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Claire Scheid[1] — Stéphane Lanteri[2]

[1] : INRIA Sophia Antipolis - Méditerranée research center, NACHOS project-team
and University of Nice - Sophia Antipolis
J.A. Dieudonné Mathematics Laboratory UMR CNRS 6621

[2] : INRIA Sophia Antipolis - Méditerranée research center, NACHOS project-team

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[2] : INRIA Sophia Antipolis - Méditerranée research center, Nachos project-team

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Abstract: This study is concerned with the solution of the time domain Maxwell's equations in a dispersive propagation media by a Discontinuous Galerkin Time Domain (DGTD) method. The Debye model is used to describe the dispersive behaviour of the media. The resulting system of equations is solved using a centered flux discontinuous Galerkin formulation for the discretization in space and a second order leap-frog scheme for the integration in time. The numerical treatment of the dispersive model relies on an Auxiliary Differential Equation (ADE) approach similary to what is adopted in the Finite Difference Time Domain (FDTD) method. Stability estimates are derived through energy estimations and the convergence is proved for both the semi-discrete and the fully discrete case.

Key-words: Maxwell's equations, time domain, dispersive medium, discontinuous Galerkin method, convergence analysis.

Centre de recherche INRIA Sophia Antipolis – Méditerranée
2004, route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex
Téléphone : +33 4 92 38 77 77 — Télécopie : +33 4 92 38 77 65

Convergence d'un schéma Galerkin Discontinu pour les équations de Maxwell en formulation mixte et temporelle en milieu dispersif

Résumé : On s'intéresse à la résolution numérique des équations de Maxwell en domaine temporel en milieu dispersif par une méthode Galerkin discontinue. Le caractère dispersif est ici pris en compte par le modèle de Debye. La méthode de résolution étudiée couple une formulation Galerkin discontinue à flux centré pour la discrétisation en espace et un schéma saute mouton du second ordre pour l'intégration en temps. Le traitement numérique du modèle dispersif repose sur une approche par équation différentielle auxiliaire à l'image de ce qui est réalisé dans la méthode de différences finies en domaine temporel. On étudie la stabilité du schéma résultant via des estimations d'énergie et prouvons la convergence des schémas semi-discrets et totalement discrets.

Mots-clés : Equations de Maxwell, domaine temporel, milieu dispersif, méthode Galerkin Discontinue, analyse de convergence.

Contents

1 Introduction

This study is concerned with the numerical modeling of the propagation of electromagnetic waves in dispersive media. Such propagation media are characterized by electromagnetic parameters (e.g. the electric permittivity) that depend on the frequency of the incident wave. There is a lot of practical problems that involve such propagation media and we are more particularly interested in situations for which one has to model the interaction of an electromagnetic wave with biological tissues. As a matter of fact, the numerical modeling of the propagation of electromagnetic waves through human tissues is at the heart of many biomedical applications such as breast imaging related to cancer tumor detection [SKVH10, KLG+10], the development of intelligent medical implants in the body [WSTI09] or microwave based hyperthermia to kill cancer cells [CBHV04, KKGU08]. Finally, one can cite the electroporation technique which consists of applying nanopulses to the tissues, permitting to only affect intracellular membranes and then envisage treatments like electrochemiotherapy or gene transfer [SKS+92, Tso91, vCB+05, MvMM00, Ser05]. In all these areas, there is a need for accurate and efficient numerical modeling techniques able to deal with the complex issues characterizing the associated propagation problems. Coming to the question of modeling electromagnetic waves leads to the study Maxwell's equations. The bibliography on Maxwell's equations is already well developed on both theoretical and numerical aspects (for a review see for example the book of P. Monk [Mon03], or the book [Bos98]). The problem of modeling electromagnetic waves propagating in human tissues however requires to taking into account the dispersive character of the propagation medium. Indeed human tissues contains a high percentage of water which gives them the properties to be dispersive. To be more precise, a dispersive medium is a medium where the speed of the propagating wave depends on the frequency.

The bibliography concerning the study of the propagation of electromagnetic waves in dispersive media is quite extended. For the numerical aspect of this problem, which is what this article is more concerned with, methodological developments have focused for quite a long time on the Finite Difference Time Domain (FDTD) method (see [Sul92, Lue90, KF90]). Numerical techniques based on the finite element method have also been investigated (see [Li06, LC06, Li07, LC08, LL08, Li11]). These studies deal with Nedelec elements and propose various proofs of convergence of semi-discrete scheme as well as in some of them fully discrete scheme with error estimates. Let us point out some of these articles. In [Li07], the mixed form of Maxwell's equations is considered, the study of a fully discrete scheme with first order accuracy is completed and some 2D numerical results are presented. In [LC06] and [LC08], the second order formulation of Maxwell's equation is considered and an example of fully discrete scheme analyzed. Finally in the very recent paper [Li11], the author considers the mixed formulation of Maxwell's equation and the dispersive character of the media is taken into account via an Auxiliary Differential Equation (ADE). The study focuses on a finite element formulation based on Nédélec elements and a leap-frog time integration scheme.

In the present work we develop such a numerical analysis for a discontinuous Galerkin discretization. Less work has been done on this topic, although a few works have been published very recently. One can cite [LZC04, LCZ05] where no convergence proof is completed, and [HL09], where error estimations are proved for the case of the second order formulation of Maxwell's equations and an interior penalty formulation for the case of cold plasma. The same interior penalty formulation is adopted in [HLY11] where furthermore some numerical tests are presented for another time scheme. In the recent article [WXZ10], an error analysis is conducted for the semi-discrete case in a unified way for many dispersive media. The scheme is written and studied in its semi-discretized version, the fully-discrete scheme is described but not analyzed. Different dispersive media are presented, considering a locally divergence free discontinuous Galerkin methods. Some numerical results are also presented to validate the theoretical findings.

This work is intended as an attempt to go even further into the analysis of a fully discrete scheme for Maxwell's equations in dispersive media. In this article we present a complete numerical analysis study of the first order Maxwell's equations in dispersive media in a discontinuous Galerkin framework with an ADE approach. The mathematical model considered is the same than in [Li11] but its numerical treatment varies. The numerical scheme presented here is also different from the one presented in [WXZ10] even if both approaches share common features. The starting point of our study is the non-dissipative discontinuous Galerkin formulation presented and studied in [FLLP05] for the Maxwell's equations in non-dispersive media. We also include here an existence result in the continuous case and the analysis of the fully discrete case using a second order leap-frog scheme in time. Theoretical results are validated using an artificial test problem for which we can construct an analytical solution. In summary, the present study differs from previous works either from the point of view of the adopted mathematical model (mixed formulation and ADE technique), or by the numerical discretization used (non-dissipative DG method with centered fluxes).

This article is organized as follows. In section 2, we set up the equations of the model and the associated notations. The problem of existence of a solution is then studied using techniques from operator theory. Finally, we study the energy and prove stability. Section 3 is devoted to the study of the semi-discrete case. The discontinuous Galerkin formulation is described and then analysed via stability and convergence analysis in section 3. Section 4 contains the analysis of the fully discrete case. A CFL condition similar to [FLLP05] is derived and convergence of the scheme then deduced. Finally section 4 gives some preliminary numerical results to validate the theory.

2 The continuous problem

In this section we state the initial and boundary value problem used to model the propagation of an electromagnetic wave in a dispersive medium. The physical behaviour of the medium is assumed to be described by the Debye model. We choose to use the ADE (Auxiliary Differential Equation) representation combined with the first order formulation of Maxwell's equations.

2.1 Formulation

Let $\Omega \subset \mathbb{R}^3$ be a bounded convex polyhedral domain. We denote by $\mathbf{n}$ the outward normal to $\partial\Omega$. Moreover, ε_0 (resp. μ_0) is the electric (resp. magnetic) permittivity in free space, ε_∞ (resp. μ_∞) the relative electric permittivity of the medium at infinite frequency and σ the conductivity of the medium. Furthermore:

$$\mathbf{L}^2(\Omega) = (L^2(\Omega))^3,$$

$$\mathcal{H}(\Omega) := \mathcal{H}(\mathbf{curl}\,,\Omega) \times \mathcal{H}(\mathbf{curl}\,,\Omega) \times \mathbf{L}^2(\Omega),$$

where $\mathcal{H}(\mathbf{curl}\,,\Omega)$ is the classical space of $\mathbf{L}^2$ fields with curl in $\mathbf{L}^2$, and:

$$\mathcal{L}(\Omega) := (\mathbf{L}^2(\Omega))^3,$$

$$\mathbf{H}^s(\Omega) = (H^s(\Omega))^3.$$

The L^2 scalar product will be denoted by $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|_{L^2(\Omega)}$. In a same way $\| \cdot \|_X$ will denote the canonical norm associated to the space X. $| \cdot |$ will denote the norm associated to the vector scalar product in $\mathbb{R}^3$ denoted by $\cdot$.

We consider the time domain formulation of Maxwell's equation in Ω. In a dispersive medium, the effect of the electric field is described by the electric displacement $\mathbf{D}$. One has:

$$\mathbf{D} = \varepsilon_0 \varepsilon_\infty \mathbf{E} + \mathbf{P},$$

where $\mathbf{P}$ is called the polarization. The constitutive relation in a linear dispersive medium is defined in the frequency domain by:

$$\hat{\mathbf{D}}(x,\omega) = \varepsilon_0 \hat{\varepsilon}_r(x,\omega) \hat{\mathbf{E}}(x,\omega),$$

where $\hat{\cdot}$ denotes the Fourier transform of the corresponding function and ε_r is called the relative permittivity. Its expression depends on the model used to describe the dispersive character of the medium (see [MPH06]). In a time domain formulation we obtain:

$$\mathbf{D}(x,t) = \varepsilon_0 \varepsilon_r(x,t) \star \mathbf{E}(x,t).$$

For a Debye type dispersive medium, the frequency variation of relative permittivity is given by:

$$\varepsilon_r(x,\omega) = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 + j\omega\tau_r}.$$

with $\varepsilon_s > \varepsilon_\infty$. This yields:

$$\hat{\mathbf{D}}(1 + j\omega\tau_r) = \varepsilon_0\varepsilon_\infty(1 + j\omega\tau_r)\hat{\mathbf{E}} + \varepsilon_0(\varepsilon_s - \varepsilon_\infty)\hat{\mathbf{E}}.$$

Taking the inverse Fourier transform, one obtains the Auxiliary Differential Equation (ADE) for $\mathbf{D}$:

$$\mathbf{D} + \tau_r \frac{\partial \mathbf{D}}{\partial t} = \varepsilon_0\varepsilon_\infty\tau_r \frac{\partial \mathbf{E}}{\partial t} + \varepsilon_0\varepsilon_s\mathbf{E}.$$

and the ADE in terms of $\mathbf{P}$ writes:

$$\mathbf{P} + \tau_r \frac{\partial \mathbf{P}}{\partial t} = \varepsilon_0(\varepsilon_s - \varepsilon_\infty)\mathbf{E}.$$

We can now state Maxwell's equation in a Debye dispersive medium. Let $T > 0$. The magnetic field $\mathbf{H}$, the electric field $\mathbf{E}$ and the polarization $\mathbf{P}$ verify the following system of equations in $[0,T]$:

$$\left\{\begin{array}{l}
\mu \dfrac{\partial \mathbf{H}}{\partial t} + \mathbf{curl}\ \mathbf{E} = 0, \\[2mm]
-\varepsilon_0\varepsilon_\infty \dfrac{\partial \mathbf{E}}{\partial t} + \mathbf{curl}\ \mathbf{H} = \dfrac{\partial \mathbf{P}}{\partial t} + \sigma\mathbf{E}, \\[2mm]
\dfrac{\partial \mathbf{P}}{\partial t} = \dfrac{1}{\tau_r}\left[\varepsilon_0\left(\varepsilon_s - \varepsilon_\infty\right)\mathbf{E} - \mathbf{P}\right],
\end{array}\right.$$

or equivalently:

$$\left\{\begin{array}{l}
\mu \dfrac{\partial \mathbf{H}}{\partial t} + \mathbf{curl}\ \mathbf{E} = 0, \\[2mm]
\varepsilon_0\varepsilon_\infty \dfrac{\partial \mathbf{E}}{\partial t} - \mathbf{curl}\ \mathbf{H} = -\dfrac{1}{\tau_r}\left[\varepsilon_0\left(\varepsilon_s - \varepsilon_\infty\right)\mathbf{E} - \mathbf{P}\right] - \sigma\mathbf{E}, \\[2mm]
\dfrac{\partial \mathbf{P}}{\partial t} = \dfrac{1}{\tau_r}\left[\varepsilon_0\left(\varepsilon_s - \varepsilon_\infty\right)\mathbf{E} - \mathbf{P}\right].
\end{array}\right. \tag{1}$$

We choose to work with *metallic boundary conditions*, i.e. $\mathbf{E} \times \mathbf{n} = 0$ and *initial conditions* are given by $\mathbf{U}_0 = (\mathbf{H}_0, \mathbf{E}_0, \mathbf{P}_0) \in \mathcal{H}(\Omega)$ such that $\mathbf{H}(0,.) = \mathbf{H}_0$, $\mathbf{E}(0,.) = \mathbf{E}_0$, $\mathbf{P}(0,.) = \mathbf{P}_0$.

