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Abstract. We present a new method for estimating the endpoint of a unidimensional sample when the distribution function decreases at a polynomial rate to zero in the neighborhood of the endpoint. The estimator is based on the use of high order moments of the variable of interest. It is assumed that the order of the moments goes to infinity, and we give conditions on its rate of divergence to get the asymptotic normality of the estimator. The good performance of the estimator is illustrated on some finite sample situations.

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1 Introduction

Let $(X_1, \ldots, X_n)$ be n independent copies of a random variable X, with bounded support $[0, \theta]$, where $\theta > 0$ is unknown. In this paper, we address the problem of estimating the (right) endpoint θ of the survival function $\overline{F}$ of X. Pioneering work on endpoint estimation includes Quenouille (1949) who introduced a jackknife estimate of the endpoint based on the naive maximum estimator. This approach was further studied by Miller (1964), Robson and Whitlock (1964), Cooke (1979) and de Haan (1981), to name a few. A well-known reference on endpoint estimation is Hall (1982), recently improved by Li and Peng (2009), in which a maximum likelihood method is used when $\overline{F}$ belongs to the Hall model, see for instance Section 5. Hall's work gave a start to the study of general threshold-based methods, together with the use of the approximation of excesses by Generalized Pareto Distributions, see for instance Smith and Weissman (1985) and Smith (1987). A general construction of estimators of the endpoint using a threshold is given in de Haan and Ferreira (2006, p. 146). Some popular estimators in this framework, called Peaks Over Threshold (POT) approach, are probability weighted moments estimators (Hosking and Wallis, 1987), moment estimators (Dekkers *et al.*, 1989) and maximum likelihood estimators (Drees *et al.*, 2003).

Other studies include Loh (1984) and Athreya and Fukuchi (1997) with a bootstrap method, Hall and Wang (1999) for a minimal-distance method, Goldenshluger and Tsybakov (2004) for endpoint estimation in presence of random errors, and Hall and Wang (2005) for a Bayesian likelihood approach. As far as detecting the presence of a finite endpoint is concerned, see Neves and Pereira (2010).

In this paper, we introduce an estimator using high moments of the variable of interest X. More precisely, the estimator is given by

$$\frac{1}{\widehat{\theta}_n} = \frac{1}{ap_n}\left[((a+1)p_n+1)\frac{\widehat{\mu}_{(a+1)p_n}}{\widehat{\mu}_{(a+1)p_n+1}} - (p_n+1)\frac{\widehat{\mu}_{p_n}}{\widehat{\mu}_{p_n+1}}\right] \tag{1}$$

where (p_n) is a nonrandom sequence such that $p_n \to \infty$, $a > 0$ and

$$\widehat{\mu}_{p_n} = \frac{1}{n}\sum_{i=1}^{n} X_i^{p_n}$$

is the classical moment estimator of $\mu_{p_n} := \mathbb{E}(X^{p_n})$. From a practical point of view, taking high order moments gives more weight to observations close to θ. From a theoretical point of view, the estimator $\widehat{\theta}_n$ converges in probability to θ without any parametric assumption on the distribution of X, see Section 3. The asymptotic normality of the estimator is established in Section 4 under a semi-parametric assumption. Some examples of distributions satisfying this assumption are provided in Section 5. Some simulations are proposed in Section 6 to illustrate the efficiency of our estimator, and to compare it with estimators of the endpoint estimation literature. Auxiliary results are postponed to Appendix A and proved in Appendix B.

2 Construction of the estimator

To justify the introduction of our estimator (1), let first Y be a random variable with survival function $\overline{G}$ defined by $\overline{G}(y) = (1 - y/\theta)^\alpha$ for all $y \in [0,\, \theta]$. We get for all $p \geq 1$,

$$M_p := \mathbb{E}(Y^p) = p\int_0^{+\infty} y^{p-1}\,\overline{G}(y)\,dy = \alpha\,\theta^p\, B(p+1,\,\alpha) \tag{2}$$

where $B(x,\,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ is the Beta function. This yields for all $p \geq 1$,

$$\frac{M_p}{M_{p+1}} = \frac{1}{\theta}\left[1 + \frac{\alpha}{p+1}\right] \tag{3}$$

leading to, for all arbitrary sequences (p_n) and all $a > 0$

$$\frac{1}{\theta} = \frac{1}{ap_n}\left[((a+1)p_n+1)\frac{M_{(a+1)p_n}}{M_{(a+1)p_n+1}} - (p_n+1)\frac{M_{p_n}}{M_{p_n+1}}\right].$$

Using the above ideas, we shall then define our estimator in two steps. First, the moment M_p is replaced by the true moment μ_p, and we set

$$\frac{1}{\Theta_n} := \frac{1}{ap_n}\left[((a+1)p_n+1)\frac{\mu_{(a+1)p_n}}{\mu_{(a+1)p_n+1}} - (p_n+1)\frac{\mu_{p_n}}{\mu_{p_n+1}}\right].$$

Second, μ_{p_n} is estimated by the corresponding empirical moment $\widehat{\mu}_{p_n}$; plugging $\widehat{\mu}_{p_n}$ in $1/\Theta_n$ yields the estimator (1) of $1/\theta$.

3 Consistency

In this section, we state and prove the consistency of our estimator in a non-parametric context. The only hypothesis is

(A_0) X is positive and the endpoint $\theta = \sup\{x \geq 0 \,|\, F(x) < 1\}$ of X is finite.

To this end, the first step is to prove a result similar to (3) for μ_{p_n}.

Proposition 1. *Under* (A_0), $\mu_{p_n}/\mu_{p_n+1} \longrightarrow 1/\theta$ *as* $n \to \infty$.

This result is a straightforward consequence of Lemma 1. The second step consists in showing that μ_{p_n} can be replaced by its empirical counterpart $\widehat{\mu}_{p_n}$. Defining $\mu_{1,\,p_n} = \mu_{p_n}/\theta^{p_n}$ as in Appendix A, we have the following result:

Proposition 2. *Assume that* (A_0) *holds. If* $n\,\mu_{1,\,p_n} \longrightarrow \infty$, *then* $\widehat{\mu}_{p_n}/\mu_{p_n} \xrightarrow{\mathbb{P}} 1$ *as* $n \to \infty$.

Proof. Let $Y_{nj} := [X_j/\theta]^{p_n}$ and $Z_{nj} := Y_{nj}/(n\,\mu_{1,\,p_n})$ for $1 \leq j \leq n$. The desired result is then tantamount to $\sum_{j=1}^n Z_{nj} \to 1$ in probability. Notice next that for all n, the $(Z_{nj})_{1\leq j\leq n}$ are positive independent random variables, and $\sum_{j=1}^n \mathbb{E}(Z_{nj}) = 1$. According to Chow and Teicher (1997, Corollary 2 p. 358), it is enough to show that

$$\forall\,\varepsilon > 0, \qquad \sum_{j=1}^n \mathbb{E}(Z_{nj}\,\mathbb{1}_{\{Z_{nj}\geq\varepsilon\}}) \to 0$$

as $n \to \infty$. The $(Z_{nj})_{1\leq j\leq n}$ being identically distributed, it is equivalent to prove that

$$\forall\,\varepsilon > 0, \qquad \frac{1}{\mu_{1,\,p_n}}\mathbb{E}(Y_{n1}\,\mathbb{1}_{\{Y_{n1}\geq\varepsilon n\,\mu_{1,\,p_n}\}}) \to 0.$$

Since $Y_{n1} \in [0,\,1]$ almost surely and $n\,\mu_{1,\,p_n} \to \infty$, we get, for sufficiently large n

$$\frac{1}{\mu_{1,\,p_n}}\mathbb{E}(Y_{n1}\,\mathbb{1}_{\{Y_{n1}\geq\varepsilon n\,\mu_{1,\,p_n}\}}) = 0$$

and the result is proved. ■

Theorem 1. *Suppose* (A_0) *holds. If* $n\,\mu_{1,\,(a+1)p_n} \to \infty$ *then* $\widehat{\theta}_n \xrightarrow{\mathbb{P}} \theta$ *as* $n \to \infty$.

Proof. Remark first that $\mu_{1,\,(a+1)p_n} \le (a+1)\mu_{1,\,p_n}$ so that $n\,\mu_{1,\,p_n} \to \infty$. Second, Lemma 1 entails $n\,\mu_{1,\,p_n+1} \to \infty$ and $n\,\mu_{1,\,(a+1)p_n+1} \to \infty$ as $n \to \infty$. We can then apply Proposition 2 to rewrite the frontier estimator as

$$\frac{1}{\widehat{\theta}_n} = \frac{1}{ap_n}\left[((a+1)p_n+1)\frac{\mu_{(a+1)p_n}}{\mu_{(a+1)p_n+1}}(1+\mathrm{o}_{\mathbb{P}}(1)) - (p_n+1)\frac{\mu_{p_n}}{\mu_{p_n+1}}(1+\mathrm{o}_{\mathbb{P}}(1))\right].$$

Using once again Lemma 1 yields $\mu_{p_n}/\mu_{p_n+1} \to 1/\theta$ and $\mu_{(a+1)p_n}/\mu_{(a+1)p_n+1} \to 1/\theta$ as $n \to \infty$. Replacing in the above equality, the conclusion follows. ■

4 Asymptotic normality

We now examine the asymptotic normality of our estimator. To this end, additional assumptions are introduced:

(A_1) $\forall\, x \in [0,\,\theta]$, $\overline{F}(x) = (1-x/\theta)^{\alpha}\, L((1-x/\theta)^{-1})$ where θ, $\alpha > 0$ and L is a slowly varying function at infinity, *i.e.* such that $L(ty)/L(y) \to 1$ as $y \to \infty$ for all $t > 0$.

(A_2) $\forall\, x \ge 1$, $L(x) = \exp\left(\int_1^x \eta(t)\, t^{-1}\, dt\right)$, where η is a Borel bounded function tending to 0 at infinity, continuously differentiable on $(1,\,+\infty)$, ultimately monotonic and non identically 0. Besides, there exists $\nu \le 0$ such that $x\,\eta'(x)/\eta(x) \to \nu$ as $x \to +\infty$.

In the general context of extreme-value theory, (A_1) entails that the distribution belongs to the Weibull max-domain of attraction with extreme-value index $-1/\alpha$, we refer the reader to de Haan and Ferreira (2006). Regarding (A_2), $L(x) = \exp\left(\int_1^x \eta(t)\, t^{-1}\, dt\right)$ is the Karamata representation for normalized slowly varying functions, see Bingham *et al.* (1987), p. 15. Under (A_2), the function $|\eta|$ is ultimately non-increasing and regularly varying at infinity with index ν, see Bingham *et al.* (1987), paragraph 1.4.2. In the extreme-value framework, ν is referred to as the second order parameter and (A_2) is a second order condition. Finally, let us note that (A_2) implies that $x\,\eta'(x) = \mathrm{O}(\eta(x))$, so that $x\,\eta'(x) \to 0$ as $x \to +\infty$.

We first show that (3) still holds, up to an error term, when M_p is replaced by μ_p.

Proposition 3. *Assume that* (A_1) *and* (A_2) *hold. Then,*

$$\frac{\mu_p}{\mu_{p+1}} = \frac{M_p}{M_{p+1}} + \mathrm{O}\left(\frac{|\eta(p)|}{p}\right).$$

Proof. Considering the change of variables $y = (1-x/\theta)^{-1}$ in (2) yields

$$M_p = p^{-\alpha}\,\theta^p\,\Gamma(\alpha+1)\,R_M(p)$$

with $\Gamma(x) = \displaystyle\int_0^{+\infty} t^{x-1}\,e^{-t}\,dt$ the Gamma function and

$$R_M(p) = 1 + \frac{I_1\,E_1(p) + I_2\,E_2(p)}{\Gamma(\alpha+1)},$$

where I_1, I_2, $E_1(p)$ and $E_2(p)$ are defined in Lemma 7 by

$$\begin{aligned} E_1(p) &= \frac{1}{I_1}\int_0^1 f_p(x)x^{-\alpha-2}\,dx - 1, & I_1 &= \int_1^{+\infty} y^\alpha e^{-y}\,dy, \\ E_2(p) &= \frac{1}{I_2}\int_1^{+\infty} g_p(x)x^{-\alpha-2}\,dx - 1, & I_2 &= \int_0^1 y^\alpha e^{-y}\,dy, \end{aligned}$$

and where f_p, g_p are the functions introduced in Lemma 6:

$$\begin{aligned} \forall\, x\in(0,\,1], \quad f_p(x) &= \left(1-\frac{1}{p}\right)^{-\alpha-1}\left(1+\frac{1}{(p-1)x}\right)^{-\alpha-2}\left(1-\frac{1}{(p-1)x+1}\right)^{p-1}, \\ \forall\, x\in[1,\,+\infty), \quad g_p(x) &= \left(1-\frac{1}{px}\right)^{p-1}. \end{aligned}$$

Similarly, the same change of variables yields

$$\mu_p = p^{-\alpha}\,\theta^p\,L(p)\,\Gamma(\alpha+1)\,\left[R_M(p)+R_\delta(p)\right] \tag{4}$$

with

$$R_\delta(p) = \frac{I_1\,\delta_1(p) + I_2\,\delta_2(p)}{\Gamma(\alpha+1)}$$

where $\delta_1(p)$ and $\delta_2(p)$ are defined in Lemma 7 by

$$\begin{aligned} \delta_1(p) &= \frac{1}{I_1}\int_0^1 f_p(x)\left[\frac{L_1((p-1)x)}{L_1(p-1)} - x\right]x^{-\alpha-3}\,dx, & L_1(x) &= xL(x+1), \\ \delta_2(p) &= \frac{1}{I_2}\int_1^{+\infty} g_p(x)\left[\frac{L_2(px)}{L_2(p)} - \frac{1}{x}\right]x^{-\alpha-1}\,dx, & L_2(x) &= L(x)/x. \end{aligned}$$

