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On simultaneous identification of a scatterer and its generalized impedance boundary condition

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Abstract: We consider the inverse scattering problem consisting in the identification of both an obstacle and two functional coefficients of a generalized boundary condition prescribed on its boundary, from far–fields due to several plane waves. After proving a uniqueness result for such inverse problem, we define and compute appropriate derivative of the far–field with respect to an obstacle with non constant impedances. A steepest descent method is then applied to retrieve both the obstacle and the functional impedances from the measured far–fields. The feasability of the method is demonstrated with the help of some 2D numerical experiments.

Key-words: Inverse scattering problem, Helmholtz equation, Generalized Impedance Boundary Conditions, Fréchet derivative, Steepest descent method.

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Identification simultanée d'un obstacle et de sa condition d'impédance généralisée.

Résumé : Nous nous penchons sur le problème inverse de diffraction qui consiste à reconstruire un obstacle et deux coefficients non–constants d'une condition d'impédance généralisée sur la frontière de celui–ci partir de la mesure du champ diffracté engendré par plusieurs ondes incidentes planes. Nous montrons dans un premier temps un résultat d'unicité pour ce problème inverse puis nous calculons la dérivée partielle du champ lointain par raport à un obstacle pour des coefficients d'impédance non–constants. Pour finir nous appliquons une méthode de descente de gradient pour retrouver l'objet et les impédances à partir des champs lointains mesurés et nous illustrons la faisabilité de notre approche à l'aide de divers exemples en dimension 2.

Mots-clés : Diffraction inverse, Equation de Helmholtz, Condition d'impédance généralisée, Différentielle de Fréchet, Méthode de gradient

1 Introduction

The identification of complex obstacles from far–field acoustic measurements has considerable interest in many applications, for example Radar applications. In this paper, the obstacle D of interest is characterized by a so-called Generalized Impedance Boundary Condition (GIBC), namely

$$\frac{\partial u}{\partial \nu} + \mathrm{div}_\Gamma(\mu \nabla_\Gamma u) + \lambda u = 0 \quad \text{on } \partial D$$

where μ and λ are complex valued functions, div_Γ and ∇_Γ are respectively the surface divergence and the surface gradient on ∂D and ν denotes the outward unit normal on ∂D. The particular case $\mu = 0$ is the well known impedance boundary condition and is used for instance to model imperfectly conducting obstacles. The cases $\mu \neq 0$ correspond to more accurate models for imperfectly conducting obstacles, or models for thin coatings or gratings (see [3, 9, 10, 11]).
The inverse problem we consider here is the identification of both the obstacle D and the impedances λ and μ from the far–fields generated by several plane waves with different directions, in the harmonic regime at a fixed frequency. Such problem in the case $\mu = 0$ has been addressed in [21, 18, 13], and in [20, 2, 22, 7] if we consider the equivalent problem for the Laplace equation in a bounded domain. For the case $\mu \neq 0$, previous studies [5, 4] are focussed on the problem of finding the impedances λ and μ for given obstacle. More precisely, in [5] some uniqueness and stability results in the case of a single incident plane wave are presented, while some numerical experiments are conducted in [4], in particular for an imperfectly known obstacle.
It seems to the authors that the present paper is a first attempt to identify both the obstacle D and the impedances λ and μ from the data. We first prove uniqueness of both D and (λ,μ) when one uses plane waves with all possible directions. We secondly compute the partial derivatives of the far–field with respect to the domain D and the impedances (λ, μ) respectively, in order to use an optimization method. The latter partial derivative was already characterized in [4] with the help of an adjoint state, the first one is the main subject of the present paper. At first glance, computing the partial derivative with respect to D is a simple generalization to the case of GIBC's of a classical shape derivative computation in the sense of Murat-Simon, as described for example in the monographs [1, 14, 23]. More precisely, our paper can be viewed as a continuation of [17] for the Neumann boundary condition and of [12] for the impedance boundary condition with constant λ, in the sense our paper is based on some integral representation of the scattered fields. However, considering some functional impedances λ and μ introduces some novel issues. In fact, since the unknown functions λ and μ are supported by the unknown boundary ∂D, the notion of partial derivative with respect to D has to be clarified. Here we adapt the usual definition of partial derivative with respect to D by extending the surface functions λ and μ to the boundary $\partial D_\varepsilon = \partial D + \varepsilon(\partial D)$, where ε is a perturbation of ∂D (see definition 4.1 hereafter). Moreover, contrary to the standard case, that λ and μ be functions implies the shape derivative depends not only on the normal part of the perturbation ε but also on the tangential part (see theorem 4.8 hereafter). It should also be noted that contrary to most contributions, we do not assume that the obstacle is star-like, which would enable us to parametrize both the obstacle and the impedances by polar angle θ. We expect that the computation of the partial derivative with respect to D could probably also be obtained by differentiating the variational formulation of the forward problem following [14], instead of using integral equations as in [17, 12] and in the present paper. Concerning the numerical reconstruction itself, and in contrast to [21, 13] for the simpler case $\mu = 0$, the forward problem is solved by using a variational formulation of the problem with the help of a finite element method and the obstacle D is updated by using a boundary variation technique (requiring a remesh of the computational domain at each step).
The outline of the paper is as follows. We describe the inverse problem in section 2. In section 3 we prove our uniqueness result, while section 4 is dedicated to the computation of the partial derivative of the far–field with respect to the obstacle, a technical lemma being postponed in an appendix. In section 5 we describe the optimization technique we use to solve the inverse problem and which is based on the partial derivative derived in the previous section. Lastly some numerical tests in $2D$ show the efficiency of our steepest descent method in section 6.

2 The statement of the inverse problem

Let D be an open bounded domain of $\mathbb{R}^d$, with $d = 2$ or 3, the boundary ∂D of which is Lipschitz continuous, such that $\Omega = \mathbb{R}^d \setminus \overline{D}$ is connected and let $(\lambda, \mu) \in (L^\infty(\partial D))^2$ be some impedance coefficients. The scattering problem with generalized impedance boundary conditions (GIBC) consists in finding $u = u^s + u^i$ such that

$$\begin{cases} \Delta u + k^2 u = 0 \quad \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \operatorname{div}_\Gamma(\mu \nabla_\Gamma u) + \lambda u = 0 \quad \text{on } \partial D \\ \displaystyle\lim_{R\to\infty} \int_{|x|=R} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds = 0. \end{cases} \tag{1}$$

Here k is the wave number, $u^i = e^{ik\hat{d}\cdot x}$ is an incident plane wave where $\hat{d}$ belongs to the unit sphere of $\mathbb{R}^d$ denoted S^{d-1}, and $u^s \in V(\Omega) := \{v \in \mathcal{D}'(\Omega), \varphi v \in H^1(\Omega) \, \forall \varphi \in \mathcal{D}(\mathbb{R}^d)$ and $v_{|\partial D} \in H^1(\partial D)\}$ is the scattered field.
The surface operators $\operatorname{div}_\Gamma$ and ∇_Γ are precisely defined in Chapter 5 of [14]. For $v \in H^1(\partial D)$ the surface gradient $\nabla_\Gamma v$ lies in $L^2_\Gamma(\partial D) := \{V \in L^2(\partial D, \mathbb{R}^d)\,,\, V\cdot\nu = 0\}$ while $\operatorname{div}_\Gamma(\mu\nabla_\Gamma u)$ is defined in $H^{-1}(\partial D)$ for $\mu \in L^\infty(\partial D)$ by

$$\langle \operatorname{div}_\Gamma(\mu\nabla_\Gamma u), v\rangle_{H^{-1}(\partial D), H^1(\partial D)} := -\int_{\partial D} \mu \nabla_\Gamma u \cdot \nabla_\Gamma v \, ds \quad \forall v \in H^1(\partial D).$$

