-00545660/en/, 2010.

[8] Cristina Di Girolami and Francesco Russo. Infinite dimensional stochastic calculus via regularization and applications. *Preprint HAL-INRIA*, http://hal.archives-ouvertes.fr/inria-00473947/fr/(Unpublished), 2010.

[9] Cristina Di Girolami and Francesco Russo. Clark-Ocone type formula for non-semimartingales with finite quadratic variation. *Comptes Rendus Mathematique*, 349(3-4):209 – 214, 2011.

[10] Nicolae Dinculeanu. *Vector integration and stochastic integration in Banach spaces.* Pure and Applied Mathematics (New York). Wiley-Interscience, New York, 2000.

[11] Nelson Dunford and Jacob T. Schwartz. *Linear operators. Part I.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1988. General theory. With the assistance of William G. Bade and Robert G. Bartle. Reprint of the 1958 original. A Wiley-Interscience Publication.

[12] Mohammed Errami and Francesco Russo. Covariation de convolution de martingales. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(5):601–606, 1998.

[13] Mohammed Errami and Francesco Russo. n-covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes. *Stochastic Process. Appl.*, 104(2):259–299, 2003.

[14] Franco Flandoli and Francesco Russo. Generalized calculus and SDEs with non regular drift. *Stoch. Stoch. Rep.*, 72(1-2):11–54, 2002.

[15] Franco Flandoli and Francesco Russo. Generalized integration and stochastic ODEs. *Ann. Probab.*, 30(1):270–292, 2002.

[16] H. Föllmer. Dirichlet processes. In *Stochastic integrals (Proc. Sympos., Univ. Durham, Durham, 1980)*, volume 851 of *Lecture Notes in Math.*, pages 476–478. Springer, Berlin, 1981.

[17] Hans Föllmer. Calcul d'Itô sans probabilités. In *Seminar on Probability, XV (Univ. Strasbourg, Strasbourg, 1979/1980) (French)*, volume 850 of *Lecture Notes in Math.*, pages 143–150. Springer, Berlin, 1981.

[18] Fausto Gozzi and Francesco Russo. Verification theorems for stochastic optimal control problems via a time dependent Fukushima-Dirichlet decomposition. *Stochastic Processes and their Applications*, 116(11):1530 – 1562, 2006.

[19] Fausto Gozzi and Francesco Russo. Weak Dirichlet processes with a stochastic control perspective. *Stochastic Processes and their Applications*, 116(11):1563 – 1583, 2006.

[20] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[21] Michel Métivier and Jean Pellaumail. *Stochastic integration.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1980. Probability and Mathematical Statistics.

[22] Jacques Neveu. *Processus aléatoires gaussiens.* Séminaire de Mathématiques Supérieures, No. 34 (Été, 1968). Les Presses de l'Université de Montréal, Montreal, Que., 1968.

[23] Francesco Russo and Pierre Vallois. The generalized covariation process and Itô formula. *Stochastic Process. Appl.*, 59(1):81–104, 1995.

[24] Francesco Russo and Pierre Vallois. Stochastic calculus with respect to continuous finite quadratic variation processes. *Stochastics Stochastics Rep.*, 70(1-2):1–40, 2000.

[25] Francesco Russo and Pierre Vallois. Elements of stochastic calculus via regularization. In *Séminaire de Probabilités XL*, volume 1899 of *Lecture Notes in Math.*, pages 147–185. Springer, Berlin, 2007.

[26] Francesco Russo, Pierre Vallois, and Jochen Wolf. A generalized class of Lyons-Zheng processes. *Bernoulli*, 7(2):363–379, 2001.

[27] Raymond A. Ryan. *Introduction to tensor products of Banach spaces*. Springer Monographs in Mathematics. Springer-Verlag London Ltd., London, 2002.

[28] Wilhelm Stannat. The theory of generalized Dirichlet forms and its applications in analysis and stochastics. *Mem. Amer. Math. Soc.*, 142(678):viii+101, 1999.