Since $\displaystyle\int_p^{p+1}\frac{\eta(t)}{t}\,dt = \mathrm{O}\left(\frac{|\eta(p)|}{p}\right)$, one clearly has

$$\frac{\mu_p}{\mu_{p+1}} - \frac{M_p}{M_{p+1}} = \frac{1}{\theta}\left[1-\frac{1}{p+1}\right]^{-\alpha}\left[\frac{R_M(p)+R_\delta(p)}{R_M(p+1)+R_\delta(p+1)} - \frac{R_M(p)}{R_M(p+1)}\right] + \mathrm{O}\left(\frac{|\eta(p)|}{p}\right), \tag{5}$$

and it is straightforward that

$$\begin{aligned} &\frac{R_M(p)+R_\delta(p)}{R_M(p+1)+R_\delta(p+1)} - \frac{R_M(p)}{R_M(p+1)} \\ &\qquad= \frac{R_\delta(p)\,R_M(p+1) - R_\delta(p+1)\,R_M(p)}{[R_M(p+1)+R_\delta(p+1)]\,R_M(p+1)} \\ &\qquad= \frac{[R_\delta(p)-R_\delta(p+1)]\,R_M(p+1) - R_\delta(p+1)[R_M(p)-R_M(p+1)]}{[R_M(p+1)+R_\delta(p+1)]\,R_M(p+1)}. \end{aligned}$$

Lemma 7 entails that $R_M\to 1$ and $R_\delta\to 0$ as $p\to\infty$ and

$$\begin{aligned} R_\delta(p+1) &= \mathrm{O}\left(|\eta(p)|\,(1+\mathcal{L}(p))\right), \\ R_M(p)-R_M(p+1) &= \mathrm{O}\left(1/p^2\right), \\ R_\delta(p)-R_\delta(p+1) &= \mathrm{O}\left(|\eta(p)|/p\right), \end{aligned}$$

where $\mathcal{L}$ is slowly varying at infinity. Consequently,

$$\frac{R_M(p)+R_\delta(p)}{R_M(p+1)+R_\delta(p+1)} - \frac{R_M(p)}{R_M(p+1)} = \mathrm{O}\left(\frac{|\eta(p)|}{p} + \frac{|\eta(p)|\,(1+\mathcal{L}(p))}{p^2}\right) = \mathrm{O}\left(\frac{|\eta(p)|}{p}\right),$$

and replacing in (5) yields the desired result. ■

Applying Proposition 3 enables us to control the bias term introduced when M_{p_n} is replaced by μ_{p_n}:

$$\frac{1}{\Theta_n} = \frac{1}{\theta} + \mathrm{O}\left(\frac{|\eta(p_n)|}{p_n}\right). \tag{6}$$

We now turn to the random term:

Theorem 2. *Assume that* (A_1) *and* (A_2) *hold. If* $n\,p_n^{-\alpha}L(p_n) \to \infty$ *then*

$$v_n \left(\frac{\widehat{\theta}_n}{\Theta_n} - 1\right) \xrightarrow{d} \mathcal{N}\left(0,\, V(\alpha,\, a)\right) \quad \text{as } n \to \infty,$$

with $v_n = \sqrt{n\,L(p_n)}\,p_n^{-\alpha/2+1}$ *and*

$$V(\alpha,\, a) = \frac{\alpha+1}{a^2\,\Gamma(\alpha)}\left[2^{-\alpha-2} - 2\frac{(a+1)^{\alpha+1}}{(a+2)^{\alpha+2}} + 2^{-\alpha-2}(a+1)^{\alpha}\right].$$

Proof. Our goal is to prove that the sequence of random variables (ξ_n) defined by

$$\xi_n = \frac{\theta}{\sqrt{V(\alpha,\, a)}}\,v_n\left(\frac{1}{\widehat{\theta}_n} - \frac{1}{\Theta_n}\right)$$

converges in distribution to a standard Gaussian random variable, Theorem 2 then being a simple consequence of this result.

The first step consists in using Lemma 9 in order to linearize ξ_n:

$$\begin{aligned}
\xi_n &= u_{n,\,a}\left[\zeta_n^{(1)} + \left(\frac{\mu_{p_n+1}}{\widehat{\mu}_{p_n+1}} - 1\right)\zeta_n^{(2)} + \left(1 + \frac{ap_n}{p_n+1}\right)\left(\frac{\mu_{(a+1)p_n+1}}{\widehat{\mu}_{(a+1)p_n+1}} - 1\right)\zeta_n^{(3)}\right](1+\mathrm{o}(1)) \\
&= u_{n,\,a}\left[\zeta_n^{(1)} + \mathrm{o}_{\mathbb{P}}(\zeta_n^{(2)}) + \mathrm{o}_{\mathbb{P}}(\zeta_n^{(3)})\right](1+\mathrm{o}(1)),
\end{aligned}$$

in view of Proposition 2. Thus, to conclude the proof, it is enough to show that

$$u_{n,\,a}\,\zeta_n^{(1)} \xrightarrow{d} \mathcal{N}(0,1), \tag{7a}$$

$$u_{n,\,a}\,\zeta_n^{(2)} \xrightarrow{d} \mathcal{N}(0,C_2), \tag{7b}$$

$$u_{n,\,a}\,\zeta_n^{(3)} \xrightarrow{d} \mathcal{N}(0,C_3), \tag{7c}$$

where C_2 and C_3 are suitable constants. Let us then write $\zeta_n^{(1)} = \sum_{k=1}^n S_{n,\,k}^{(1)}$, where

$$\begin{aligned}
S_{n,\,k}^{(1)} &= \frac{1}{n}\,A_n^t\left[X_k^{p_n},\ X_k^{p_n+1},\ X_k^{(a+1)p_n},\ X_k^{(a+1)p_n+1}\right]^t, \\
A_n &= \left[a_{n,\,0}^{(1)},\ a_{n,\,1}^{(1)},\ a_{n,\,2}^{(1)},\ a_{n,\,3}^{(1)}\right]^t, \\
a_{n,\,0}^{(1)} &= -1, \\
a_{n,\,1}^{(1)} &= \frac{\mu_{p_n}}{\mu_{p_n+1}}, \\
a_{n,\,2}^{(1)} &= \left(1 + \frac{ap_n}{p_n+1}\right)\frac{\mu_{p_n+1}}{\mu_{(a+1)p_n+1}}, \\
a_{n,\,3}^{(1)} &= -\left(1 + \frac{ap_n}{p_n+1}\right)\frac{\mu_{p_n+1}\,\mu_{(a+1)p_n}}{\mu_{(a+1)p_n+1}^2},
\end{aligned}$$

with A^t standing for the transposed matrix of A. In order to use Lyapounov's central limit theorem (see e.g. Billingsley, 1979, p. 312), it remains to prove that

$$\frac{1}{[\mathrm{Var}(\zeta_n^{(1)})]^{3/2}} \sum_{k=1}^{n} \mathbb{E}|S_{n,\,k}^{(1)}|^3 \to 0 \tag{8}$$

as $n \to \infty$, which requires to control $\mathrm{Var}(\zeta_n^{(1)})$ and $\mathbb{E}|S_{n,\,1}^{(1)}|^3$.

To compute an equivalent for $\mathrm{Var}(\zeta_n^{(1)})$, remark that $\mathrm{Var}(\zeta_n^{(1)}) = \dfrac{1}{n} A_n^t\, M_n\, A_n$ with

$$M_n = \begin{bmatrix} \mu_{2p_n} & \mu_{2p_n+1} & \mu_{(a+2)p_n} & \mu_{(a+2)p_n+1} \\ \mu_{2p_n+1} & \mu_{2p_n+2} & \mu_{(a+2)p_n+1} & \mu_{(a+2)p_n+2} \\ \mu_{(a+2)p_n} & \mu_{(a+2)p_n+1} & \mu_{(2a+2)p_n} & \mu_{(2a+2)p_n+1} \\ \mu_{(a+2)p_n+1} & \mu_{(a+2)p_n+2} & \mu_{(2a+2)p_n+1} & \mu_{(2a+2)p_n+2} \end{bmatrix}.$$

Let us now rewrite that as

$$\begin{aligned} \mathrm{Var}(\zeta_n^{(1)}) &= \frac{1}{n}\left[w(p_n,\, p_n) - 2\left(1 + \frac{ap_n}{p_n+1}\right) \frac{\mu_{p_n+1}}{\mu_{(a+1)p_n+1}}\, w(p_n,\, (a+1)p_n) \right. \\ &\qquad \left. + \left(1 + \frac{ap_n}{p_n+1}\right)^2 \frac{\mu_{p_n+1}^2}{\mu_{(a+1)p_n+1}^2}\, w((a+1)p_n,\, (a+1)p_n) \right] \end{aligned}$$

where

$$w(up_n,\, vp_n) = \left[-1,\ \frac{\mu_{up_n}}{\mu_{up_n+1}}\right] \begin{bmatrix} \mu_{(u+v)p_n} & \mu_{(u+v)p_n+1} \\ \mu_{(u+v)p_n+1} & \mu_{(u+v)p_n+2} \end{bmatrix} \left[-1,\ \frac{\mu_{vp_n}}{\mu_{vp_n+1}}\right]^t.$$

We now apply Proposition 3, and use (4) together with Lemma 7 to obtain, after some cumbersome but elementary computations,

$$w(up_n,\, vp_n) = \frac{\Gamma(\alpha+1)\,\alpha(\alpha+1)}{(u+v)^{\alpha+2}}\, \theta^{(u+v)p_n}\, p_n^{-\alpha-2}\, L(p_n)\,(1+\mathrm{o}(1)).$$

Taking into account that

$$\left(1 + \frac{ap_n}{p_n+1}\right) \frac{\mu_{p_n+1}}{\mu_{(a+1)p_n+1}} = \frac{(a+1)^{\alpha+1}}{\theta^{ap_n}}(1+\mathrm{o}(1)) \tag{9}$$

we get

$$\mathrm{Var}(\zeta_n^{(1)}) = a^2\,\Gamma^2(\alpha+1) V(\alpha,\, a) \frac{1}{n}\, \theta^{2p_n}\, p_n^{-\alpha-2}\, L(p_n)\,(1+\mathrm{o}(1)). \tag{10}$$

To show (8), it then suffices to prove that $\mathbb{E}|S_{n,\,1}^{(1)}|^3 = \mathrm{O}(n^{-3}\,\theta^{3p_n}\,p_n^{-\alpha-3}\,L(p_n))$. To this aim, let us introduce $Y_1 = X_1/\theta$ and the associated survival function $\overline{F}_1(x) = (1-x)^\alpha\, L((1-x)^{-1})$, $\forall\, x \in [0,\, 1]$. Hölder's inequality leads to

$$\frac{\mathbb{E}|S_{n,\,1}^{(1)}|^3}{n^{-3}\,\theta^{3p_n}} \le 4\ \mathbb{E}|Y_1^{p_n}(a_{n,\,0}^{(1)} + a_{n,\,1}^{(1)}\,\theta\, Y_1)|^3 + 4\ \mathbb{E}|Y_1^{(a+1)p_n}(a_{n,\,2}^{(1)}\,\theta^{ap_n} + a_{n,\,3}^{(1)}\,\theta^{ap_n+1}\,Y_1)|^3.$$

Setting

$$\begin{aligned}
H_{n,0}^{(1)}(u) &= -1, \\
H_{n,1}^{(1)}(u) &= \alpha u, \\
H_{n,2}^{(1)}(u) &= \left(1+\frac{ap_n}{p_n+1}\right)\theta^{ap_n}\,\frac{\mu_{p_n+1}}{\mu_{(a+1)p_n+1}}, \\
H_{n,3}^{(1)}(u) &= -\left(1+\frac{ap_n}{p_n+1}\right)\theta^{ap_n}\,\frac{\mu_{p_n+1}}{\mu_{(a+1)p_n+1}}\cdot\frac{\alpha u}{a+1},
\end{aligned}$$

some more straightforward albeit burdensome computations show that there exist two sequences of Borel functions $(\chi_n^{(1,1)})$ and $(\chi_n^{(1,2)})$ uniformly converging to 0 on $[0, 1]$ such that for all $u \in [0, 1]$,

$$\begin{aligned}
a_{n,0}^{(1)} + a_{n,1}^{(1)}\,\theta u &= H_{n,0}^{(1)}(u)(1-u) + \frac{H_{n,1}^{(1)}(u)+\chi_n^{(1,1)}(u)}{p_n}, \\
a_{n,2}^{(1)}\,\theta^{ap_n} + a_{n,3}^{(1)}\,\theta^{ap_n+1}\,u &= H_{n,2}^{(1)}(u)(1-u) + \frac{H_{n,3}^{(1)}(u)+\chi_n^{(1,2)}(u)}{p_n}.
\end{aligned}$$

Recalling (9), we obtain that $H_{n,j}^{(1)}$ are Borel uniformly bounded functions on $[0, 1]$, so that we can use Lemma 10 twice to obtain the desired bound for $\mathbb{E}|S_{n,1}^{(1)}|^3$. Finally, applying Lyapounov's central limit theorem and using the condition $n\,p_n^{-\alpha}L(p_n)\to\infty$ concludes the proof of (7a).

