Generalized IFS for Signal Processing

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1 Introduction

Several methods have been proposed to estimate the Hölder exponents of signals [1, 4]. In this paper, we propose a new approach, based on a generalization of iterated functions system (IFS), which is well adapted to irregular continuous 1D signals. We also use these generalized IFS to build parsimonious models of complex signals and to perform segmentation on them. This paper is organized as follows : in section 2 we recall the definition of generalized iterated functions systems (GIFS) and the definition of the Hölder exponent of a nowhere differentiable continuous function. In section 3, we recall a result obtained in [2] concerning the Hölder exponents of the attractors of GIFS. We then propose a method to solve the inverse problem for GIFS and we use it to estimate the Hölder exponents of discrete signals. We present results obtained on generalized Weierstrass functions. In section 4, we develop a synthesis scheme for a given signal. The parameters of the GIFS, along with the Hölder exponents at different resolutions of the signal, allow to give a functional representation of discrete data. This in turn permits signal segmentation. An application on a residual speech signal is presented.

2 Background on generalized IFS

Consider a collection of sets $(F^k)_{k\in\mathbb{N}^*}$, where each F^k is a non-empty finite set of contractions S_i^k on a $[0\,;1]$, for $i = 0, \ldots, N_k - 1$, where $N_k \geq 1$ is an integer, which denotes the cardinal of F^k. We denote by c_i^k the contraction ratio of S_i^k, for $i = 0, \ldots, N_k - 1$, and $k \in \mathbb{N}^*$.
Consider the operator W^k, defined on every compact $A \subset [0\,;1]$, by :

$$W^k(A) = \bigcup_{i=0}^{N_k-1} S_i^k(A).$$

Then, under some conditions [3], there exists a unique compact G such that :

$$\lim_{k\to\infty} W^k \circ \ldots \circ W^1(A) = G \text{ for every compact } A \subset [0\,;1].$$

We call G the attractor of the GIFS $(K, \{F^k\}_{k\in\mathbb{N}^*})$ and $W^k \circ \ldots \circ W^1(A)$ its attractor at scale k (which depends on A).
In the following we consider the special case of IFS where the S_i^k's are affine functions and the number of functions at step k is m^k, for some integer $m \geq 2$. Let F^k be defined as the set of affine transformations $S_i^k\,(0 \leq i < m^k)$ represented in matrix notation by:

$$S_i^k \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1/m & 0 \\ a_i^k & c_i^k \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} + \begin{pmatrix} i/m^k \\ b_i^k \end{pmatrix}$$

We suppose $0 \leq t \leq 1$, $1/m < |c_i| < 1$ and that each S_i^k is operating only on $[im^{-k+1}\,; (i+1)m^{-k+1}]$ if i is even, and on $[(i-1)m^{-k+1}\,; im^{-k+1}]$ if i is odd. Every S_i maps to $[im^{-k}\,; (i+1)m^{-k}]$. Suppose, also, that we want to interpolate the points $(\frac{i}{m}, y_i)$, for $i = 0, \ldots, m$, and $y_i \in \mathbb{R}$. Then, it was shown in [2] that if $s(t)$ is function from $[0\,;1]$ to $[0\,;1]$, which is a lower limit of a sequence of continuous functions, there exists c_i^k's such that the attractor of the IFS defined above is the graph of a continuous function f that verifies :

$$f\left(\frac{i}{m}\right) = y_i \;\; \forall\, i = 0, \ldots, m$$

and

$$\alpha_f(t) = s(t) \;\; \forall\, t \in [0\,;1]$$

where $\alpha_f(t)$ is the Hölder exponent of f at the point t, and is defined as follows :

$$\alpha_f(t) = \liminf_{h\to 0} \frac{\log(|f(t+h) - f(t)|)}{\log|h|}.$$

In the rest of the paper we will call α_f the Hölder function of f.

3 Estimation of Hölder exponents using GIFS

In this section, we develop a technique to estimate the Hölder exponents of discrete signals. This technique requires an investigation on the inverse problem of GIFS. For this purpose, we need to recall a result proved in [2] :