Remark 2.1 *The same analysis can be done with absorbing boundary conditions. The additional difficulties are given by some additional calculus like in [FLLP05].*

We now state a weak formulation for the solution of system (1). Let:

$$\beta := \frac{\varepsilon_0}{\tau_r}\left(\varepsilon_s - \varepsilon_\infty\right) \quad \text{and} \quad \alpha := \beta + \sigma.$$

We seek the solution $\mathbf{U} = (\mathbf{H}, \mathbf{E}, \mathbf{P})$ in $\mathcal{C}([0,T], \mathcal{H}(\Omega))$ such that $\forall(\varphi, \psi, \phi) \in \mathcal{H}(\Omega)$:

$$\left\{\begin{array}{l} \displaystyle\int_\Omega \mu \frac{\partial \mathbf{H}}{\partial t} \cdot \varphi + \int_\Omega \mathbf{curl}\ \mathbf{E} \cdot \varphi = 0, \\ \displaystyle\int_\Omega \varepsilon_0 \varepsilon_\infty \frac{\partial \mathbf{E}}{\partial t} \cdot \psi - \int_\Omega \mathbf{curl}\ \mathbf{H} \cdot \psi = \frac{1}{\tau_r} \int_\Omega \mathbf{P} \cdot \psi - \alpha \int_\Omega \mathbf{E} \cdot \psi, \\ \displaystyle\int_\Omega \frac{\partial \mathbf{P}}{\partial t} \cdot \phi = -\frac{1}{\tau_r} \int_\Omega \mathbf{P} \cdot \phi + \beta \int_\Omega \mathbf{E} \cdot \phi. \end{array}\right. \tag{2}$$

2.2 Existence of a solution

In order to prove the existence of a solution to the problem we consider, we will use operator theory (see [CH90, Paz83] for details). One writes the equations (in a strong formulation) in a system form. If $\mathcal{U} = (U_1, U_2, U_3)$, with $U_1 = \mathbf{H}$, $U_2 = \mathbf{E}$, $U_3 = \mathbf{P}$:

$$\left\{\begin{array}{l} \displaystyle\Lambda \frac{d\mathcal{U}}{dt} = \mathcal{I}(\mathcal{U}) + \mathcal{K}(\mathcal{U}), \\ \mathcal{U}(0) = U^0, \end{array}\right. \tag{3}$$

with:

$$\mathcal{I}(\mathcal{U}) = (-\mathbf{curl}\ U_2\ ,\ \mathbf{curl}\, U_1\ ,\ 0),$$

$$\mathcal{K}(\mathcal{U}) = (0\ ,\ -\alpha U_2 + \frac{1}{\tau_r} U_3\ ,\ \frac{1}{\tau_r} U_2 - \frac{1}{\varepsilon_0(\varepsilon_s - \varepsilon_\infty)\tau_r} U_3),$$

and:

$$\Lambda = \begin{bmatrix} \mu I_3 & 0 & 0 \\ 0 & \varepsilon_0 \varepsilon_\infty I_3 & 0 \\ 0 & 0 & \dfrac{1}{\varepsilon_0(\varepsilon_s - \varepsilon_\infty)} I_3 \end{bmatrix}.$$

The unbounded operator $\mathcal{I}$ is defined on $D(\mathcal{I}) = \mathcal{H}(\mathbf{curl}\,, \Omega) \times \mathcal{H}_0(\mathbf{curl}\,, \Omega) \times \mathbf{L}^2(\Omega)$ which is dense in $\mathcal{L}(\Omega)$. Furthermore:

$$\langle \mathcal{I}(\mathcal{U}), \mathcal{V} \rangle = -\langle \mathcal{U}, \mathcal{I}(\mathcal{V}) \rangle, \quad \forall (\mathcal{U}, \mathcal{V}) \in D(\mathcal{I})^2.$$

This yields that:

$$\mathcal{I}^* = -\mathcal{I}, \quad D(\mathcal{I}^*) = D(\mathcal{I}),$$

and:

$$\langle \mathcal{I}(\mathcal{U}), \mathcal{U} \rangle = 0, \forall \mathcal{U} \in D(\mathcal{I}).$$

This implies that $\mathcal{I}$ is dissipative in $\mathcal{L}(\Omega)$.

The following lemma is central to prove our result of existence:

Lemma 2.2 *Let $\nu > 0$ be fixed. Then, the operator $\mathcal{I} - \nu \Lambda I$ is maximal-dissipative (for notations see [Paz83]) and is the infinitesimal generator of a $\mathcal{C}_0$ semigroup on $\mathcal{L}(\Omega)$.*

Proof. For all $\mathcal{U} \in D(\mathcal{I})$, $\langle \mathcal{I}(\mathcal{U}) - \nu \Lambda \mathcal{U}, \mathcal{U} \rangle = -\nu \|\Lambda^{\frac{1}{2}} \mathcal{U}\|^2_{\mathcal{L}(\Omega)} \leq -C \|\mathcal{U}\|^2_{\mathcal{L}(\Omega)}$. Same equality holds for $\mathcal{I}^*$. Thus $\mathcal{I} - \nu \Lambda I$ and $\mathcal{I}^* - \nu \Lambda I$ are also dissipative. Furthermore, since $\langle \mathcal{U} - (\mathcal{I}(\mathcal{U}) - \nu \Lambda \mathcal{U}), \mathcal{U} \rangle = \|\mathcal{U}\|^2_{\mathcal{L}(\Omega)} + \nu \|\Lambda^{\frac{1}{2}} \mathcal{U}\|^2_{\mathcal{L}(\Omega)}$, with the same equality for $\mathcal{I}^*$, one concludes that $\mathcal{I} - \nu \Lambda I$ and $\mathcal{I}^* - \nu \Lambda I$ are maximal dissipative. Results from semi-group theory as in [Paz83], allow to conclude. $\square$

Introducing $\mathcal{V}(t)$ such that $\mathcal{U}(t) = \exp(\nu t) \mathcal{V}(t)$, $\mathcal{V}(t)$ satisfies the following equation:

$$\left\{\begin{array}{l} \displaystyle\Lambda \frac{d\mathcal{V}}{dt} - \mathcal{I}(\mathcal{V}) + \nu \Lambda \mathcal{V} - \mathcal{K}(\mathcal{V}) = 0, \\ \mathcal{V}(0) = \mathcal{U}^0. \end{array}\right.$$

The operator $\mathcal{K}$ is linear and bounded on $\mathcal{L}(\Omega)$, with Theorem 1.1. from [Paz83], Chapter 3, one deduces that $\mathcal{I}-\nu\Lambda+\mathcal{K}$ is also the infinitesimal generator of a $\mathcal{C}_0$ semigroup on $\mathcal{L}(\Omega)$. Then *there exists a unique solution* in the sense of [Paz83]. Therefore there exists a unique mild solution $\mathcal{V} \in \mathcal{C}^0([0,T],\mathcal{L}(\Omega))$. If $\mathcal{U} \in D(\mathcal{I})$ then it is a classical solution of the initial value problem.

One can precise the regularity in time for the solutions. We have that $\mathcal{W} = \dfrac{d\mathcal{V}}{dt}$ is solution of:

$$\begin{cases} \Lambda \dfrac{d\mathcal{W}}{dt} - \mathcal{I}(\mathcal{W}) + \nu\mathcal{W} - \mathcal{K}(\mathcal{W}) = 0, \\ \mathcal{W}^0 = \Lambda^{-1}(\mathcal{I}(\mathcal{W}) - \nu\mathcal{W} + \mathcal{K}(\mathcal{W})). \end{cases}$$

If $\mathcal{U}^0 \in D(\mathcal{I})$, then $\mathcal{W}^0 \in \mathbf{L}^2(\Omega)$, and using the same arguments than before, one deduces that there exists a unique weak solution $\mathcal{W}$ to the previous equations with $\mathcal{W} \in \mathcal{C}^0([0,T],\mathcal{L}(\Omega))$. Then one can conclude that $\dfrac{d\mathcal{U}}{dt} = \dfrac{d}{dt}\left[\exp(\nu t)\mathcal{V}(t)\right]$ is a weak solution of the problem considered initially.
We can now turn to the corresponding solution $\mathcal{U}$ to conclude.

Theorem 2.3 *If* $(\mathbf{H}^0,\mathbf{E}^0,\mathbf{P}^0) \in D(\mathcal{I})$*, then there exists a unique weak solution:*

$$(\mathbf{H},\mathbf{E},\mathbf{P}) \in \mathcal{C}^0([0,T],\mathcal{H}(\Omega)) \cap \mathcal{C}^1([0,T],\mathcal{L}(\Omega)),$$

of the equation (2).

If more regularity is needed, one has to suppose more regularity on the initial conditions, especially on $(\mathbf{H},\mathbf{E})$. The regularity for $\mathbf{P}$ is obtained even with $\mathbf{L}^2(\Omega)$ initial data. In the rest of the paper, we will for our needs suppose that furthermore $(\mathbf{H}^0,\mathbf{E}^0,\mathbf{P}^0) \in (H^{s+1}(\Omega)\times H^{s+1}(\Omega)\times L^2(\Omega))\cap\mathcal{H}(\Omega)$, for $s > 2$.

2.3 The energy

Let $\mathcal{E}$ define the energy of the system for $t \in [0,T]$:

$$\mathcal{E}(t) = \frac{1}{2}\left(\mu\|\mathbf{H}\|^2_{\mathbf{L}^2(\Omega)}(t) + \varepsilon_0\varepsilon_\infty\|\mathbf{E}\|^2_{\mathbf{L}^2(\Omega)}(t) + \frac{1}{\varepsilon_0(\varepsilon_s-\varepsilon_\infty)}\|\mathbf{P}\|^2_{\mathbf{L}^2(\Omega)}(t)\right). \tag{4}$$

We have the following result:

Proposition 2.4 *The energy is decreasing, i.e.* $\forall t \in [0,T]$*:*

$$\mathcal{E}(t) \le \mathcal{E}(0).$$

Proof. Taking $\varphi = \mathbf{H}$, $\psi = \mathbf{E}$ et $\phi = \mathbf{P}$, as test functions in (2), one obtains:

$$\begin{cases} \mu\langle\dfrac{\partial\mathbf{H}}{\partial t},\mathbf{H}\rangle + \langle\mathbf{curl}\ \mathbf{E},\mathbf{H}\rangle = 0, \\ \varepsilon_0\varepsilon_\infty\langle\dfrac{\partial\mathbf{E}}{\partial t},\mathbf{E}\rangle - \langle\mathbf{H},\mathbf{curl}\ \mathbf{E}\rangle = -\displaystyle\int_{\partial\Omega}(\mathbf{H}\times\mathbf{E})\cdot n + \frac{1}{\tau_r}\langle\mathbf{P},\mathbf{E}\rangle - \alpha\|\mathbf{E}\|^2_{\mathbf{L}^2(\Omega)}, \\ \dfrac{1}{\varepsilon_0(\varepsilon_s-\varepsilon_\infty)}\langle\dfrac{\partial\mathbf{P}}{\partial t},\mathbf{P}\rangle = -\dfrac{1}{\varepsilon_0(\varepsilon_s-\varepsilon_\infty)\tau_r}\|\mathbf{P}\|^2_{\mathbf{L}^2(\Omega)} + \dfrac{1}{\tau_r}\langle\mathbf{E},\mathbf{P}\rangle, \end{cases}$$

This yields:

$$\begin{aligned} \frac{1}{2}\left[\mu\tfrac{d}{dt}\|\mathbf{H}\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon_0\varepsilon_\infty\frac{d}{dt}\|\mathbf{E}\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{\varepsilon_0(\varepsilon_s-\varepsilon_\infty)}\frac{d}{dt}\|\mathbf{P}\|^2_{\mathbf{L}^2(\Omega)}\right] &= -\frac{1}{\varepsilon_0(\varepsilon_s-\varepsilon_\infty)\tau_r}\|\mathbf{P}\|^2_{\mathbf{L}^2(\Omega)} \\ &+ \frac{2}{\tau_r}\langle\mathbf{E},\mathbf{P}\rangle - \alpha\|\mathbf{E}\|^2_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

so that:

$$\frac{d}{dt}\mathcal{E}(t) = -\frac{1}{\tau_r\varepsilon_0(\varepsilon_s-\varepsilon_\infty)}\|\mathbf{P}-\varepsilon_0(\varepsilon_s-\varepsilon_\infty)\mathbf{E}\|^2_{\mathbf{L}^2(\Omega)} - \sigma\|\mathbf{E}\|^2_{\mathbf{L}^2(\Omega)}$$

which gives the result. □

3 Semi-discretization by a DG method

3.1 Formulation

We discretize the system (2) using a discontinuous Galerkin method formulated on simplicial meshes. We follow the approach of [FLLP05]. Let Ω_h be a discretization of the computational domain Ω, using a triangulation $\mathcal{T}_h$ as follows: $\Omega_h \equiv \Omega$, $\mathcal{T}_h = \cup_{i=1}^{N_\tau}\tau_i$. We assume that the mesh is shape regular. An internal face is denoted by $a_{ik} = \tau_i \cap \tau_k$, for $i \neq k$ and $\mathbf{n}_{ik}$ is the unitary normal vector oriented from τ_i to τ_k. For a given element τ_i, $i \in [0, N_\tau]$, one denotes by $\mathcal{V}_i$ the set of the indices of all neighboring elements of τ_i.