Proofs of (7b) and (7c) are completely similar since $\zeta_n^{(2)}$ and $\zeta_n^{(3)}$ can be rewritten as

$$\begin{aligned}
\zeta_n^{(2)} &= \sum_{k=1}^n S_{n,k}^{(2)} \quad \text{with} \quad S_{n,k}^{(2)} = \frac{1}{n}\left[a_{n,0}^{(2)},\ a_{n,1}^{(2)}\right]\left[X_k^{p_n},\ X_k^{p_n+1}\right]^t, \\
\zeta_n^{(3)} &= \sum_{k=1}^n S_{n,k}^{(3)} \quad \text{with} \quad S_{n,k}^{(3)} = \frac{1}{n}\left[a_{n,0}^{(3)},\ a_{n,1}^{(3)}\right]\left[X_k^{(a+1)p_n},\ X_k^{(a+1)p_n+1}\right]^t
\end{aligned}$$

with clear definitions of the sequences $a_{n,i}^{(j)}$, $i = 0, 1$, $j = 2, 3$. Applying Lemma 10 with

$$\begin{aligned}
H_{n,0}^{(2)}(u) &= -1, \\
H_{n,1}^{(2)}(u) &= \alpha u, \\
H_{n,0}^{(3)}(u) &= \theta^{ap_n}\,\frac{\mu_{p_n+1}}{\mu_{(a+1)p_n+1}}, \\
H_{n,1}^{(3)}(u) &= -\theta^{ap_n}\,\frac{\mu_{p_n+1}}{\mu_{(a+1)p_n+1}}\cdot\frac{\alpha u}{a+1}
\end{aligned}$$

yields $\mathbb{E}|S_{n,1}^{(j)}|^3 = \mathrm{O}(n^{-3}\,\theta^{3p_n}\,p_n^{-\alpha-3}\,L(p_n))$, $j = 2, 3$. Using Lyapounov's central limit theorem then allows us to complete the proof of Theorem 2. ■

Noticing that $\widehat{\theta}_n - \theta = \Theta_n\left[\frac{\widehat{\theta}_n}{\Theta_n} - 1\right] + [\Theta_n - \theta]$, the asymptotic normality of $\widehat{\theta}_n$ centered on the true endpoint θ is a consequence of (6) and Theorem 2.

Theorem 3. *Assume that* (A_1) *and* (A_2) *hold. If* $n\,p_n^{-\alpha}L(p_n)\to\infty$ *and* $n\,p_n^{-\alpha}L(p_n)\eta^2(p_n)\to 0$, *then*

$$v_n\left(\frac{\widehat{\theta}_n}{\theta} - 1\right) \xrightarrow{d} \mathcal{N}(0,\, V(\alpha,\, a)) \quad \textit{as } n\to\infty,$$

with the notations of Theorem 2.

In view of Theorem 3, it may be interesting to estimate the unknown parameter α. From (3), the following estimator is considered:

$$\widehat{\alpha}_n = (p_n + 1)\left[\widehat{\theta}_n \frac{\widehat{\mu}_{p_n}}{\widehat{\mu}_{p_n+1}} - 1\right].$$

Proposition 4. *Under the assumptions of Theorem 3, $\widehat{\alpha}_n = \alpha + \mathrm{O}_{\mathbb{P}}(p_n/v_n)$.*

Proof. Let us introduce $\alpha_n = (p_n + 1)\left[\Theta_n \frac{\mu_{p_n}}{\mu_{p_n+1}} - 1\right]$ and focus first on the random term

$$\frac{v_n}{p_n}(\widehat{\alpha}_n - \alpha_n) = v_n \left[\left[\widehat{\theta}_n - \Theta_n\right] \frac{\widehat{\mu}_{p_n}}{\widehat{\mu}_{p_n+1}} - \Theta_n \frac{\mu_{p_n+1}}{\widehat{\mu}_{p_n+1}} \cdot \frac{\zeta_n^{(2)}}{\mu_{p_n+1}}\right](1 + \mathrm{o}(1))$$

with notations of Lemma 9. Recall that, from Proposition 1, $\mu_{p_n}/\mu_{p_n+1} \to 1/\theta$, from Proposition 2, $\mu_{p_n}/\widehat{\mu}_{p_n} \xrightarrow{\mathbb{P}} 1$ and from (6), $\Theta_n \to \theta$ as $n \to \infty$ so that

$$\frac{v_n}{p_n}(\widehat{\alpha}_n - \alpha_n) = v_n \left(\widehat{\theta}_n - \Theta_n\right)\left[\frac{1}{\theta} + \mathrm{o}_{\mathbb{P}}(1)\right] - \theta v_n \frac{\zeta_n^{(2)}}{\mu_{p_n+1}}(1 + \mathrm{o}_{\mathbb{P}}(1)).$$

Besides, applying Theorem 2 yields $v_n\left(\widehat{\theta}_n - \Theta_n\right) = \mathrm{O}_{\mathbb{P}}(1)$. Now,

$$v_n \frac{\zeta_n^{(2)}}{\mu_{p_n+1}} = \frac{v_n}{\mu_{p_n+1} u_{n,\,a}} u_{n,\,a}\, \zeta_n^{(2)} = \mathrm{O}_{\mathbb{P}}(1),$$

from Lemma 8 and since $u_{n,\,a}\,\zeta_n^{(2)}$ is asymptotically Gaussian (see (7b)). As a preliminary conclusion, we have

$$\frac{v_n}{p_n}(\widehat{\alpha}_n - \alpha_n) = \mathrm{O}_{\mathbb{P}}(1).$$

Turning to the bias term, (6) and Proposition 3 yield

$$\alpha_n = \alpha + (p_n + 1)\,\mathrm{O}\left(\frac{|\eta(p_n)|}{p_n}\right) = \alpha + \mathrm{o}\left(\frac{p_n}{v_n}\right),$$

which completes the proof. ■

By plugging $\widehat{\alpha}_n$ in the asymptotic variance of Theorem 3, classical arguments thus yield:

Corollary 1. *Under the assumptions of Theorem 3,*

$$v_n \sqrt{\frac{1}{V(\widehat{\alpha}_n,\, a)}}\left(\frac{\widehat{\theta}_n}{\theta} - 1\right) \xrightarrow{d} \mathcal{N}(0, 1) \quad \text{as } n \to \infty.$$

Confidence intervals for θ may then be built using this result.

5 Examples

In this section, we highlight some cases where our hypotheses hold. Since $\eta(x) = xL'(x)/L(x)$, one can see that (A_1) and (A_2) are satisfied in the general context of:

1. The Hall model (see Hall, 1982), namely $L(x) = C + Dx^{-\beta}(1+\delta(x))$ for all sufficiently large x, where C, $\beta > 0$, $D \in \mathbb{R} \setminus \{0\}$ and δ is a Borel bounded twice continuously differentiable function on $(1, +\infty)$ such that $\delta(x) \to 0$, $x\delta'(x) \to 0$ and $x^2\delta''(x) \to 0$ as $x \to +\infty$. Here, $\nu = -\beta < 0$.

2. The case where $L(x) = f(\ln x)$, where f is a rational function. Here, $\nu = 0$.

Let us now focus on two particular distributions that are also used for the numerical experiments of Section 6. Both of them have an endpoint $\theta = 1$. The first distribution has survival function

$$\overline{F}(x) = \left[1 + \left(\frac{1}{x} - 1\right)^{-\tau_1}\right]^{-\tau_2}, \quad x \in (0, 1), \tag{11}$$

with $\tau_1, \tau_2 > 0$. Remark that, if X is distributed from (11), then it can be rewritten as $X = 1 - 1/(1+Y)$ where Y is Burr$(1, \tau_1, \tau_2)$ distributed, namely, Y has survival function $\overline{G}(y) = (1 + y^{\tau_1})^{-\tau_2}$ for $y \geq 0$. It can be shown that (A_1) is verified with $\alpha = \tau_1 \tau_2$ and

$$\forall\, y \geq 1, \qquad L(y) = \left[\frac{y^{\tau_1}}{1 + (y-1)^{\tau_1}}\right]^{\tau_2}.$$

L is clearly $\mathcal{C}^\infty$ on $(1, +\infty)$ and one readily obtains

$$\forall\, y > 1, \qquad \eta(y) := y\frac{L'(y)}{L(y)} = \tau_1\, \tau_2\, \frac{1 - (y-1)^{\tau_1 - 1}}{1 + (y-1)^{\tau_1}}.$$

As a result, η is continuously differentiable on $(1, +\infty)$, ultimately monotonic and non identically 0. Besides,

$$y\frac{\eta'(y)}{\eta(y)} = -y\left[(\tau_1 - 1)\frac{(y-1)^{\tau_1-2}}{1-(y-1)^{\tau_1-1}} + \tau_1\, \frac{(y-1)^{\tau_1-1}}{1+(y-1)^{\tau_1}}\right] \to -\min(\tau_1, 1) < 0,$$

as $y \to +\infty$ and thus (A_2) holds with $\nu = -\min(\tau_1, 1)$. Note that one can also show that L belongs to the Hall class. The second considered distribution has survival function

$$\overline{F}(x) = \frac{1}{\Gamma(b)} \int_{-\ln(1-x)}^{\infty} (\lambda t)^{b-1}\, \lambda e^{-\lambda t}\, dt, \quad x \in (0, 1), \tag{12}$$

with $b \geq 1$ and $\lambda > 0$. Here, when X is distributed from (12), it can be rewritten as $X = 1 - e^{-Y}$ where Y is Gamma(b, λ) distributed. Note that, if $b = 1$, then X has survival function $\overline{F}(x) = (1-x)^\lambda$, namely $L \equiv 1$, and (A_1), (A_2) straightforwardly hold. If $b > 1$, then (A_1) holds with $\alpha = \lambda$,

$$\begin{aligned} L(x) &= \frac{\lambda^{b-1}}{\Gamma(b)} \ln^{b-1}(x)[1 + \delta(x)] \\ \delta(x) &= \frac{1}{x^{-\lambda}\, \lambda^{b-1} \ln^{b-1}(x)} \left[\int_{\ln x}^{\infty} (\lambda t)^{b-1}\, \lambda e^{-\lambda t}\, dt\right] - 1 = (b-1) \int_1^\infty u^{b-2}\, e^{-\lambda(u-1)\ln x}\, du. \end{aligned}$$

Note that δ is $\mathcal{C}^\infty$ on $(1, +\infty)$ and goes to 0 at infinity. Therefore, L is slowly varying and $\mathcal{C}^\infty$ on $(1, +\infty)$. Now

$$\begin{aligned}
\eta(x) := x\frac{L'(x)}{L(x)} &= \frac{b-1}{\ln x} + x\delta'(x)(1+\mathrm{o}(1)) \\
&= \frac{b-1}{\ln x} - \lambda(b-1)\int_1^\infty (u-1)u^{b-2}\, e^{-\lambda(u-1)\ln x}\, du(1+\mathrm{o}(1)) \\
&= \frac{b-1}{\ln x} + \mathrm{o}(1/\ln x),
\end{aligned}$$

so that η is slowly varying and positive in a neighborhood of $+\infty$. Finally, noting that

$$\frac{d}{dx}\Big[x\delta'(x)\Big] = \frac{\lambda^2(b-1)}{x}\int_1^\infty (u-1)^2 u^{b-2}\, e^{-\lambda(u-1)\ln x}\, du = \mathrm{o}\left(\frac{1}{x\ln^2 x}\right)$$

it follows that $\eta'(x) = \dfrac{(1-b)}{x\ln^2 x}(1+\mathrm{o}(1))$ entailing that η is ultimately non-increasing and that $x\,\eta'(x)/\eta(x) \to 0$ as $x \to +\infty$. As a conclusion, (A_2) holds with $\nu = 0$.

6 Numerical experiments

In this section, we shall examine the performances of our estimator on samples with size $n = 500$ on eight situations obtained by considering the models (11) and (12) with four different sets of parameters, see the first column of Table 1. We choose $p_n = n^{1/\alpha}/\ln\ln n$ in order to satisfy the assumptions in Theorem 3 and a set $\mathcal{A} = \{0.2, 0.6, 1.0, \ldots, 21\}$ of different values of a is tested. In each of the eight situations, $N = 1000$ replications of the sample are generated and the average L^1-error is computed:

$$E(a) = \frac{1}{N}\sum_{j=1}^{N} |\varepsilon(j, a)|, \quad \text{where} \quad \varepsilon(j, a) = \widehat{\theta}^{(j,a)} - \theta$$

with $\widehat{\theta}^{(j,a)}$ being the estimator computed on the $j-$th replication with $a \in \mathcal{A}$ and $\theta = 1$. Then, the "optimal" value of a is retained: $a^\star = \operatorname{argmin}\{E(a),\, a \in \mathcal{A}\}$. For the sake of comparison, the same procedure has been applied to the extreme-value moment estimator, see for instance de Haan and Ferreira (2006, Remark 4.5.5), which depends on a parameter $k \in \{2, 3, \ldots, n-1\}$. The naive maximum estimator has also been considered. Note that, since the maximum estimator does not depend on any parameter, the associated function E is constant. Numerical results are summarized in Table 1, where $E(a^\star)$ is displayed. In the upper part of the table, it appears that, for the distribution (11), performance of all these estimators decrease as $|\nu|$ decreases. This phenomenon can be explained since ν drives the bias of most extreme-value estimators. For instance, when $|\nu|$ is small, η converges slowly to 0 and Proposition 3 shows that the approximation error of μ_p/μ_{p+1} by M_p/M_{p+1} is large. Besides, the lower part of Table 1 shows that, for the distribution (12), when α increases, performance of all these estimators decrease as well, since the simulated points are getting more and more distant from the endpoint. Let us highlight that, in all the considered situations, our estimator yields slightly better (optimal) results than the maximum estimator and the extreme-value moment estimator.

