On efficiency of peer-to-peer file streaming with random contacts

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***Abstract*—In peer-to-peer streaming the corresponding file is is divided into many small chunks that are obtained, from other peers, in a fixed order forming a stream of information. We assume a 'video on demand'-type scenario, all peers wish to obtain the entire file. We use a continuous model, where the set of chunks is replaced by a real variable in a finite segment. The file streaming is then depicted as movement of a peer along this segment. The peers form a network, where any peer can download from any other. We consider an open system with continuous inflow of peers. Peers select their targets for downloading in a random fashion, thus forming a random overlay for streaming. We take into account that multiple dowloaders get bandwidth inversely proportional to their number. Our aim is to see whether such strategy can work. We make simulation for somehow realistic system and found that when ignoring possible delay of the search of peers, this system can provide quite good service that seems to scale well with number of peers. Making some simplifications we found a system that can be solved in a 'mean-field' approximation when the characteristics of the contact graph are approximated by some mean values. This solution supports simulations showing linear scaling of the system with arrival rate in the large system size limit. Further, we anylze the effect of multiple contact and suggest that the system has almost ideal scaling, meaning that peers can download with almost maximal speed when having just few simultaneous contacts.**

I. Introduction

File sharing using the peer-to-peer techniques has shown to be very effective and popular among Internet users. Such systems like Bit-Torrent [1] and eDonkey are well known. These systems are typically used to obtain a file in small chunks that can be obtained in any order. A relatively new addition in this row of applications, is streaming of the file, which means that the chunks must be obtained in a fixed order and without substantial delays. The simplest form of streaming looks like watching a TV-channel, a peer that joins the network starts to obtain the file not from the beginning, but rather from the position available at that moment and then proceeds in obtaing a stream of chunks and forwards analogous stream, if requested, to other peers. For this case good models are found, see in [2].

We suggest a, possibly novel, modeling scheme that is intermediate between TV-like live streaming and traditional peer-to-peer file sharing in the respect that peers can start the streaming at any time from the beginning of the file. In our model, there is only one permanent peer, the 'seed', that holds the entire file. The peers can immediately leave the system after obtaing the entire file, or they can be more 'generous' and contribute the system even after they have obtained the whole file. We compare these both scenarios. The file is obtained in a streaming mode, that is in a linear order. The scenario could be that a peer downloads a file or movie, and watches it simultaneously. Thus the downloading rate of the file must exceed the bit rate of the application. In real systems, the file is divided into many small parts, called the chunks. Peers arrive into the system with constant rate, and upon arriving start to download chunks. Chunks can be downloaded from any other peer that has the needed chunks. We assume that the selecting of peers for downloading is done in a uniformly random fashion. In our previous studies and experiments we studied the plausibility of such approach for Bit-Torrent type, non-streaming, file sharing, [3], [4], [5], [6]. The random contact approach allows decentralized schemes without coordinators like 'trackers' that are used in current Bit-Torrents. There are several practical and decentralized ways of achieving the uniform sampling of the peers for the random contacts, see in [5].

In our streaming model, we assume a simplified continuous model where the chunks are approximated by a continuous variable $x \in [0,1]$, corresponding to very short and numerous chunks. The linear order in the real segment $[0,1]$, corresponds to the fixed order in which the file is downloaded, from the start, ($x=0$), until the end, ($x=1$). The downloading process itself is depicted as a continuous movement or walk along this real segment with some non-constant speed. In the model, the graph representing contacts between downloaders and uploaders is approximated by a certain random graph process, taking into account the effect of degree of a peer on downloading rate. We assume, that peers with many downloaders will provide slower rates to their individual downloaders, thus we have a 'push'-model where the uploader decides what to give to its downloaders. The mean-field approach means that we estimate the corresponding degrees by their expected values, taken from the graph process, instead of random. This system has a steady state with nonuniform distribution of peers along the segment $[0,1]$, with monotonous growth of the number of peers towards the end of the segment due to decrease of downloading rate towards the end of the file. This results in a system that has a relatively good performance, the system has an accumulation of more peers near the end of the file, making them almost as seeds. We get indications that the system scales up to any sizes having just one seed. We analyze the effect of multiple contacts and argue that just few parallel contacts may improve the performance of the system making it nearly optimal . We have two variants of the algorithm with respect to 'generosity' of the peers that finish their downloading. In a generous model such peers provide service to their clients while in the ungenerous case, they disappear the system, and the corresponding clients must find new hosts. This circumstance has a notable effect on throughput. However, the difference is only quantitave. The ungenerous scheme can also provide substantial service level, especially when enhanced with multiple contacts.

All in all, our model has two dimensions, the elapsed time and the parameter $x \in [0, 1]$ indicating the position of a peer in the downloading of the file. The state of the system is characterized by time-dependent distribution of peers on $[0, 1]$ and the random graph corresponding to the downloading-uploading relations. Notably the state space is simpler that that for the non-streaming chunk based protocols, where a peer can have any combination of chunks. Thus, streaming may be practically more demanding, but theoretically somehow simpler to deal with. Indeed, if the number of chunks is m, then for Bit-Torrent-like system the number of possible peer's states, is 2^m, and for the streaming system it is equal to $m+1$, the number of possible stages of file downloading.

