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Abstract. We present an approach for signal enhancement based on the analysis of the local Hölder regularity. The method does not make explicit assumptions on the type of noise or on the global smoothness of the original data, but rather supposes that signal enhancement is equivalent to increasing the Hölder regularity at each point. Such a scheme is well adapted to the case where the signal to be recovered is itself very irregular, e.g. nowhere differentiable with rapidly varying local regularity. In particular, we show an application to SAR image denoising where this technique yields good results compared to other algorithms.

1. Introduction. A large of number of techniques have been proposed for signal enhancement. The basic frame is as follows. One observes a signal Y which is some combination $F(X, B)$ of the signal of interest X and a "noise" B. Making various assumptions on the noise, the structure of X and the function F, one then tries to derive a method to obtain an estimate $\hat{X}$ of the original signal which is optimal in some sense. Most commonly, B is assumed to be independent of X, and, in the simplest case, is taken to be white, Gaussian and centered. F usually amounts to convoluting X with a low pass filter and adding the noise. Assumptions on X are almost always related to its regularity, e.g. X is supposed to be piecewise C^n for some $n \geq 1$. Techniques proposed in this setting resort to two domains: functional analysis and statistical theory. In particular, wavelet based approaches, developed in the last ten years, may be considered from both points of view [1, 4].

Our approach in this work is different from previous ones in several respects. First, we do not make explicit assumptions on the type of noise and the coupling between X and B through F. However, if some information of this type is available, it can readily be used in our method. Second, we do not require that X belongs to a given *global* smoothness class but rather concentrate on its *local* regularity. More precisely, we view enhancement as equivalent to increasing the Hölder function α_Y (see next section for definitions) of the observations. Indeed, it is generally true that the local regularity of the noisy observations is smaller than the one of the original signal, so that in any case, $\alpha_{\hat{X}}$ should be greater than α_Y. If the Hölder function of X happens to be known, it may serve as a target for the algorithm. If this is not the case, it can be estimated from Y provided sufficient information on F and B is available (e.g. independent additive noise of known law). More generally, the largest $\alpha_{\hat{X}}$, the more regular the

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estimate will be, and the smoother it will look. We thus define our estimate $\hat{X}$ to be the signal "closest" to the observations which has the desired Hölder function. Note that since the Hölder exponent is a local notion, this procedure is naturally adapted for signals which have sudden changes in regularity, like discontinuities. From a broader perspective, such a scheme is appropriate when one tries to recover signals which are highly irregular and for which it is important that the restauration procedure yields the right regularity structure (i.e. preserves the evolution of α_X along the path). An example of this situation is when denoising is to be followed by image segmentation based on textural information: Suppose we wish to differentiate highly textured zones (appearing for instance in MR or radar imaging) in a noisy image. Applying an enhancement technique which assumes that the original signal is, say, piecewise C^1, will induce a loss of the information which is precisely the one needed for segmentation, since the denoised image will not contain much texture. The same difficulty occurs in other situations such as change detection from noisy sequences of aerial images, automatic monitoring of the evolution of lung diseases from scintigraphic images, turbulence data analysis or the characterization of non-voiced parts of voice signals: in all these cases, the decision criterion is often based on a variation of the irregularity in certain regions, and one needs to preserve this information.

Our denoising technique is thus well suited to the case where the original signal X displays the following features:

- X is everywhere irregular.
- The regularity of X (as measured by its Hölder function) may vary varies rapidly in time/space.
- The Hölder function of X bears essential information for subsequent processing.

The remaining of this paper is organized as follows. Section 2 recalls some basic facts about Hölder regularity analysis, which is the basis of our approach. The denoising method is explained in Section 3. Numerical results on both 1D and 2D signals are displayed in Section 4.

2. Hölder regularity analysis. There are two main paths for measuring the local regularity of a function $f : K \to \mathbf{R}$ where K is a bounded subset of $\mathbf{R}^n$. The first one is geometrical, and consists in evaluating local fractional dimensions of its graph Γ. The second, analytical, way of evaluating the regularity of f is to consider a family of nested functional spaces, and to determine the ones it actually belongs to. A popular choice is to consider Hölder spaces, either in their local or pointwise version. We will focus in this paper on the Hölder characterizations of regularity. To simplify notations, we assume that our signals are nowhere differentiable. Generalisation to other signals simply requires to introduce polynomials in the definitions [6].