To further compare the behavior of the estimators in the "optimal" case, boxplots of the associated errors $\varepsilon(j,\, a^*)$ are displayed on Figure 1 and Figure 2. Clearly, the maximum as well as our estimator underestimate the endpoint. However, the error associated to our estimator is smaller than the error of the maximum. Besides, our estimator has a smaller variance than both the maximum estimator and the extreme-value moment estimator.

A graphical comparison on both models of the functions E associated to the three estimators is proposed on Figure 3–6. On model (12), the shape of the curves associated to our estimator and to the extreme-value moment estimator are similar, see Figure 5 and Figure 6. On the contrary, it appears on Figure 3 and Figure 4 that, on model (11), the functions E associated to the extreme-value moment estimator and our estimator have very different shapes, even though they have similar minima. The error associated to the extreme-value moment estimator is very sensitive to the choice of the parameter k whereas the error associated to our estimator is stable for a large panel of a values.

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7 Appendix A: Auxiliary results

Let us set $\overline{F}_1(y) := \overline{F}(\theta y)$ and $\mu_{1,\,p_n} := \mu_{p_n}/\theta^{p_n}$. The first result deals with the behavior of the moment $\mu_{1,\,p_n}$.

Lemma 1. *If (A_0) holds, then $\mu_{1,\,p_n}/\mu_{1,\,p_n+1} \to 1$ as $n \to \infty$.*

As it has been mentioned before, (A_2) implies that $x\,\eta'(x) \to 0$ as $x \to \infty$. The next lemma establishes some consequences of this property.

Lemma 2. *Let φ be a continuously differentiable function on $(1,\,+\infty)$ such that $x\,\varphi'(x) \to 0$ as $x \to +\infty$. Then,*

(i) $t \sup_{x \geq 1} |\varphi(tx) - \varphi((t+1)x)| \to 0$ *as* $t \to \infty$.

(ii) For all $q > 0$, $t \sup_{x \in (0,\,1]} x^q\, |\varphi(tx) - \varphi((t+1)x)| \to 0$ *as* $t \to \infty$.

Before proceeding, let us introduce some more notations. For all $k \in \mathbb{R}$, let P_k be the set of collections of Borel functions $(f_p)_{p \geq 1}$ on $(0,\,1]$ such that

1. $\exists\, p_k \geq 1, \quad \exists\, C_k \geq 0, \quad \forall\, p \geq p_k, \quad \forall\, x \in (0,\,1], \quad |f_p(x)| \leq C_k\, x^k,$
2. $\exists\, p_k \geq 1, \quad \exists\, C_k \geq 0, \quad \forall\, p \geq p_k, \quad \forall\, x \in (0,\,1], \quad p^2 |f_{p+1} - f_p|(x) \leq C_k\, x^k,$
3. $\forall\, x \in (0,\,1], \quad p^2 |f_{p+2} - 2f_{p+1} + f_p|(x) \to 0$ as $p \to \infty$.

Let $P = \bigcap_{k \geq 0} P_k$. Besides, let U be the set of collections of Borel functions $(f_p)_{p \geq 1}$ on $[1,\,+\infty)$ such that

1. $\sup_{x \geq 1} |f_p(x)| = \mathrm{O}(1)$ as $p \to \infty$,
2. $p^2 \sup_{x \geq 1} |f_{p+1} - f_p|(x) = \mathrm{O}(1)$ as $p \to \infty$,
3. $p^2 \sup_{x \geq 1} |f_{p+2} - 2f_{p+1} + f_p|(x) \to 0$ as $p \to \infty$.

These sets will reveal useful to study the asymptotic properties of $\widehat{\theta}_n$ since this estimator is based on increments of sequences of functions. A stability property of the set P is given in the next lemma.

Lemma 3. *Let (f_p), (g_p) be two collections of Borel functions. If for some $k \in \mathbb{R}$, $(f_p) \in P_k$ and $(g_p) \in P$, then $(f_p\, g_p) \in P$.*

We now give a continuity property of some integral transforms defined on P and U.

Lemma 4. *Let $(f_p) \in P$, $(g_p) \in U$ and (u_p), (v_p) be two collections of Borel functions such that $f_p(x) \to f(x)$ for all $x \in (0,\,1]$,*

$$\sup_{x \geq 1} |g_p(x) - g(x)| \to 0, \quad \sup_{0 < x \leq 1} |u_p(x) - u(x)| \to 0 \text{ and } \sup_{x \geq 1} |v_p(x) - v(x)| \to 0 \text{ as } p \to \infty,$$

where f, g, u, v are four Borel functions such that f and u (resp. g and v) are defined on $(0,\, 1]$ (resp. $[1,\, +\infty)$). Assume further that u and v are bounded. Then, for all $k > 1$,

$$\begin{aligned}\int_0^1 x^{-k} f_p(x)\, u_p(x)\, dx &\to \int_0^1 x^{-k} f(x)\, u(x)\, dx,\\ \int_1^{+\infty} x^{-k} g_p(x)\, v_p(x)\, dx &\to \int_1^{+\infty} x^{-k} g(x)\, v(x)\, dx\end{aligned}$$

as $p \to \infty$.

The following lemma provides sufficient conditions on collections of functions to belong to the previous sets.

Lemma 5. *Let (f_p), (g_p) be two collections of Borel functions. Assume that there exist Borel functions F_i and Borel bounded functions G_i, $0 \le i \le 2$, such that*

$$\begin{aligned}\forall\, x \in (0,\, 1], \quad p^2 \left| f_p(x) - \sum_{k=0}^{2} p^{-k} F_k(x) \right| &\to 0 \text{ as } p \to \infty,\\ p^2 \sup_{x \ge 1} \left| g_p(x) - \sum_{k=0}^{2} p^{-k} G_k(x) \right| &\to 0 \text{ as } p \to \infty.\end{aligned}$$

Then, for all $x \in (0,\, 1]$, $p^2 |f_{p+2} - 2 f_{p+1} + f_p|(x) \to 0$ as $p \to \infty$, and $(g_p) \in U$.

We are now in position to exhibit two particular elements of P and U:

Lemma 6. *Let (f_p) and (g_p), $p \ge 1$ be two collections of Borel functions defined by*

$$\begin{aligned}\forall\, x \in (0,\, 1], \quad f_p(x) &= \left(1 - \frac{1}{p}\right)^{-\alpha-1} \left(1 + \frac{1}{(p-1)x}\right)^{-\alpha-2} \left(1 - \frac{1}{(p-1)x+1}\right)^{p-1},\\ \forall\, x \in [1,\, +\infty), \quad g_p(x) &= \left(1 - \frac{1}{px}\right)^{p-1}.\end{aligned}$$

Then $(f_p) \in P$, $(g_p) \in U$ and

$$\forall\, x \in (0,\, 1], \quad f_p(x) \to e^{-1/x} \text{ and } \sup_{x \ge 1} \left| g_p(x) - e^{-1/x} \right| \to 0 \text{ as } p \to \infty. \tag{13}$$

Lemma 7 is the key tool for establishing precise expansions of the moments μ_p and M_p.

Lemma 7. *Let $(f_p) \in P$ and $(g_p) \in U$ such that (13) holds and define*

$$\begin{aligned}E_1(p) &= \frac{1}{I_1} \int_0^1 f_p(x) x^{-\alpha-2}\, dx - 1, & I_1 &= \int_1^{+\infty} y^{\alpha} e^{-y}\, dy,\\ E_2(p) &= \frac{1}{I_2} \int_1^{+\infty} g_p(x) x^{-\alpha-2}\, dx - 1, & I_2 &= \int_0^1 y^{\alpha} e^{-y}\, dy,\\ \delta_1(p) &= \frac{1}{I_1} \int_0^1 f_p(x) \left[\frac{L_1((p-1)x)}{L_1(p-1)} - x \right] x^{-\alpha-3}\, dx, & L_1(x) &= x L(x+1),\\ \delta_2(p) &= \frac{1}{I_2} \int_1^{+\infty} g_p(x) \left[\frac{L_2(px)}{L_2(p)} - \frac{1}{x} \right] x^{-\alpha-1}\, dx, & L_2(x) &= L(x)/x,\end{aligned}$$

where L is a slowly varying function at infinity. Then, for all $i = 1,\, 2$, *(i)* $E_i(p) \to 0$ *as* $p \to \infty$,

(ii) $p^2(E_i(p+1) - E_i(p)) = \mathrm{O}(1)$,

(iii) $p^2(E_i(p+2) - 2\,E_i(p+1) + E_i(p)) \to 0$ *as* $p \to \infty$,

(iv) $\delta_i(p) \to 0$ *as* $p \to \infty$.

Moreover, if L *satisfies* (A_2)*, then*

(v) There exists a slowly varying function $\mathcal{L}$ *such that* $\delta_1(p) = \mathrm{O}\left(|\eta(p)|\,\mathcal{L}(p)\right)$,

(vi) $\delta_2(p) = \mathrm{O}\left(|\eta(p)|\right)$,

(vii) For all $i = 1,\, 2$, $\delta_i(p+1) - \delta_i(p) = \mathrm{O}\left(|\eta(p)|/p\right)$,

(viii) For all $i = 1,\, 2$, $p^2(\delta_i(p+2) - 2\,\delta_i(p+1) + \delta_i(p)) \to 0$ *as* $p \to \infty$.

Sometimes, a first order expansion of the moment μ_p is sufficient:

Lemma 8. *If* (A_1) *holds then, as* $p \to \infty$,

$$\mu_p = p^{-\alpha}\,\theta^p\,L(p)\,\Gamma(\alpha+1)(1+\mathrm{o}(1)).$$

The next result consists in linearizing the quantity ξ_n appearing in the proof of Theorem 2:

Lemma 9. *Let* $p_n \to \infty$ *and* $\nu_p = \widehat{\mu}_p - \mu_p$*. If* (A_1) *is satisfied, then*

$$\xi_n = u_{n,\,a}\left[\zeta_n^{(1)} + \left(\frac{\mu_{p_n+1}}{\widehat{\mu}_{p_n+1}} - 1\right)\zeta_n^{(2)} + \left(1 + \frac{a p_n}{p_n+1}\right)\left(\frac{\mu_{(a+1)p_n+1}}{\widehat{\mu}_{(a+1)p_n+1}} - 1\right)\zeta_n^{(3)}\right](1+\mathrm{o}(1)),$$

where

$$\begin{aligned}
\zeta_n^{(1)} &= \zeta_n^{(2)} + \left[1 + \frac{a p_n}{p_n+1}\right]\zeta_n^{(3)}, \\
\textit{with}\quad \zeta_n^{(2)} &= -\nu_{p_n} + \frac{\mu_{p_n}}{\mu_{p_n+1}}\nu_{p_n+1}, \\
\zeta_n^{(3)} &= \frac{\mu_{p_n+1}}{\mu_{(a+1)p_n+1}}\left[\nu_{(a+1)p_n} - \frac{\mu_{(a+1)p_n}}{\mu_{(a+1)p_n+1}}\,\nu_{(a+1)p_n+1}\right] \\
\textit{and}\quad u_{n,\,a} &= \frac{1}{a\,\Gamma(\alpha+1)}\sqrt{\frac{1}{V(\alpha,\,a)}}\,\frac{p_n^{\alpha} v_n}{\theta^{p_n}\,L(p_n)}.
\end{aligned}$$

The final lemma of this section provides an asymptotic bound of the third-order moments appearing in the proof of Theorem 2.

Lemma 10. *Let* $k \in \mathbb{N}$ *and* $p_n \to \infty$*. Let* $(H_{n,\,j})_{0 \le j \le k}$ *be sequences of Borel uniformly bounded functions on* $[0,\,1]$ *and*

$$\forall\, u \in [0,\,1], \qquad h_n(u) = \sum_{j=0}^{k} \frac{H_{n,\,j}(u)}{p_n^j}(1-u)^{k-j}.$$

If Y *is a random variable with survival function* $\overline{G}(x) = (1-x)^{\alpha}\,L((1-x)^{-1})$ *where* $\alpha > 0$ *and* L *is a Borel slowly varying function at infinity, then*

$$\mathbb{E}|Y^{p_n}\,h_n(Y)|^3 = \mathrm{O}(p_n^{-\alpha-3k}\,L(p_n)).$$

8 Appendix B: Proofs

Proof of Lemma 1. Let $I_{p_n} := \mu_{1,\,p_n}/p_n$ and $\varepsilon > 0$. The integral I_{p_n} is expanded as

$$I_{p_n} = \int_{1-\varepsilon}^{1} y^{p_n-1}\overline{F_1}(y)\,dy \left[1 + \frac{\displaystyle\int_0^{1-\varepsilon} y^{p_n-1}\overline{F_1}(y)\,dy}{\displaystyle\int_{1-\varepsilon}^{1} y^{p_n-1}\overline{F_1}(y)\,dy} \right]$$

where

$$0 \le \frac{\displaystyle\int_0^{1-\varepsilon} y^{p_n-1}\overline{F_1}(y)\,dy}{\displaystyle\int_{1-\varepsilon}^{1} y^{p_n-1}\overline{F_1}(y)\,dy} \le \frac{1-\varepsilon}{\displaystyle\int_{1-\varepsilon}^{1} \left[\frac{y}{1-\varepsilon}\right]^{p_n-1} \overline{F_1}(y)\,dy} \le \frac{1-\varepsilon}{\displaystyle\left[\frac{1-\varepsilon/2}{1-\varepsilon}\right]^{p_n-1} \int_{1-\varepsilon/2}^{1} \overline{F_1}(y)\,dy}.$$

Since $\left[\dfrac{1-\varepsilon/2}{1-\varepsilon}\right]^{p_n-1} \to \infty$ as $n \to \infty$, it follows that

$$I_{p_n} = \int_{1-\varepsilon}^{1} y^{p_n-1}\overline{F_1}(y)\,dy\,(1+\mathrm{o}(1)). \tag{14}$$

In view of

$$1 \le \int_{1-\varepsilon}^{1} y^{p_n-1}\overline{F_1}(y)\,dy \Big/ \int_{1-\varepsilon}^{1} y^{p_n}\overline{F_1}(y)\,dy \le \frac{1}{1-\varepsilon}$$

and (14), one thus has $I_{p_n}/I_{p_n+1} \to 1$ as $n \to \infty$ and Lemma 1 is proved. ■

Proof of Lemma 2. If φ' is identically 0, then φ is constant on $[1, +\infty)$ and the results are straightforward. Otherwise, let us consider (i) and (ii) separately.