General setting. One seeks an approximation $(\mathbf{H}_h, \mathbf{E}_h, \mathbf{P}_h)$ of $(\mathbf{H}, \mathbf{E}, \mathbf{P})$ such that on each finite element τ_i, $\mathbf{H}_h$, $\mathbf{E}_h$, $\mathbf{P}_h$ are polynomials whose degree can in all generality depend on the triangle τ_i. For i given, we denote by d_i the number of degrees of freedom associated to the finite element τ_i and $\mathcal{P}_i$, the associated polynomial space. Moreover, $(\varphi_{i,j})_{1\leq j\leq d_i}$ defines a set of linearly independent functions such that $\mathcal{P}_i = \text{Span}\{\varphi_{i,j}, 1 \leq j \leq d_i\}$. Then $\mathbf{H}_h$, $\mathbf{E}_h$, $\mathbf{P}_h$ are defined by: $(\mathbf{E}_h)_{/\tau_i} = \mathbf{E}_i$, $(\mathbf{H}_h)_{/\tau_i} = \mathbf{H}_i$, $(\mathbf{P}_h)_{/\tau_i} = \mathbf{P}_i$, where $(\mathbf{H}_i, \mathbf{E}_i, \mathbf{P}_i) \in \mathcal{P}_i^9$ and:

$$\mathbf{V}_h = \left\{\mathbf{W}_h \in \mathbf{L}^2(\Omega) \quad | \quad (\mathbf{W}_h)_{/\tau_i} \in (\mathcal{P}_i)^3, \quad \forall \tau_i \in \mathcal{T}_h\right\},$$

the corresponding approximation space.
If $\mathbf{W}_h \in \mathbf{V}_h$:

- $\mathbf{W}_i$ denotes its restriction to the element τ_i.
- We define its average through any internal face a_{ik}, for given (i, k):

$$\{\mathbf{W}_h\}_{ik} = \frac{\mathbf{W}_{i/a_{ik}} + \mathbf{W}_{k/a_{ik}}}{2}, \tag{5}$$

  where $\mathbf{W}_{i/a_{ik}}$ denotes the restriction of $\mathbf{W}_i$ to the face a_{ik}.

- $[\![\mathbf{W}_h]\!]_{ik}$ stands for the tangential jump through any internal face a_{ik}:

$$[\![\mathbf{W}_h]\!]_{ik} = \left(\mathbf{W}_{k/a_{ik}} - \mathbf{W}_{i/a_{ik}}\right) \times \mathbf{n}_{ik}. \tag{6}$$

  In the approximation, we choose to use completely centered fluxes, i.e. the value of the field at an internal face is approximated by its average on this face. Finally $\mathcal{F}_h$ denotes the union of all the faces of the simplicial mesh $\mathcal{T}_h$ and $\mathcal{F}_h^{int}$ the union of all internal faces. A term $\displaystyle\int_{\mathcal{F}_h^{int}}$ will be understood as $\displaystyle\sum_{a_{ik}\in\mathcal{F}_h^{int}}\int_{a_{ik}}$.

In what follows, we consider the particular case where the discretization space is given by discontinuous piecewise polynomials of degree at most k in each tetrahedron. In τ_i, it is denoted by $\mathbb{P}_k(\tau_i)$. Then:

$$\mathbf{V}_h = \left\{\mathbf{W}_h \in \mathbf{L}^2(\Omega) \quad | \quad (\mathbf{W}_h)_{/\tau_i} \in (\mathbb{P}_k(\tau_i))^3, \quad \forall \tau_i \in \mathcal{T}_h\right\}.$$

For what concern the treatment of the boundary condition on a metallic wall, for each of the corresponding boundary face (still denoted a_{ik}), we set $\mathbf{E}_{k/a_{ik}} = -\mathbf{E}_{i/a_{ik}}$, and $\mathbf{H}_{k/a_{ik}} = \mathbf{H}_{i/a_{ik}}$.

Taking the dot product of the equations of (1) by vectorial test functions φ, ψ, ϕ and integrating by part over τ_i, one has:

$$\left\{\begin{array}{l} \mu \displaystyle\int_{\tau_i} \frac{\partial \mathbf{H}}{\partial t} \cdot \varphi + \int_{\tau_i} \mathbf{E} \cdot \mathbf{curl}\, \varphi = \int_{\partial \tau_i} \varphi \cdot (\mathbf{E} \times \mathbf{n}), \\ \varepsilon_0 \varepsilon_\infty \displaystyle\int_{\tau_i} \frac{\partial \mathbf{E}}{\partial t} \cdot \psi - \int_{\tau_i} \mathbf{H} \cdot \mathbf{curl}\, \psi = -\int_{\partial \tau_i} \psi \cdot (\mathbf{H} \times \mathbf{n}) + \frac{1}{\tau_r} \int_{\tau_i} \mathbf{P} \cdot \psi - \alpha \int_{\tau_i} \mathbf{E} \cdot \psi, \\ \displaystyle\int_{\tau_i} \frac{\partial \mathbf{P}}{\partial t} \cdot \phi = -\frac{1}{\tau_r} \int_{\tau_i} \mathbf{P} \cdot \phi + \beta \int_{\tau_i} \mathbf{E} \cdot \phi. \end{array}\right. \tag{7}$$

One seeks the semi-discrete solution $\mathbf{U}_h = (\mathbf{H}_h, \mathbf{E}_h, \mathbf{P}_h) \in \mathcal{C}^1(0, T, \mathbf{V}_h^3)$ as a solution of the following weak formulation, $\forall (\varphi_h, \psi_h, \phi_h) \in \mathbf{V}_h^3$, $\forall t \in [0, T]$, $\forall i \in [0, N_\tau]$:

$$\left\{\begin{array}{l} \mu \displaystyle\int_{\tau_i} \frac{\partial \mathbf{H}_h}{\partial t} \cdot \varphi_h + \int_{\tau_i} \mathbf{E}_h \cdot \mathbf{curl}\, \varphi_h = \sum_{k \in V_i} \int_{a_{ik}} \varphi_h \cdot (\{\mathbf{E}_h\}_{ik} \times \mathbf{n}), \\ \varepsilon_0 \varepsilon_\infty \displaystyle\int_{\tau_i} \frac{\partial \mathbf{E}_h}{\partial t} \cdot \psi_h - \int_{\tau_i} \mathbf{H}_h \cdot \mathbf{curl}\, \psi_h = -\sum_{k \in V_i} \int_{a_{ik}} \psi_h \cdot (\{\mathbf{H}_h\}_{ik} \times \mathbf{n}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \displaystyle + \frac{1}{\tau_r} \int_{\tau_i} \mathbf{P}_h \cdot \psi_h - \alpha \int_{\tau_i} \mathbf{E}_h \cdot \psi_h, \\ \displaystyle\frac{1}{\varepsilon_0(\varepsilon_s - \varepsilon_\infty)} \int_{\tau_i} \frac{\partial \mathbf{P}_h}{\partial t} \cdot \phi_h = -\frac{1}{\varepsilon_0(\varepsilon_s - \varepsilon_\infty)\tau_r} \int_{\tau_i} \mathbf{P}_h \cdot \phi_h + \frac{1}{\tau_r} \int_{\tau_i} \mathbf{E}_h \cdot \phi_h. \end{array}\right. \tag{8}$$

We consider the following initial conditions:

$$\mathbf{H}_h(0) = \pi_h(\mathbf{H}_0), \quad \mathbf{E}_h(0) = \pi_h(\mathbf{E}_0), \quad \mathbf{P}_h(0) = \pi_h(\mathbf{P}_0),$$

where π_h is the orthogonal L^2-projection onto $\mathbf{V}_h$.

Global formulation. Let $\mathbf{U} = (X, Y, Z)$ and $\mathbf{U}' = (X', Y', Z')$ and let us define:

$$\left\{\begin{array}{l} m(\mathbf{U}, \mathbf{U}') = \mu \displaystyle\int_\Omega X \cdot X' + \varepsilon_0 \varepsilon_\infty \int_\Omega Y \cdot Y' + \frac{1}{\varepsilon_0(\varepsilon_s - \varepsilon_\infty)} \int_\Omega Z \cdot Z', \\ a(\mathbf{U}, \mathbf{U}') = \displaystyle\int_\Omega (X \cdot \mathbf{curl}_h Y' - Y \mathbf{curl}_h X') - \alpha \int_\Omega Y \cdot Y' + \frac{1}{\tau_r} \int_\Omega Y \cdot Z' \\ \qquad\qquad \displaystyle + \frac{1}{\tau_r} \int_\Omega Z \cdot Y' - \frac{1}{\varepsilon_0(\varepsilon_s - \varepsilon_\infty)\tau_r} \int_\Omega Z \cdot Z', \\ b(\mathbf{U}, \mathbf{U}') = \displaystyle\int_{\mathcal{F}_h^{int}} \{X\} [\![Y']\!] - \int_{\mathcal{F}_h^{int}} \{Y\} [\![X']\!] + \int_{\partial\Omega} Y' \cdot (X \times n), \end{array}\right. \tag{9}$$

where $\mathbf{curl}_h$ is the piecewise curl operator, such that $(\mathbf{curl}_h V)_{/\tau_i} = \mathbf{curl}\,(V_{/\tau_i})$.

Summing up all the contributions of (8) on each element, one obtains:

$$m(\frac{\partial}{\partial t} \mathbf{U}_h, \mathbf{U}'_h) = a(\mathbf{U}_h, \mathbf{U}'_h) + b(\mathbf{U}_h, \mathbf{U}'_h) \text{ for all } \mathbf{U}'_h = (\varphi_h, \psi_h, \phi_h) \in \mathbf{V}_h^3. \tag{10}$$

Existence. One chooses a basis for the DG space and writes the equation in a matricial form. Then, solving these equations requires the inversion of the DG mass matrix and the resolution of a system of ordinary differential equations. This yields the existence of the semi-discrete solution.

Proposition 3.1 *The weak formulation is consistent. In other words, if* $\mathbf{U}$ *is the exact solution then:*

$$m(\frac{\partial}{\partial t}\mathbf{U}, \mathbf{U}'_h) = a(\mathbf{U}, \mathbf{U}'_h) + b(\mathbf{U}, \mathbf{U}'_h),$$

for all $\mathbf{U}'_h = (\varphi_h, \psi_h, \phi_h) \in \mathbf{V}_h^3$.

3.2 Stability analysis

Like in the continuous case, one defines the semi-discrete energy $\mathcal{E}_h$:

$$\mathcal{E}_h(t) = \frac{1}{2}\left(\mu\|\mathbf{H}_h\|^2_{\mathbf{L}^2(\Omega)}(t) + \varepsilon_0\varepsilon_\infty\|\mathbf{E}_h\|^2_{\mathbf{L}^2(\Omega)}(t) + \frac{1}{\varepsilon_0(\varepsilon_s - \varepsilon_\infty)}\|\mathbf{P}_h\|^2_{\mathbf{L}^2(\Omega)}(t)\right). \tag{11}$$

Proposition 3.2 $\mathcal{E}_h$ *is a decreasing function in time i.e.* $\mathcal{E}_h(t) \leq \mathcal{E}_h(0)$.