II. The file streaming algorithm and simulations

Consider an open system, where peers arrive with constant rate $\lambda > 0$. The peers aim to obtain the file in a certain fixed order, from the very beginning. This is modeled with a real variable $x \in [0, 1]$, with $x(t)$ indicating the portion of file obtained so far at the moment of time t. Thus $x = 0$ means the starting phase of the process and $x = 1$, the final stage. We assume that as soon as peers reach the final stage, they depart the system. However, there is one peer, the seed, that is constantly in the system and holds the entire file.

As the peer arrives, it makes a random contact to one node in the system, including the seed as possible target. All peers have the same constant bandwidth equal to one, symmetrically for downloading and uploading. For simplicity, the uploading and downloading is considered as a deterministic with constant rate conditionally to the degree of the host. A peer divides evenly its uploading capacity among its uploaders. So, if a peer has 3 downloaders, each of them gets the file with speed $\frac{1}{3}$, and so on. We are not looking at, practically important, issues like incentives for cooperation or free riding, assuming full cooperation without any malicious behavior. The effect of 'churn' is also ignored.

In the case that a peer looses the host from which it downloads, it must find a new host. Lost of target peer corresponds to a host peer that reaches the finish-line, $x = 1$, and departures. To find a new host, corresponding peer makes again a new random contact. The possible targets are now those peers that have greater x-coordinates than the peer seeking a new contact, this is because only those have the content required by this peer. Finding such peers may be demanding. However, such search is done only when the host is lost. There is also a natural search mechanism that could help: find a random peer, if its coordinate is large enough, take it as a host. If not, one can contact the peer from which that contacted peer is currently downloading, since that peer must be ahead of the contacted peer, and so on until a host with large enough coordinate is found. However, we do not analyze the delay caused by the search and assume instant search. It is assumed that the downloading process dominates the overall delay. It may also happen that the downloader 'catches up', in x-coordinate, the host from which it downloads. At that moment it reduces its downloading speed in such a way that it remains just behind the host. This is natural, because a peer cannot download parts of the file that the host does not have.

The contact process results in a kind of directed dynamical random tree, where links point from the uploader to the downloader and seed being the root node (the most advanced node must download from the seed). Notably, the out-degrees of nodes tend to grow with the coordinate. This is because, number of potential peers that contact some peer, grows monotonously with its coordinate. This has an effect of slowing down the movement of peers as they get closer to the end, since they must contact a peer nearer and nearer to the 'finish line'. This can be beneficial, since those peers that are at the final stages have most of the file. The concentration of peers that are close to finish is larger than at the starting phase, thus it may be that the seed is needed only for a very short amount of time just before the end.

We simulated our algorithm in several instances. The system starts from the state with just the seed node. Then the peers arrive according to Poisson process with intensity λ. After a relaxation period, the system seems to reach a steady-like state, where the system size fluctuates around some average, n. In Fig.1, the steady state empirical distribution of peers along $x \in [0, 1]$ is shown. As we anticipated, the concentration of peers grows near the end of the segment. The system seems to pose stability in the sense that it relaxes to a certain kind of state starting from the empty state contains only the seed, as shown in Fig.2. Note that the distribution in Fig.1 corresponds to the same set of simulations taken at the fixed time $t = 3000$. We made two more longer runs to point that steady state is reached. The ratio of average system size$/\lambda$, is between 2 and 3. This means that the average downloading rate is around 0.5.

III. Mean-field model of streaming

We want to build a tractable and simple model to allow analytical calculations. The idea is to replace crucial parameters with their 'mean' values. Similarly to [7], [8] used to model Bit-Torrent-like systems, we also make several abstractions, like replace the chunks by a continuous variable. We hope that such an approach helps us to point out some key aspects of the streaming protocols.

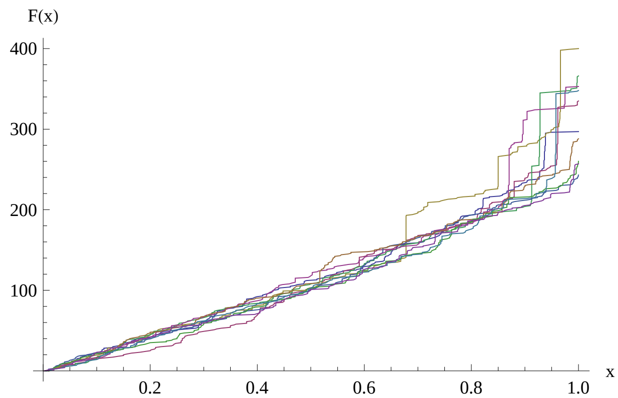

Fig. 1. 10 instances of simulations of random system, the empirical distribution $F(x)$ of the number of peers with coordinate less than x. The arrival rate was, $\lambda = 100$. This distribution corresponds to the steady state, reached after relaxation. The system has higher concentration of peers near $x = 1$, because the function grows, in mean, faster than the linear function, which would indicate the even distribution.