Proposition 1 *The attractor of the IFS defined above is the graph of a continuous function f such that :*

$$f\left(\frac{i}{m}\right) = y_i \;\; \forall\, i = 0, \ldots, m$$

and

$$\alpha_f(t) = \min(\alpha_1, \alpha_2, \alpha_3)$$

where

$$\begin{cases} \alpha_1 = \liminf\limits_{k \to +\infty} \frac{\log(c^k_{m^{k-1}i_1+m^{k-2}i_2+\ldots+mi_{k-1}+i_k} \ldots c^2_{mi_1+i_2} c^1_{i_1})}{\log(m^{-k})} \\ \alpha_2 = \liminf\limits_{k \to +\infty} \frac{\log(c^k_{m^{k-1}j_1+m^{k-2}j_2+\ldots+mj_{k-1}+j_k} \ldots c^2_{mj_1+j_2} c^1_{j_1})}{\log(m^{-k})} \\ \alpha_3 = \liminf\limits_{k \to +\infty} \frac{\log(c^k_{m^{k-1}l_1+m^{k-2}l_2+\ldots+ml_{k-1}+l_k} \ldots c^2_{ml_1+l_2} c^1_{l_1})}{\log(m^{-k})} \end{cases} \quad (1)$$

and where, for any positive integer k, the k-uples $(j_1, ..., j_k)$ and $(l_1, ..., l_k)$ of non negative integers strictly smaller than m, are uniquely determined by :

$$t_k = m^{-k}[m^k t]$$

$$\textit{if}\; t_k + m^{-k} < 1\,, t_k^+ = t_k + m^{-k} = \sum_{p=1}^{k} j_p m^{-p} \;\textit{else}\; t_k^+ = t_k$$

$$\textit{if}\; t_k - m^{-k} > 0\,, t_k^- = t_k - m^{-k} = \sum_{p=1}^{k} l_p m^{-p} \;\textit{else}\; t_k^- = t_k$$

This yields a non parametric method to estimate the Hölder function of a signal. Indeed, if for a given signal, we are able to determine a GIFS whose attractor is the graph of the signal (inverse problem), then, using proposition 1, we get an expression of the Hölder exponent at every location of the signal. However, there still remain some limitations. The c_i^k's belong to $]\frac{1}{m}\,;1[$, which implies that the Hölder exponents computed with this method belong to $[0\,;1]$. This means that the attractor of a GIFS is necessarily the graph of a nowhere differentiable continuous function. The scheme we propose below to solve the inverse problem thus yields better results on irregular signals.
Let $F = \{f(i), i = 0, ..., 2^J\}$ be a given discrete signal. For $j = 1, ..., J$, consider the set P_j defined by :

$$P_j = \{f(i2^{J-j}), i = 0, ..., 2^j\}.$$

Namely, P_j is the signal sub-sampled with a step equal to 2^{J-j}.
For a given $j \in \{1, ..., J-1\}$, the set $\{c_i^j, i = 0, ..., 2^j - 1\}$ is simply obtained as the set of contraction ratios of the 2^j affine functions that permit to map the polygon defined by P_j to the one defined by P_{j+1}. To clarify, we show in Fig. 5 an example with $J = 3$. The signal samples are represented by circles. For $j = 1, 2$, the GIFS coefficients are obtained as follows :

$$c_i^j = \frac{w_i^j}{w^{j-1}_{[\frac{i}{2}]}} \;\forall i = 0, ..., 2^j - 1$$

Fig. 1 displays the function defined on $[0\,;1]$ by :

$$W(t) = \sum_{k=0}^{\infty} 3^{-ks(t)} sin(3^k t),$$

where $s(t) = |sin(5\pi t)|)$. This function is a generalization of the Weierstrass function and was shown [2] to have Hölder function $\alpha_W(t) = s(t)$.
Fig. 2 displays the corresponding estimated Hölder function using our method and a method based on time-frequency energy distributions [4]. The distribution used here is a continuous wavelet (Morlet wavelet) transform. The theoretical Hölder function is the thick line, the GIFS estimation is the thin line and the continuous wavelet transform estimation is the dotted line. The results obtained using our method, in this case, are satisfactory. Other examples and comparison to other methods will be shown in the extended version.

4 Signal modeling using GIFS

In this section, we show how the c_i^k's obtained by solving the inverse problem corresponding to F can be used to develop a parsimonious synthesis model of the data. The idea is to keep only some of the c_i^k's, which correspond in the time domain to some sub-sampling of the signal. We then determine the missing c_i^k's from the Hölder function of the signal at different scales. Remark that, for a given $j \in \{1, ..., J-1\}$, the set $\{c_i^k\,; k = 1, ..., J-j, i = 0, ..., 2^k-1\}$ determines the GIFS whose attractor at scale j is P_{J-j+1}, when starting with P_1. Let j_0 be a fixed integer, and suppose that we want to keep only $\{c_i^k\,; k = 1, ..., J - j_0, i = 0, ..., 2^k - 1\}$. This corresponds to keeping one sample out of 2^{j_0} samples. Then, for every $j = j_0 - 1, ..., 0$ we compute the Hölder function s_j of the attractor at scale j. We then perform a moving average on each s_j in order to obtain a smooth function s_j^* which can be defined with a few parameters. Then, we put :