Proof. Using the previously introduced bilinear forms, one has by (10):

$$m(\frac{\partial}{\partial t}\mathbf{U}_h, \mathbf{U}_h) = a(\mathbf{U}_h, \mathbf{U}_h) + b(\mathbf{U}_h, \mathbf{U}_h).$$

Simplifying the expressions one finds:

$$\frac{d}{dt}\mathcal{E}_h \leq -\frac{1}{\tau_r\varepsilon_0(\varepsilon_s - \varepsilon_\infty)}\|\mathbf{P} - \varepsilon_0(\varepsilon_s - \varepsilon_\infty)\mathbf{E}\|^2_{\mathbf{L}^2(\Omega)} - \sigma\|\mathbf{E}\|^2_{\mathbf{L}^2(\Omega)} \leq 0.$$

Remark 3.3 *This result follows directly from ideas of [FLLP05]. One just needs to consider in addition the contributions due to* $\mathbf{E}$ *and* $\mathbf{P}$ *which are of the same form than in the continuous case.*

□

3.3 Convergence

In this section, we first recall some classical approximation results and then elaborate on the convergence of the introduced semi-discrete scheme.

3.3.1 Classical approximation results

Lemma 3.4 *Let* $\tau \in \mathcal{T}_h$ *and* Π_h *a linear continuous projector from* $H^{s+1}(\tau)$ *to* $\mathbb{P}_k(\tau)$*, for* $s \geq 0$ *and* $k \geq 1$*. Then if* $u \in H^{s+1}(\tau)$*:*

$$|u - \Pi_h(u)|_{m,\tau} \leq C h_\tau^{min(s,k)+1-m}\|u\|_{s+1,\tau}, \quad m = 0, 1 \tag{12}$$

and:

$$\|u - \Pi_h(u)\|_{0,\partial\tau} \leq C h_\tau^{min(s,k)+\frac{1}{2}}\|u\|_{s+1,\tau}. \tag{13}$$

Lemma 3.5 (Inverse inequalities) *Let* $\tau \in \mathcal{T}_h$ *and* $k \geq 1$*, then for all* $p \in \mathbb{P}_k(\tau)$*:*

$$\|p\|_{0,\partial\tau} \leq C h_\tau^{-\frac{1}{2}}\|p\|_{0,\tau}, \tag{14}$$

$$\|p\|_{1,\tau} \leq C h_\tau^{-1}\|p\|_{0,\tau}. \tag{15}$$

3.3.2 Convergence result

We have the following result.

Theorem 3.6 *Let* $(\mathbf{H}, \mathbf{E}, \mathbf{P})$ *be the exact solution of (1) and* $(\mathbf{H}_h, \mathbf{E}_h, \mathbf{P}_h) \in \mathcal{C}^1([0,T], \mathbf{V}_h^3)$ *be the semi-discrete solution of (7). If* $(\mathbf{H}, \mathbf{E}, \mathbf{P}) \in \mathcal{C}^0([0,T], H^{s+1}(\Omega)^9)$ *for* $s \geq 0$*, then there exists a constant* $C > 0$ *independent of* h *such that:*

$$\begin{aligned} \max_{t\in[0,T]} \left(\|\pi_h(\mathbf{H}) - \mathbf{H}_h\|^2_{\mathbf{L}^2(\Omega)} + \|\pi_h(\mathbf{E}) - \mathbf{E}_h\|^2_{\mathbf{L}^2(\Omega)} + \|\pi_h(\mathbf{P}) - \mathbf{P}_h\|^2_{\mathbf{L}^2(\Omega)} \right)^{\frac{1}{2}} \\ \leq C h^{min(s,k)} \|(\mathbf{H}, \mathbf{E}, \mathbf{P})\|_{\mathcal{C}^0([0,T],(\mathbf{H}^{s+1}(\Omega))^3)} \end{aligned} \tag{16}$$

where π_h *is the orthogonal* L^2*-projection on* $\mathbf{V}_h$*.*

Proof. One needs an estimation of:

$$\|\pi_h(\mathbf{E}) - \mathbf{E}_h\|^2_{\mathbf{L}^2(\Omega)} + \|\pi_h(\mathbf{H}) - \mathbf{H}_h\|^2_{\mathbf{L}^2(\Omega)} + \|\pi_h(\mathbf{P}) - \mathbf{P}_h\|^2_{\mathbf{L}^2(\Omega)}.$$

By abusing the notation, we will consider that $\pi_h(\mathbf{U}) = (\pi_h(\mathbf{H}), \pi_h(\mathbf{E}), \pi_h(\mathbf{P}))$. If:

$$\varepsilon(t) = \frac{1}{2} m(\pi_h(\mathbf{U}) - \mathbf{U}_h, \pi_h(\mathbf{U}) - \mathbf{U}_h),$$

then there exists a constant $C > 0$ such that:

$$\varepsilon(t) \geq C \left[\|\pi_h(\mathbf{E}) - \mathbf{E}_h\|^2_{\mathbf{L}^2(\Omega)} + \|\pi_h(\mathbf{H}) - \mathbf{H}_h\|^2_{\mathbf{L}^2(\Omega)} + \|\pi_h(\mathbf{P}) - \mathbf{P}_h\|^2_{\mathbf{L}^2(\Omega)} \right].$$

From the continuous equations and the semi-discrete ones, we can write:

$$\begin{cases} m(\frac{\partial}{\partial t}\mathbf{U}, \mathbf{U}_h') & = & a(\mathbf{U}, \mathbf{U}_h') + b(\mathbf{U}, \mathbf{U}_h') \\ m(\frac{\partial}{\partial t}\mathbf{U}_h, \mathbf{U}_h') & = & a(\mathbf{U}_h, \mathbf{U}_h') + b(\mathbf{U}_h, \mathbf{U}_h'). \end{cases}$$

This gives:

$$m(\frac{\partial}{\partial t}(\mathbf{U} - \mathbf{U}_h), \mathbf{U}_h') = a(\mathbf{U} - \mathbf{U}_h, \mathbf{U}_h') + b(\mathbf{U} - \mathbf{U}_h, \mathbf{U}_h'). \tag{17}$$

Applying this relation to $\mathbf{U}_h' = \pi_h(\mathbf{U}) - \mathbf{U}_h$ yields:

$$m(\frac{\partial}{\partial t}(\mathbf{U} - \mathbf{U}_h), \pi_h(\mathbf{U}) - \mathbf{U}_h) = a(\mathbf{U} - \mathbf{U}_h, \pi_h(\mathbf{U}) - \mathbf{U}_h) + b(\mathbf{U} - \mathbf{U}_h, \pi_h(\mathbf{U}) - \mathbf{U}_h) = 0.$$

Furthermore:

$$\begin{aligned} m(\frac{\partial}{\partial t}\pi_h(\mathbf{U}) - \frac{\partial}{\partial t}\mathbf{U}_h, \pi_h(\mathbf{U}) - \mathbf{U}_h) &= m(\frac{\partial}{\partial t}\pi_h(\mathbf{U}) - \frac{\partial}{\partial t}\mathbf{U} + \frac{\partial}{\partial t}\mathbf{U} - \frac{\partial}{\partial t}\mathbf{U}_h, \pi_h(\mathbf{U}) - \mathbf{U}_h) \\ &= m(\pi_h(\frac{\partial}{\partial t}\mathbf{U}) - \frac{\partial}{\partial t}\mathbf{U}, \pi_h(\mathbf{U}) - \mathbf{U}_h) \\ &+ m(\frac{\partial}{\partial t}\mathbf{U} - \frac{\partial}{\partial t}\mathbf{U}_h, \pi_h(\mathbf{U}) - \mathbf{U}_h). \end{aligned} \tag{18}$$

Since π_h is the orthogonal L^2-projection on $\mathbf{V}_h$:

$$m(\pi_h(\frac{\partial}{\partial t}\mathbf{U}) - \frac{\partial}{\partial t}\mathbf{U}, \pi_h(\mathbf{U}) - \mathbf{U}_h) = 0,$$

and
$$\begin{array}{rcl} a(\pi_h(\mathbf{U})-\mathbf{U},\pi_h(\mathbf{U})-\mathbf{U}_h) & = & \langle \pi_h(\mathbf{E})-\mathbf{E},\mathbf{curl}_h(\pi_h(\mathbf{H})-\mathbf{H}_h)\rangle \\ & - & \langle \pi_h(\mathbf{H})-\mathbf{H},\mathbf{curl}_h(\pi_h(\mathbf{E})-\mathbf{E}_h)\rangle \\ & - & \alpha\langle \pi_h(\mathbf{E})-\mathbf{E},\pi_h(\mathbf{E})-\mathbf{E}_h\rangle \\ & - & \dfrac{1}{\varepsilon_0(\varepsilon_s-\varepsilon_\infty)\tau_r}\langle \pi_h(\mathbf{P})-\mathbf{P},\pi_h(\mathbf{P})-\mathbf{P}_h\rangle \\ & + & \dfrac{1}{\tau_r}\langle \pi_h(\mathbf{P})-\mathbf{P},\pi_h(\mathbf{E})-\mathbf{E}_h\rangle \\ & + & \dfrac{1}{\tau_r}\langle \pi_h(\mathbf{E})-\mathbf{E},\pi_h(\mathbf{P})-\mathbf{P}_h\rangle = 0, \end{array}$$

for the same reason. One then deduces from the last remark and (17):

$$\begin{array}{rcl} m(\dfrac{\partial}{\partial t}\pi_h(\mathbf{U})-\dfrac{\partial}{\partial t}\mathbf{U}_h,\pi_h(\mathbf{U})-\mathbf{U}_h) & = & m(\dfrac{\partial}{\partial t}\mathbf{U}-\dfrac{\partial}{\partial t}\mathbf{U}_h,\pi_h(\mathbf{U})-\mathbf{U}_h) \\ & = & a(\mathbf{U}-\mathbf{U}_h,\pi_h(\mathbf{U})-\mathbf{U}_h)+b(\mathbf{U}-\mathbf{U}_h,\pi_h(\mathbf{U})-\mathbf{U}_h) \\ & = & a(\mathbf{U}-\pi_h(\mathbf{U})+\pi_h(\mathbf{U})-\mathbf{U}_h,\pi_h(\mathbf{U})-\mathbf{U}_h) \qquad (19) \\ & + & b(\mathbf{U}-\mathbf{U}_h,\pi_h(\mathbf{U})-\mathbf{U}_h) \\ & = & a(\pi_h(\mathbf{U})-\mathbf{U}_h,\pi_h(\mathbf{U})-\mathbf{U}_h)+b(\mathbf{U}-\mathbf{U}_h,\pi_h(\mathbf{U})-\mathbf{U}_h) \\ & = & a(\pi_h(\mathbf{U})-\mathbf{U}_h,\pi_h(\mathbf{U})-\mathbf{U}_h)+b(\pi_h(\mathbf{U})-\mathbf{U}_h,\pi_h(\mathbf{U})-\mathbf{U}_h) \\ & + & b(\mathbf{U}-\pi_h(\mathbf{U}),\pi_h(\mathbf{U})-\mathbf{U}_h). \end{array}$$

One can quite easily show (as in previous sections and in [FLLP05]) that for any $\mathbf{U}'_h=(X'_h,Y'_h,Z'_h)$ in $(\mathbf{V}_h)^3$:

$$a(\mathbf{U}'_h,\mathbf{U}'_h)+b(\mathbf{U}'_h,\mathbf{U}'_h)=-\alpha\|Y'_h\|^2_{\mathbf{L}^2(\Omega)}+\frac{2}{\tau_r}\langle Y'_h,Z'_h\rangle-\frac{1}{\varepsilon_0(\varepsilon_s-\varepsilon_\infty)\tau_r}\|Z'_h\|^2_{\mathbf{L}^2(\Omega)}.$$

Same argument than in previous proofs, gives that

$$a(\mathbf{U}'_h,\mathbf{U}'_h)+b(\mathbf{U}'_h,\mathbf{U}'_h)\le 0.$$

Then, since:

$$\varepsilon(t)=\frac{1}{2}\int_0^t m(\partial_s(\pi_h(\mathbf{U})-\mathbf{U}_h),\pi_h(\mathbf{U})-\mathbf{U}_h)ds,$$

one can write:

$$\varepsilon(t)\ \le\ \frac{1}{2}\int_0^t b(\mathbf{U}-\pi_h(\mathbf{U}),\pi_h(\mathbf{U})-\mathbf{U}_h).$$

It only remains to estimate the last term involving b. We will refer to [FLLP05], since this follows exactly their proof of theorem 3.5. on page 1163. One obtains:

$$\begin{array}{c} b(\pi_h(\mathbf{U})-\mathbf{U},\pi_h(\mathbf{U})-\mathbf{U}_h) \\ \le Ch^{\min(s,k)}\left(\|\pi_h(\mathbf{E})-\mathbf{E}_h\|^2_{\mathbf{L}^2(\Omega)}+\|\pi_h(\mathbf{H})-\mathbf{H}_h\|^2_{\mathbf{L}^2(\Omega)}\right)^{\frac{1}{2}}\|(\mathbf{H},\mathbf{E})\|_{(\mathbf{H}^{s+1}(\Omega))^2}. \end{array}$$

Let:

$$\gamma(t) = \|\pi_h(\mathbf{H}) - \mathbf{H}_h\|^2_{\mathbf{L}^2(\Omega)} + \|\pi_h(\mathbf{E}) - \mathbf{E}_h\|^2_{\mathbf{L}^2(\Omega)} + \|\pi_h(\mathbf{P}) - \mathbf{P}_h\|^2_{\mathbf{L}^2(\Omega)},$$

and:

$$\delta = \max_{t\in[0,T]} \gamma(t)^{\frac{1}{2}}.$$

δ is well defined due to the regularity hypotheses on the solutions $(\mathbf{H}, \mathbf{E}, \mathbf{P})$ and $(\mathbf{H}_h, \mathbf{E}_h, \mathbf{P}_h)$. Then

$$\varepsilon(t) \leq Ch^{min(s,k)} \int_0^t \gamma(u)^{\frac{1}{2}} \|(\mathbf{H}, \mathbf{E})\|_{(\mathbf{H}^{s+1}(\Omega))^2} du$$

One deduces for all $t \in [0, T]$:

$$\frac{\gamma(t)}{\delta} \leq Ch^{\min(s,k)} \|(\mathbf{H}, \mathbf{E})\|_{\mathcal{C}^0([0,T],(\mathbf{H}^{s+1}(\Omega))^2)}, \tag{20}$$

which leads to the result. □

Error estimation is then easily deduced from (16). Indeed one makes use of:

$$\mathbf{U} - \mathbf{U}_h = \mathbf{U} - \pi_h(\mathbf{U}) + \pi_h(\mathbf{U}) - \mathbf{U}_h,$$

and the classical approximation results recalled previously. The approximation error then keeps the same order.