DEFINITION 2.1. *Pointwise Hölder exponent*
Let $\alpha \in (0,1)$, *and* $x_0 \in K \subset \mathbf{R}$. *A function* $f : K \to \mathbf{R}$ *is in* $C^{\alpha}_{x_0}$ *if for all* x *in a neighbourhood of* x_0,

$$|f(x) - f(x_0)| \leq c|x - x_0|^{\alpha} \tag{2.1}$$

where c is a constant.

The pointwise Hölder exponent of f at x_0, denoted $\alpha_p(x_0)$, is the supremum of the α for which (2.1) holds.

Let us now introduce the local Hölder exponent: Let $\alpha \in (0,1)$, $\Omega \subset \mathbf{R}$. One says that $f \in C^{\alpha}_{l}(\Omega)$ if:

$$\exists\, C : \; \forall\, x, y \in \Omega : \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} \leq C \; .$$

Let: $\alpha_l\left(f, x_0, \rho\right) = \sup\left\{\alpha : f \in C^{\alpha}_{l}\left(B\left(x_0, \rho\right)\right)\right\}$ where $B\left(x_0, \rho\right)$ is the ball centered at x_0 with radius ρ. $\alpha_l\left(f, x_0, \rho\right)$ is non increasing as a function of ρ.

We are now in position to give the definition of the local Hölder exponent:

DEFINITION 2.2. *Let* f *be a continuous function. The* local Hölder exponent *of* f *at* x_0 *is the real number:*

$$\alpha_l\left(f, x_0\right) = \lim_{\rho \to 0} \alpha_l\left(f, x_0, \rho\right) \; .$$

Since α_p and α_l are defined at each point, we may associate to f two functions $x \to \alpha_p(x)$ and $x \to \alpha_l(x)$ which are two different ways of measuring the evolution of its regularity.

These regularity characterizations are widely used in fractal analysis because they have direct interpretations both mathematically and in applications. It has been shown for instance that α_p indeed corresponds to the auditive perception of smoothness for voice signals. Similarly, simply computing the Hölder exponent at each point of an image already gives a good idea of its structure, as for instance its edges [8]. More generally, in many applications, it is desirable to model, synthesize or process signals which are highly irregular, and for which the relevant information lies in the singularities more than in the amplitude. In such cases, the study of the Hölder functions is of obvious interest. From a theoretical point of view, the structure of α_p is known [2]: the class of pointwise Hölder functions of continuous signals is exactly the set of lower limits of continuous functions. This allows to handle a large variety of situations. However, the pointwise Hölder characterization has a number of drawbacks, a major one being that it is not stable under the action of (pseudo) differential operators. This precludes, for instance, the use of the Hilbert transform, commonly used in signal processing. In the same way, knowing the pointwise Hölder exponent of a function at a point x_0 is not sufficient to predict the Hölder exponent of its derivative at the same point. Finally, this exponent is generally hard to compute numerically, in particular in the case of multifractal signals. On the other hand, α_l is stable under differentiation or integration. Moreover, it is easier to estimate than the pointwise Hölder exponent. Its main drawback is that $\alpha_l(x)$ is a lower semi continuous function [5], i.e.,

$$\forall\, x_0 \in \mathbf{R},\ \forall\, \epsilon,\ \exists\, \eta : y \in B(x_0, \eta) \Rightarrow \alpha_l(y) > \alpha_l(x_0) - \epsilon\ .$$

Recalling that a lower semi continuous function is determined by its values on a certain everywhere dense set, we see that the local Hölder exponent is a less versatile notion than its pointwise counterpart. In particular, it is not well suited for multifractal analysis: in general, α_l will be constant for e.g. multinomial measures or IFS.

In [10], a theoretical approach for signal denoising based on the use of the pointwise Hölder exponent and the associated multifractal spectrum was investigated. We develop here a practical enhancement technique under the simplifying assumption that the local and pointwise Hölder functions coincide. This technique is simple from an algorithmic point of view, and yields good results on several kind of images. Since from now on we will assume that $\alpha_l = \alpha_p$, we shall simply write α.