$$c_i^j = 2^{-js_j^*(i2^{-j})} \;\; \forall\, i = 0, ..., 2^j - 1.$$

This is justified by the fact that, if $s(t)$ is continuous function from $[0\,;1]$ to $[0\,;1]$, then a GIFS whose contraction ratios are of the form :

$$c_i^k = 2^{-ks(i2^{-k})},$$

has an attractor which is the graph of a continuous function with Hölder exponent at each point t equal to $s(t)$.
Hence, we build a signal model which is all the more parsimonious than j_0 is large. The choice of j_0 depends on the desired synthesis quality.
Fig. 3 shows a residual speech signal obtained by subtraction of the harmonic part to the original speech signal. The harmonic part was extracted using the HNS model [5] developed by J. Laroche, Y. Stilyanou and E. Moulines.
Fig. 4 shows the reconstructed residual signal using our synthesis scheme with $j_0 = 2$.
The method can be generalized to perform signal segmentation based on a split and merge scheme. The associated energy is a function of the c_i^k's. This will be detailed in the extended version of the paper.

References

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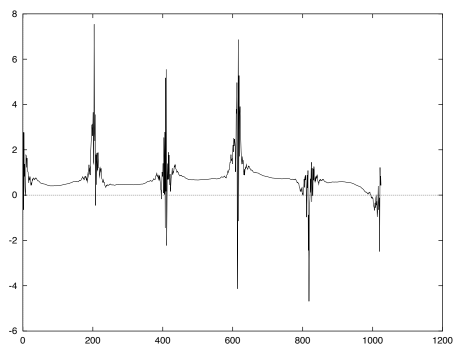

Fig. 1 Generalized Weierstrass function with $s(t) = |sin(5\pi t)|$

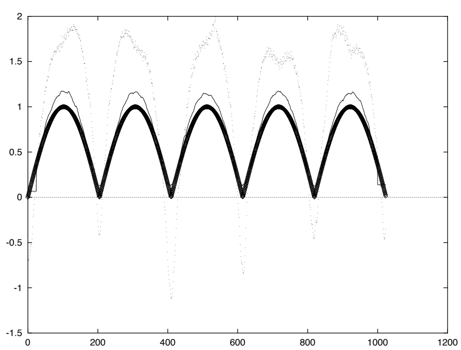

Fig. 2 Estimation of the Hölder function

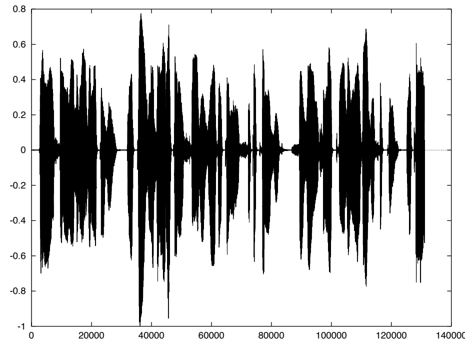

Fig. 3 Original residual speech signal

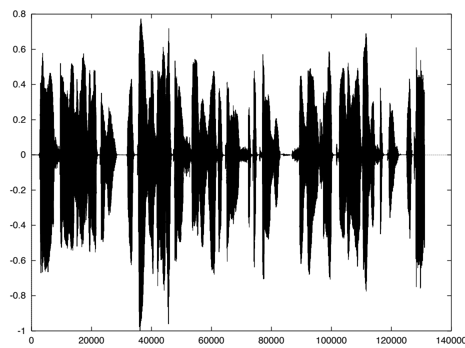

Fig. 4 Reconstructed signal

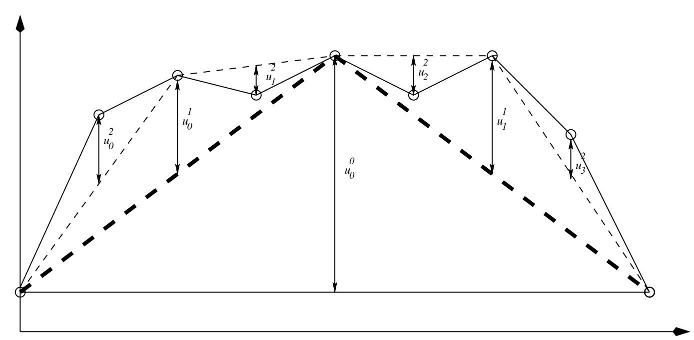

Fig. 5 Computation of GIFS coefficients