Corollary 3.7 *Under the same assumptions than those of theorem 3.6, there exists a constant $C > 0$ such that:*

$$\left(\|\mathbf{H} - \mathbf{H}_h\|^2_{\mathcal{C}^0([0,T],\mathbf{L}^2(\Omega))} + \|\mathbf{E} - \mathbf{E}_h\|^2_{\mathcal{C}^0([0,T],\mathbf{L}^2(\Omega))} + \|\mathbf{P} - \mathbf{P}_h\|^2_{\mathcal{C}^0([0,T],\mathbf{L}^2(\Omega))}\right)^{\frac{1}{2}}$$
$$\leq Ch^{\min(s,k)} \|(\mathbf{H}, \mathbf{E}, \mathbf{P})\|_{\mathcal{C}^0([0,T],(\mathbf{H}^{s+1}(\Omega))^3)}.$$

4 Fully discrete study

The semi-discrete scheme is discretized in time using the leap-frog or Störmer Verlet scheme (for a formulation see [HLW06]). This scheme has the advantages of being explicit. On the other hand, the price to pay is that a CFL condition has to be satisfied.

4.1 Formulation

We consider a fixed time step Δt. The unknowns for the electric field and the polarization are approximated at integer time station $t^n = n\Delta t$ with notation $\mathbf{E}^n$ and $\mathbf{P}^n$. The unknowns for the magnetic field are approximated at half-integer time station $t^{n+\frac{1}{2}} = (n + \frac{1}{2})\Delta t$ with notation $\mathbf{H}^{n+\frac{1}{2}}$. Let $\mathbf{E}^{[n+\frac{1}{2}]} = \dfrac{\mathbf{E}^n + \mathbf{E}^{n+1}}{2}$ and $\mathbf{P}^{[n+\frac{1}{2}]} = \dfrac{\mathbf{P}^n + \mathbf{P}^{n+1}}{2}$.

One seeks $(\mathbf{H}_h^{n+\frac{3}{2}}, \mathbf{E}_h^{n+1}, \mathbf{P}_h^{n+1})$ such that $\forall \varphi_h \in \mathbf{V}_h$, $\forall \psi_h \in \mathbf{V}_h$, $\forall \phi_h \in \mathbf{V}_h$:

$$\mu \int_{\tau_i} \frac{\mathbf{H}_i^{n+\frac{3}{2}} - \mathbf{H}_i^{n+\frac{1}{2}}}{\Delta t} \cdot \varphi_h = \sum_{k\in\mathcal{V}_i} \int_{a_{ik}} \varphi_h \cdot \left(\left\{\mathbf{E}_h^{n+1}\right\}_{ik} \times \mathbf{n}_{ik}\right) - \int_{\tau_i} \mathbf{curl}\, \varphi_h \cdot \mathbf{E}_i^{n+1}, \tag{21}$$

$$\begin{aligned}\varepsilon_0\varepsilon_\infty \int_{\tau_i} \frac{\mathbf{E}_i^{n+1} - \mathbf{E}_i^{n}}{\Delta t} \cdot \psi_h &= -\sum_{k\in\mathcal{V}_i} \int_{a_{ik}} \psi_h \cdot \left(\left\{\mathbf{H}_h^{n+\frac{1}{2}}\right\}_{ik} \times \mathbf{n}_{ik}\right) + \int_{\tau_i} \mathbf{curl}\, \psi_h \cdot \mathbf{H}_i^{n+\frac{1}{2}} \\ &+ \frac{1}{\tau_r}\int_{\tau_i} \mathbf{P}_i^{[n+\frac{1}{2}]} \cdot \psi_h - \alpha \int_{\tau_i} \mathbf{E}_i^{[n+\frac{1}{2}]} \cdot \psi_h, \end{aligned} \tag{22}$$

$$\frac{1}{\varepsilon_0(\varepsilon_s - \varepsilon_\infty)} \int_{\tau_i} \frac{\mathbf{P}_i^{n+1} - \mathbf{P}_i^{n}}{\Delta t} \cdot \phi_h = -\frac{1}{\varepsilon_0(\varepsilon_s - \varepsilon_\infty)\tau_r} \int_{\tau_i} \mathbf{P}_i^{[n+\frac{1}{2}]} \cdot \phi_h + \frac{1}{\tau_r} \int_{\tau_i} \mathbf{E}_i^{[n+\frac{1}{2}]} \cdot \phi_h. \tag{23}$$

Initial conditions are given by $\mathbf{H}_h^0 = \mathbf{H}_h(0)$, $\mathbf{E}_h^0 = \mathbf{E}_h(0)$, $\mathbf{P}_h^0 = \mathbf{P}_h(0)$.

The fact that the DG mass matrix is invertible uniquely defines, under the conditions:

$$1 + \frac{\Delta t}{2\tau_r} \neq 0 \text{ and } \varepsilon_0\varepsilon_\infty + \frac{\alpha}{2}\Delta t - \frac{\beta}{2\tau_r + \Delta t} \neq 0,$$

the iterates $(\mathbf{H}^{n+\frac{1}{2}}, \mathbf{E}^{n+1}, \mathbf{P}^{n+1})$.

Like in the semi-discrete case, we now turn to stability analysis via the study of the associated discrete energy.

4.2 Discrete energy

One defines:

$$\mathcal{E}^n = \sum_{i\in[0,N_\tau]} \mathcal{E}_i^n, \tag{24}$$

with:

$$\mathcal{E}_i^n = \frac{1}{2}\left(\varepsilon_0\varepsilon_\infty \langle \mathbf{E}_i^n, \mathbf{E}_i^n\rangle_{/\tau_i} + \mu\langle \mathbf{H}_i^{n-\frac{1}{2}}, \mathbf{H}_i^{n+\frac{1}{2}}\rangle_{/\tau_i} + \frac{1}{\varepsilon_0(\varepsilon_s - \varepsilon_\infty)}\langle \mathbf{P}_i^n, \mathbf{P}_i^n\rangle_{/\tau_i}\right), \tag{25}$$

where $\langle ., .\rangle_{/\tau_i}$ denotes the scalar product on τ_i.

Positivity of the energy. With the hypotheses made on the mesh elements τ_i, we can assume that there exists a generic constant $C > 0$ such that $\forall p \in \mathbb{P}_k$:

$$\|\mathbf{curl}\, p\|_{\tau_i} \leq Ch^{-1}\|p\|_{\tau_i},$$

and:

$$\|p\|^2_{a_{ik}} \leq Ch^{-1}\|p\|^2_{\tau_i}.$$

One can then follow exactly [FLLP05] (§ 2.4, the added term in the energy we consider does not affect the proof) to prove the following result:

Proposition 4.1 (CFL condition) *The discrete energy (24)-(25) defines a quadratic form of the unknowns $\mathbf{E}^n$, $\mathbf{H}^{n-\frac{1}{2}}$ and $\mathbf{P}^n$ if:*

$$\frac{1}{\sqrt{\varepsilon_0\varepsilon_\infty\mu}}\Delta t < Ch. \tag{26}$$

Variation of the energy. We now prove the following result.

Proposition 4.2 *The discrete energy (24)-(25) is decreasing i.e. there exists $C > 0$ such that:*

$$\mathcal{E}^n \leq \mathcal{E}^0.$$

Proof. One writes the difference $\mathcal{E}_i^{n+1} - \mathcal{E}_i^n$ and uses the fully discrete version of the equations with particular test functions. Equation (21) is used at time stations $n+\frac{1}{2}$ and at time $n-\frac{1}{2}$ with test function given by $\mathbf{H}_i^{n+\frac{1}{2}}$. Equation (22) is used at time station n with test function given by $\mathbf{E}_i^{[n+\frac{1}{2}]}$. Equation (23) is used at time stationn, with test function given by $\mathbf{P}_i^{[n+\frac{1}{2}]}$. This yields:

$$\begin{aligned}
\mathcal{E}_i^{n+1} - \mathcal{E}_i^n &= -\Delta t \sum_{k\in\mathcal{V}_i} \int_{a_{ik}} \mathbf{E}_i^{[n+\frac{1}{2}]} \cdot \left(\left\{ \mathbf{H}_h^{n+\frac{1}{2}} \right\}_{a_{ik}} \times \mathbf{n}_{ik} \right) + \Delta t \int_{\tau_i} \mathbf{curl}\ \mathbf{E}_i^{[n+\frac{1}{2}]} \cdot \mathbf{H}_i^{n+\frac{1}{2}} \\
&+ \frac{\Delta t}{\tau_r} \int_{\tau_i} \mathbf{P}_i^{[n+\frac{1}{2}]} \cdot \mathbf{E}_i^{[n+\frac{1}{2}]} - \alpha \Delta t \int_{\tau_i} |\mathbf{E}_i^{[n+\frac{1}{2}]}|^2 \\
&+ \Delta t \sum_{k\in\mathcal{V}_i} \int_{a_{ik}} \mathbf{H}_i^{n+\frac{1}{2}} \cdot \left(\left\{ \mathbf{E}_h^{[n+\frac{1}{2}]} \right\}_{a_{ik}} \times \mathbf{n}_{ik} \right) - \Delta t \int_{\tau_i} \mathbf{curl}\ \mathbf{H}_i^{n+\frac{1}{2}} \cdot \mathbf{E}_i^{[n+\frac{1}{2}]} \\
&- \frac{\Delta t}{\varepsilon_0(\varepsilon_s - \varepsilon_\infty)\tau_r} \int_{\tau_i} |\mathbf{P}_i^{[n+\frac{1}{2}]}|^2 + \frac{\Delta t}{\tau_r} \int_{\tau_i} \mathbf{E}_i^{[n+\frac{1}{2}]} \cdot \mathbf{P}_i^{[n+\frac{1}{2}]}.
\end{aligned} \tag{27}$$

Integrating by parts yields:

$$\begin{aligned}
\mathcal{E}_i^{n+1} - \mathcal{E}_i^n &= -\frac{\Delta t}{2} \sum_{k\in\mathcal{V}_i} \int_{a_{ik}} \left(\mathbf{E}_i^{[n+\frac{1}{2}]} \times \mathbf{H}_k^{n+\frac{1}{2}} + \mathbf{E}_k^{[n+\frac{1}{2}]} \times \mathbf{H}_i^{n+\frac{1}{2}} \right) \cdot \mathbf{n}_{ik} \\
&+ \frac{\Delta t}{\tau_r} \int_{\tau_i} \mathbf{P}_i^{[n+\frac{1}{2}]} \cdot \mathbf{E}_i^{[n+\frac{1}{2}]} - \alpha \Delta t \int_{\tau_i} |\mathbf{E}_i^{[n+\frac{1}{2}]}|^2 \\
&- \frac{\Delta t}{\varepsilon_0(\varepsilon_s - \varepsilon_\infty)\tau_r} \int_{\tau_i} |\mathbf{P}_i^{[n+\frac{1}{2}]}|^2 + \frac{\Delta t}{\tau_r} \int_{\tau_i} \mathbf{E}_i^{[n+\frac{1}{2}]} \cdot \mathbf{P}_i^{[n+\frac{1}{2}]}.
\end{aligned}$$

Taking into account that we are dealing with a metallic boundary condition only, and adding all the contributions:

$$\begin{aligned}
\mathcal{E}^{n+1} - \mathcal{E}^n &= \sum_{i\in[0,N_T]} \Bigg[\frac{2\Delta t}{\tau_r} \int_{\tau_i} \mathbf{P}_i^{[n+\frac{1}{2}]} \cdot \mathbf{E}_i^{[n+\frac{1}{2}]} - \alpha \Delta t \int_{\tau_i} |\mathbf{E}_i^{[n+\frac{1}{2}]}|^2 - \\
&\qquad \frac{\Delta t}{\varepsilon_0(\varepsilon_s - \varepsilon_\infty)\tau_r} \int_{\tau_i} |\mathbf{P}_i^{[n+\frac{1}{2}]}|^2 + \Bigg].
\end{aligned}$$

This gives like in the continuous and semi-discrete case:

$$\mathcal{E}^{n+1} - \mathcal{E}^n \quad \leq \quad 0$$

$\square$

We can now turn to the analysis of the convergence of the fully discrete scheme.