The last equation in (1) is the classical Sommerfeld radiation condition. The proof for well–posedness of problem (1) and the numerical computation of its solution can be done using the so–called Dirichlet–to–Neumann map so that we can give an equivalent formulation of (1) in a bounded domain $\Omega_R = \Omega \cap B_R$ where B_R is the ball of radius R such that $D \subset B_R$. The Dirichlet–to–Neumann map, $S_R : H^{1/2}(\partial B_R) \mapsto H^{-1/2}(\partial B_R)$ is defined for $g \in H^{1/2}(\partial B_R)$ by $S_R g := \partial u^e/\partial r|_{\partial B_R}$ where $u^e \in V(\mathbb{R}^d \setminus \overline{B_R})$ is the radiating solution of the Helmholtz equation outside B_R and $u^e = g$ on ∂B_R.
Solving (1) is equivalent to find u in $V_R := \{v \in H^1(\Omega_R);\, v_{|\partial D} \in H^1(\partial D)\}$ such that:

$$\begin{cases} \Delta u + k^2 u = 0 \quad \text{in } \Omega_R \\ \dfrac{\partial u}{\partial \nu} + \operatorname{div}_\Gamma(\mu \nabla_\Gamma u) + \lambda u = 0 \quad \text{on } \partial D \\ \dfrac{\partial u}{\partial r} - S_R(u) = \dfrac{\partial u^i}{\partial r} - S_R(u^i) \quad \text{on } \partial B_R. \end{cases} \tag{2}$$

We introduce the assumption

Assumption 2.1. *The coefficients $(\lambda, \mu) \in (L^\infty(\partial D))^2$ are such that*

$$\Im m(\lambda) \geq 0,\ \Im m(\mu) \leq 0 \qquad \text{a.e. in } \partial D$$

and there exists $c > 0$ such that

$$\Re e(\mu) \geq c \qquad \text{a.e. in } \partial D.$$

Well–posedness of problem (2) is established in the following theorem, the proof of which is classical and given in [4].

Theorem 2.2. *With assumption 2.1 the problem (2) has a unique solution u in V_R.*

In order to define the inverse problem, we recall now the definition of the far–field associated to a scattered field. From [8], the scattered field has the asymptotic behaviour:

$$u^s(x) = \frac{e^{ikr}}{r^{(d-1)/2}} \left(u^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{r}\right) \right) \qquad r \longrightarrow +\infty$$

uniformly for all the directions $\hat{x} = x/r \in S^{d-1}$ with $r = |x|$, and the far–field $u^\infty \in L^2(S^{d-1})$ has the following integral representation on the boundary ∂D:

$$u^\infty(\hat{x}) = \int_{\partial D} \left(u^s(y) \frac{\partial \Phi^\infty(\hat{x}, y)}{\partial \nu(y)} - \frac{\partial u^s(y)}{\partial \nu} \Phi^\infty(\hat{x}, y) \right) ds(y) \quad \forall \hat{x} \in S^{d-1}. \tag{3}$$

Here $\Phi^\infty(\cdot, y)$ is the far–field associated with the Green function $\Phi(\cdot, y)$ of the Helmholtz equation. The function $\Phi(\cdot, y)$ is defined in $\mathbb{R}^2$ by $\Phi(x, y) = (i/4)H_0^1(k|x - y|)$, where H_0^1 is the Hankel function of the first kind and of order 0, and in $\mathbb{R}^3$ by $e^{ik|x-y|}/(4\pi|x - y|)$. The associated far–fields are defined in S^1 by $(e^{i\pi/4}/\sqrt{8\pi k})e^{-iky\cdot\hat{x}}$ and in S^2 by $(1/4\pi)e^{-iky\cdot\hat{x}}$ respectively. The second integral in (3) has to be understood as a duality pairing between $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D)$. We are now in a position to define the far–field map

$$T : \quad (\lambda, \mu, \partial D) \to u^\infty$$

where u^∞ is the far–field associated with the scattered field $u^s = u - u^i$ and u is the unique solution of problem (1) with obstacle D and impedances (λ, μ) on ∂D.
The general inverse problem we are interested in is the following: given several incident plane waves of direction $\hat{d} \in S^{d-1}$, is it possible to reconstruct the obstacle D as well as the impedances λ and μ defined on ∂D from the corresponding far–field $u^\infty = T(\lambda, \mu, \partial D)$? The first question of interest is the identifiability of $(\lambda, \mu, \partial D)$ from the far–field data u^∞, that is uniqueness.

3 A uniqueness result

In this section, we provide a uniqueness result concerning identification of both the obstacle D and the impedances (λ, μ) from the far–fields associated to plane waves with all incident directions $\hat{d} \in S^{d-1}$. In this respect we denote by $u^\infty(\hat{x}, \hat{d})$ the far–field in the $\hat{x}$ direction that is associated to the plane wave with direction $\hat{d}$. In the following, we introduce some regularity assumptions for the obstacle D and the impedances λ, μ.

Assumption 3.1. *The boundary ∂D is C^2, and the impedances satisfy $\lambda \in C^0(\partial D)$ and $\mu \in C^1(\partial D)$.*

The main result is the following theorem, which is a generalization of the uniqueness result for $\mu = 0$ proved in [6, theorem 4.7].

Theorem 3.2. *Assume that $(\lambda_1, \mu_1, \partial D_1)$ and $(\lambda_2, \mu_2, \partial D_2)$ satisfy assumptions 2.1 and 3.1, and the corresponding far–fields $u_1^\infty = T(\lambda_1, \mu_1, \partial D_1)$ and $u_2^\infty = T(\lambda_2, \mu_2, \partial D_2)$ satisfy $u_1^\infty(\hat{x}, \hat{d}) = u_2^\infty(\hat{x}, \hat{d})$ for all $\hat{x} \in S^{d-1}$ and $\hat{d} \in S^{d-1}$. Then $D_1 = D_2$ and $(\lambda_1, \mu_1) = (\lambda_2, \mu_2)$.*

The proof of the above theorem is based on several results, the first one is the mixed reciprocity lemma and does not require the regularity assumption 3.1.

Lemma 3.3. *Let $w^\infty(\cdot, z)$ be the far–field associated to the incident field $\Phi(\cdot, z)$ with $z \in \Omega$, and $u^s(\cdot, \hat{x})$ be the scattered field associated to the plane wave of direction $\hat{x} \in S^{d-1}$. Then*

$$w^\infty(-\hat{x}, z) = c(d)\, u^s(z, \hat{x}),$$

with $c(2) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}$ and $c(3) = \frac{1}{4\pi}$.

Proof. For two incident fields u_1^i and u_2^i, the associated total fields u_1 and u_2 satisfy, by using the boundary condition on ∂D,

$$\int_{\partial D} \left(u_1 \frac{\partial u_2}{\partial \nu} - u_2 \frac{\partial u_1}{\partial \nu} \right) ds$$
$$= \int_{\partial D} (\mu \nabla_\Gamma u_1 \cdot \nabla_\Gamma u_2 - \lambda u_1 u_2 - \mu \nabla_\Gamma u_1 \cdot \nabla_\Gamma u_2 + \lambda u_1 u_2)\, ds = 0.$$

By using the decomposition $u_1 = u_1^i + u_1^s$ and $u_2 = u_2^i + u_2^s$, that the incident fields solve the Helmholtz equation inside D, that the scattered fields solve the Helmholtz equation outside D as well as the radiation condition, we obtain

$$\int_{\partial D} \left(u_1^s \frac{\partial u_2^i}{\partial \nu} - u_2^i \frac{\partial u_1^s}{\partial \nu} \right) ds = \int_{\partial D} \left(u_2^s \frac{\partial u_1^i}{\partial \nu} - u_1^i \frac{\partial u_2^s}{\partial \nu} \right) ds. \tag{4}$$

Now we use the following integral representation on the boundary ∂D for $u^s(\cdot, \hat{x})$: for $z \in \Omega$ and $\hat{x} \in S^{d-1}$,

$$u^s(z, \hat{x}) = \int_{\partial D} \left(u^s(y, \hat{x}) \frac{\partial \Phi(y, z)}{\partial \nu(y)} - \frac{\partial u^s(y, \hat{x})}{\partial \nu} \Phi(y, z) \right) ds(y).$$

By applying equation (4) when u_1^i is the plane wave of direction $\hat{x}$ and u_2^i is the point source $\Phi(\cdot, z)$, it follows that

$$u^s(z, \hat{x}) = \int_{\partial D} \left(w^s(y, z) \frac{\partial e^{ik\hat{x}.y}}{\partial \nu(y)} - \frac{\partial w^s(y, z)}{\partial \nu(y)} e^{ik\hat{x}.y} \right) ds(y).$$

Lastly, from the integral representation (3) and the above equation we obtain

$$c(d)\, u^s(z, \hat{x}) = w^\infty(-\hat{x}, z).$$

□

The second lemma is a density result and does not require the regularity assumption 3.1 either. Since it is a slightly more general version of lemma 4 in [5], the proof is omitted.