3. Signal enhancement. We adopt in this paper a functional analysis point of view. This means that we do not make any assumption about the noise structure, nor the way it interacts with the data. Rather, we seek a regularized version of the observed data that fulfills some constraints. A statistical approach, based on risk minimization, is presented elsewhere [9].

Let X denote the original signal and Y the degraded observations. We seek a regularized version $\hat{X}$ of Y that meets the following constraints:

1. $\hat{X}$ is close to Y in the L^2 sense,
2. the (local or pointwise) Hölder function of $\hat{X}$ is prescribed.

If α_X is known, we choose $\alpha_{\hat{X}} = \alpha_X$. In some situations, α_X is not known but can be estimated from Y (see below). Otherwise, we just set $\alpha_{\hat{X}} = \alpha_Y + \delta$, where δ is a user-defined positive function, so that the regularity of $\hat{X}$ will be everywhere larger than the one of the observations.

Two problems must be solved in order to obtain $\hat{X}$. First, we need a procedure that estimates the local Hölder function of a signal from discrete observations. Second, we need to be able to manipulate the data so as to impose a specific regularity. A third difficulty arises from the following analysis: Assume the simplest case of an L^2 signal corrupted by independent white Gaussian noise. It is easy to check that almost surely $\alpha_Y = -\frac{1}{2}$ everywhere, because a) $-\frac{1}{2}$ is the regularity of the noise, b) $\alpha_X \geq 0$ since $X \in L^2$, c) the regularity of the sum of two signals which have everywhere different Hölder exponents is the minimum of the two regularities. Thus α_Y does not depend on X, and one cannot go back from α_Y to α_X. This fact casts doubts on the efficiency on the whole approach, since the information it is based on is degenerate in this case. The situation appears even worse when one considers that, for any L^2 signal Y, any $\epsilon > 0$ and any admissible regularity function α such that $\alpha < \alpha_Y$, there exists a signal $\hat{X} \in L^2$ such that $\alpha_{\hat{X}} = \alpha$ and $||\hat{X} - Y|| < \epsilon$. Indeed, let B be any signal with Hölder function α . Then, for all $\mu \neq 0$, $\hat{X} = Y + \mu B$ has regularity α and μ can be taken small enough to have $||\hat{X} - Y|| < \epsilon$. Thus we may find an $\hat{X}$ *independent of the original signal*, with the right regularity, and as close as we wish to the observations.

All these problems are solved once one realizes that the mathematical notion of Hölder regularity is an abstraction that makes sense only asymptotically. One needs to analyze carefully how it should be adapted to a finite setting, much in the same way as what is done for abstract white noise. In particular, we are interested in a perceptual notion of regularity: If two 2D functions A and B are such that $\alpha_A < \alpha_B$, but an imaging of A and B at a given resolution yields the contrary visual impression that A is smoother than B, then of course our algorithm should go with the perceptual information. In other words, in practical applications we are not interested in the asymptotic behavior, but in the scales which are really present in the image, and our estimate of α will reflect this fact. This means precisely one thing: the estimation/denoising procedure should yield results in agreement with what is perceived, and not care for the "true" α, which may or may not be accessible from the finite data. To go back to the example above, while it is true that at infinite resolution the sum "signal + white noise" would locally look much the same as white noise as far as regularity is concerned, this is not the case at finite resolution, where the influence of the signal is still perceptible.

Remark 1. Since our procedure is differential, i.e. we wish to impose $\alpha_{\hat{X}} = \alpha_Y + \delta$ or $\alpha_{\hat{X}} - \alpha_X = 0$, for *estimated* α_X and α_Y, we do not have to care about constant bias.