4.3 Convergence of the fully discrete scheme

In this section we prove the following result.

Proposition 4.3 *Let* $(\mathbf{H},\mathbf{E},\mathbf{P}) \in \mathcal{C}^3([0,T],\mathcal{L}(\Omega)) \cap \mathcal{C}^0([0,T],(\mathbf{H}^{s+1}(\Omega))^3)$. *Under the CFL condition (26), the following error estimate holds:*

$$\max_{n=0..N}\left(\mu\left\|\mathbf{H}(t_{n+\frac{1}{2}})-\mathbf{H}_h^{n+\frac{1}{2}}\right\|^2_{\mathcal{C}([0,T],\mathbf{L}^2(\Omega))} + \varepsilon_0\varepsilon_\infty\left\|\mathbf{E}(t_n)-\mathbf{E}_h^n\right\|^2_{\mathcal{C}([0,T],\mathbf{L}^2(\Omega))} + \frac{1}{\varepsilon_0(\varepsilon_s-\varepsilon_\infty)}\left\|\mathbf{P}(t_n)-\mathbf{P}_h^n\right\|^2_{\mathcal{C}([0,T],\mathbf{L}^2(\Omega))}\right)^{\frac{1}{2}}$$
$$\leq C\left(\Delta t^2 + h^{\min(s,k)}\right)\left(\|(\mathbf{H},\mathbf{E},\mathbf{P})\|_{\mathcal{C}^3([0,T],\mathcal{L}(\Omega))} + \|(\mathbf{H},\mathbf{E},\mathbf{P})\|_{\mathcal{C}^0([0,T],(\mathbf{H}^{s+1}(\Omega))^3)}\right).$$

Proof. The proof uses a result of consistency. It involves the estimation of the local consistency error given by:

$$\varepsilon_h^n = \left(\|\mathbf{E}_h(t_{n+1})-\tilde{\mathbf{E}}_h^{n+1}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{H}_h(t_{n+\frac{3}{2}})-\tilde{\mathbf{H}}_h^{n+\frac{3}{2}}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{P}_h(t_{n+1})-\tilde{\mathbf{P}}_h^{n+1}\|^2_{\mathbf{L}^2(\Omega)}\right)^{\frac{1}{2}},$$

where $\tilde{\mathbf{E}}_h^{n+1}$, $\tilde{\mathbf{H}}_h^{n+\frac{3}{2}}$, $\tilde{\mathbf{P}}_h^{n+1}$ have been computed from equations (21), (22), (23) as follows:

$$\mu\int_{\tau_i}\frac{\tilde{\mathbf{H}}_i^{n+\frac{3}{2}}-\mathbf{H}_i(t_{n+\frac{1}{2}})}{\Delta t}\cdot\varphi_h = \sum_{k\in\mathcal{V}_i}\int_{a_{ik}}\varphi_h\cdot\{\mathbf{E}_h(t_{n+1})\}_{ik}\times\mathbf{n}_{ik} \quad (28)$$
$$- \int_{\tau_i}\mathbf{curl}\,\varphi_h\cdot\mathbf{E}_i(t_{n+1}), \quad (29)$$
$$\varepsilon_0\varepsilon_\infty\int_{\tau_i}\frac{\tilde{\mathbf{E}}_i^{n+1}-\mathbf{E}_i(t_n)}{\Delta t}\cdot\psi_h = \sum_{k\in\mathcal{V}_i}\int_{a_{ik}}\psi_h\cdot\left\{\mathbf{H}_h(t_{n+\frac{1}{2}})\right\}_{ik}\times\mathbf{n}_{ik} \quad (30)$$
$$+ \int_{\tau_i}\mathbf{curl}\,\psi_h\cdot\mathbf{H}_i(t_{n+\frac{1}{2}}) + \int_{\tau_i}\frac{\mathbf{P}_i(t_{n+1})+\mathbf{P}_i(t_n)}{2\tau_r}\cdot\psi_h \quad (31)$$
$$- \alpha\int_{\tau_i}\frac{\mathbf{E}_i(t_{n+1})+\mathbf{E}_i(t_n)}{2}\cdot\psi_h, \quad (32)$$
$$\int_{\tau_i}\frac{\tilde{\mathbf{P}}_i^{n+1}-\mathbf{P}_i(t_n)}{\Delta t}\cdot\phi_h = -\frac{1}{\tau_r}\int_{\tau_i}\frac{\mathbf{P}_i(t_{n+1})+\mathbf{P}_i(t_n)}{2}\cdot\phi_h \quad (33)$$
$$+ \beta\int_{\tau_i}\frac{\mathbf{E}_i(t_{n+1})+\mathbf{E}_i(t_n)}{2}\cdot\phi_h, \quad (34)$$

For the theory on ordinary differential equations, we follow [HLW06, Dem99]. Using Taylor formulas and approximation properties, one can prove that the consistency error is of order 2, by which we mean that there exists $C>0$, such that:

$$|\varepsilon_h^n| \leq C\Delta t^3\|\mathbf{U}\|_{\mathcal{C}^3([0,T],\mathbf{L}^2(\Omega))}, \quad (35)$$

Indeed, one uses equalities like:

$$\mathbf{E}_h(t_{n+1})-\mathbf{E}_h(t_n) = \Delta t\frac{\partial}{\partial t}\mathbf{E}_h(t_{n+\frac{1}{2}}) + \frac{\Delta t^3}{28}\left(\frac{\partial^3}{\partial t^3}\mathbf{E}_h(c_{n+1}) + \frac{\partial^3}{\partial t^3}\mathbf{E}_h(c_n)\right)$$

and:

$$\frac{\mathbf{E}_h(t_{n+1})+\mathbf{E}_h(t_n)}{2} = \mathbf{E}(t_{n+\frac{1}{2}}) + \frac{\Delta t^2}{16}\left(\frac{\partial^2}{\partial t^2}\mathbf{E}_h(d_{n+1}) + \frac{\partial^2}{\partial t^2}\mathbf{E}_h(d_n)\right).$$

$(c_{n+1},c_n)\in[t_{n+\frac{1}{2}},t_{n+1}]\times[t_n,t_{n+\frac{1}{2}}]$ and $(d_{n+1},d_n)\in[t_{n+\frac{1}{2}},t_{n+1}]\times[t_n,t_{n+\frac{1}{2}}]$.

Writing the local error, one obtains $\int_\Omega (\mathbf{E}_h(t_{n+1}) - \tilde{\mathbf{E}}_h^{n+1}) \cdot \psi_h$ in terms of integrals of the type $\Delta t^3 \int_\Omega \frac{\partial^3}{\partial t^3}\mathbf{E}_h \cdot \psi_h$, $\Delta t^3 \int_\Omega \frac{\partial^2}{\partial t^2}\mathbf{E}_h \cdot \psi_h$, $\Delta t^3 \int_\Omega \frac{\partial^2}{\partial t^2}\mathbf{P}_h \cdot \psi_h$. This leads to the estimation:

$$\|\mathbf{E}_h(t_{n+1}) - \tilde{\mathbf{E}}_h^{n+1}\|_{\mathbf{L}^2(\Omega)} \leq C\Delta t^3 \left(\|\frac{\partial^3}{\partial t^3}\mathbf{E}_h\|_{\mathbf{L}^2(\Omega)} + \|\frac{\partial^2}{\partial t^2}\mathbf{E}_h\|_{\mathbf{L}^2(\Omega)} + \|\frac{\partial^2}{\partial t^2}\mathbf{P}_h\|_{\mathbf{L}^2(\Omega)} \right).$$

With same arguments for the remaining two equations, one obtains (35).

With this notation $(\mathbf{H}_h, \mathbf{E}_h, \mathbf{P}_h)$ verifies the following set of equations:

$$\begin{aligned}
\mu \int_{\tau_i} \frac{\mathbf{H}_i(t_{n+\frac{3}{2}}) - \mathbf{H}_i(t_{n+\frac{1}{2}})}{\Delta t} \cdot \varphi_h &= \sum_{k\in\mathcal{V}_i} \int_{a_{ik}} \varphi_h \cdot \{\mathbf{E}_h(t_{n+1})\}_{ik} \times \mathbf{n}_{ik} - \int_{\tau_i} \mathbf{curl}\,\varphi_h \cdot \mathbf{E}_i(t_{n+1}) \\
&+ \int_{\tau_i} \frac{\mu}{\Delta t}\left(\mathbf{H}_i(t_{n+\frac{3}{2}}) - \tilde{\mathbf{H}}_i^{n+\frac{3}{2}}\right) \cdot \varphi_h, \\
\varepsilon \int_{\tau_i} \frac{\mathbf{E}_i(t_{n+1}) - \mathbf{E}_i(t_n)}{\Delta t} \cdot \psi_h &= \sum_{k\in\mathcal{V}_i} \int_{a_{ik}} \psi_h \cdot \left\{\mathbf{H}_h(t_{n+\frac{1}{2}})\right\}_{ik} \times \mathbf{n}_{ik} + \int_{\tau_i} \mathbf{curl}\,\psi_h \cdot \mathbf{H}_i(t_{n+\frac{1}{2}}) \\
&+ \int_{\tau_i} \frac{\mathbf{P}_i(t_n) + \mathbf{P}_i(t_{n+1})}{2\tau_r} \cdot \psi_h - \alpha \int_{\tau_i} \frac{\mathbf{E}_i(t_n) + \mathbf{E}_i(t_{n+1})}{2} \cdot \psi_h \\
&+ \int_{\tau_i} \frac{\varepsilon_0\varepsilon_\infty}{\Delta t}(\mathbf{E}_i(t_{n+1}) - \tilde{\mathbf{E}}_i^{n+1}) \cdot \psi_h, \\
\int_{\tau_i} \frac{\mathbf{P}_i(t_{n+1}) - \mathbf{P}_i(t_n)}{\Delta t} \cdot \phi_h &= -\frac{1}{\tau_r} \int_{\tau_i} \frac{\mathbf{P}_i(t_n) + \mathbf{P}_i(t_{n+1})}{2} \cdot \phi_h + \beta \int_{\tau_i} \frac{\mathbf{E}_i(t_n) + \mathbf{E}_i(t_{n+1})}{2} \cdot \phi_h \\
&+ \int_{\tau_i} \frac{1}{\Delta t}(\mathbf{P}_i(t_{n+1}) - \tilde{\mathbf{P}}_i^{n+1}) \cdot \phi_h,
\end{aligned}$$