Lemma 3.4. *Let $u(\cdot, \hat{d})$ denote the solution of (1) associated to the incident wave $u^i(x) = e^{ik\hat{d}.x}$ and assume that for some $f \in H^{-1}(\partial D)$,*

$$< u(\cdot, \hat{d}), f >_{H^1(\partial D), H^{-1}(\partial D)} = 0, \quad \forall \hat{d} \in S^{d-1}.$$

Then $f = 0$.

We are now in a position to prove theorem 3.2.

Proof of theorem 3.2. The first step of the proof consists in proving that $D_1 = D_2$, following the method of [16, 15]. Let us denote $\tilde{\Omega}$ the unbounded connected component of $\mathbb{R}^d \setminus \overline{D_1 \cup D_2}$. From Rellich's lemma and unique continuation, we obtain that

$$u_1^s(z, \hat{d}) = u_2^s(z, \hat{d}), \qquad \forall z \in \tilde{\Omega},\, \forall \hat{d} \in S^{d-1}. \tag{5}$$

Using the mixed reciprocity lemma 3.3, we obtain that

$$u_1^\infty(-\hat{d}, z) = u_2^\infty(-\hat{d}, z), \qquad \forall \hat{d} \in S^{d-1},\, \forall z \in \tilde{\Omega},$$

where $u_1^\infty(\cdot, z)$ and $u_2^\infty(\cdot, z)$ are the far–fields associated to the incident field $\Phi(\cdot, z)$ with $z \in \tilde{\Omega}$. By using again Rellich's lemma and unique continuation, it follows that

$$u_1^s(x, z) = u_2^s(x, z), \qquad \forall (x, z) \in \tilde{\Omega} \times \tilde{\Omega}. \tag{6}$$

Assume that $D_1 \not\subset D_2$. Since $\mathbb{R}^d \setminus \overline{D_2}$ is connected, there exists some non empty open set $\Gamma_* \subset (\partial D_1 \cap \partial \tilde{\Omega}) \setminus \overline{D_2}$. We now consider some point $x_* \in \Gamma_*$ and the sequence

$$x_n = x_* + \frac{\nu_1(x_*)}{n}.$$

For sufficiently large n, $x_n \in \tilde{\Omega}$. From (6), we hence have by denoting $P_1 v := \partial v / \partial \nu + \mathrm{div}_\Gamma(\mu_1 \nabla_\Gamma v) + \lambda_1 v$,

$$P_1 u_2^s(\cdot, x_n) = P_1 u_1^s(\cdot, x_n) \quad \text{on} \quad \Gamma_*.$$

Using boundary condition on ∂D_1 for $u_1 = u_1^s + \Phi(\cdot, x_n)$, this implies that

$$P_1 u_2^s(\cdot, x_n) = -P_1 \Phi(\cdot, x_n) \quad \text{on} \quad \Gamma_*.$$

Using assumption 3.1 and the fact that u_2^s is smooth in the neighborhood of Γ_*, we obtain

$$\lim_{n\to+\infty} P_1 u_2^s(\cdot, x_n) = \frac{\partial u_2^s}{\partial \nu}(\cdot, x_*) + \mu_1 \Delta_\Gamma u_2^s(\cdot, x_*) + \nabla_\Gamma \mu_1 \cdot \nabla_\Gamma u_2^s(\cdot, x_*) + \lambda_1 u_2^s(\cdot, x_*)$$

in $L^2(\Gamma_*)$. On the other hand, for $x_1 \in \Gamma_* \setminus \{x_*\}$, we have pointwise convergence

$$\lim_{n\to+\infty} P_1 \Phi(x_1, x_n) = \frac{\partial \Phi}{\partial \nu}(x_1, x_*) + \mu_1 \Delta_\Gamma \Phi(x_1, x_*) + \nabla_\Gamma \mu_1 \cdot \nabla_\Gamma \Phi(x_1, x_*) + \lambda_1 \Phi(x_1, x_*).$$

We hence obtain that

$$\frac{\partial \Phi}{\partial \nu}(\cdot, x_*) + \mathrm{div}_\Gamma(\mu_1 \nabla_\Gamma \Phi)(\cdot, x_*) + \lambda_1 \Phi(\cdot, x_*) \in L^2(\Gamma_*). \tag{7}$$

Now we consider some reals $R_* > r_* > 0$ such that $\partial D \cap B(x_*, R_*) \subset \Gamma_*$, a function $\phi \in C_0^\infty(B(x_*, R_*))$ with $\phi = 1$ on $\overline{B(x_*, r_*)}$, and $w_*^s := \phi\, \Phi(\cdot, x_*)$. The function w_*^s satisfies

$$\begin{cases} \Delta w_*^s + k^2 w_*^s = f \quad \text{in } \Omega_1 \\ \dfrac{\partial w_*^s}{\partial \nu} + \mathrm{div}_\Gamma(\mu_1 \nabla_\Gamma w_*^s) + \lambda_1 w_*^s = g \quad \text{on } \partial D_1 \\ \displaystyle\lim_{R\to\infty} \int_{|x|=R} \left| \frac{\partial w_*^s}{\partial r} - ikw_*^s \right|^2 ds = 0, \end{cases} \tag{8}$$

with

$$\begin{aligned} f =& (\Delta\phi)\Phi(\cdot, x_*) + 2\nabla\phi \cdot \nabla\Phi(\cdot, x_*), \\ g =& \phi \left(\frac{\partial \Phi}{\partial \nu}(\cdot, x_*) + \mathrm{div}_\Gamma(\mu_1 \nabla_\Gamma \Phi)(\cdot, x_*) + \lambda_1 \Phi(\cdot, x_*) \right) \\ &+ \Phi(\cdot, x_*) \left(\frac{\partial \phi}{\partial \nu} + \nabla_\Gamma \mu_1 \cdot \nabla_\Gamma \phi + \mu_1 \Delta_\Gamma \phi \right) + 2\mu_1 \nabla_\Gamma \Phi(\cdot, x_*) \cdot \nabla_\Gamma \phi. \end{aligned}$$

Since $\phi = 1$ in the neighborhood of x_* and by using (7), we have $f \in L^2(\Omega_1)$ and $g \in L^2(\partial D_1)$. With the help of a variational formulation for the auxiliary problem (8) as in [4], we conclude that $w_*^s \in H^1(B_R \setminus \overline{D_1})$, hence $\Phi(\cdot, x_*) \in H^1(\Omega_1 \cap B(x_*, r_*))$. Since ∂D is C^2, we can find a finite cone C_* of apex x_*, angle θ_*, radius r_* and axis directed by $\xi_* = \nu_1(x_*)$, such that $C_* \subset \Omega_1 \cap B(x_*, r_*)$. Hence $\Phi(\cdot, x_*) \in H^1(C_*)$.
In the case $d = 3$ (the case $d = 2$ is similar), we have

$$\nabla\Phi(\cdot, x_*) = -\frac{e^{ik|x-x_*|}}{4\pi|x - x_*|^2} \left(\frac{1}{|x - x_*|} - ik \right) (x - x_*),$$

and by using spherical coordinates (r, θ, ϕ) centered at x_*,

$$\int_{C^*} \frac{dx}{|x - x_*|^4} = \int_0^{r_*} \int_0^{\theta_*} \int_0^{2\pi} \frac{r^2 \sin\theta\, dr d\theta d\phi}{r^4} = +\infty,$$

which contradicts the fact that $\Phi(\cdot, x_*) \in H^1(C_*)$. Then $D_1 \subset D_2$. We prove the same way that $D_2 \subset D_1$, and then $D_1 = D_2 = D$.
The second step of the proof consists in proving that $(\lambda_1, \mu_1) = (\lambda_2, \mu_2)$. In this view we denote $\lambda = \lambda_1 - \lambda_2$ and $\mu = \mu_1 - \mu_2$. From equality (5), the total fields associated with the plane waves of direction $\hat{d}$ satisfy

$$u(x, \hat{d}) := u_1(x, \hat{d}) = u_2(x, \hat{d}) \qquad \forall x \in \mathbb{R}^d \setminus \overline{D},\ \forall \hat{d} \in S^{d-1}.$$