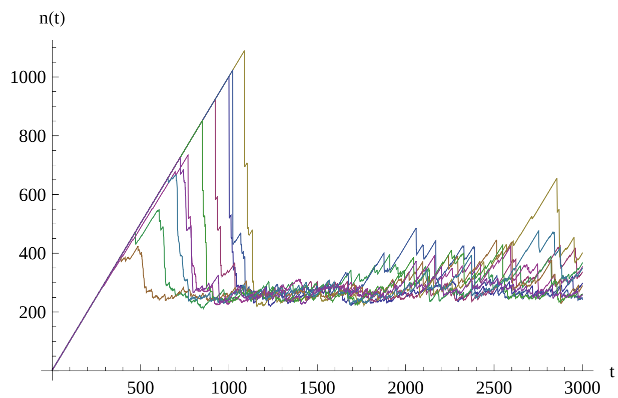

Fig. 2. The same 10 instances of simulations as in Fig.1. The system size n as a function of time. The system seems to relax toward some equilibrium level, although fluctuations are rather profound. The arrival rate was taken as 100/time unit, meaning that there were about 300 thousand arrivals during a single simulation.

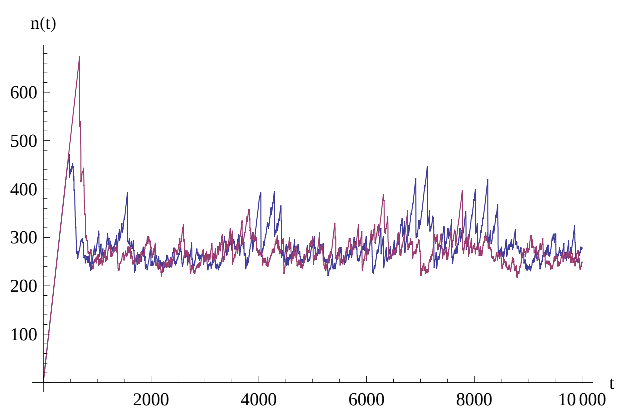

Fig. 3. Two longer runs, with $\lambda = 100$ (flow of approximately one million peers), to check stability of the steady state.

By $t \in [0, \infty)$, we denote the time, $x \in [0, 1]$ is the stage of downloading parameter, as above. Our aim is to find steady-state scaling of the expected system size in the large input rate, λ, limit. We call our approach 'mean-field' because the random downloading rates are replaced by their expectations.

We consider two algorithms that are different from the 'generosity' point of view. In first one the peers make a random contact to one node in the system as they arrive into the system. This contact remains the same throughout the downloading process. When the contacted peer finishes, it still provides stream to all those peers contacted to it. That is why, we call this the 'Generous model'. However, such peer with all chunks collected, does not accept any new contacts. This is similar to that simulated already, except some optimistic assumption and generosity of the leaving peers. Then we remove this generosity, by assuming that peers disappear as soon as they finish and the corresponding downloaders find new random contacts. Thus this model is closer to one we simulated in previous section. This model we call 'Ungenerous'. We assume that in this system there is a permanent peer, the seed with the whole file in its possession.

A. *Generous model*

Let us formulate the Generous model. We make the following assumption about the contacts. We take the average characteristics of the contact graphs from a graph process $T_n(t)$, which is similar to the contact graph of the Generous process and we conjecture that in the large system size the erros vanish. The graph has constant size n, the nodes are enumerated as $\{1, 2, \cdots, n\}$. The time is discrete $t \in N$. The process starts from a graph without links, with n isolated vertices. Then at each time-step, a new node is added to the left and all others are shifted by 1, step to the right, say node k is transformed to node $k + 1$, except the rightmost node n, which disappears from the system. Simultaneously upon arriving the new node makes a uniformly random contact to one node $\{2, 3, \cdots n\}$ in the system. As the nodes are shifted, their link structure is also shifted parallel and unchanged. If a node x points to n with degree δ, at some moment of time t, we agree that in subsequent time steps $t + k, k \le n - x$, the node $x + k$, at time $t + k$ has a link to some fictious node with the same degree δ, inherited from n at time t. This agreement corresponds to generosity of leaving peer.

The random out-degrees of nodes (in-degree is always 1) are denoted as D_k. In the large enough times, the effect of the initial state $T_n(1)$ disappears, because after $2n + 1$ time steps all dependences on it have disappeared. Then the stationary distribution of degrees can be written Obviously, $D_k \cong Bin(k - 1, 1/(n - 1))$, stating that it is binomially distributed. This is because newly arriving peer makes uniform link with probability $1/(n - 1)$, and until the corresponding node has taken the position k, it has been a potential target for such contacts for $k - 1$ times.

Returning to the Generous model, the service model is the following, a peer with D downloaders, gives a constant rate $1/D$ for streaming to every contact. This contains an optimistic assumption, since the peer can not 'overtake' the peer they are downloading. Unfortunately, we must ignore this natural circumstance due to difficult dependences it would bring into the model. Thus, the constant rate $1/D$ is speed at which a peer moves along the segment $[0,1]$ and is denoted by V. We use the same notions also for the auxiliary graph-process. We want to estimate the same random variable $1/D$, for the graph process. For a node $1 \leq k \leq n$, the speed is denoted as V_k, such node can have contact with any node $s \in \{k+1, k+2, \cdots, n\}$ that are in the system, with the probability $1/(n-1)$ or it can have contact with a node that has already disappeared from the system, with probability $(n-k)/n$. Thus such a node s has an out-degree at least one, but other nodes $\{1, 2, \cdots, s-1\} - \{k\}$, $s-2$ in number, made independent attempts to contact the same node s and with the same probability $1/(n-1)$. Thus the node s, contacted by the node k has out-degree $1 + \Delta_s$, where is a binomial random variable $Bin(s-2, 1/(n-1))$ for $s > 2$ and the corresponding rate is $1/(1+\Delta_s)$. In the special case $s = 2$, we have $\Delta_2 = 0$. If the node k has contacted, upon arriving, any node from $n-k+1, n-k+2, \cdots n$, those nodes have already left the system and give the rate $1/(1+\Delta_{ext})$, where the random variable Δ_{ext} is distributed as $Bin(n-2, 1/(n-1))$. As a result we have for our graph process:

Proposition 1

$$\mathbb{E}(V_k) = \frac{1}{n-1}\sum_{s=k+1}^{n} \mathbb{E}\left(\frac{1}{1+\Delta_s}\right) + \frac{k}{n}\mathbb{E}\left(\frac{1}{1+\Delta_{ext}}\right)$$

$$= \sum_{s=k+1}^{n} \frac{1}{(s-1)}\left(1 - \left(1 - \frac{1}{n-1}\right)^{s-1}\right) + \frac{k}{n}\left(1 - \left(1 - \frac{1}{n-1}\right)^{n-1}\right),$$

where we used the formula $\mathbb{E}(1/(1+X)) = \frac{1-(1-p)^{n+1}}{(1+n)p}$, for a $Bin(n,p)$ distributed r.v. X.

Now we want to combine the graph process with the original model with movement on the segment.

'Mean-field' assumption

We make a simplifying 'mean-field' assumption by saying that the peers, in Generous model, move with average speed $v_k \equiv \mathbb{E}V_k$, taken from the previous proposition for the graph process. This is a hypothesis about the large size steady state of the Generous model. In such circumstances the system would be in some approximate sense similar to the graph process with the same number of nodes, n. By this we mean that since inflow and outflow of peers are equal in mean in the steady state, this would roughly corresponds to one in - and one out - feature of the graph process. The constant size of the graph corresponds to the circumstance that in a large scale steady state of the generous model the system size is assumed to be large and almost a constant (fluctuations are small compared to mean), denoted by n. Of course, we ignore such differences like that the average size of the steady state of the Generous model is not an integer. We are interested in asymptotic ration of arrival rate and the average size of the steady state, and such things are unrelevant.

After such assumptions, the condition for the steady state in the mean-field approximation is:

$$\frac{1}{\lambda}\sum_{k=1}^{n} v_k = 1, \tag{1}$$

where n is now unknown, that should be such that it matches the above condition for the give λ. Thus we have the following relation in the limit of $\lambda \to \infty$, in the mean-field approximation:

Proposition 2

$$\frac{n(\lambda)}{\lambda} \to c,$$

with

$$c = \left(P\int_0^1 dx \int_x^1 dy \frac{1}{y}(1-e^{-y}) + \frac{1}{2}(1-e^{-1})\right)^{-1} \approx 1.462,$$

where P, stands for the Cauchy's principal value.

Proof: Indeed, using the Proposition 1 and relation (1), we get:

$$\frac{1}{\lambda}\sum_{k=1}^{n-1}\sum_{s=k+1}^{n}\frac{1}{s-1}\left(1 - e^{(s-1)\ln(1-1/(n-1))}\right) + \frac{1}{\lambda}\sum_{k=1}^{n}\frac{k}{n}\left(1 - \left(1 - \frac{1}{n-1}\right)^{n-1}\right) = 1,$$

in the sense of limit of $n \to \infty$, the left hand side can be written:

$$\frac{n}{\lambda}\sum_{k=1}^{n}\frac{1}{n}\sum_{s=k}^{n}\frac{1}{n}\frac{1}{s/n}\left(1 - e^{-s/n}\right) + \frac{n}{\lambda}\sum_{k=1}^{n}\frac{1}{n}\frac{k}{n}(1-e^{-1}) = 1,$$

which can be seen as an integral sum with constant step $1/n$ of two variables, $(x,y) \in [1/n, 1] \times [1/n, 1]$. Since the corresponding integral exists, due to well enough behaving integrand, $(\frac{1}{y}(1-e^{-y}) + x(1-e^{-1}))dxdy$, and the cut-off at the lower bound of the integral of the type, $\epsilon = 1/n \to 0$, giving the P-value. Thus the limit of this sum exists and coincides with result given in the Prop. 2. ∎

The value of the coefficient, $c = 1.462...$, means that the downloading of the file happens at the speed c times slower than the maximal value, which is 1. In Fig.4 we plotted the expected download rates for a system of 100 peers. The download rate decays as the peers proceed. This means also that the 'concentration' of peers grows also toward the end of the segment $[0,1]$. This is somehow beneficial, since such peers possess more chunks than those at the beginning of their downloading process. For comparison, the simulated system showed a worse scaling, since the corresponding coefficient is almost twice as large, see the Figure 3. This is probably due to

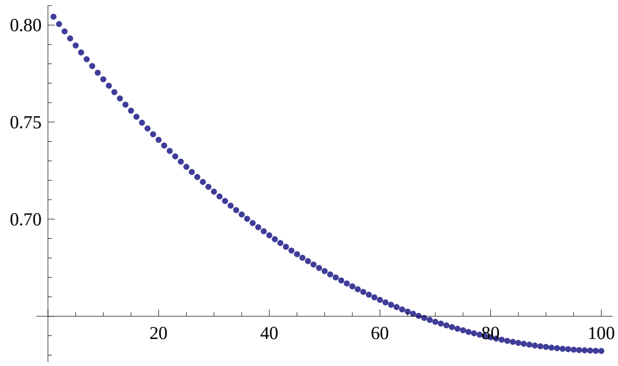

Fig. 4. The sequence of expected download rates $v_1, v_2, \cdots, v_{100}$, for one hundred peers.