So that if one defines:

$$\begin{aligned}
\mathfrak{h}_i^n(\varphi_h) &= \int_{\tau_i} \frac{\mu}{\Delta t}(\mathbf{H}_i(t_{n+\frac{3}{2}}) - \tilde{\mathbf{H}}_i^{n+\frac{3}{2}}) \cdot \varphi_h, \\
\mathfrak{e}_i^n(\psi_h) &= \int_{\tau_i} \frac{\varepsilon_0\varepsilon_\infty}{\Delta t}(\mathbf{E}_i(t_{n+1}) - \tilde{\mathbf{E}}_i^{n+1}) \cdot \psi_h, \\
\mathfrak{p}_i^n(\phi_h) &= \int_{\tau_i} \frac{1}{\Delta t}(\mathbf{P}_i(t_{n+1}) - \tilde{\mathbf{P}}_i^{n+1}) \cdot \phi_h,
\end{aligned}$$

one has:

$$\begin{aligned}
\mu \int_{\tau_i} \frac{\mathbf{H}_i(t_{n+\frac{3}{2}}) - \mathbf{H}_i(t_{n+\frac{1}{2}})}{\Delta t} \cdot \varphi_h &= \sum_{k\in\mathcal{V}_i} \int_{a_{ik}} \varphi_h \cdot \{\mathbf{E}_h(t_{n+1})\}_{ik} \times \mathbf{n}_{ik} \\
&- \int_{\tau_i} \mathbf{curl}\,\varphi_h \cdot \mathbf{E}_i(t_{n+1}) + \mathfrak{h}_i^n(\varphi_h), \\
\varepsilon_0\varepsilon_\infty \int_{\tau_i} \frac{\mathbf{E}_i(t_{n+1}) - \mathbf{E}_i(t_n)}{\Delta t} \cdot \psi_h &= \sum_{k\in\mathcal{V}_i} \int_{a_{ik}} \psi_h \cdot \left\{\mathbf{H}_h(t_{n+\frac{1}{2}})\right\}_{ik} \times \mathbf{n}_{ik} \\
&+ \int_{\tau_i} \mathbf{curl}\,\psi_h \cdot \mathbf{H}_i(t_{n+\frac{1}{2}}) + \int_{\tau_i} \frac{\mathbf{P}_i(t_n) + \mathbf{P}_i(t_{n+1})}{2\tau_r} \cdot \psi_h \\
&- \alpha \int_{\tau_i} \frac{\mathbf{E}_i(t_n) + \mathbf{E}_i(t_{n+1})}{2} \cdot \psi_h + \mathfrak{e}_i^n(\psi_h), \\
\int_{\tau_i} \frac{\mathbf{P}_i(t_{n+1}) - \mathbf{P}_i(t_n)}{\Delta t} \cdot \phi_h &= -\frac{1}{\tau_r} \int_{\tau_i} \frac{\mathbf{P}_i(t_n) + \mathbf{P}_i(t_{n+1})}{2} \cdot \phi_h \\
&+ \beta \int_{\tau_i} \frac{\mathbf{E}_i(t_n) + \mathbf{E}_i(t_{n+1})}{2} \cdot \phi_h + \mathfrak{p}_i^n(\phi_h),
\end{aligned}$$

In a global formulation:

$$\mathfrak{h}_h^n(\varphi_h) = \int_\Omega \frac{\mu}{\Delta t}(\mathbf{H}_h(t_{n+\frac{3}{2}}) - \tilde{\mathbf{H}}_h^{n+\frac{3}{2}}) \cdot \varphi_h,$$

$$\mathfrak{e}_h^n(\psi_h) = \int_\Omega \frac{\varepsilon_0\varepsilon_\infty}{\Delta t}(\mathbf{E}_h(t_{n+1}) - \tilde{\mathbf{E}}_h^{n+1}) \cdot \psi_h,$$

$$\mathfrak{p}_h^n(\phi_h) = \int_\Omega \frac{1}{\Delta t}(\mathbf{P}_h(t_{n+1}) - \tilde{\mathbf{P}}_h^{n+1}) \cdot \phi_h.$$

and

$$|\mathfrak{h}_h^n(\varphi_h)| \leq \frac{C}{\Delta t}\left\|\mathbf{H}_h(t_{n+\frac{3}{2}}) - \tilde{\mathbf{H}}_h^{n+\frac{3}{2}}\right\|_{\mathbf{L}^2(\Omega)} \|\varphi_h\|_{\mathbf{L}^2(\Omega)},$$

$$|\mathfrak{e}_h^n(\psi_h)| \leq \frac{C}{\Delta t}\left\|\mathbf{E}_h(t_{n+1}) - \tilde{\mathbf{E}}_h^{n+1}\right\|_{\mathbf{L}^2(\Omega)} \|\psi_h\|_{\mathbf{L}^2(\Omega)},$$

$$|\mathfrak{p}_h^n(\phi_h)| \leq \frac{C}{\Delta t}\left\|\mathbf{P}_h(t_{n+1}) - \tilde{\mathbf{P}}_h^{n+1}\right\|_{\mathbf{L}^2(\Omega)} \|\phi_h\|_{\mathbf{L}^2(\Omega)}.$$

Using (35), one has:

$$|||\mathfrak{h}_h^n||| + |||\mathfrak{e}_h^n||| + |||\mathfrak{p}_h^n||| \leq C\Delta t^2 \|\mathbf{U}\|_{\mathcal{C}^3([0,T],\mathbf{L}^2(\Omega))}, \tag{36}$$

where $|||\cdot|||$ is these linear forms norm if one considers the linear form on $\mathbf{L}^2(\Omega)$. With this at hand, one can study the error. Let:

$$\eta_h^n = \mathbf{U}_h(t_n) - \mathbf{U}_h^n = (\mathbb{H}_h^{n-\frac{1}{2}}, \mathbb{E}_h^n, \mathbb{P}_h^n),$$

and:

$$\mathbb{H}_h^{n-\frac{1}{2}} = \mathbf{H}_h(t_{n-\frac{1}{2}}) - \mathbf{H}_h^{n-\frac{1}{2}},$$

$$\mathbb{E}_h^n = \mathbf{E}_h(t_n) - \mathbf{E}_h^n \quad , \quad \mathbb{P}_h^n = \mathbf{P}_h(t_n) - \mathbf{P}_h^n.$$

Define the error energy:

$$\mathfrak{E}_i^n = \frac{1}{2}\left(\varepsilon_0\varepsilon_\infty \|\mathbb{E}_i^n\|^2_{\mathbf{L}^2(\tau_i)} + \mu\langle \mathbb{H}_i^{n+\frac{1}{2}}, \mathbb{H}_i^{n-\frac{1}{2}}\rangle_{/\tau_i} + \frac{1}{\varepsilon_0(\varepsilon_s - \varepsilon_\infty)}\|\mathbb{P}_i^n\|^2_{\mathbf{L}^2(\tau_i)}\right). \tag{37}$$

The fields $(\mathbb{H}_h^n, \mathbb{E}_h^n, \mathbb{P}_h^n)$ verify the equations:

$$\begin{aligned}
\mu\int_{\tau_i} \frac{\mathbb{H}_i^{n+\frac{3}{2}} - \mathbb{H}_i^{n+\frac{1}{2}}}{\Delta t} \cdot \varphi_h &= \sum_{k\in\mathcal{V}_i} \int_{a_{ik}} \varphi_h \cdot \left\{\mathbb{E}_h^{n+1}\right\}_{ik} \times \mathbf{n}_{ik} \\
&- \int_{\tau_i} \mathbf{curl}\,\varphi_h \cdot \mathbb{E}_i^{n+1} + \mathfrak{h}_i^n(\varphi_h), \\
\varepsilon\int_{\tau_i} \frac{\mathbb{E}_i^{n+1} - \mathbb{E}_i^n}{\Delta t} \cdot \psi_h &= \sum_{k\in\mathcal{V}_i} \int_{a_{ik}} \psi_h \cdot \left\{\mathbb{H}_h^{n+\frac{1}{2}}\right\}_{ik} \times \mathbf{n}_{ik} \\
&+ \int_{\tau_i} \mathbf{curl}\,\psi_h \cdot \mathbb{H}_i^{n+\frac{1}{2}} + \int_{\tau_i} \frac{\mathbb{P}_i^n + \mathbb{P}_i^{n+1}}{2\tau_r} \cdot \psi_h \\
&- \alpha\int_{\tau_i} \frac{\mathbb{E}_i^n + \mathbb{E}_i^{n+1}}{2} \cdot \psi_h + \mathfrak{e}_i^n(\psi_h), \\
\int_{\tau_i} \frac{\mathbb{P}_i^{n+1} - \mathbb{P}_i^n}{\Delta t} \cdot \phi_h &= -\frac{1}{\tau_r}\int_{\tau_i} \frac{\mathbb{P}_i^n + \mathbb{P}_i^{n+1}}{2} \cdot \phi_h \\
&+ \beta\int_{\tau_i} \frac{\mathbb{E}_i^n + \mathbb{E}_i^{n+1}}{2} \cdot \phi_h + \mathfrak{p}_i^n(\phi_h).
\end{aligned}$$

Reasonning similarly than for the discrete energy:

$$\begin{aligned}\mathfrak{E}^{n+1}-\mathfrak{E}^{n} &= \sum_{i\in[0,N_T]}\left[\frac{\Delta t}{\tau_r}\int_{\tau_i}\mathbb{P}_h^{[n+\frac{1}{2}]}\cdot\mathbb{E}_h^{[n+\frac{1}{2}]}-\alpha\Delta t\int_{\tau_i}|\mathbb{E}_h^{[n+\frac{1}{2}]}|^2-\right.\\ &\qquad\left.\frac{\Delta t}{\varepsilon_0(\varepsilon_s-\varepsilon_\infty)\tau_r}\int_{\tau_i}|\mathbb{E}_h^{[n+\frac{1}{2}]}|^2+\frac{\Delta t}{\tau_r}\int_{\tau_i}\mathbb{E}_h^{[n+\frac{1}{2}]}\cdot\mathbb{P}_h^{[n+\frac{1}{2}]}\right]\\ &+\frac{1}{2}\left(\mathfrak{h}_h^n(\mathbb{H}^{n+\frac{1}{2}})+\mathfrak{h}_h^{n+1}(\mathbb{H}^{n+\frac{1}{2}})\right)+\mathfrak{e}_h^n(\mathbb{E}^{[n+\frac{1}{2}]})+\mathfrak{p}_h^n(\mathbb{P}^{[n+\frac{1}{2}]}).\end{aligned}\tag{38}$$

One can then prove that under a CFL condition of the same type than (26),

$$\left(\|\mathbb{H}_h^n\|^2_{\mathbf{L}^2(\Omega)}+\|\mathbb{E}_h^n\|^2_{\mathbf{L}^2(\Omega)}+\|\mathbb{P}_h^n\|^2_{\mathbf{L}^2(\Omega)}\right)^{\frac{1}{2}}\leq\Delta t^2\|\mathbf{U}\|_{\mathcal{C}^3([0,T],\mathbf{L}^2(\Omega))}.\tag{39}$$

This gives the result. □

Using the triangle inequality and results already established in the semi-discrete case, one can deduce an error estimate:

$$\begin{aligned}&\max_{n=0..N}\left(\mu\left\|\mathbf{H}(t_{n+\frac{1}{2}})-\mathbf{H}_h^{n+\frac{1}{2}}\right\|^2_{\mathcal{C}([0,T],\mathbf{L}^2(\Omega))}+\varepsilon_0\varepsilon_\infty\left\|\mathbf{E}(t_n)-\mathbf{E}_h^n\right\|^2_{\mathcal{C}([0,T],\mathbf{L}^2(\Omega))}+\right.\\ &\qquad\left.\frac{1}{\varepsilon_0(\varepsilon_s-\varepsilon_\infty)}\left\|\mathbf{P}(t_n)-\mathbf{P}_h^n\right\|^2_{\mathcal{C}([0,T],\mathbf{L}^2(\Omega))}\right)^{\frac{1}{2}}\\ \leq\;& C\Delta t^2\left\|(\mathbf{H},\mathbf{E},\mathbf{P})\right\|_{\mathcal{C}^3([0,T],\mathbf{L}^2(\Omega))}+Ch^{\min(s,k)}\left\|(\mathbf{H},\mathbf{E},\mathbf{P})\right\|_{\mathcal{C}([0,T],(\mathbf{H}^{s+1}(\Omega)))^3}\\ \leq\;& C\left(\Delta t^2+h^{\min(s,k)}\right)\left(\|(\mathbf{H},\mathbf{E},\mathbf{P})\|_{\mathcal{C}^3([0,T],\mathbf{L}^2(\Omega))}+\|(\mathbf{H},\mathbf{E},\mathbf{P})\|_{\mathcal{C}^0([0,T],(\mathbf{H}^{s+1}(\Omega))^3)}\right).\end{aligned}\tag{40}$$

□

Finally, one can conclude that the order of the convergence of the scheme is $\mathcal{O}(\Delta t^2+h^{\min(s,k)})$.