From the boundary condition on ∂D for u_1 and u_2 it follows that

$$\mathrm{div}_\Gamma(\mu\nabla_\Gamma u(\cdot,\hat{d})) + \lambda u(\cdot,\hat{d}) = 0 \qquad \text{on } \partial \mathrm{D}, \qquad \forall \hat{\mathrm{d}} \in \mathrm{S}^{\mathrm{d}-1}.$$

For some $\phi \in H^1(\partial D)$, multiplying the above equation with integration by parts leads to

$$< u(\cdot,\hat{d}), \mathrm{div}_\Gamma(\mu\nabla_\Gamma\phi) + \lambda\phi >_{H^1(\partial D),H^{-1}(\partial D)} = 0, \qquad \forall \hat{d} \in S^{d-1}.$$

With the help of lemma 3.4, we obtain that

$$\mathrm{div}_\Gamma(\mu\nabla_\Gamma\phi) + \lambda\phi = 0 \qquad \text{on } \partial D, \quad \forall \phi \in H^1(\partial D).$$

Choosing $\phi = 1$ in the above equation leads to $\lambda = 0$. The above equation also implies that

$$\int_{\partial D} \mu |\nabla_\Gamma \phi|^2 \, ds = 0, \quad \forall \phi \in H^1(\partial D).$$

Assume that $\mu(x_0) \neq 0$ for some $x_0 \in \partial D$, then for example $\Re e(\mu)(x_0) > 0$ without loss of generality. Since μ is continuous there exists $\varepsilon > 0$ such that $\Re e(\mu)(x) > 0$ for all $x \in \partial D \cap B(x_0,\varepsilon)$. Let us choose ϕ as a smooth and compactly supported function in $\partial D \cap B(x_0,\varepsilon)$. We obtain that

$$\int_{\partial D \cap B(x_0,\varepsilon)} \Re e(\mu) |\nabla_\Gamma \phi|^2 \, ds = 0,$$

and then $\nabla_\Gamma \phi = 0$ on ∂D, that is ϕ is a constant on ∂D, which is a contradiction. We hence have $\mu = 0$ on ∂D, which completes the proof. □

As illustrated by theorem 3.2, if all plane waves are used as incident fields, then it is possible to retrieve both the obstacle and the impedances, with reasonable assumptions on such unknowns. In the sequel, we consider an effective method to retrieve both the obstacle and the impedances in the case we use several plane waves. Such method will be based on a standard steepest descent method and in this view, we need to compute the partial derivative of the far–field with respect to the obstacle, the impedances being fixed. This is the aim of next section. The computation of the partial derivative with respect to the impedances is already known and given in [4]. The adopted approach is the one used in [17] for the Neumann boundary condition and in [12] for the classical impedance boundary condition with constant λ.

4 Differentiation of far–field with respect to the obstacle

Throughout this section, we assume that the boundary of the obstacle and the impedances are smooth, typically ∂D is C^4, $\lambda \in C^2(\partial D)$ and $\mu \in C^3(\partial D)$, which ensures that the solution to problem (2) belongs to $H^4(\Omega_R)$. In order to compute the partial derivative of the far–field associated to the solution of problem (1) with respect to the obstacle, we consider a perturbed obstacle D_ε and some impedances $(\lambda_\varepsilon, \mu_\varepsilon)$ that correspond to the impedances (λ, μ) once transported on the perturbed boundary ∂D_ε.
More precisely, we consider some mapping $\varepsilon \in C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ with $C^{1,\infty} := C^1 \cap W^{1,\infty}$ equipped with the norm $||\varepsilon|| := ||\varepsilon||_{W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)}$. From [14, section 5.2.2], if we assume that $||\varepsilon|| < 1$, the mapping $f_\varepsilon := Id + \varepsilon$ is a C^1–diffeomorphism of $\mathbb{R}^d$. The perturbed obstacle D_ε is defined from D by

$$\partial D_\varepsilon = \{x + \varepsilon(x), \, x \in \partial D\},$$

while the transported impedances $(\lambda_\varepsilon, \mu_\varepsilon)$ on ∂D_ε are defined from (λ, μ) by

$$\lambda_\varepsilon = \lambda \circ f_\varepsilon^{-1}, \quad \mu_\varepsilon = \mu \circ f_\varepsilon^{-1}. \tag{9}$$

We now define the partial derivative of the far–field with respect to the obstacle.

Definition 4.1. *We say that the far–field operator $T : (\lambda, \mu, \partial D) \to u^\infty$ is differentiable with respect to ∂D if there exists a continuous linear operator $T'_{\lambda,\mu}(\partial D) : C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \to L^2(S^{d-1})$ and a function $o(||\varepsilon||) : C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \to L^2(S^{d-1})$ such that*

$$T(\lambda_\varepsilon, \mu_\varepsilon, \partial D_\varepsilon) - T(\lambda, \mu, \partial D) = T'_{\lambda,\mu}(\partial D) \cdot \varepsilon + o(||\varepsilon||),$$

where λ_ε and μ_ε are defined by (9) and $\lim_{||\varepsilon|| \to 0} o(||\varepsilon||)/||\varepsilon|| = 0$ in $L^2(S^{d-1})$.

Remark 1. *Note that if λ and μ are constants, the above definition coincides with the classical notion of Fréchet diffentiability with respect to an obstacle.*

In the following, we make use of the following definitions

$$J_\varepsilon := |\det(\nabla f_\varepsilon)|, \quad J_\varepsilon^\nu := J_\varepsilon |(\nabla f_\varepsilon)^{-T} \nu|, \quad P_\varepsilon := (\nabla f_\varepsilon)^{-1} (\nabla f_\varepsilon)^{-T},$$

where $\det(B)$ stands for the determinant of matrix B, while B^{-T} stands for the transposition of the inverse of invertible matrix B.
Now we denote by u_ε the solution of problem (1) with obstacle D_ε instead of obstacle D and impedances $(\lambda_\varepsilon, \mu_\varepsilon)$ instead of impedances (λ, μ). We assume in addition that $\overline{D} \subset D_\varepsilon$. We have the following integral representation for $u_\varepsilon^s - u^s$:

Lemma 4.2. *For $x \in \mathbb{R}^d \setminus \overline{D}_\varepsilon$,*

$$u_\varepsilon^s(x) - u^s(x) = \int_{\partial D_\varepsilon} u_\varepsilon \left\{ \frac{\partial w}{\partial \nu_\varepsilon}(\cdot, x) + \operatorname{div}_{\Gamma_\varepsilon}(\mu_\varepsilon \nabla_{\Gamma_\varepsilon} w)(\cdot, x) + \lambda_\varepsilon w(\cdot, x) \right\} ds_\varepsilon,$$

where $w(\cdot, x)$ is the solution of problem (1) with incident wave $\Phi(\cdot, x)$.

Proof. Let $x \in \mathbb{R}^d \setminus \overline{D_\varepsilon}$. By using the Green formula in D for plane wave u^i and point source $\Phi(\cdot, x)$, we have

$$\int_{\partial D} \left(u^i \frac{\partial \Phi}{\partial \nu}(\cdot, x) - \frac{\partial u^i}{\partial \nu} \Phi(\cdot, x) \right) ds = 0,$$

then obtain the representation formula

$$u^s(x) = \int_{\partial D} \left(u \frac{\partial \Phi}{\partial \nu}(\cdot, x) - \frac{\partial u}{\partial \nu} \Phi(\cdot, x) \right) ds. \tag{10}$$

By using the boundary condition for $w(\cdot, x)$ and Green Formula on ∂D, we obtain

$$\int_{\partial D} \left(u \frac{\partial \Phi}{\partial \nu}(\cdot, x) - \frac{\partial u}{\partial \nu} \Phi(\cdot, x) \right) ds = -\int_{\partial D} \left(u \frac{\partial w^s}{\partial \nu}(\cdot, x) - \frac{\partial u}{\partial \nu} w^s(\cdot, x) \right) ds.$$

By using again Green Formula outside D and the radiation condition for u^s and w^s, we obtain

$$u^s(x) = \int_{\partial D} \left(\frac{\partial u^i}{\partial \nu} w^s(\cdot, x) - u^i \frac{\partial w^s}{\partial \nu}(\cdot, x) \right) ds.$$

We now use the Green Formula in $D_\varepsilon \setminus \overline{D}$ and find

$$u^s(x) = \int_{\partial D_\varepsilon} \left(\frac{\partial u^i}{\partial \nu_\varepsilon} w^s(\cdot, x) - u^i \frac{\partial w^s}{\partial \nu_\varepsilon}(\cdot, x) \right) ds_\varepsilon.$$