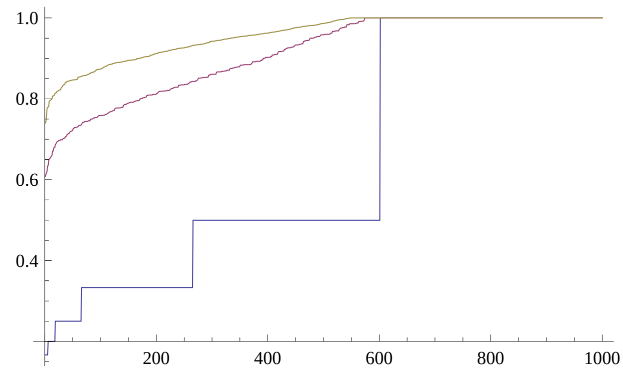

Fig. 5. Simulated empirical distribution, with 1000 instances, of V for $d = 1, 5, 10$, the upmost line corresponds to $d = 10$, the next is for $d = 5$ showing that the small rate contacts can be avoided with such large d.

more realistic scenario where overtaking are prohibited, thus reducing the downloading rate. The other factor is that the peers are not as generous, since after finishing, they disappear from the system.

This algorithm is not practical in the sense that a peer that is 'unlucky' and starts to download from a peer that has several downloaders, will suffer from a very low download rate throughout the process. This can be avoided by using several contacts. What happens can be understood from the following heuristical reasoning. A more rigorous analysis is postponed for latter work. Let $d \in \{1, 2, \cdots, \}$ be the number of contacts a peer makes upon arrivals. Let assume that in other respects the algorithm remains the same as in the case of $d = 1$. As an example assume, as a worst case scenario, that all contacted peers are near the borderline of the downloading with $x = 1$, such peers tend to have largest degrees and thus provide minimal bandwidth. Assume olsa that the degree of such a peers are Poisson distributed with parameter d, independent random variables. This would probably be a good approximation in a large system. Then the bandwidth allocated to such peer would be:

$$V = \left(\frac{1}{1 + X_1} + \frac{1}{1 + X_2} + \cdots + \frac{1}{1 + X_d} \right) \wedge 1.$$

Where $\wedge$, stands for infimum. Note that this is a pessimistic estimate, since not necessarily all peers that are encountered in r.v.'s $X_1, X_2, \cdots, X_d$ require their full share of bandwidth. Since $\mathbb{E}(1 + X_i)^{-1} = (1 - e^{-d})/d$, the sum of r.v.'s in the above formula is, typically, close to $\frac{d}{d}(1 - e^{-d}) \approx 1$, for large d. Thus, it seems that it is possible to get almost full speed of downloading, because for other contacts with smaller expected degrees, this relation is even more clear. Thus there is some minimal d, that would give sufficient download rates for almost all peers, thus it seems to be possible to decrease c near 1. To illustrate our point, we depicted empirical distributions of V for $d = 1, 5, 10$ in Fig. 5. Using the mean field approach, the upper bound for the large system size can be found, similarly as in Proposition 2. the result would be:

Conjecture 3 For a system with $d = 1, 2, \cdots$ contacts:

$$\frac{n(\lambda)}{\lambda} \to c_d, \lambda \to \infty$$

with

$$c_d \leq \left(P \int_0^1 dx \int_x^1 dy \frac{1}{y}(1 - e^{-dy}) \wedge 1 + \frac{1}{2}(1 - e^{-d}) \right)^{-1},$$

with already for $d = 2$, the upper bound is numerically $c_2 \leq 1.089$.

B. *Ungenerous model*

In a more realistic scenario, peers would not be as generous after finishing their job. Indeed, if the movie-like files are very large and peers that finish their own downloading and still serving all their 'clients' would have unreasonable delays. That is why we assume that after the peers finish, they disappear from the system, and the peers attached to them, must find new uniformly random hosts. To have our mean field approximation we wish to find expected degrees of peers in a large system of size n. We also assume that there is one extra peer, the seed, with entire file, visible only for the peer that has no other peers ahead. This peer is called a seed. Thus our model relies only weakly on the seed. As a result, the most advanced peer has a determistic download rate 1. However, this is not a big issue since in a large system the effect of one peer is very small in overall performance.