5 Numerical results

We present here some numerical results validating the theoretical analysis of the previous sections. We consider solving the following system of equations where one adds an artificial source current $\mathbf{J}$ in the equation of $\mathbf{E}$ which is such that we can deduce the exact solution of the problem:

$$\left\{\begin{array}{l}\mu\dfrac{\partial\mathbf{H}}{\partial t}+\mathbf{curl}\;\mathbf{E}=0,\\ \varepsilon_0\varepsilon_\infty\dfrac{\partial\mathbf{E}}{\partial t}-\mathbf{curl}\;\mathbf{H}=-\dfrac{1}{\tau_r}\left[\varepsilon_0\left(\varepsilon_s-\varepsilon_\infty\right)\mathbf{E}-\mathbf{P}\right]-\sigma\mathbf{E}+\mathbf{J},\\ \dfrac{\partial\mathbf{P}}{\partial t}=\dfrac{1}{\tau_r}\left[\varepsilon_0\left(\varepsilon_s-\varepsilon_\infty\right)\mathbf{E}-\mathbf{P}\right].\end{array}\right.\tag{41}$$

The type of analysis used in previous sections extend to this case. Indeed, one can for example prove that:

$$\frac{d}{dt}\mathcal{E}(t)\leq C\|\mathbf{J}\|_{\mathbf{L}^2(\Omega)}\sqrt{\mathcal{E}(t)}$$

which gives:

$$\sqrt{\mathcal{E}(t)}\leq\sqrt{\mathcal{E}(0)}+CT\|\mathbf{J}\|_{\mathcal{C}([0,T],\mathbf{L}^2(\Omega))}$$

so that the energy is bounded, but not decreasing. This type of argument is also used to prove an analoguous result on the boundedness on the semi-discrete energy. Furthermore, up to considering the projection of the source current on the finite element space, theorem 3.6 is still valid. We furthermore modify the fully-discrete scheme in accordance with the introduction of the source current. The modified scheme is as follows:

$$\mu_\infty \int_{\tau_i} \frac{\mathbf{H}_i^{n+\frac{3}{2}} - \mathbf{H}_i^{n+\frac{1}{2}}}{\Delta t} \cdot \varphi_h = \sum_{k\in\mathcal{V}_i} \int_{a_{ik}} \varphi_h \cdot \left(\left\{\mathbf{E}_h^{n+1}\right\}_{ik} \times \mathbf{n}_{ik}\right) - \int_{\tau_i} \mathbf{curl}\, \varphi_h \cdot \mathbf{E}_i^{n+1}, \tag{42}$$

$$\begin{aligned} \varepsilon_\infty \int_{\tau_i} \frac{\mathbf{E}_i^{n+1} - \mathbf{E}_i^{n}}{\Delta t} \cdot \psi_h &= -\sum_{k\in\mathcal{V}_i} \int_{a_{ik}} \psi_h \cdot \left(\left\{\mathbf{H}_h^{n+\frac{1}{2}}\right\}_{ik} \times \mathbf{n}_{ik}\right) + \int_{\tau_i} \mathbf{curl}\, \psi_h \cdot \mathbf{H}_i^{n+\frac{1}{2}} \\ &+ \frac{1}{\tau_r}\int_{\tau_i} \mathbf{P}_i^{[n+\frac{1}{2}]} \cdot \psi_h - \alpha \int_{\tau_i} \mathbf{E}_i^{[n+\frac{1}{2}]} \cdot \psi_h + \int_{\tau_i} \mathbf{J}_i^{n+\frac{1}{2}} \cdot \psi_h, \end{aligned} \tag{43}$$

$$\int_{\tau_i} \frac{\mathbf{P}_i^{n+1} - \mathbf{P}_i^{n}}{\Delta t} \cdot \phi_h = -\frac{1}{\tau_r}\int_{\tau_i} \mathbf{P}_i^{[n+\frac{1}{2}]} \cdot \phi_h + \beta \int_{\tau_i} \mathbf{E}_i^{[n+\frac{1}{2}]} \cdot \phi_h. \tag{44}$$

with $\mathbf{J}_i^{n+\frac{1}{2}} = \mathbf{J}_i(t_{n+\frac{1}{2}})$. The choice of considering $\mathbf{J}^{n+\frac{1}{2}}$ in the equations ensures that one keeps a scheme of second order accuracy. We choose to give the equation in an adimensional form with:

$$\tilde{\mathbf{H}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\mathbf{H}, \quad \tilde{\sigma} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\sigma, \quad \tilde{\mathbf{E}} = \mathbf{E}, \quad \tilde{\mathbf{P}} = \frac{\mathbf{P}}{\varepsilon_0},$$

$$\tilde{t} = \frac{1}{\sqrt{\varepsilon_0\mu_0}}t, \quad \tilde{\tau}_r = \frac{1}{\varepsilon_0\mu_0}\tau_r, \quad \tilde{\beta} = \frac{\varepsilon_s - \varepsilon_\infty}{\tilde{\tau}_r}, \quad \tilde{\alpha} = \tilde{\beta} + \tilde{\sigma}$$

Furthermore, for this numerical test, we choose to work with Maxwell's equations in 2D for transverse magnetic (TMz) waves. In this case, one has that $H_z = E_x = E_y = P_x = P_y = 0$ and the fields do not depend on the z coordinate. The corresponding renormalized equations are then given by:

$$\begin{cases} \mu_\infty \dfrac{\partial \tilde{H}_x}{\partial \tilde{t}} + \dfrac{\partial \tilde{E}_z}{\partial y} = 0, \\ \mu_\infty \dfrac{\partial \tilde{H}_y}{\partial \tilde{t}} - \dfrac{\partial \tilde{E}_z}{\partial x} = 0, \\ \varepsilon_\infty \dfrac{\partial \tilde{E}_z}{\partial \tilde{t}} - \dfrac{\partial \tilde{H}_y}{\partial x} + \dfrac{\partial \tilde{H}_x}{\partial y} = \dfrac{1}{\tilde{\tau}_r}\tilde{P}_z - \tilde{\alpha}\tilde{E}_z + \tilde{J}_z, \\ \dfrac{\partial \tilde{P}_z}{\partial \tilde{t}} = -\dfrac{1}{\tilde{\tau}_r}\tilde{P}_z + \tilde{\beta}\tilde{E}_z. \end{cases} \tag{45}$$

where $\tilde{J}_z$ denotes the source current.

In the rest of this section, we will omit the $\tilde{.}$ for clarity. The computational domain is the square $[0,1]\times[0,1]$. In our simulations, we choose $\mu_\infty = 1$, $\varepsilon_\infty = 1$, $\varepsilon_s = 5$, $\tau_r = 2.82\times 10^{-3}$. The exact solution is given by:

$$\begin{cases} H_x = -\dfrac{1}{\sqrt{2}}\sin(\pi x)\cos(\pi y)\sin(\sqrt{2}\pi t), \\ H_y = \dfrac{1}{\sqrt{2}}\cos(\pi x)\sin(\pi y)\sin(\sqrt{2}\pi t), \\ E_z = \sin(\pi x)\sin(\pi y)\cos(\sqrt{2}\pi t), \\ P_z = \beta\sin(\pi x)\sin(\pi y)\left[\dfrac{1}{4\pi^2 + \dfrac{1}{\tau_r^2}}\left(\dfrac{1}{\tau_r}\cos(\sqrt{2}\pi t) + \sqrt{2}\pi\sin(\sqrt{2}\pi t)\right)\right], \end{cases}$$

if we impose:

$$J_z = -\frac{\beta}{\tau_r}\sin(\pi x)\sin(\pi y)\left[\frac{1}{4\pi^2+\frac{1}{\tau_r^2}}\left(\frac{1}{\tau_r}\cos(\sqrt{2}\pi t)+\sqrt{2}\pi\sin(\sqrt{2}\pi t)\right)\right] \quad (46)$$

$$+ \quad \alpha\sin(\pi x)\sin(\pi y)\cos(\sqrt{2}\pi t). \quad (47)$$

The above exact solution verifies the boundary conditions imposed to **H** and **E** on the border of the square. The simulation time has been fixed to $T = 6$ m (renormalized unit). The square is meshed with triangles by a subdivision of a finite different grid defined by the number of discretization points along the x and y axes. Our numerical tests are obtained using a Fortran 77 code. On figure 1, we show the exact and computed solutions at the point with coordinates $(0.45, 0.45)$ in the square. In table 1 and table 2, we present the energy errors obtained for the DGTD-$\mathbb{P}_p$ for $p = 1$ and $p = 2$ respectively. We choose the time step Δt so that it verifies the CFL condition. In this manner, the convergence rate obtained corresponds to the space convergence rate. Indeed, since the theoretical convergence rate is $O(\Delta t^2 + h^p)$, for a $\mathbb{P}_p$ interpolation with $p \leq 2$, the global convergence rate should be dominated by the one in space. This is what we observe in the present simulation results. Considering $p > 3$ will lead to a rate of convergence of 2 since then the latter is dominated by the one in time. The numerical results are then in accordance with the prediction of the theoretical analysis.

Mesh size	Energy error	CPU time (seconds)	Convergence rate
1/8	0.062489	0.05	-
1/16	0.028823	0.22	1.1164
1/32	0.014248	1.63	1.0164
1/64	0.007102	13.49	1.0045
1/128	0.003546	128.77	1.0018

Table 1: Energy errors obtained with $\Delta t = \text{CFL} \times h$ for DGTD-$\mathbb{P}_1$ approximation.

Mesh size	Energy error	CPU time (seconds)	Convergence rate
1/8	0.00405208	0.08	-
1/16	0.000898612	0.53	2.1729
1/32	0.000220122	4.14	2.0294
1/64	5.51758e-05	35.87	1.9962
1/128	1.38308e-05	510.31	1.9961

Table 2: Energy errors obtained with $\Delta t = \text{CFL} \times h$ for DGTD-$\mathbb{P}_2$ approximation.

6 Conclusion

In this work we have conducted a complete study of a discontinuous Galerkin finite element formulation for Maxwell's equation in dispersive media. Some numerical results validate the theoretical study. Several such studies, either with continuous finite element methods or with discontinuous Galerkin methods, have been published in the recent years denoting an increased interest in this modeling context especially for the simulation of problems involving the interaction of electromagnetic waves with biological media. This work is a necessary first step in the study of Maxwell's equation in dispersive media, before exploiting this in future work, especially on more realistic 3D simulations. A typical application context that we will consider in the sequel is the numerical modelling of human head tissue exposure to electromagnetic waves from cellular phones [SCL$^+$06].

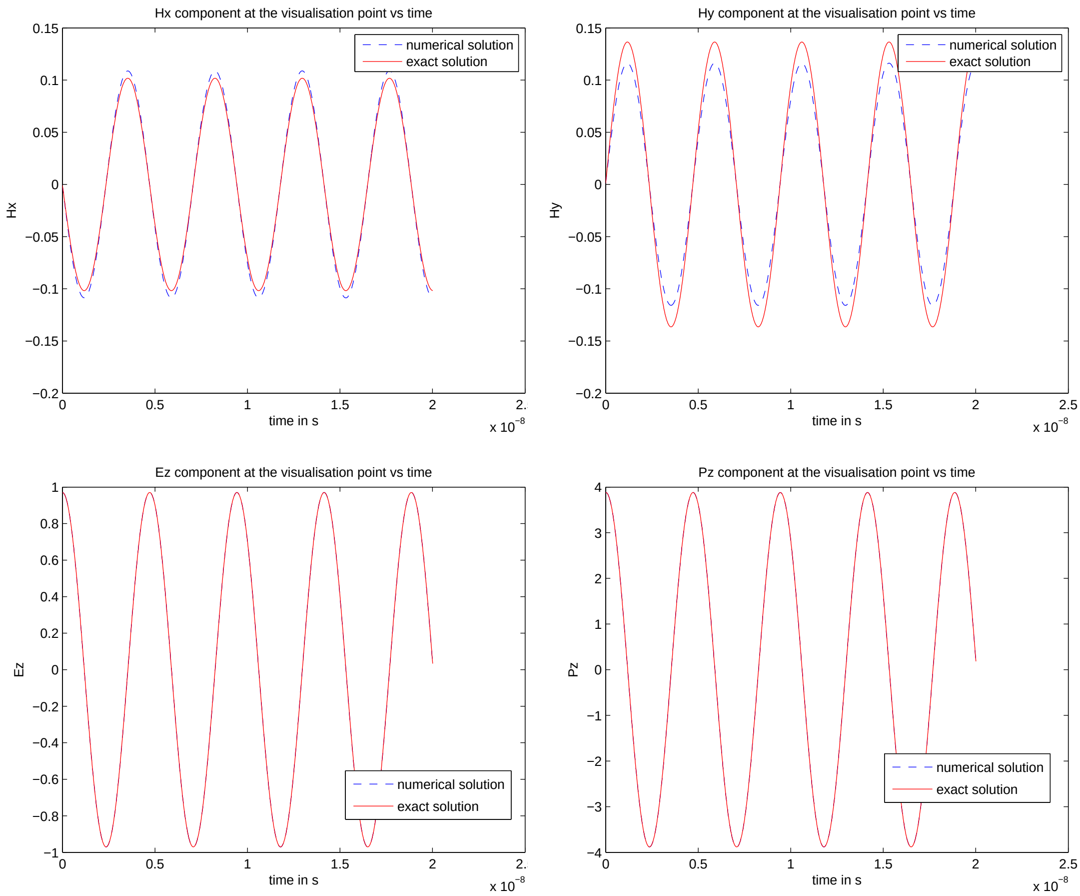

Figure 1: Comparison between exact and numerical solutions with $h = 1/64$ and a DGTD-$\mathbb{P}_2$ approximation.

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Centre de recherche INRIA Sophia Antipolis – Méditerranée
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

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