We will use a wavelet based procedure for estimating and controlling the Hölder function. In general, estimating the Hölder exponents at a given x from the wavelet coefficients is not an easy task, because one needs to take into account all the coefficients in a neighbourhood of x. More precisely, let $\{\psi_{j,k}\}_{j,k}$ be an orthonormal wavelet basis, where as usual j denotes scale and k position, and assumes that ψ is regular enough and has sufficiently many vanishing moments. Let $(d_{j,k})_{j,k}$ denote the wavelet coefficients. Then one has, for the pointwise exponent:

PROPOSITION 3.1. *[6] Assume that* $f \in C^{\alpha}_{x_0}$. *If* $|k2^{-j} - x_0| \leq 1/2$, *then*

$$|d_{j,k}| \leq C2^{-j(\alpha+1/2)}(1 + 2^j|k2^{-j} - x_0|)^{\alpha+1/2}. \tag{3.1}$$

Conversely, if (3.1) holds for all (j,k)*'s such that* $|k2^{-j} - x_0| \leq 2^{-j/(\log j)^2}$, *and if* $f \in C^{\log}$, *then there exists a constant* C *and a polynomial* P *of degree at most* $[\alpha]$ *such that*

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha (\log(|x - x_0|))^2. \tag{3.2}$$

C^{log} is the class of functions f whose wavelet coefficients verify

$$|d_{j,k}| \leq C 2^{-\frac{j}{\log j}} .$$

This regularity condition is stronger than uniform continuity, but does not imply a uniform Hölder continuity.

The local exponent is characterized as follows:

PROPOSITION 3.2. *[5, 11] The local Hölder exponent $\alpha_l(x_0)$ at x_0 verifies:*

$$\alpha_l(x_0) + \frac{1}{2} = \lim_{\eta \to 0} (\sup\{s / \exists C, \mathrm{supp}(\psi_{j,k}) \subset B(x_0, \eta) \Rightarrow |d_{j,k}| \leq C 2^{-sj}\}). \tag{3.3}$$

As said above, we make in this work a simplifying assumption: We assume that, for all x, $\alpha_p(x) = \alpha_l(x)$. This implies in particular that "large" wavelet coefficients for x, i.e. the ones that control its exponents by turning inequality (3.1) into an equality, are located *above* x. As a consequence, one can have access to an estimate of α simply by performing a linear regression of $\log(|d_{j,k}|)$ w.r.t. to the scale j (log denotes base-2 logarithm) considering only those indices (j,k) such that the support of $\psi_{j,k}$ is centered above t. Of course this will only be approximate, but since (3.1) is optimal, and if the discretization is fine enough so that many wavelet coefficients are "large", we may hope to obtain results sufficient for our purpose. Indeed, Figure 1 shows that though the estimated values of α are far from being perfect on a Weierstrass signal[1], the difference of the estimates for two such functions is correct. According to Remark 1 above, this is all we need.

Two points are essential in this estimation procedure:

- The estimation is obtained through a regression on a finite number of scales, defined as a subset of the scales available on the discrete data. This avoids the pathologies described above concerning the regularity of the sum of two signals. In particular, it is possible to express the Hölder function of the noisy signal $Y = X$ + Gaussian white noise as a function of α_X, and thus to estimate conversely α_X from α_Y [9].
- The use of (orthonormal) wavelets allows to perform the reconstruction in a simple way: Starting from the coefficient $(d_{j,k})$ of the observations, we shall define a procedure that modifies them to

[1] Recall that the Weierstrass function, which is defined as $W(t) = \sum_{n=0}^{\infty} \lambda^{-n\alpha} sin(\lambda^n t)$, $\lambda \geq 2$, $\alpha \in (0,1)$ has local and pointwise Hölder exponents α at each t.