(i) Let $t, x \ge 1$. The mean value theorem shows that there exists $h_1(t, x) \in (0, 1)$ such that

$$\begin{aligned} t\,|\varphi(tx) - \varphi((t+1)x)| &= \frac{t}{t+h_1(t,\,x)}\,|[(t+h_1(t,\,x))x]\;\varphi'\,[(t+h_1(t,\,x))x]| \\ &\le \sup_{y\ge t} |y\,\varphi'(y)| \to 0 \end{aligned}$$

uniformly in $x \ge 1$, as $t \to +\infty$.

(ii) Let $t \ge 1$ and $x \in (0, 1]$, $q > 0$, $\varepsilon > 0$ and

$$c(\varepsilon) := \left[\frac{\varepsilon}{2} \cdot \frac{1}{\sup_{y>1} |y\,\varphi'(y)|}\right]^{1/q}.$$

Applying the mean value theorem again shows that there exists $h_2(t, x) \in (0, 1)$ such that

$$\begin{aligned} tx^q\;|\varphi(tx) - \varphi((t+1)x)| &= x^q\,\frac{t}{t+h_2(t,\,x)}\,|[(t+h_2(t,\,x))x]\;\varphi'\,[(t+h_2(t,\,x))x]| \\ &\le x^q \sup_{y>1} |y\,\varphi'(y)| \mathbb{1}_{\{0<x<c(\varepsilon)\}} + \sup_{y\ge t\,c(\varepsilon)} |y\,\varphi'(y)| \mathbb{1}_{\{c(\varepsilon)\le x\le 1\}} \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all t large enough, uniformly in $x \in (0, 1]$, which concludes the proof of Lemma 2. ■

Proof of Lemma 3. This result easily follows from the identities

$$\begin{aligned}
(fg)_{p+1}-(fg)_p &= f_{p+1}(g_{p+1}-g_p)+g_p(f_{p+1}-f_p),\\
(fg)_{p+2}-2(fg)_{p+1}+(fg)_p &= (f_{p+2}-2f_{p+1}+f_p)\,g_{p+2}+(f_{p+1}-f_p)\,(g_{p+2}-g_p)\\
&+ f_{p+1}\,(g_{p+2}-2g_{p+1}+g_p),
\end{aligned}$$

and from the properties of (f_p) and (g_p). ■

Proof of Lemma 4. Remark that, for p large enough,

$$\forall\, x\in(0,\,1],\quad x^{-k}\,|f_p(x)|\,|u_p(x)|\le C_k\big\{|u(x)|+r(x)\big\}$$

where r is a bounded Borel function on $(0,\,1]$. The upper bound is an integrable function on $(0,\,1]$, so that the dominated convergence theorem yields

$$\int_0^1 x^{-k} f_p(x)\,u_p(x)\,dx\to\int_0^1 x^{-k} f(x)\,u(x)\,dx$$

as $p\to\infty$, which proves the first part of the lemma.

Since v is bounded on $[1,\,+\infty)$, $(g_p\,v_p)$ converges uniformly to gv on $[1,\,+\infty)$. The function $x\mapsto x^{-k}$ being integrable on $[1,\,+\infty)$, the dominated convergence theorem yields

$$\int_1^{+\infty} x^{-k} g_p(x)\,v_p(x)\,dx\to\int_1^{+\infty} x^{-k} g(x)\,v(x)\,dx$$

as $p\to\infty$, which concludes the proof of Lemma 4. ■

Proof of Lemma 5. Remark that

$$\frac{1}{p+1}-\frac{1}{p}=\mathrm{O}\left(\frac{1}{p^2}\right)\ \text{and}\ \frac{1}{p+2}-\frac{2}{p+1}+\frac{1}{p}=\mathrm{O}\left(\frac{1}{p^3}\right)$$

to obtain the result. ■

Proof of Lemma 6. – It is clear that for all $x\in(0,\,1]$, $f_p(x)\to e^{-1/x}$ as $p\to\infty$.
– In order to prove that $(f_p)\in P$, let us rewrite $f_p(x)$ as $f_p(x)=\sigma_p\,\varphi_p(x)\,\psi_p(x)$ where

$$\sigma_p=\left(1-\frac{1}{p}\right)^{-\alpha-1},\quad \varphi_p(x)=\left(1+\frac{1}{(p-1)x}\right)^{-\alpha-2},\quad \psi_p(x)=\left(1-\frac{1}{(p-1)x+1}\right)^{p-1},$$

for all $x\in(0,\,1]$, and prove that $(\sigma_p)\in P_0$, $(\varphi_p)\in P_{-1}$ and $(\psi_p)\in P$. First, note that

$$\sigma_p=1+\frac{\alpha+1}{p}+\frac{(\alpha+1)(\alpha+2)}{2}\,\frac{1}{p^2}+\mathrm{o}\left(\frac{1}{p^2}\right)$$

so that the collection of constant functions (σ_p) lies in P_0. Second, we have

$$\forall\, p>1,\quad \forall x\in(0,\,1],\quad |\varphi_p(x)|\le 1\le x^{-1}. \tag{15}$$

Moreover,

$$[\varphi_{p+1} - \varphi_p](x) = \varphi_p(x) \left[\left(1 - \frac{1}{p}\right)^{-\alpha-2} \left(1 - \frac{x}{px+1}\right)^{\alpha+2} - 1 \right],$$

and since $\forall\, x \in (0,\, 1]$, $x/(px+1) \leq 1/p$, Taylor expansions yield, uniformly in $x \in (0,\, 1]$,

$$[\varphi_{p+1} - \varphi_p](x) = \varphi_p(x) \left[\frac{\alpha+2}{p(px+1)} + \mathrm{O}\left(\frac{1}{p^2}\right) \right].$$

It follows that there exists a positive constant $C^{(1)}$ such that for p large enough,

$$p^2 |\varphi_{p+1} - \varphi_p|(x) \leq C^{(1)}\, x^{-1}. \tag{16}$$

Third, let $x \in (0,\, 1]$, and consider a pointwise Taylor expansion of φ_p to get

$$\varphi_p(x) = 1 - \frac{\alpha+2}{px} + \frac{\alpha+2}{p^2 x}\left(-1 + \frac{\alpha+3}{2x}\right) + \mathrm{o}\left(\frac{1}{p^2}\right).$$

Using (15), (16) and applying Lemma 5 therefore shows that $(\varphi_p) \in P_{-1}$.

Let $x \in (0,\, 1]$, $k \geq 0$, $\Psi_x(p) = (1 - 1/(px+1))^p$, so that $\psi_p(x) = \Psi_x(p-1)$. Routine calculations show that $\Psi_x(p)$ is a positive non-increasing function of p. Consequently, for all sufficiently large p and for all $x \in (0,\, 1]$, $\psi_p(x) \leq \psi_{k+1}(x)$. Remarking that $\psi_{k+1}(x) \leq k^k\, x^k$ for all $x \in (0,\, 1]$, it follows that

$$\forall\, k \geq 0, \quad \exists\, p_k \geq 1, \quad \exists\, C_k \geq 0, \quad \forall\, p \geq p_k, \quad \forall\, x \in (0,\, 1], \quad |\psi_p(x)| \leq C_k\, x^k. \tag{17}$$

Recall that Ψ_x is non-increasing and write

$$|\psi_{p+1} - \psi_p|(x) = \psi_p(x) \left[1 - \left(1 - \frac{1}{px+1}\right)\left(1 + \frac{1}{p-1}\right)^{p-1} \left(1 - \frac{x}{px+1}\right)^{p-1} \right].$$

Taylor expansions of the logarithm function at 1 and of the exponential function at 0 imply that, uniformly in $x \in (0,\, 1]$,

$$e\left(1 - \frac{x}{px+1}\right)^{p-1} = \exp\left(\frac{1}{px+1}\right)\left[1 + \frac{x}{px+1} - \frac{p}{2}\left(\frac{x}{px+1}\right)^2 + \mathrm{O}\left(\frac{1}{p^2}\right)\right].$$

Since for all $x \in (0,\, 1]$, $0 \leq 1/(px+1) \leq 1$, applying the mean value theorem to the function $h \mapsto (1-h)e^h$ gives

$$\left| \left[1 - \frac{1}{px+1}\right] \exp\left[\frac{1}{px+1}\right] - 1 \right| \leq \frac{e}{(px+1)^2}.$$

A Taylor expansion of $\left[1 + \dfrac{1}{p-1}\right]^{p-1}$ then yields, uniformly in $x \in (0,\, 1]$,

$$|\psi_{p+1} - \psi_p|(x) \leq \psi_p(x) \left[\left(e + \frac{1}{2p}\right) \frac{1}{(px+1)^2} + \mathrm{O}\left(\frac{1}{p^2}\right) \right].$$

Therefore, there exists $C^{(3)} \geq 0$ such that, for all p large enough,

$$p^2\, |\psi_{p+1} - \psi_p|\,(x) \leq \psi_p(x)\, C^{(3)}\, x^{-2}.$$

Taking (17) into account, this entails

$$\forall\, k \geq 0, \quad \exists\, p_k \geq 1, \quad \exists\, C_k \geq 0, \quad \forall\, p \geq p_k, \quad \forall\, x \in (0,\, 1], \quad p^2 |\psi_{p+1} - \psi_p|(x) \leq C_k\, x^k. \qquad (18)$$

A pointwise Taylor expansion of ψ_p finally gives

$$\psi_p(x) = e^{-1/x} \left[1 + \frac{1}{2px^2} + \frac{1}{p^2 x^2} \left(\frac{1}{2} - \frac{1}{3x} + \frac{1}{8x^2} \right) + \mathrm{o}\left(\frac{1}{p^2} \right) \right].$$

Using (17), (18) and applying Lemma 5 shows that $(\psi_p) \in P$. Lemma 3 therefore shows that $(f_p) \in P$.

– Finally, a Taylor expansion entails

$$p^2 \sup_{x \geq 1} \left| g_p(x) - e^{-1/x} \left[1 + \frac{1}{px} \left(1 - \frac{1}{2x} \right) + \frac{1}{p^2 x^2} \left(\frac{1}{2} - \frac{5}{6x} + \frac{1}{8x^2} \right) \right] \right| \to 0$$

as $p \to \infty$. It follows that $g_p(x) \to e^{-1/x}$ as $p \to \infty$ uniformly on $[1,\, +\infty)$. Lemma 5 then shows that $(g_p) \in U$. ■

Proof of Lemma 7. (i), (ii) and (iii) are simple consequences of $(f_p) \in P$, $(g_p) \in U$, (13) and of the dominated convergence theorem.

(iv) Let us introduce

$$\forall\, x \in (0,\, 1],\ Q_p^{(1)}(x) = \frac{L_1((p-1)x)}{L_1(p-1)} - x \ \text{ and } \ \forall\, x \geq 1,\ Q_p^{(2)}(x) = \frac{L_2(px)}{L_2(p)} - \frac{1}{x},$$

so that

$$I_1\, \delta_1(p) = \int_0^1 f_p(x) Q_p^{(1)}(x) x^{-\alpha-3}\, dx \ \text{ and } \ I_2\, \delta_2(p) = \int_1^{+\infty} g_p(x) Q_p^{(2)}(x) x^{-\alpha-1}\, dx.$$

First, remark that $x \mapsto L(x+1)$ is a slowly varying function, so that according to Bingham *et al.* (1987), Proposition 1.3.6(iii), L_1 is regularly varying with index 1. Bingham *et al.* (1987), Theorem 1.5.2 thus entails $Q_p^{(1)}(x) \to 0$ uniformly in $x \in (0,\, 1]$ as $p \to \infty$. Applying Lemma 4 yields $\delta_1(p) \to 0$ as $p \to \infty$. Second, since L_2 is regularly varying with index -1, using again Bingham *et al.* (1987), Theorem 1.5.2 leads to $Q_p^{(2)}(x) \to 0$ as $p \to \infty$ uniformly in $x \geq 1$. Applying Lemma 4 again entails $\delta_2(p) \to 0$ as $p \to \infty$.