We assume that the system reaches a steady state, where the inflow of peers equals to the outflow. We again use a discrete model, related to the original model. The unproven hypothesis is that as $n \to \infty$, the results are equivalent. We assume a following random graph process $\mathcal{G}_n(t), t \in N$. Fix integer n and start from an empty graph, at $t = 0$. At $t = 1$ an isolated node is added. Then at each time step $t = 2, 3, \cdots$, a new node is added and this node makes a uniformly and random link to one node already in the system. At moment of time $t = n$, there are n nodes in the system. At times $t > n$, along with arrival of a new node the 'oldest' node disappears from the system as well as all links pointing to it and further all nodes that loose their link in this way, will make one uniformly random link among those that that are older. By definition until time $n+1$ we have so called 'uniformly random recursive tree'(urrt) $\mathcal{T}_t$, see eg. [9], [10], or , where all possible elementary outcomes, recursive trees (sometimes such trees are also called the 'increasing Cayley trees', [11])of order t, are taken with equal probability. Thus, a recursive tree of order k is a rooted, non-plane and labeled tree, with labels from set $\{1, 2, \cdots, k\}$, the root has label 1, and labels along any branch stemming from the root form an increasing sequence. For k nodes there are $(k-1)!$, recursive trees of order of k that are possible outcomes of $\mathcal{G}_n(k)$, and every such outcome has equal probability. It appears that such Markov chain is a sequence of random graphs with urrt distribution for all times t. For $t < n+1$, this is true by definition, for $t > n$, we have:

Proposition 4

For all times $t \geq n$, $\mathcal{G}_n(t) \sim \mathcal{T}_n$.

Proof:

This means that links are drawn independently and that the link probabilities are $\mathbb{P}((k,s) \in E_t) = \frac{1}{n-k}$ if $1 \leq k < s \leq n$ and 0 other wisely, and where E_t is the link set of graph at time t. This is true, by assumption, for $t = n$, as was stated above. This is the start for the induction, assume that the claim is true until some time $t_0 \geq n$. Is it true for $t = t_0 + 1$? There is a link $(x, y) \in E_{t_0+1}$ if $(x-1, y-1) \in E_{t_0}$, this has probability $1/(n-x+1)$, by the induction assumption. The other possibility is that $(x-1, n) \in E_{t_0}$ and at the current time step $t_0 + 1$, (x, y) was selected as a new link, this event has probability $\frac{1}{n-x+1}\frac{1}{n-x}$. Where the first multiplier comes from induction assumption and the last from the process assumption that a new link is drawn uniformly to nodes having larger labels. As a result

$$\mathbb{P}((x,y) \in E_{t_0+1}) = \frac{1}{n-x+1} + \frac{1}{n-x+1}\frac{1}{n-x} = \frac{1}{n-x+1}\frac{n-x+1}{n-x} = \frac{1}{n-x}.$$

■

Corollary 5

The expected degree of node $1 < y \leq n$

$$d_y \equiv \mathbb{E}D_y = \sum_{k=1}^{y-1} \frac{1}{n-k}.$$

In the large n limit we use the 'Poisson Approximation', [12], see also the 'Poisson Paradigm' in [13]: if X is a sum of n indicators of independent or 'almost' independent random variables, then $X \approx Po(\mathbb{E}X)$, in some approximate sense, and assume that $D_y \cong Po(\mathbb{E}D_y)$. In this circumstance a useful distance between distributions is the total variance distance, which is in our case:

$$d_{TV}(X,Y) = \frac{1}{2}\sum_k \mid \mathbb{P}(X=k) - \mathbb{P}(Y=k) \mid .$$

In the case of independent indicator summands $\{I_\alpha\}$, $X = \sum_\alpha I_\alpha$, $\lambda \equiv EX$, $\pi_\alpha \equiv EI_\alpha$, we have the Theorem 6.22 in [12] in the form:

Theorem

$$d_{TV}(X, Po(\lambda) \leq (\lambda^{-1} \wedge 1)\sum_\alpha \pi_\alpha^2.$$

From this Theorem, we can deduce the following

Lemma 6 For large enough n, we have

$$d_{TV}(D_k, X_k) \leq \frac{\pi^2}{\ln n}$$

where $X_k \cong Po(d_k), 1 < k \leq n$, we do not include the variable $D_1 = 0$, for which we have a trivial coincidence with $X_1 \cong Po(0)$.

Proof: In notions of Theorem 6.22 we have $\pi_i = \frac{1}{n-i}$, for $1 \leq i < k$. From the right-hand side of above Theorem we get in our cases the following upper bounds:

$$(d_k^{-1} \wedge 1)\sum_{1 \leq j < k}\left(\frac{1}{n-j}\right)^2 \leq \frac{\sum_{1 \leq j<k}\left(\frac{1}{n-j}\right)^2}{d_k} = \\ = \frac{\sum_{1 \leq j<k}\left(\frac{1}{n-j}\right)^2}{\sum_{1 \leq j<k}\left(\frac{1}{n-j}\right)} \equiv f_k.$$