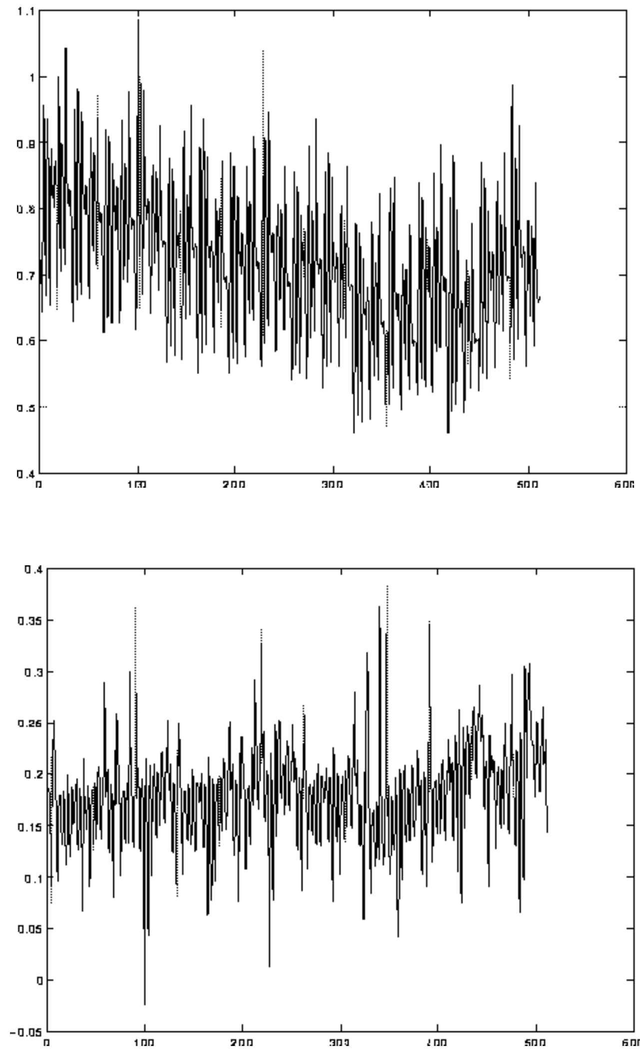

FIG. 1. *Top: Hölder exponent estimate on a Weierstrass function with theoretical α equal to 0.7. The mean of the estimates along the curve is 0.72, and the standard deviation is 0.1. Bottom: difference of the estimates on two Weierstrass functions of exponents 0.7 and 0.9. The mean is 0.19 and the standard deviation 0.05.*

obtain coefficients $(c_{j,k})$ that verify (3.1) with the desired α, and then reconstruct $\hat{X}$ form the $(c_{j,k})$.

Note that a much better estimate of $\alpha_X(t)$ can be obtained by measuring the *oscillations* of X in balls centered at t and of radii ϵ_k, and then regressing the logarithm of these oscillations w.r.t. the logarithm of the ϵ_k.

However, this procedure does not lead to a simple algorithm for the inverse problem, i.e. obtaining a signal with prescribed regularity.

We may now reformulate our program as follows: For a given set of observations $Y = (Y_1, \dots, Y_{2^n})$ and a target Hölder function α, find $\hat{X}$ such that $||\hat{X} - Y||_{L^2}$ is minimum and the regression of the logarithm of the wavelet coefficients of $\hat{X}$ above any point i w.r.t. scale is $-(\alpha(i) + \frac{1}{2})$. Note that we must adjust the wavelet coefficients in a global way. Indeed, each coefficient at scale j subsumes information about roughly 2^{n-j} points. Thus we cannot consider each point i sequentially and modify the wavelet coefficients above it to obtain the right regularity, because point $i+1$, which shares many coefficients with i, requires different modifications. The right way to control the regularity is to write the regression contraints simultaneously for all points. This yields a system which is linear in the logarithm of the coefficients:

$$\Delta L = A$$

where Δ is a $(2^n, 2^{n+1} - 1)$ matrix of rank 2^n, and

$$L = (\log|c_{1,1}|, \log|c_{2,1}|, \log|c_{2,2}|, \dots \log|c_{n,2^n}|),$$
$$A = -\frac{n(n-1)(n+1)}{12}\left(\alpha(1) + \frac{1}{2}, \dots, \alpha(2^n) + \frac{1}{2}\right) .$$

Since we use an orthonormal wavelet basis, the requirements on the $(c_{j,k})$ may finally be written as:

minimize: $\displaystyle\sum_{j,k} (d_{j,k} - c_{j,k})^2$

subject to:

$$\text{(i)} \qquad \forall\ i = 1, \dots, 2^n, \sum_{j=1}^{n} s_j \log(|c_{j,E((i-1)2^{j+1-n})}|) = -M_n(\alpha(i) + \frac{1}{2})$$

where $E(x)$ denotes the integer part of x and the coefficients $s_j = j - \frac{n+1}{2}$, $M_n = \frac{n(n-1)(n+1)}{12}$ and Equation (i) are deduced from the requirement that the linear regression of the wavelet coefficients of $\hat{X}$ above position i should equal $-(\alpha(i) + \frac{1}{2})$.