(v) Let p be large enough so that $|\eta|$ is non-increasing in $[p,\, +\infty)$. Pick $s > 1 - \nu$ and let $Q_p^{(1,\,1)}(x) := x^s Q_p^{(1)}(x)$. Using the ideas of the proof of Lemma 4, one has

$$I_1\, \delta_1(p) = \int_0^1 f_p(x)\, Q_p^{(1,\,1)}(x) x^{-\alpha-s-3}\, dx = \mathrm{O}\left(\sup_{0<x\leq 1} x^{-1} \left| Q_p^{(1,\,1)}(x) \right| \right).$$

Introducing

$$R_p^{(1)}(x) := \int_{(p-1)x+1}^{p} \frac{\eta(t)}{t}\, dt,$$

(A_2) and the well-known inequality $|e^u - 1| \leq |u| e^{|u|}$ for all $u \in \mathbb{R}$ yield

$$\begin{aligned} \sup_{0<x\leq 1} x^{-1} \left| Q_p^{(1,\,1)}(x) \right| &= \mathrm{O}\left(\sup_{0<x\leq 1} \left\{ x^s \left| 1 - \exp\left(-R_p^{(1)}(x) \right) \right| \right\} \right) \\ &= \mathrm{O}\left(\sup_{0<x\leq 1} \left\{ x^s \left| R_p^{(1)}(x) \right| \exp\left| R_p^{(1)}(x) \right| \right\} \right). \qquad (19) \end{aligned}$$

Letting $\widetilde{\eta}(t) = (t+1)^{1-\nu}\,\eta(t+1)$, we get

$$x^s\left|R_p^{(1)}(x)\right| \le x^{s+1}\int_{p-1}^{(p-1)/x}\left[\left|\frac{\widetilde{\eta}(ux)}{\widetilde{\eta}(u)} - x\right| + x\right]\cdot|\widetilde{\eta}(u)|\frac{du}{(ux+1)^{2-\nu}}.$$

Remarking that $\widetilde{\eta}$ is regularly varying with index 1, Bingham *et al.* (1987), Theorem 1.5.2 implies that for p large enough,

$$\sup_{\substack{0<x\le 1\\ u\ge p-1}}\left|\frac{\widetilde{\eta}(ux)}{\widetilde{\eta}(u)} - x\right| \le 1.$$

Moreover, for all $u>0$ and $x\in(0,\,1]$, one has $x/(ux+1)\le 1/u$, so that, for p large enough,

$$\begin{aligned}
x^s\left|R_p^{(1)}(x)\right| &\le 2\,x^{s-(1-\nu)}\int_{p-1}^{(p-1)/x}|\widetilde{\eta}(u)|\left[\frac{x}{ux+1}\right]^{2-\nu}du\\
&\le 2\,x^{s-(1-\nu)}\int_{p-1}^{(p-1)/x}|\eta(u+1)|\frac{(u+1)^{1-\nu}}{u^{2-\nu}}\,du\\
&\le 2^{2-\nu}\,|\eta(p)|\,x^{s-(1-\nu)}\int_{p-1}^{(p-1)/x}\frac{du}{u}\\
&\le 2^{2-\nu}\,|\eta(p)|\sup_{x\in(0,\,1]}\left|x^{s-(1-\nu)}\,\ln x\right|\\
&= \mathrm{O}\left(|\eta(p)|\right), \qquad (20)
\end{aligned}$$

uniformly in $x\in(0,\,1]$ since $x\mapsto x^{s-(1-\nu)}\ln x$ is bounded on $(0,\,1]$. Let us now consider $\mathcal{L}(y) = \exp\left(\int_1^y|\eta(t)|\,t^{-1}\,dt\right)$. Clearly, $\mathcal{L}$ is slowly varying at infinity and $\exp|R_p^{(1)}(x)|\le\mathcal{L}(p)$. Consequently, in view of (19) and (20), it follows that

$$\sup_{0<x\le 1}x^{-1}\left|Q_p^{(1,\,1)}(x)\right| = \mathrm{O}(|\eta(p)|\,\mathcal{L}(p)), \qquad (21)$$

and therefore $\delta_1(p) = \mathrm{O}(|\eta(p)|\,\mathcal{L}(p))$.

(vi) Similarly, for all $x\ge 1$ and large p, we have

$$\left|Q_p^{(2)}(x)\right| \le x^{-1}\left|\int_p^{px}\frac{\eta(t)}{t}\,dt\right|\exp\left|\int_p^{px}\frac{\eta(t)}{t}\,dt\right| \le |\eta(p)|\cdot\ln x\cdot x^{|\eta(p)|-1}. \qquad (22)$$

Let p be so large that $|\eta(p)|\le 1$. Since $x\mapsto x^{-\alpha-1}\ln x$ is integrable on $[1,\,+\infty)$, the arguments of the proof of Lemma 4 entail

$$I_2\,\delta_2(p) = \int_1^{+\infty}g_p(x)Q_p^{(2)}(x)x^{-\alpha-1}\,dx \le |\eta(p)|\int_1^{+\infty}g_p(x)\,x^{-\alpha-1}\ln x\,dx = \mathrm{O}\left(|\eta(p)|\right).$$

(vii) Keeping in mind that $s>1-\nu$, the following expansion holds

$$\begin{aligned}
I_1\left[\delta_1(p+1)-\delta_1(p)\right] &= \int_0^1 f_p(x)\,[Q_{p+1}^{(1,\,1)} - Q_p^{(1,\,1)}](x)x^{-\alpha-s-3}\,dx \qquad (23)\\
&+ \int_0^1[f_{p+1}-f_p](x)\,Q_{p+1}^{(1,\,1)}(x)x^{-\alpha-s-3}\,dx. \qquad (24)
\end{aligned}$$

Let us first focus on (23). In view of (A_2), and considering

$$R_p^{(2)}(x) := \int_{(p-1)x+1}^{px+1}\frac{\eta(t)}{t}\,dt,$$

for $x \in (0,\, 1]$, one obtains

$$[Q^{(1,\,1)}_{p+1} - Q^{(1,\,1)}_{p}](x) = x^s\, \frac{L_1(px)}{L_1(p-1)}\, \frac{p-1}{p} \left\{ \exp\left(-R^{(2)}_p(1)\right) - \exp\left(-R^{(2)}_p(x)\right) \right\}.$$

Mimicking the proof of (v), we thus get for p large enough

$$\begin{aligned} x^s \left| R^{(2)}_p(x) \right| \;&\leq\; x^{s+1} \int_{p-1}^{p} \left[\left| \frac{\widetilde{\eta}(ux)}{\widetilde{\eta}(u)} - x \right| + x \right] \cdot |\widetilde{\eta}(u)| \frac{du}{(ux+1)^{2-\nu}} \\ &\leq\; 2^{2-\nu}\, |\eta(p)|\, \ln\left(1 + \frac{1}{p-1}\right) \\ &=\; \mathrm{O}\left(|\eta(p)|/p\right), \end{aligned} \tag{25}$$

uniformly in $x \in (0,\, 1]$. A Taylor expansion of the exponential function at 0 then entails

$$\exp\left(-R^{(2)}_p(x)\right) = 1 - \frac{1}{x^s} \left\{ x^s R^{(2)}_p(x) \right\} \left[1 + \rho\left(R^{(2)}_p(x) \right) \right]$$

where ρ is locally bounded on $\mathbb{R}$. Since

$$\sup_{\substack{0<x\leq 1\\ p\geq 1}} \left| R^{(2)}_p(x) \right| \leq \sup_{t\geq 1} |\eta(t)| \quad \Rightarrow \quad \sup_{\substack{0<x\leq 1\\ p\geq 1}} \left| \rho\left(R^{(2)}_p(x) \right) \right| < +\infty, \tag{26}$$

it follows that

$$\sup_{0<x\leq 1} \left\{ x^s \left| \exp\left(-R^{(2)}_p(x)\right) - 1 \right| \right\} = \mathrm{O}\left(|\eta(p)|/p\right).$$

Applying Bingham *et al.* (1987), Theorem 1.5.2 to L_1 yields

$$\sup_{0<x\leq 1} |Q^{(1,\,1)}_{p+1} - Q^{(1,\,1)}_{p}|(x) = \mathrm{O}\left(|\eta(p)|/p\right)$$

and consequently,

$$\int_0^1 f_p(x)\, [Q^{(1,\,1)}_{p+1} - Q^{(1,\,1)}_{p}](x) x^{-\alpha-s-3}\, dx = \mathrm{O}\left(|\eta(p)|/p\right). \tag{27}$$

Focusing on (24), for all $0 < x \leq 1$, because $(f_p) \in P$, we get for all sufficiently large p

$$p^2\, x^{-\alpha-s-2}\, |f_{p+1} - f_p|(x) \leq C_{\alpha+s+2}$$

which is integrable on $(0,\, 1]$. Consequently, in view of (21),

$$\int_0^1 [f_{p+1} - f_p](x)\, Q^{(1,\,1)}_{p+1}(x) x^{-\alpha-s-3}\, dx = \mathrm{O}\left(\frac{|\eta(p)|\, \mathcal{L}(p)}{p^2} \right) = \mathrm{O}\left(|\eta(p)|/p\right). \tag{28}$$

Collecting (27) and (28) yields $\delta_1(p+1) - \delta_1(p) = \mathrm{O}\left(|\eta(p)|/p\right)$. Let us remark that

$$I_2\, [\delta_2(p+1) - \delta_2(p)] \;=\; \int_1^{+\infty} g_p(x)\, [Q^{(2)}_{p+1} - Q^{(2)}_{p}](x)\, x^{-\alpha-1}\, dx \tag{29}$$

$$+\; \int_1^{+\infty} [g_{p+1} - g_p](x)\, Q^{(2)}_{p+1}(x)\, x^{-\alpha-1}\, dx, \tag{30}$$

and consider first (29). From (A_2), we have

$$[Q^{(2)}_{p+1} - Q^{(2)}_{p}](x) = \frac{L_2((p+1)x)}{L_2(p)}\, \frac{p+1}{p} \left[\exp\left(-\int_p^{p+1} \frac{\eta(t)}{t}\, dt \right) - \exp\left(-\int_p^{p+1} \frac{\eta(tx)}{t}\, dt \right) \right].$$

Since for all $x \geq 1$, we have $p \left| \int_p^{p+1} \eta(tx)\, t^{-1}\, dt \right| \leq |\eta(p)|$, and recalling that, as $p \to \infty$

$$\sup_{x \geq 1} \left| \frac{L_2((p+1)x)}{L_2(p)} - \frac{1}{x} \right| \to 0,$$

a Taylor expansion of the exponential function at 0 yields

$$\sup_{x \geq 1} |Q^{(2)}_{p+1} - Q^{(2)}_p|(x) = \mathrm{O}\left(|\eta(p)|/p\right). \tag{31}$$

Taking into account that $(g_p) \in U$ and using (22), it follows that

$$\int_1^{+\infty} [g_{p+1} - g_p](x)\, Q^{(2)}_{p+1}(x) x^{-\alpha-1}\, dx = \mathrm{O}\left(|\eta(p)|/p\right).$$

Moreover, from (31), the uniform convergence of (g_p) to $x \mapsto e^{-1/x}$ on $[1, +\infty)$ and the dominated convergence theorem, we get

$$\int_1^{+\infty} g_p(x)\, [Q^{(2)}_{p+1} - Q^{(2)}_p](x) x^{-\alpha-1}\, dx = \mathrm{O}\left(|\eta(p)|/p\right).$$

This eventually leads to $\delta_2(p+1) - \delta_2(p) = \mathrm{O}\left(|\eta(p)|/p\right)$ and establishes (vii).