The upper bounds $\{f_k\}$ appear monotonously growing: $f_1 \equiv 0 \leq f_1 \leq f_2 \leq \cdots \leq f_n$. Indeed, denote $q_k = \sum_{1 \leq j<k}\left(\frac{1}{n-j}\right)^2$ and $w_k = \sum_{1 \leq j<k}\left(\frac{1}{n-j}\right)$, we have $q_{k+1} = q_k + \frac{1}{(n-k)^2}$ and $w_{k+1} = w_k + \frac{1}{(n-k)}$ and thus

$$f_{k+1} - f_k = \frac{q_{k+1}}{w_{k+1}} - \frac{q_k}{w_k} = \frac{q_k + \frac{1}{(n-k)^2}}{w_k + \frac{1}{(n-k)}} - \frac{q_k}{w_k} = \\ \frac{\frac{1}{n-k}\sum_{s<k}\left(\frac{1}{(n-s)(n-k)} - \frac{1}{(n-s)^2}\right)}{w_k w_{k+1}} \geq \\ \frac{\frac{1}{n-k}\sum_{s<k}\left(\frac{1}{(n-s)^2} - \frac{1}{(n-s)^2}\right)}{w_k w_{k+1}} = 0.$$

As a result the upper bound f_n is a uniform upperboud for all variables:

$$d_{TV}(D_k, X_k) \leq f_n,$$

and we just need to estimate f_n. By definition:

$$f_n = \frac{\sum_{1 \leq j<n}\frac{1}{j^2}}{\sum_{1 \leq j<n}\frac{1}{j}},$$

The denominator is a monotonously growing and convergent sum, with limit equal to $\pi^2/6$, and thus it is less than, say, $\pi^2/2$ for all n. The nominator is a harmonic sum, which diverges as $\ln n$ and is larger than $\frac{1}{2}\ln n$, for large enough n. As result we have $f_n \leq \pi^2/\ln n$, starting from some, large enough n. ■

Lemma 7

$$\mid E\left(\frac{1}{1+D_k}\right) - E\left(\frac{1}{1+X_k}\right) \mid \leq \frac{2\pi^2}{\ln n}.$$

Proof: Indeed, the left-hand side equals to:

$$| \sum_{j\geq 0} \frac{1}{1+j} \left(\mathbb{P}(D_k = j) - \mathbb{P}(X_k = j)\right) | \leq$$
$$\sum_{j\geq 0} \frac{1}{1+j} \mid \mathbb{P}(D_k = j) - \mathbb{P}(X_k = j) \mid \leq$$
$$\sum_{j\geq 0} \mid \mathbb{P}(D_k = j) - \mathbb{P}(X_k = j) \mid = 2d_{TV}(D_k, X_k)$$

from which the claim follows using the Lemma 6. ■

Our aim is to find expectation of the V_k, the rate for peer k. By definition

$$EV_k = \frac{1}{n-k} \sum_{s=k+1}^{n} E\left(\frac{1}{D_s} \mid (k,s)\right),$$

where the condition (k, s), means that the expectation is calculated under the condition that peer k has a link to peer s. In the large system limit, such conditions have a small effect, because we have:

Lemma 8 For $s < k$

$$E\frac{1}{1+D_{k-1}} \geq E\left(\frac{1}{D_k} \mid (s,k)\right) \geq E\frac{1}{1+D_k}$$

Proof: By assumption $D_k = \sum_{j=1}^{k-1} Y_j$, $Y_j \sim Be\left(\frac{1}{n-j}\right), j \in \{1, 2, \cdots n-1\}$, are independent Bernoulli variables. As a result

$$E\left(\frac{1}{D_k} \mid (s,k)\right) = E\left(\frac{1}{1+D_k - Y_s}\right).$$

It is clear that

$$E\frac{1}{1+D_k - Y_{k-1}} \geq E\frac{1}{1+D_k - Y_s} \geq E\frac{1}{1+D_k},$$

from which the claim follows since $D_k - Y_{k-1} \sim D_{k-1}$. ■

Lemma 9 The expected degree d_s, $1 < s < n$, for large enough n, is bounded as:

$$\ln\left(\frac{1}{1-\frac{s-1}{n}}\right) \leq d_s \leq \ln\left(\frac{1}{1-\frac{s+1/2}{n}}\right).$$

and

$$\ln n/2 \leq d_n \leq 2\ln n$$

Proof: First consider $1 < s < n$. By definition

$$d_s = \sum_{j=1}^{s-1} \frac{1}{n-j} = \sum_{j=1}^{s-1} \frac{1}{n}\left(\frac{1}{1-j/n}\right) \leq$$
$$\int_0^{\frac{s-1}{n}} dx \frac{1}{1-x-\frac{1}{n}} = \ln\left(\frac{1-\frac{1}{n}}{1-\frac{s}{n}}\right) =$$
$$\ln\left(\frac{1}{(1-\frac{s}{n})(1+\frac{1}{n}+(\frac{1}{n})^2+\cdots)}\right) =$$
$$\ln\left(\frac{1}{1-\frac{s-1+s/n+O(1/n)}{n}}\right) \leq \ln\left(\frac{1}{1-\frac{s+O(1/n)}{n}}\right) \leq$$
$$\ln\left(\frac{1}{1-\frac{s+1/2}{n}}\right),$$

where the first inequality follows from the fact that d_s is a lower integral sum of the integral that follows. W The last inequality is true for all n, larger than some fixed value after which $O(1/n) < 1/2$. The lower bound is found from the inequality

$$d_s \geq \int_0^{\frac{s-1}{n}} dx \frac{1}{1-x} = \ln\left(\frac{1}{1-\frac{s-1}{n}}\right),$$

which is true because, d_s is now an upper integral sum of the integral involved. For the case $s = n$, we notice that d_n is increasing as sum of the harmonic sum, which is diverging as $\ln n$, thus the bounds are true for large enough n. ■