Using again the Green formula outside D_ε and the radiation condition for u_ε^s and w^s, we obtain

$$u^s(x) = \int_{\partial D_\varepsilon} \left(\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} w^s(\cdot, x) - u_\varepsilon \frac{\partial w^s}{\partial \nu_\varepsilon}(\cdot, x) \right) ds_\varepsilon.$$

That u_ε satisfies the boundary condition on ∂D_ε implies

$$-u^s(x) = \int_{\partial D_\varepsilon} u_\varepsilon \left(\frac{\partial w^s}{\partial \nu_\varepsilon}(\cdot, x) + \mathrm{div}_{\Gamma_\varepsilon}(\mu_\varepsilon \nabla_{\Gamma_\varepsilon} w^s)(\cdot, x) + \lambda_\varepsilon w^s(\cdot, x) \right) ds_\varepsilon.$$

Lastly, we use formula (10) for u_ε and D_ε, as well as the boundary condition of u_ε on ∂D_ε, and obtain that for $x \in \mathbb{R}^d \setminus \overline{D_\varepsilon}$,

$$u^s_\varepsilon(x) = \int_{\partial D_\varepsilon} u_\varepsilon \left(\frac{\partial \Phi}{\partial \nu_\varepsilon}(\cdot, x) + \mathrm{div}_{\Gamma_\varepsilon}(\mu_\varepsilon \nabla_{\Gamma_\varepsilon} \Phi)(\cdot, x) + \lambda_\varepsilon \Phi(\cdot, x) \right) ds.$$

We complete the proof by adding the two last equations, given $w(\cdot, x) = w^s(\cdot, x) + \Phi(\cdot, x)$. □

We continue our computation by replacing u_ε by u in the integral representation of lemma 4.2 at first order for $||\varepsilon||$, uniformly for x in some compact subset $K \subset \mathbb{R}^d \setminus \overline{D}$.

Lemma 4.3. *We have*

$$u^s_\varepsilon(x) - u^s(x) = \int_{\partial D_\varepsilon} u \left\{ \frac{\partial w}{\partial \nu_\varepsilon}(\cdot, x) + \mathrm{div}_{\Gamma_\varepsilon}(\mu_\varepsilon \nabla_{\Gamma_\varepsilon} w)(\cdot, x) + \lambda_\varepsilon w(\cdot, x) \right\} ds_\varepsilon + \mathcal{O}(||\varepsilon||^2),$$

uniformly for x in some compact subset $K \subset \mathbb{R}^d \setminus \overline{D}$.

Proof. We have

$$\int_{\partial D_\varepsilon} (u_\varepsilon - u) \left\{ \frac{\partial w}{\partial \nu_\varepsilon}(\cdot, x) + \mathrm{div}_{\Gamma_\varepsilon}(\mu_\varepsilon \nabla_{\Gamma_\varepsilon} w)(\cdot, x) + \lambda_\varepsilon w(\cdot, x) \right\} ds_\varepsilon$$
$$= \int_{\partial D_\varepsilon} (u_\varepsilon - u) \frac{\partial w}{\partial \nu_\varepsilon} ds_\varepsilon - \int_{\partial D_\varepsilon} \mu_\varepsilon \nabla_{\Gamma_\varepsilon}(u_\varepsilon - u) \cdot \nabla_{\Gamma_\varepsilon} w \, ds_\varepsilon + \int_{\partial D_\varepsilon} \lambda_\varepsilon (u_\varepsilon - u) w \, ds_\varepsilon.$$

We consider now each term of the above sum separately. By denoting $\tilde{u}_\varepsilon = u_\varepsilon \circ f_\varepsilon$, $\hat{u}_\varepsilon = u \circ f_\varepsilon$ and $\hat{w}_\varepsilon = w \circ f_\varepsilon$, the change of variable $x_\varepsilon = f_\varepsilon(x)$ in the third integral (see [14, proposition 5.4.3]) implies

$$\int_{\partial D_\varepsilon} \lambda_\varepsilon (u_\varepsilon - u) w \, ds_\varepsilon = \int_{\partial D} \lambda (\tilde{u}_\varepsilon - \hat{u}_\varepsilon) \hat{w}_\varepsilon J^\nu_\varepsilon \, ds.$$

As a consequence,

$$\int_{\partial D_\varepsilon} \lambda_\varepsilon (u_\varepsilon - u) w \, ds_\varepsilon - \int_{\partial D} \lambda (\tilde{u}_\varepsilon - \hat{u}_\varepsilon) w \, ds = \int_{\partial D} \lambda (\tilde{u}_\varepsilon - \hat{u}_\varepsilon)(\hat{w}_\varepsilon J^\nu_\varepsilon - w) \, ds$$

and then

$$\left| \int_{\partial D_\varepsilon} \lambda_\varepsilon (u_\varepsilon - u) w \, ds_\varepsilon - \int_{\partial D} \lambda (\tilde{u}_\varepsilon - \hat{u}_\varepsilon) w \, ds \right| \leq ||\lambda||_{L^\infty(\partial D)} ||\tilde{u}_\varepsilon - \hat{u}_\varepsilon||_{L^2(\partial D)} U_\varepsilon(x)$$

with

$$U_\varepsilon(x) = ||((w \circ f_\varepsilon) J^\nu_\varepsilon - w)(\cdot, x)||_{L^2(\partial D)}.$$

Concerning the second integral,

$$\int_{\partial D_\varepsilon} \mu_\varepsilon \nabla_{\Gamma_\varepsilon}(u_\varepsilon - u) \cdot \nabla_{\Gamma_\varepsilon} w \, ds_\varepsilon = \int_{\partial D_\varepsilon} (\mu \circ f_\varepsilon^{-1}) \nabla_{\Gamma_\varepsilon}((\tilde{u}_\varepsilon - \hat{u}_\varepsilon) \circ f_\varepsilon^{-1}) \cdot \nabla_{\Gamma_\varepsilon}(\hat{w}_\varepsilon \circ f_\varepsilon^{-1}) \, ds_\varepsilon$$

We may prove (see [4, proof of lemma 3.4]) that for $z \in H^1(\partial D)$, $x \in \partial D$ and $x_\varepsilon = f_\varepsilon(x)$,

$$\nabla_{\Gamma_\varepsilon}(z \circ f_\varepsilon^{-1})(x_\varepsilon) = (\nabla f_\varepsilon(x))^{-T} \nabla_\Gamma z(x).$$

As a consequence

$$\int_{\partial D_\varepsilon} \mu_\varepsilon \nabla_{\Gamma_\varepsilon}(u_\varepsilon - u) \cdot \nabla_{\Gamma_\varepsilon} w \, ds_\varepsilon = \int_{\partial D} \mu \nabla_\Gamma (\tilde{u}_\varepsilon - \hat{u}_\varepsilon) \cdot P_\varepsilon \cdot \nabla_\Gamma \hat{w}_\varepsilon J^\nu_\varepsilon \, ds,$$

and

$$\int_{\partial D_\varepsilon} \mu_\varepsilon \nabla_{\Gamma_\varepsilon}(u_\varepsilon - u)\cdot \nabla_{\Gamma_\varepsilon} w \, ds_\varepsilon - \int_{\partial D} \mu \nabla_\Gamma(\tilde{u}_\varepsilon - \hat{u}_\varepsilon)\cdot \nabla_\Gamma w \, ds$$
$$= \int_{\partial D} \mu \nabla_\Gamma(\tilde{u}_\varepsilon - \hat{u}_\varepsilon)\cdot (J_\varepsilon^\nu P_\varepsilon \cdot \nabla_\Gamma \hat{w}_\varepsilon - \nabla_\Gamma w)\, ds.$$

Then

$$\left| \int_{\partial D_\varepsilon} \mu_\varepsilon \nabla_{\Gamma_\varepsilon}\ (u_\varepsilon - u)\cdot \nabla_{\Gamma_\varepsilon} w \, ds_\varepsilon - \int_{\partial D} \mu \nabla_\Gamma(\tilde{u}_\varepsilon - \hat{u}_\varepsilon)\cdot \nabla_\Gamma w \, ds \right|$$
$$\le ||\mu||_{L^\infty(\partial D)} ||\tilde{u}_\varepsilon - \hat{u}_\varepsilon||_{H^1(\partial D)} V_\varepsilon(x)$$

with

$$V_\varepsilon(x) = ||(J_\varepsilon^\nu P_\varepsilon \cdot \nabla_\Gamma(w\circ f_\varepsilon) - \nabla_\Gamma w)(\cdot, x)||_{L^2(\partial D)}.$$