(viii) Let $q > 1 - \nu$ and $Q^{(1,\,2)}_p(x) = x^{2q+1} Q^{(1)}_p(x)$ so that

$$I_1\, \delta_1(p) = \int_0^1 f_p(x)\, Q^{(1,\,2)}_p(x) x^{-\alpha-2q-4}\, dx$$

and the following expansion holds

$$\begin{aligned}
&I_1 \left\{\delta_1(p+2) - 2\,\delta_1(p+1) + \delta_1(p)\right\} \\
&\quad = \int_0^1 [f_{p+1} - f_p](x)\, [Q^{(1,\,2)}_{p+2} - Q^{(1,\,2)}_p](x) x^{-\alpha-2q-4}\, dx && (32)\\
&\quad + \int_0^1 f_{p+1}(x)\, [Q^{(1,\,2)}_{p+2} - 2Q^{(1,\,2)}_{p+1} + Q^{(1,\,2)}_p](x) x^{-\alpha-2q-4}\, dx && (33)\\
&\quad + \int_0^1 [f_{p+2} - 2f_{p+1} + f_p](x)\, Q^{(1,\,2)}_{p+2}(x) x^{-\alpha-2q-4}\, dx. && (34)
\end{aligned}$$

Considering (32), arguments given in the proof of (vii) show that

$$\int_0^1 [f_{p+1} - f_p](x)\, [Q^{(1,\,2)}_{p+2} - Q^{(1,\,2)}_p](x) x^{-\alpha-2q-4}\, dx = \mathrm{o}\left(1/p^2\right). \tag{35}$$

Let us now focus on (33). From (25), (26) and (A_2), a Taylor expansion yields

$$\begin{aligned}
&[Q^{(1,\,2)}_{p+2} - 2Q^{(1,\,2)}_{p+1} + Q^{(1,\,2)}_p](x) \\
&\quad = x^{2q+1} \frac{L_1((p+1)x)}{L_1(p-1)} \frac{p-1}{p+1} \left\{ R^{(2)}_p(1) - R^{(2)}_{p+1}(1) + R^{(2)}_{p+1}(x) - R^{(2)}_p(x) \right\} + \mathrm{o}(1/p^2),
\end{aligned}$$

uniformly in $x \in (0,\, 1]$. Let $x \in (0,\, 1]$: using the inequality $x/(tx+1) \leq 1/t$, we obtain

$$\begin{aligned}
&x^{2q+1} \left| R^{(2)}_{p+1}(x) - R^{(2)}_p(x) \right| \\
&\quad = x^{2q+2} \left| \int_{p-1}^{p} \frac{\eta((t+1)x+1) - \eta(tx+1)}{(t+1)x+1} - \frac{x\,\eta(tx+1)}{(tx+1)((t+1)x+1)}\, dt \right| \\
&\quad \leq x^q \left[\sup_{t \geq p-1} \left\{ t \left| \eta((t+1)x+1) - \eta(tx+1) \right| \right\} + \sup_{t \geq p-1} \left\{ x \left| \eta(tx+1) \right| \right\} \right] \cdot \frac{1}{p(p-1)}.
\end{aligned}$$

Moreover, since, as $p \to \infty$

$$\sup_{t \ge p-1} \left\{ x^{q+1} \left| \eta(tx+1) \right| \right\} \le (p-1)^{-1-1/q} \sup_{t \ge 1} |\eta(t)| + \sup_{t \ge (p-1)^{1-1/q}} |\eta(t)| \to 0,$$

Lemma 2(ii) entails that, as $p \to \infty$,

$$p^2 \sup_{0 < x \le 1} |Q_{p+2}^{(1,2)} - 2Q_{p+1}^{(1,2)} + Q_p^{(1,2)}|(x) \to 0.$$

The dominated convergence theorem then yields

$$\int_0^1 f_{p+1}(x) \, [Q_{p+2}^{(1,2)} - 2Q_{p+1}^{(1,2)} + Q_p^{(1,2)}](x) x^{-\alpha-2q-4} \, dx = \mathrm{o}\left(1/p^2\right). \tag{36}$$

Let us finally consider (34). Since $(f_p) \in P$ and in view of the triangular inequality, we have, for p large enough,

$$p^2 \, x^{-\alpha-2q-4} \, |f_{p+2} - 2f_{p+1} + f_p|(x) \le C_{\alpha+2q+4}.$$

Because $(f_p) \in P$, the dominated convergence theorem yields

$$\int_0^1 [f_{p+2} - 2f_{p+1} + f_p](x) \, Q_{p+2}^{(1,2)}(x) x^{-\alpha-2q-4} \, dx = \mathrm{o}\left(1/p^2\right). \tag{37}$$

Collecting (35), (36) and (37), it follows that $p^2(\delta_1(p+2) - 2\,\delta_1(p+1) + \delta_1(p)) \to 0$ as $p \to \infty$. Similarly,

$$\begin{aligned}
I_2 \left\{ \delta_2(p+2) - 2\,\delta_2(p+1) + \delta_2(p) \right\} &= \int_1^{+\infty} [g_{p+1} - g_p](x) [Q_{p+2}^{(2)} - Q_p^{(2)}](x) x^{-\alpha-1} \, dx & (38) \\
&+ \int_1^{+\infty} g_{p+1}(x) \, [Q_{p+2}^{(2)} - 2Q_{p+1}^{(2)} + Q_p^{(2)}](x) x^{-\alpha-1} \, dx & (39) \\
&+ \int_1^{+\infty} [g_{p+2} - 2g_{p+1} + g_p](x) \, Q_{p+2}^{(2)}(x) x^{-\alpha-1} \, dx, & (40)
\end{aligned}$$

and the three terms are considered separately. First, ideas similar to those developed in the proof of (vii) allow us to control (38):

$$\int_1^{+\infty} [g_{p+1} - g_p](x) \, [Q_{p+2}^{(2)} - Q_p^{(2)}](x) x^{-\alpha-1} \, dx = \mathrm{o}\left(1/p^2\right). \tag{41}$$

Second, since $p \left| \int_p^{p+1} \eta(tx) \, t^{-1} \, dt \right| \to 0$ as $p \to \infty$ uniformly in $x \ge 1$, (A_2) entails

$$\begin{aligned}
[Q_{p+2}^{(2)} - 2Q_{p+1}^{(2)} + Q_p^{(2)}](x) &= \frac{L_2((p+2)x)}{L_2(p)} \, \frac{p+2}{p} \left\{ \int_p^{p+1} \frac{\eta(t)}{t} \, dt - \int_{p+1}^{p+2} \frac{\eta(t)}{t} \, dt \right. \\
&\qquad \left. + \int_{p+1}^{p+2} \frac{\eta(tx)}{t} \, dt - \int_p^{p+1} \frac{\eta(tx)}{t} \, dt \right\} + \mathrm{o}(1/p^2)
\end{aligned}$$

uniformly in $x \ge 1$. Remarking that

$$\begin{aligned}
\int_p^{p+1} \frac{\eta(tx)}{t} \, dt - \int_{p+1}^{p+2} \frac{\eta(tx)}{t} \, dt &= \int_p^{p+1} \frac{t[\eta(tx) - \eta((t+1)x)] + \eta(tx)}{t(t+1)} \, dt \\
&\le \frac{1}{p^2} \left[\sup_{t \ge p} \left\{ t \cdot \left| \eta(tx) - \eta((t+1)x) \right| \right\} + |\eta(p)| \right],
\end{aligned}$$

Lemma 2(i) implies that $p^2\,|Q^{(2)}_{p+2} - 2Q^{(2)}_{p+1} + Q^{(2)}_{p}|(x) \to 0$, uniformly in $x \geq 1$ as $p \to \infty$, in view of Bingham *et al.* (1987), Theorem 1.5.2. The uniform convergence of (g_p) to $x \mapsto e^{-1/x}$ on $[1, +\infty)$ and the dominated convergence theorem yield the following bound for (39):

$$\int_1^{+\infty} g_{p+1}(x)\,[Q^{(2)}_{p+2} - 2Q^{(2)}_{p+1} + Q^{(2)}_{p}](x)x^{-\alpha-1}\,dx = \mathrm{o}\left(1/p^2\right). \tag{42}$$

Finally, recalling that $(g_p) \in U$ and the uniform convergence of $(Q^{(2)}_p)$ to 0 on $[1, +\infty)$, (40) is controlled as

$$\int_1^{+\infty} [g_{p+2} - 2g_{p+1} + g_p](x)\,Q^{(2)}_{p+2}(x)x^{-\alpha-1}\,dx = \mathrm{o}\left(1/p^2\right). \tag{43}$$

Collecting (41), (42) and (43), it follows that $p^2(\delta_2(p+2) - 2\,\delta_2(p+1) + \delta_2(p)) \to 0$ as $p \to \infty$ and the lemma is proved. ■

Proof of Lemma 8. It is a direct consequence of the expansion (4) and Lemma 7(i), (iv). ■

Proof of Lemma 9. Let us remark that, from Lemma 8,

$$\xi_n = \frac{\mu_{p_n+1}}{p_n+1}\,u_{n,\,a}\cdot ap_n\left(\frac{1}{\widehat{\theta}_n} - \frac{1}{\Theta_n}\right)(1+\mathrm{o}(1)), \tag{44}$$

and consider the expansion

$$\begin{aligned}
ap_n\left(\frac{1}{\widehat{\theta}_n} - \frac{1}{\Theta_n}\right) &= ((a+1)p_n+1)\frac{\widehat{\mu}_{(a+1)p_n}\,\mu_{(a+1)p_n+1} - \mu_{(a+1)p_n}\,\widehat{\mu}_{(a+1)p_n+1}}{\widehat{\mu}_{(a+1)p_n+1}\,\mu_{(a+1)p_n+1}}\\
&\quad -(p_n+1)\frac{\widehat{\mu}_{p_n}\,\mu_{p_n+1} - \mu_{p_n}\,\widehat{\mu}_{p_n+1}}{\widehat{\mu}_{p_n+1}\,\mu_{p_n+1}}\\
&=: \Delta^{(1)}_n - \Delta^{(2)}_n
\end{aligned}$$

with

$$\begin{aligned}
\frac{\mu_{p_n+1}}{p_n+1}\Delta^{(1)}_n &:= \left[1 + \frac{ap_n}{p_n+1}\right]\frac{\mu_{p_n+1}}{\mu_{(a+1)p_n+1}}\left\{\frac{\mu_{(a+1)p_n+1}}{\widehat{\mu}_{(a+1)p_n+1}}\left[\nu_{(a+1)p_n} - \frac{\mu_{(a+1)p_n}}{\mu_{(a+1)p_n+1}}\,\nu_{(a+1)p_n+1}\right]\right\},\\
\frac{\mu_{p_n+1}}{p_n+1}\Delta^{(2)}_n &:= \frac{\mu_{p_n+1}}{\widehat{\mu}_{p_n+1}}\left[\nu_{p_n} - \frac{\mu_{p_n}}{\mu_{p_n+1}}\,\nu_{p_n+1}\right].
\end{aligned}$$

Replacing in (44), Lemma 9 follows. ■

Proof of Lemma 10. Hölder's inequality yields

$$\mathbb{E}|Y^{p_n}\,h_n(Y)|^3 \leq (k+1)^2\left[\sum_{j=0}^{k}\frac{1}{p_n^{3j}}\sup_{\substack{[0,\,1]\\ n\in\mathbb{N}\setminus\{0\}}}|H_{n,\,j}|^3\,\mathbb{E}\left[Y^{p_n}\,(1-Y)^{k-j}\right]^3\right].$$

It then suffices to prove that $\forall\,j \in \{0,\ldots,k\}$, $\mathbb{E}\left[Y^{p_n}\,(1-Y)^{k-j}\right]^3 = \mathrm{O}(p_n^{-\alpha-(3k-3j)}\,L(p_n))$. Let $\lambda,\,\mu \geq 0$. The function

$$(y,\,\omega) \mapsto \frac{d}{dy}\left[y^\lambda\,(1-y)^\mu\right]\mathbb{1}_{\{y\leq Y(w)\}}$$

being Lebesgue $\otimes\,\mathbb{P}-$integrable, Fubini's theorem entails

$$\begin{aligned}
\mathbb{E}\left[Y^{p_n}\,(1-Y)^{k-j}\right]^3 &= \int_0^1 \frac{d}{dy}\left[y^{3p_n}\,(1-y)^{3k-3j}\right]\overline{G}(y)\,dy \\
&\leq \int_0^1 3p_n\,y^{3p_n-1}(1-y)^{3k-3j}\,\overline{G}(y)\,dy.
\end{aligned}$$

Finally, if (s_n) is a real sequence tending to $+\infty$ and $d\geq 0$, we have, from Lemma 8,

$$\begin{aligned}
\int_0^1 y^{s_n}(1-y)^d\,\overline{G}(y)\,dy &= \int_0^1 y^{s_n}(1-y)^{d+\alpha}\,L((1-y)^{-1})\,dy \\
&= s_n^{-d-\alpha-1}\,L(s_n)\,\Gamma(d+\alpha+1)(1+\mathrm{o}(1)).
\end{aligned}$$

Replacing in the inequality above and recalling that L is slowly varying at infinity, it follows that $\mathbb{E}\left[Y^{p_n}\,(1-Y)^{k-j}\right]^3 = \mathrm{O}(p_n^{-\alpha-(3k-3j)}\,L(p_n))$, which establishes Lemma 10. ■

Distribution	Maximum	Extreme-value moment estimator	High-order moments estimator
$1 - \dfrac{1}{1 + \mathrm{Burr}(1,\, \tau_1,\, \tau_2)}$			
$(\tau_1,\, \tau_2) = (3/2,\, 2/3)$ $\Rightarrow (\alpha,\, \nu) = (1,\, -1)$	$2.0 \cdot 10^{-3}$	$1.7 \cdot 10^{-3}$	$1.6 \cdot 10^{-3}$
$(\tau_1,\, \tau_2) = (5/6,\, 6/5)$ $\Rightarrow (\alpha,\, \nu) = (1,\, -5/6)$	$2.0 \cdot 10^{-3}$	$2.1 \cdot 10^{-3}$	$1.7 \cdot 10^{-3}$
$(\tau_1,\, \tau_2) = (2/3,\, 3/2)$ $\Rightarrow (\alpha,\, \nu) = (1,\, -2/3)$	$2.1 \cdot 10^{-3}$	$2.0 \cdot 10^{-3}$	$1.8 \cdot 10^{-3}$
$(\tau_1,\, \tau_2) = (1/2,\, 2)$ $\Rightarrow (\alpha,\, \nu) = (1,\, -1/2)$	$2.3 \cdot 10^{-3}$	$2.4 \cdot 10^{-3}$	$2.0 \cdot 10^{-3}$
$1 - \exp(-\mathrm{Gamma}(b,\, \lambda))$			
$(b,\, \lambda) = (2,\, 1)$ $\Rightarrow (\alpha,\, \nu) = (1,\, 0)$	$2.3 \cdot 10^{-4}$	$1.9 \cdot 10^{-4}$	$1.9 \cdot 10^{-4}$
$(b,\, \lambda) = (2,\, 5/4)$ $\Rightarrow (\alpha,\, \nu) = (5/4,\, 0)$	$1.1 \cdot 10^{-3}$	$9.2 \cdot 10^{-4}$	$8.4 \cdot 10^{-4}$
$(b,\, \lambda) = (2,\, 5/3)$ $\Rightarrow (\alpha,\, \nu) = (5/3,\, 0)$	$5.7 \cdot 10^{-3}$	$4.5 \cdot 10^{-3}$	$3.9 \cdot 10^{-3}$
$(b,\, \lambda) = (2,\, 5/2)$ $\Rightarrow (\alpha,\, \nu) = (5/2,\, 0)$	$3.0 \cdot 10^{-2}$	$2.1 \cdot 10^{-2}$	$1.8 \cdot 10^{-2}$

Table 1: Mean L^1-errors associated to the estimators in the eight situations.