Searching the most general solution to the program above does not seem to be an easy task. We consider instead in this paper the following special case: We impose that, for all (j, k),

$$\text{(ii)} \qquad c_{j,k} = B_j d_{j,k}$$

where the multipliers B_j are real numbers belonging to the interval $(0, 1]$. The main motivation for the restriction on the form of the $c_{j,k}$ is of course that it leads to a simple solution. The choice of the range of the B_j parallels an idea at work in classical denoising by wavelet shrinkage, namely that we seek to reduce the variance of the estimator by decreasing the absolute value of the coefficients. Let α_Y denote the estimated regularity of the observations. By definition:

$$\text{(iii)} \qquad \sum_{j=1}^{n} s_j \log(|d_{j,E((i-1)2^{j+1-n})}|) = -M_n(\alpha_Y(i) + \frac{1}{2}) \ .$$

Subtracting (iii) to (i) and using (ii), we get:

$$\forall\, i = 1, \ldots, 2^n, \qquad \sum_{j=1}^{n} s_j \log(B_j) = M_n(\alpha_Y(i) - \alpha(i)) \ .$$

Thus the ansatz (ii) imposes that the desired increase in regularity is uniform along the path, i.e. $\delta(i) = \delta =$ constant. This restriction can be weakened by a classical block technique, i.e. by imposing (ii) on subparts of the wavelet tree. Assuming we are ready to accept this constraint,we are now left with the following program:

$$\text{minimize: } \sum_{j=1}^{n} e_j^2 (B_j - 1)^2$$

$$\text{subject to: } \sum_{j=1}^{n} s_j \log(B_j) = \beta \ \text{ and } \ 0 < B_j \leq 1$$

where $\beta = M_n(\alpha_Y(i) - \alpha(i))$ and $e_j^2 = \sum_k (d_{j,k})^2$ is the energy of the observations at scale j. It is easily proved that this program has a unique solution $B^* = (B_1^*, \ldots, B_n^*)$, and that $B_i^* = 1$ for $i \leq \frac{n+1}{2}$. The other values are classically found using a Lagrange multiplier. Remark that our procedure thus accounts automatically for the adjustment made in classical wavelet shrinkage, namely that the large scale coefficients are usually left untouched (i.e. the $d_{j,k}$ are not thresholded for j smaller than a j_0 set in a heuristic way). More generally, it is interesting to compare our scheme with the soft-thresholding policy. Recall that soft-thresholding replaces the noisy coefficient $d_{j,k}$ by $e_{j,k} = \text{sgn}(d_{j,k})(|d_{j,k}| - \lambda)_+$, where λ is a threshold than depends, among other things, on n and the type of noise. Denoting $\beta_j = -\log(B_j)$, we see that in our case:

$$\log(|c_{j,k}|) = \log(|d_{j,k}|) - \beta_j, \text{with } \ \text{sgn}(c_{j,k}) = \text{sgn}(d_{j,k}) \ .$$

The regularity based enhancement is thus a kind of shrinkage on the logarithm of the wavelet coefficients, and the restriction (ii) may be interpreted as a requirement that the threshold must depend only on scale and not on position. Another point of view is obtained by noting that since all the coefficients at a given scale are multiplied by the same number, this procedure is equivalent to a sub-band filtering. Indeed, assuming (ii), the