It remains to consider the first integral.

$$\int_{\partial D_\varepsilon} (u_\varepsilon - u)\frac{\partial w}{\partial \nu_\varepsilon}\, ds_\varepsilon = \int_{\partial D_\varepsilon} ((\tilde{u}_\varepsilon - \hat{u}_\varepsilon)\circ f_\varepsilon^{-1})\nabla(\hat{w}_\varepsilon \circ f_\varepsilon^{-1})\cdot \nu_\varepsilon \, ds_\varepsilon$$
$$= \int_{\partial D_\varepsilon} ((\tilde{u}_\varepsilon - \hat{u}_\varepsilon)\circ f_\varepsilon^{-1})(\nabla f_\varepsilon)^{-T}\cdot (\nabla \hat{w}_\varepsilon \circ f_\varepsilon^{-1})\cdot \nu_\varepsilon ds_\varepsilon$$
$$= \int_{\partial D} (\tilde{u}_\varepsilon - \hat{u}_\varepsilon)\nabla \hat{w}_\varepsilon \cdot (\nabla f_\varepsilon)^{-1}\cdot (\nu_\varepsilon \circ f_\varepsilon) J_\varepsilon^\nu \, ds.$$

Then

$$\int_{\partial D_\varepsilon} (u_\varepsilon - u)\frac{\partial w}{\partial \nu_\varepsilon}\, ds_\varepsilon - \int_{\partial D} (\tilde{u}_\varepsilon - \hat{u}_\varepsilon)\frac{\partial w}{\partial \nu}\, ds = \int_{\partial D} (\tilde{u}_\varepsilon - \hat{u}_\varepsilon)(J_\varepsilon^\nu \nabla \hat{w}_\varepsilon \cdot (\nabla f_\varepsilon)^{-1}\cdot (\nu_\varepsilon\circ f_\varepsilon) - \nabla w\cdot \nu)\, ds,$$

hence

$$\left| \int_{\partial D_\varepsilon} (u_\varepsilon - u)\frac{\partial w}{\partial \nu_\varepsilon}\, ds_\varepsilon - \int_{\partial D} (\tilde{u}_\varepsilon - \hat{u}_\varepsilon)\frac{\partial w}{\partial \nu}\, ds \right| \le ||\tilde{u}_\varepsilon - \hat{u}_\varepsilon||_{H^{1/2}(\partial D)}\, W_\varepsilon(x)$$

with

$$W_\varepsilon(x) = ||J_\varepsilon^\nu \nabla(w\circ f_\varepsilon)\cdot (\nabla f_\varepsilon)^{-1}\cdot (\nu_\varepsilon \circ f_\varepsilon) - \nabla w \cdot \nu||_{H^{-1/2}(\partial D)}.$$

By using the fact that

$$J_\varepsilon^\nu(x) = 1 + \mathcal{O}(||\varepsilon||), \quad P_\varepsilon(x) = Id(1+\mathcal{O}(||\varepsilon||)),$$

we conclude that

$$U_\varepsilon(x), V_\varepsilon(x), W_\varepsilon(x) = \mathcal{O}(||\varepsilon||),$$

uniformly for x in some compact subset $K \subset \mathbb{R}^d \setminus \overline{D}$.
On the other hand,

$$||\tilde{u}_\varepsilon - \hat{u}_\varepsilon||_{H^1(\partial D)} \le ||\tilde{u}_\varepsilon - u||_{H^1(\partial D)} + ||\hat{u}_\varepsilon - u||_{H^1(\partial D)}.$$

We have

$$||\tilde{u}_\varepsilon - u||_{H^1(\partial D)} = \mathcal{O}(||\varepsilon||), \quad ||\hat{u}_\varepsilon - u||_{H^1(\partial D)} = \mathcal{O}(||\varepsilon||).$$

The first estimate is a consequence of theorem 3.3 in [4], that is continuity of the solution of problem (2) with respect to the scatterer D. The second one comes from the fact that $\hat{u}_\varepsilon = u\circ f_\varepsilon$. We remark that due to boundary condition satisfied by $w(\cdot, x)$ on ∂D, we have

$$0 = \int_{\partial D} (\tilde{u}_\varepsilon - \hat{u}_\varepsilon)\frac{\partial w}{\partial \nu}\, ds - \int_{\partial D} \mu \nabla_\Gamma(\tilde{u}_\varepsilon - \hat{u}_\varepsilon)\cdot \nabla_\Gamma w\, ds + \int_{\partial D} \lambda(\tilde{u}_\varepsilon - \hat{u}_\varepsilon) w\, ds$$
$$= \int_{\partial D_\varepsilon} (u_\varepsilon - u)\frac{\partial w}{\partial \nu_\varepsilon}\, ds_\varepsilon - \int_{\partial D_\varepsilon} \mu_\varepsilon \nabla_{\Gamma_\varepsilon}(u_\varepsilon - u)\cdot \nabla_{\Gamma_\varepsilon} w\, ds_\varepsilon + \int_{\partial D_\varepsilon} \lambda_\varepsilon (u_\varepsilon - u) w\, ds_\varepsilon + \mathcal{O}(||\varepsilon||^2),$$

and conclude that

$$\int_{\partial D_\varepsilon}(u_\varepsilon - u)\left\{\frac{\partial w}{\partial \nu_\varepsilon}(\cdot,x) + \mathrm{div}_{\Gamma_\varepsilon}(\mu_\varepsilon \nabla_{\Gamma_\varepsilon} w)(\cdot,x) + \lambda_\varepsilon w(\cdot,x)\right\} ds_\varepsilon = \mathcal{O}(||\varepsilon||^2),$$

which completes the proof in view of lemma 4.2. □

To continue the computation of the partial derivative of solution u of problem (1) with respect to the domain D, we need to extend the definitions of some surface quantifies, essentially the outward normal ν on ∂D and the surface gradient ∇_Γ. inside the volumic domain $D_\varepsilon \setminus \overline{D}$. In this view, for $x_0 \in \partial D$, by definition of a domain of class C^1 there exist a function ϕ of class C^1 and two open sets $U \subset \mathbb{R}^{d-1}$ and $V \subset \mathbb{R}^d$ which are neighborhood of 0 and x_0 respectively, such that $\phi(0) = x_0$ and

$$\partial D \cap V = \{\phi(\xi)\,;\xi \in U\}.$$

Defining now for $t \in [0,1]$,

$$f_t := Id + t\varepsilon, \quad \phi_t := f_t \circ \phi,$$

ϕ_t is a parametrization of $\partial D_t = (Id + t\varepsilon)(\partial D)$, and hence the tangential vectors of ∂D_t at $x_0^t = f_t(x_0)$ are

$$e_j^t = \frac{\partial \phi_t}{\partial \xi_j} = (Id + t\nabla\varepsilon)\frac{\partial \phi}{\partial \xi_j} = (Id + t\nabla\varepsilon)e_j, \quad \text{for} \quad j = 1, d-1. \tag{11}$$

To such basis we associate the covariant basis (e_t^i) of ∂D_t at point x_0^t (see for example [19, section 2.5]) by

$$e_t^i \cdot e_j^t = \delta_j^i, \quad \text{for} \quad i,j = 1, d-1. \tag{12}$$

With these definitions, the outward normal of ∂D_t at point x_0^t is given by

$$\nu_t = \frac{e_1^t \times e_2^t}{|e_1^t \times e_2^t|},$$

while the tangential gradient of function $w \in H^1(\partial D_t)$ is given, denoting $\tilde{w}_t = w \circ \phi_t$, by

$$\nabla_{\Gamma_t} w(x_0^t) = \sum_{i=1}^{d-1} \frac{\partial \tilde{w}_t}{\partial \xi_i}(0) e_t^i. \tag{13}$$

It is hence possible to consider in domain $D_\varepsilon \setminus \overline{D}$ an extended outward normal ν_t and an extended tangential gradient $\nabla_{\Gamma_t} w$ by using parametrization (ξ_i, t) for $i = 1, d-1$. In the same spirit, the impedances (λ, μ) are extended to (λ_t, μ_t), that is

$$\lambda_t = \lambda \circ f_t^{-1}, \quad \mu_t = \mu \circ f_t^{-1}.$$

We are now in a position to transform the integral representation of $u_\varepsilon^s - u^s$ on ∂D_ε in lemma 4.3 into an integral representation on ∂D. We have the following proposition.