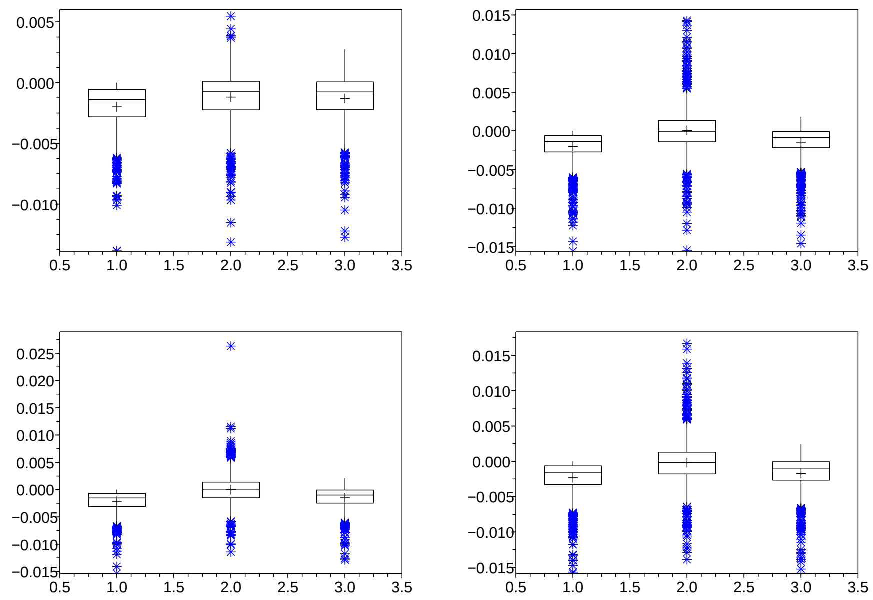

Figure 1: Boxplots of $\varepsilon(j,\, a^*)$ on model (11). Left: maximum estimator, middle: extreme-value moment estimator, right: high-order moments estimator. Top left: $(\tau_1,\, \tau_2) = (3/2,\, 2/3)$; top right: $(\tau_1,\, \tau_2) = (5/6,\, 6/5)$; bottom left: $(\tau_1,\, \tau_2) = (2/3,\, 3/2)$; bottom right: $(\tau_1,\, \tau_2) = (1/2,\, 2)$.

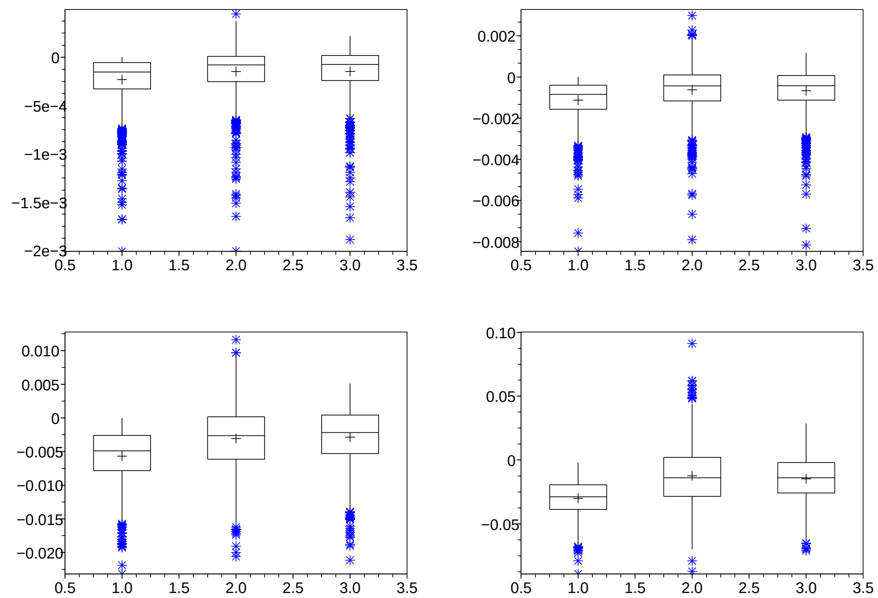

Figure 2: Boxplots of $\varepsilon(j,\, a^*)$ on model (12). Left: maximum estimator, middle: extreme-value moment estimator, right: high-order moments estimator. Top left: $(b,\, \lambda) = (2,\, 1)$; top right: $(b,\, \lambda) = (2,\, 5/4)$; bottom left: $(b,\, \lambda) = (2,\, 5/3)$; bottom right: $(b,\, \lambda) = (2,\, 5/2)$.

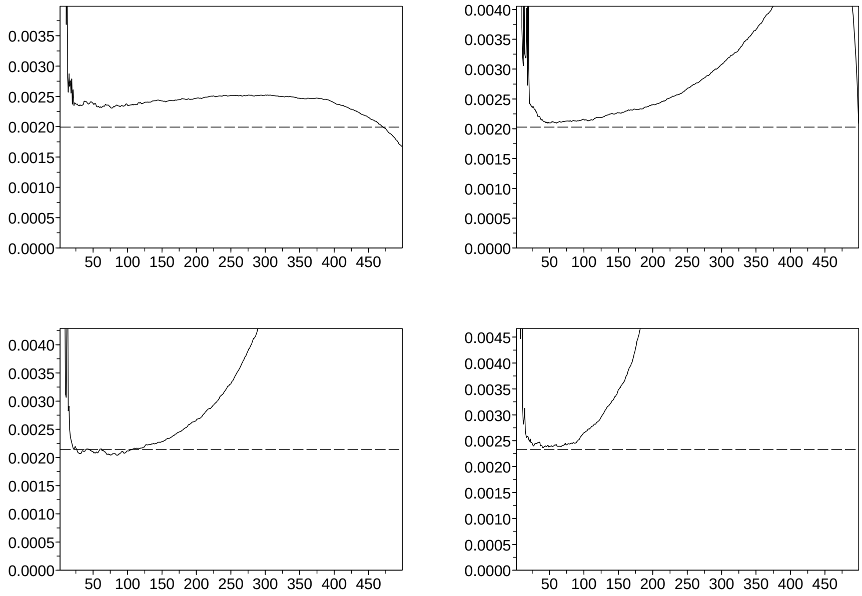

Figure 3: Comparison of the maximum and extreme-value moment estimators on model (11). Horizontally: parameter k, vertically: error E, dashed line: maximum estimator, solid line: extreme-value moment estimator. Top left: $(\tau_1, \tau_2) = (3/2, 2/3)$; top right: $(\tau_1, \tau_2) = (5/6, 6/5)$; bottom left: $(\tau_1, \tau_2) = (2/3, 3/2)$; bottom right: $(\tau_1, \tau_2) = (1/2, 2)$.

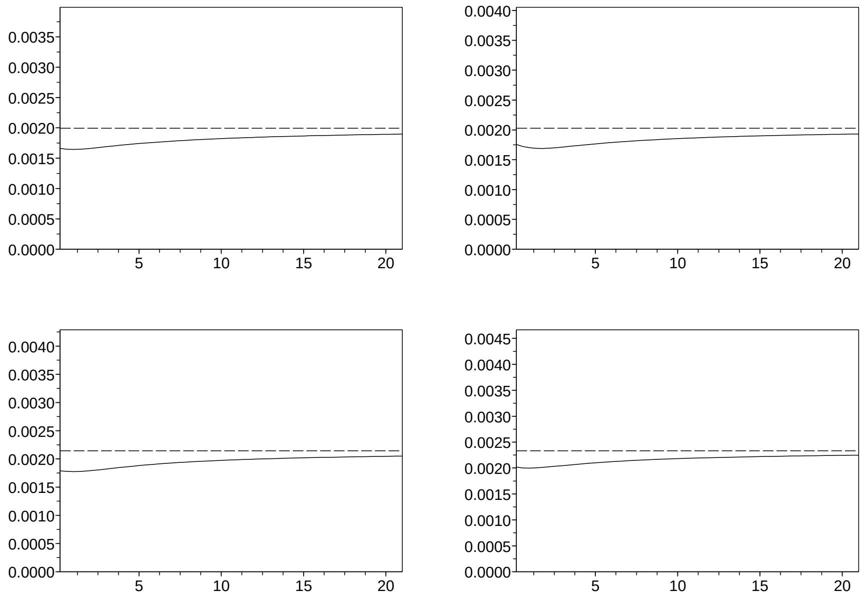

Figure 4: Comparison of the maximum and high-order moments estimators on model (11). Horizontally: parameter a, vertically: error E, dashed line: maximum estimator, solid line: high-order moments estimator. Top left: $(\tau_1, \tau_2) = (3/2, 2/3)$; top right: $(\tau_1, \tau_2) = (5/6, 6/5)$; bottom left: $(\tau_1, \tau_2) = (2/3, 3/2)$; bottom right: $(\tau_1, \tau_2) = (1/2, 2)$.

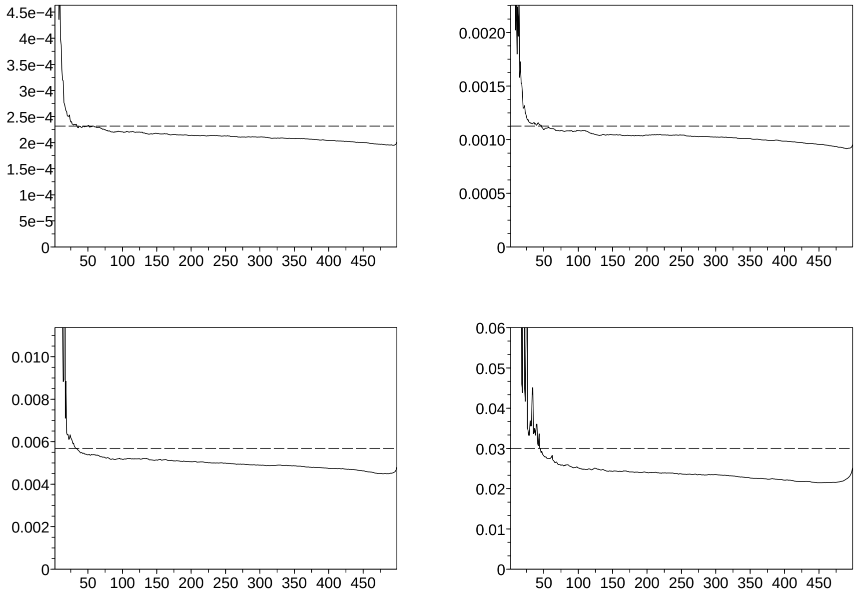

Figure 5: Comparison of the maximum and extreme-value moment estimators on model (12). Horizontally: parameter k, vertically: error E, dashed line: maximum estimator, solid line: extreme-value moment estimator. Top left: $(b,\,\lambda) = (2,\,1)$, top right: $(b,\,\lambda) = (2,\,5/4)$, bottom left: $(b,\,\lambda) = (2,\,5/3)$, bottom right: $(b,\,\lambda) = (2,\,5/2)$.

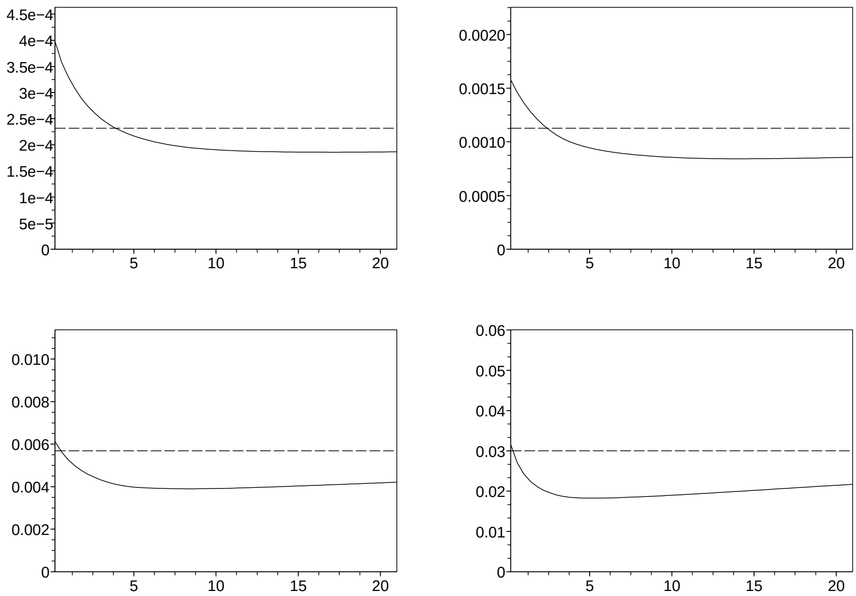

Figure 6: Comparison of the maximum and high-order moments estimators on model (12). Horizontally: parameter a, vertically: error E, dashed line: maximum estimator, solid line: high-order moments estimator. Top left: $(b,\,\lambda) = (2,\,1)$, top right: $(b,\,\lambda) = (2,\,5/4)$, bottom left: $(b,\,\lambda) = (2,\,5/3)$, bottom right: $(b,\,\lambda) = (2,\,5/2)$.