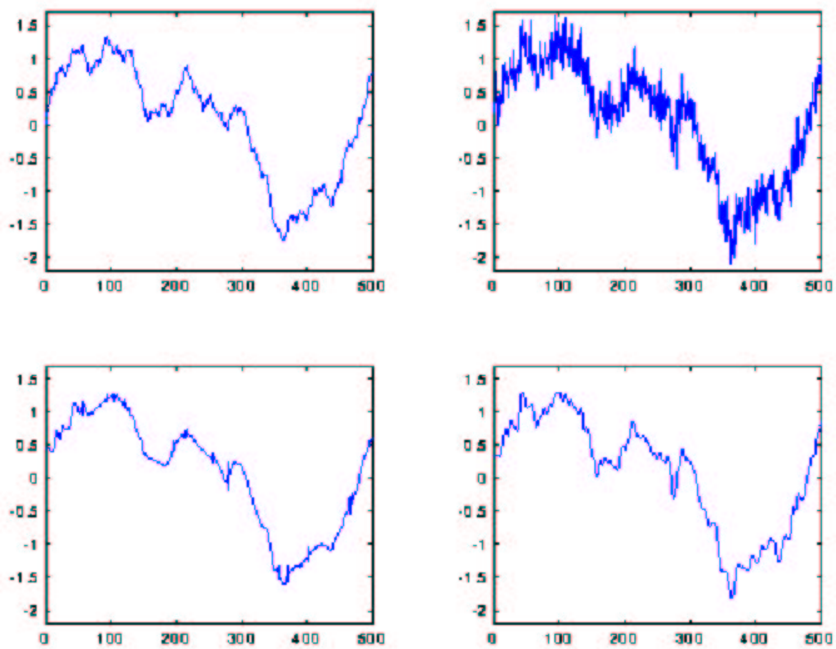

FIG. 2. *Weierstrass function of exponent 0.8 (top left), noisy version with additive Gaussian noise (top right), denoised version using wavelet shrinkage (bottom left), enhanced version using the Hölder regularity based procedure (bottom right).*

algorithm may be implemented directly in the Fourier domain, without the help of the wavelet transform. The wavelet coefficients are however needed to compute the values of the multipliers and to interpret them as a way to control the local regularity.

4. Numerical experiments. We first show a result of enhancement on synthetic 1D data. The original signal is a Weierstrass function with exponent $H = 0.8$ (i.e. $\alpha_X(t) = 0.8$ for all t), which has been corrupted with additive Gaussian white noise. Figure 2 shows the original signal, the noisy one, and the result of the enhancement procedure. For comparison, a denoising using a classical wavelet shrinkage is also displayed. For both procedures, the parameters were set so as to obtain the best fit to the known original signal. It is seen that, for irregular signals such as Weierstrass functions, the Hölder based enhancement yields more satisfactory results. In particular, notice how our method preserves a roughly constant regularity along the path, while the wavelet shrinkage yields a signal with both quite smooth and very irregular regions.

FIG. 3. *Original SAR image (courtesy IRD).*

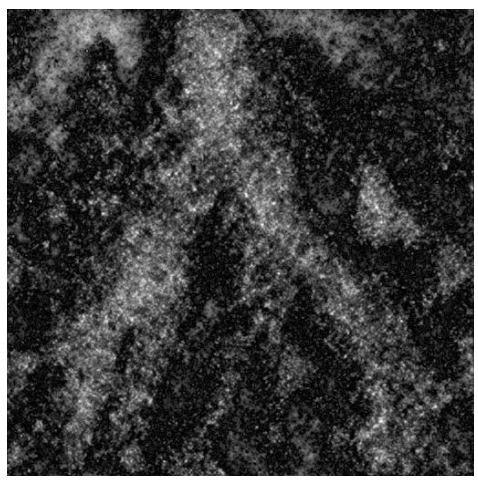

FIG. 4. *Image denoised using the Hölder regularity scheme.*

Our second example deals with a synthetic aperture radar (SAR) image. A huge literature has been devoted to the difficult problem of enhancing these images, where the noise, called speckle, is non Gaussian, correlated and multiplicative. A fine analysis of the physics of the speckle suggests that it follows a K-distribution [7]. Classical techniques specifically designed for SAR image denoising include geometric filtering and

FIG. 5. *Image denoised using soft thresholding.*

Kuan filtering. Wavelet shrinkage methods have also been adapted to this case [3].

SAR imaging of natural landscapes is a good test for our technique, since we are in a frame where the noise and its interaction with the data is complex, and, in addition, the original signal is itself irregular. Figures 3, 4, and 5 show an original SAR image, its denoising with the Hölder method and with the soft thresholding procedure. Again, the parameters of both methods have been set so as to obtain the most visually pleasing results. The application here is the automated monitoring of the hydrographic network in an African region. Notice how the river, which assumes a "Λ" shape in the middle of the scene, is not easily seen on the original image, but is nicely uncovered by the regularity based enhancement.

Let us mention finally that the above procedure is implemented in FracLab, a software toolbox for image and signal processing with fractals. FracLab is available at `www-rocq.inria.fr/fractales/`

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