Proposition 4.4. *We have*

$$\begin{aligned} &u_\varepsilon^s(x) - u^s(x) \\ = &\int_{\partial D} (\varepsilon \cdot \nu)\, \mathrm{div}\left(\, -\mu_t \nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w(\cdot,x)\nu_t + u\nabla w(\cdot,x) + \lambda_t u w(\cdot,x)\nu_t\right)|_{t=0} ds + \mathcal{O}(||\varepsilon||^2), \end{aligned}$$

uniformly for x in some compact subset $K \subset \mathbb{R}^d \setminus \overline{D}$.

Proof. The proof relies on the Stokes formula and on a change of variable. We have by using the extension of fields which is described above and remarking that $\nu^0 = \nu$ and $\nu^1 = \nu_\varepsilon$,

$$\int_{\partial D_\varepsilon} \left\{ u\frac{\partial w}{\partial \nu_\varepsilon} - \mu_\varepsilon \nabla_{\Gamma_\varepsilon} u \cdot \nabla_{\Gamma_\varepsilon} w + \lambda_\varepsilon uw \right\} ds_\varepsilon - \int_{\partial D} \left\{ u\frac{\partial w}{\partial \nu} - \mu \nabla_\Gamma u \cdot \nabla_\Gamma w + \lambda uw \right\} ds$$
$$= \int_{D_\varepsilon \setminus \overline{D}} \operatorname{div}\left\{ u\nabla w - \mu_t (\nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w)\,\nu_t + \lambda_t uw\,\nu_t \right\} dx$$
$$= \int_{\partial D} \int_0^1 \operatorname{div}\left\{ u\nabla w - \mu_t (\nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w)\,\nu_t + \lambda_t uw\,\nu_t \right\} (\varepsilon \cdot \nu)\,dtds + \mathcal{O}(||\varepsilon||^2).$$

Here we have used the change of variable $(x_{\partial D}, t) \to x_{\partial D} + t\varepsilon(x_{\partial D})$ for $x_{\partial D} \in \partial D$ and $t \in [0,1]$, the determinant of the associated Jacobian matrix at first order being

$$J = (\varepsilon \cdot \nu) + t\,\mathcal{O}(||\varepsilon||^2),$$

as well as the fact that $(\varepsilon \cdot \nu) \geq 0$. Lastly, by using a first order approximation of the integrand as in [17], we obtain

$$\int_{\partial D_\varepsilon} \left\{ u\frac{\partial w}{\partial \nu_\varepsilon} - \mu_\varepsilon \nabla_{\Gamma_\varepsilon} u \cdot \nabla_{\Gamma_\varepsilon} w + \lambda_\varepsilon uw \right\} ds_\varepsilon - \int_{\partial D} \left\{ u\frac{\partial w}{\partial \nu} - \mu \nabla_\Gamma u \cdot \nabla_\Gamma w + \lambda uw \right\} ds$$
$$= \int_{\partial D} (\varepsilon \cdot \nu) \operatorname{div}\left\{ u\nabla w - \mu_t (\nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w)\,\nu_t + \lambda_t uw\,\nu_t \right\}|_{t=0} ds + \mathcal{O}(||\varepsilon||^2).$$

We complete the proof by using the boundary condition satisfied by u on ∂D and the result of lemma 4.3. □

The remainder of the section consists in computing the divergence term. In order to do that we need the following technical lemma, the proof of which is postponed in an appendix.

Lemma 4.5. *We have*

$$(\varepsilon \cdot \nu)(\nabla \lambda_t \cdot \nu_t)|_{t=0} = -(\nabla_\Gamma \lambda \cdot \varepsilon),$$
$$(\operatorname{div}\nu_t)|_{t=0} = \operatorname{div}_\Gamma \nu,$$

and by denoting $\varepsilon_\Gamma = \varepsilon - (\varepsilon \cdot \nu)\nu$,

$$(\varepsilon \cdot \nu)\nabla(\nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w) \cdot \nu_t|_{t=0} = -\varepsilon_\Gamma \cdot \nabla_\Gamma(\nabla_\Gamma u \cdot \nabla_\Gamma w)$$
$$+ \nabla_\Gamma(\varepsilon_\Gamma \cdot \nabla_\Gamma u + (\nabla u \cdot \nu)(\varepsilon \cdot \nu)) \cdot \nabla_\Gamma w + \nabla_\Gamma u \cdot \nabla_\Gamma(\nabla_\Gamma w \cdot \varepsilon_\Gamma + (\nabla w \cdot \nu)(\varepsilon \cdot \nu))$$
$$- \nabla_\Gamma u \cdot (\nabla \varepsilon + (\nabla \varepsilon)^T) \cdot \nabla_\Gamma w.$$

In order to handle reasonable expressions we split the computation of the divergence term in proposition 4.4 in two terms that we compute separately.

Proposition 4.6. *We have*

$$(\varepsilon \cdot \nu) \operatorname{div}(u\nabla w(\cdot,x) + \lambda_t uw(\cdot,x)\nu_t)|_{t=0} =$$
$$(\varepsilon \cdot \nu)\left(\nabla_\Gamma u \cdot \nabla_\Gamma w(\cdot,x) + M_\mu u\, M_\mu w(\cdot,x) - (k^2 + \lambda^2 - \lambda(\operatorname{div}_\Gamma \nu) uw(\cdot,x))\right)$$
$$- (\nabla_\Gamma \lambda \cdot \varepsilon) uw(\cdot,x),$$

where we have used the short notation $M_\mu \cdot = \operatorname{div}_\Gamma(\mu \nabla_\Gamma \cdot)$*, uniformly for* x *in some compact subset* $K \subset \mathbb{R}^d \setminus \overline{D}$.

Proof. We have

$$\operatorname{div}(u\nabla w + \lambda_t uw\nu_t) = \nabla u \cdot \nabla w + u\Delta w + uw(\nabla \lambda_t \cdot \nu_t) + \lambda_t w \nabla u \cdot \nu_t + \lambda_t u \nabla w \cdot \nu_t + \lambda_t uw(\operatorname{div}\nu_t).$$

By using the equation $\Delta w + k^2 w = 0$ and the decomposition of gradient into its normal and tangential parts, we obtain

$$\begin{aligned}\operatorname{div}(u\nabla w + \lambda_t uw\nu_t)|_{t=0} =& \nabla_\Gamma u \cdot \nabla_\Gamma w + (\nabla u\cdot\nu)(\nabla w\cdot\nu) - (k^2 - \lambda(\operatorname{div}\nu_t)|_{t=0})uw \\ &+ uw(\nabla\lambda_t\cdot\nu_t)|_{t=0} + \lambda(\nabla u\cdot\nu)w + \lambda u(\nabla w\cdot\nu).\end{aligned}$$

We can now replace $\nabla u\cdot\nu$ and $\nabla w\cdot\nu$ by $(-M_\mu u - \lambda u)$ and $(-M_\mu w - \lambda w)$ respectively, which leads to

$$\begin{aligned}\operatorname{div}(u\nabla w + \lambda_t uw\nu_t)|_{t=0} =& \nabla_\Gamma u\cdot\nabla_\Gamma w + M_\mu u\, M_\mu w - (k^2 + \lambda^2 - \lambda(\operatorname{div}\nu_t)|_{t=0})uw \\ &+ uw(\nabla\lambda_t\cdot\nu_t)|_{t=0}.\end{aligned}$$

We complete the proof by using lemma 4.5. □

Proposition 4.7. *We have*

$$\begin{aligned}&(\varepsilon\cdot\nu)\operatorname{div}\left(\mu_t\nabla_{\Gamma_t}u\cdot\nabla_{\Gamma_t}w(\cdot,x)\nu_t\right)|_{t=0} = \\ &-(\nabla_\Gamma\mu\cdot\varepsilon)(\nabla_\Gamma u\cdot\nabla_\Gamma w(\cdot,x)) + \mu(\varepsilon\cdot\nu)(\nabla_\Gamma u\cdot\nabla_\Gamma w(\cdot,x))(\operatorname{div}_\Gamma\nu) \\ &+\mu(\varepsilon\cdot\nu)\nabla_\Gamma(\nabla u\cdot\nu)\cdot\nabla_\Gamma w(\cdot,x) + \mu(\varepsilon\cdot\nu)\nabla_\Gamma u\cdot\nabla_\Gamma(\nabla w\cdot\nu)(\cdot,x) \\ &+\mu(\nabla u\cdot\nu)\nabla_\Gamma(\varepsilon\cdot\nu)\cdot\nabla_\Gamma w(\cdot,x) + \mu(\nabla w\cdot\nu)(\cdot,x)\nabla_\Gamma(\varepsilon\cdot\nu)\cdot\nabla_\Gamma u \\ &- 2\mu(\varepsilon\cdot\nu)(\nabla_\Gamma u\cdot\nabla_\Gamma\nu\cdot\nabla_\Gamma w(\cdot,x)),\end{aligned}$$

uniformly for x in some compact subset $K \subset \mathbb{R}^d\setminus\overline{D}$.