We have the following asymptotic relation, for $n \to \infty$:

Proposition 10

$$\frac{v}{n} := \frac{1}{n} \sum_{0\leq k<n} EV_k \to r,$$

with

$$r = P\int_0^1 dx \frac{1}{1-x} \int_x^1 dy \frac{-y}{\ln(1-y)} = \frac{1}{2}.$$

Proof: From Lemma 8 we have

$$\sum_{k=1}^{n-1} \frac{1}{n-k} \sum_{s=k+1}^{n} E\frac{1}{1+D_s} \leq v \leq$$
$$\sum_{k=1}^{n-1} \frac{1}{n-k} \sum_{s=k}^{n-1} E\frac{1}{1+D_s}.$$

Let us find the asymptotic of the lower bound. First we use the Lemma 7:

$$\sum_{k=1}^{n-1} \frac{1}{n-k} \sum_{s=k+1}^{n} E\frac{1}{1+D_s} \geq$$
$$\sum_{k=1}^{n-1} \frac{1}{n-k} \sum_{s=k+1}^{n} E\frac{1}{1+X_s} - \frac{2\pi^2 n}{\ln n} =$$
$$n\left(\sum_{k=1}^{n-1} \frac{1}{n} \frac{1}{1-\frac{k}{n}} \sum_{s=k+1}^{n} \frac{1}{n} E\frac{1}{1+X_s} - \frac{2\pi^2}{\ln n}\right).$$

Thus we want to find limit of the

$$u_n \equiv \sum_{k=1}^{n-1} \frac{1}{n} \frac{1}{1-\frac{k}{n}} \sum_{s=k+1}^{n} \frac{1}{n} E\frac{1}{1+X_s} - \frac{2\pi^2}{\ln n}$$

as $n \to \infty$. The last term can be dropped since it goes to 0. $X_s \cong Po(d_s)$, and thus:

$$E\frac{1}{1+X_s} = \frac{1-e^{-d_s}}{d_s}$$

Thus

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{1}{n} \frac{1}{1-\frac{k}{n}} \sum_{s=k+1}^{n} \frac{1}{n} \left(\frac{1-e^{-d_s}}{d_s}\right).$$

Using the lemma 9, we see that, assuming that the contribution of X_n goes to zero: $\widetilde{v} := \lim_{n\to\infty} u_n$

$$\widetilde{v} \geq \lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{1}{n} \frac{1}{1-\frac{k}{n}} \sum_{s=k+1}^{n-1} \frac{1}{n} \left(\frac{1-e^{-\ln\left(\frac{1}{1-\frac{s-1}{n}}\right)}}{\ln\left(\frac{1}{1-\frac{s+1/2}{n}}\right)} \right) =$$

$$\lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{1}{n} \frac{1}{1-\frac{k}{n}} \sum_{s=k+1}^{n-1} \frac{1}{n} \frac{\frac{s-1}{n}}{-\ln\left(1-\frac{s+1/2}{n}\right)} =$$

$$P \int_0^1 dx \frac{1}{1-x} \int_x^1 dy \frac{-y}{\ln(1-y)} = r.$$

Thus the we get the lower bound of the limit $v \geq rn$. By examining the upper bound, we get in similar way, that $v \leq rn$ and as a result $v = nr$. Finally we show that the contribution of X_n to $\widetilde{v}$ is zero, by definition this contribution is

$$\sum_{k=1}^{n-1} \frac{1}{n} \frac{1}{1-\frac{k}{n}} \frac{1}{n} \frac{1-\exp(-d_n)}{d_n} \leq$$

$$\sum_{k=1}^{n-1} \frac{1}{n-k} \frac{1}{n} \frac{1-\exp(-2\ln n)}{\ln n/2} \leq$$

$$\frac{2}{n \ln n} \sum_{j=1}^{n-1} \frac{1}{j} \leq \frac{4}{n}$$

where we used the Lemma 9. ∎

Interestingly, this is closer to our previous simulations, only slightly lower value of simulations which is around 2.5. This could be explained by the error coming from using an asymptotic and the optimistic assumption ignoring overtaking of peers.

We can also generalize the result in the case of many, $d \in \{1, 2, \cdots\}$, contacts, as was done in the previous model.
Conjecture 11 in the case of d contacts the ungenerous mode yields the lower bounds of

$$\frac{v}{n} \to r_d,$$

with

$$r_d \geq P \int_0^1 dx \frac{1}{1-x} \int_x^1 dy \left(\frac{-(1-(1-y)^d)}{\ln(1-y)} \wedge 1 \right) = \frac{1}{1+\frac{1}{d}}$$

However, the sequence of lower bounds, converges to 1 much slower than in the generous model. This could indicate, that in a more realistic system, a substantial degree of downloading is beneficial despite the burden of maintaining several contacts. Interestingly, even the BitTorrent system uses several, 5 in numbers, flows of chunks, to get downstream flow maximal. This could be reasonable in the case of streaming as well.

Conclusions We analyzed the effect of bandwidth sharing in file streaming systems where random contacts are used to find targets for downloading. Our result seems to indicate good behavior of such systems in the limit of large arrival rates.

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