Proof. We have

$$\begin{aligned}&\operatorname{div}\left(\mu_t\nabla_{\Gamma_t}u\cdot\nabla_{\Gamma_t}w(\cdot,x)\nu_t\right)|_{t=0} \\ &= (\nabla\mu_t\cdot\nu_t)|_{t=0}(\nabla_\Gamma u\cdot\nabla_\Gamma w) + \mu(\nabla_\Gamma u\cdot\nabla_\Gamma w)(\operatorname{div}\nu_t)|_{t=0} + \mu\nabla(\nabla_{\Gamma_t}u\cdot\nabla_{\Gamma_t}w)\cdot\nu_t|_{t=0}.\end{aligned}$$

By using lemma 4.5, we obtain that

$$\begin{aligned}(\varepsilon\cdot\nu)\operatorname{div}\left(\mu_t\nabla_{\Gamma_t}u\cdot\nabla_{\Gamma_t}w(\cdot,x)\nu_t\right)|_{t=0} =& -(\nabla_\Gamma\mu\cdot\varepsilon)(\nabla_\Gamma u\cdot\nabla_\Gamma w) \\ &+ \mu(\varepsilon\cdot\nu)(\nabla_\Gamma u\cdot\nabla_\Gamma w)(\operatorname{div}_\Gamma\nu) - \mu\,\varepsilon_\Gamma\cdot\nabla_\Gamma(\nabla_\Gamma u\cdot\nabla_\Gamma w) \\ +\mu\nabla_\Gamma(\varepsilon_\Gamma\cdot\nabla_\Gamma u + (\nabla u\cdot\nu)(\varepsilon\cdot\nu))\cdot\nabla_\Gamma w &+ \mu\nabla_\Gamma u\cdot\nabla_\Gamma(\nabla_\Gamma w\cdot\varepsilon_\Gamma + (\nabla w\cdot\nu)(\varepsilon\cdot\nu)) \\ &- \mu\nabla_\Gamma u\cdot(\nabla\varepsilon + (\nabla\varepsilon)^T)\cdot\nabla_\Gamma w.\end{aligned}$$

In the following, for a surface vector a_Γ, we denote by $\nabla_\Gamma a_\Gamma$ the $(d-1)\times(d-1)$ tensor defined by

$$\nabla_\Gamma a_\Gamma\cdot e_j = \frac{\partial a_\Gamma}{\partial\xi_j}, \quad j = 1,..,d-1.$$

The third line of the equation above, using $\nabla_\Gamma(a_\Gamma\cdot b_\Gamma) = (\nabla_\Gamma a_\Gamma)^T\cdot b_\Gamma + a_\Gamma\cdot\nabla_\Gamma b_\Gamma$, can be expressed as

$$\begin{aligned}\mu\nabla_\Gamma(\varepsilon_\Gamma\cdot\nabla_\Gamma u +& (\nabla u\cdot\nu)(\varepsilon\cdot\nu))\cdot\nabla_\Gamma w + \mu\nabla_\Gamma u\cdot\nabla_\Gamma(\nabla_\Gamma w\cdot\varepsilon_\Gamma + (\nabla w\cdot\nu)(\varepsilon\cdot\nu)) \\ =& \mu\,\varepsilon_\Gamma\cdot\nabla_\Gamma(\nabla_\Gamma u)\cdot\nabla_\Gamma w + \mu\nabla_\Gamma u\cdot\nabla_\Gamma\varepsilon_\Gamma\cdot\nabla_\Gamma w \\ &+ \mu(\varepsilon\cdot\nu)\nabla_\Gamma(\nabla u\cdot\nu)\cdot\nabla_\Gamma w + \mu(\nabla u\cdot\nu)\nabla_\Gamma(\varepsilon\cdot\nu)\cdot\nabla_\Gamma w \\ &+ \mu\nabla_\Gamma u\cdot\nabla_\Gamma(\nabla_\Gamma w)\cdot\varepsilon_\Gamma + \mu\nabla_\Gamma u\cdot(\nabla_\Gamma\varepsilon_\Gamma)^T\cdot\nabla_\Gamma w \\ &- \mu(\varepsilon\cdot\nu)\nabla_\Gamma u\cdot\nabla_\Gamma(\nabla w\cdot\nu) + \mu(\nabla w\cdot\nu)\nabla_\Gamma u\cdot\nabla_\Gamma(\varepsilon\cdot\nu).\end{aligned}$$

Gathering the two above expressions and using the fact that

$$\varepsilon_\Gamma\cdot\nabla_\Gamma(\nabla_\Gamma u\cdot\nabla_\Gamma w) = \varepsilon_\Gamma\cdot\nabla_\Gamma(\nabla_\Gamma u)\cdot\nabla_\Gamma w + \nabla_\Gamma u\cdot\nabla_\Gamma(\nabla_\Gamma w)\cdot\varepsilon_\Gamma,$$

we obtain that

$$
\begin{aligned}
(\varepsilon\cdot\nu)\mathrm{div}\,(\mu_t\nabla_{\Gamma_t}u\cdot\nabla_{\Gamma_t}w(\cdot,x)\nu_t)\,|_{t=0} &= -(\nabla_\Gamma\mu\cdot\varepsilon)(\nabla_\Gamma u\cdot\nabla_\Gamma w)+\mu(\varepsilon\cdot\nu)(\nabla_\Gamma u\cdot\nabla_\Gamma w)(\mathrm{div}_\Gamma\nu)\\
&+\mu(\varepsilon\cdot\nu)\nabla_\Gamma(\nabla u\cdot\nu)\cdot\nabla_\Gamma w+\mu(\nabla u\cdot\nu)\nabla_\Gamma(\varepsilon\cdot\nu)\cdot\nabla_\Gamma w\\
&+\mu(\varepsilon\cdot\nu)\nabla_\Gamma u\cdot\nabla_\Gamma(\nabla w\cdot\nu)+\mu(\nabla w\cdot\nu)\nabla_\Gamma u\cdot\nabla_\Gamma(\varepsilon\cdot\nu)\\
&-\mu\nabla_\Gamma u\cdot(\nabla_\Gamma\varepsilon+(\nabla_\Gamma\varepsilon)^T)\cdot\nabla_\Gamma w+\mu\nabla_\Gamma u\cdot(\nabla_\Gamma\varepsilon_\Gamma+(\nabla_\Gamma\varepsilon_\Gamma)^T)\cdot\nabla_\Gamma w.
\end{aligned}
$$

Now we need to evaluate $\nabla_\Gamma\varepsilon-\nabla_\Gamma\varepsilon_\Gamma$. Since $\varepsilon-\varepsilon_\Gamma=(\varepsilon\cdot\nu)\nu$, we have

$$\nabla_\Gamma\varepsilon-\nabla_\Gamma\varepsilon_\Gamma=\nabla_\Gamma((\varepsilon\cdot\nu)\nu)=(\varepsilon\cdot\nu)\nabla_\Gamma\nu+\nu\otimes\nabla_\Gamma(\varepsilon\cdot\nu),$$

where for a surface field a, we denote by $\nu\otimes\nabla_\Gamma a$ the $d\times d$ tensor M defined by

$$M\cdot e_j=\frac{\partial a}{\partial\xi_j}\nu\quad j=1,..,d-1,\quad M\cdot\nu=0.$$

This implies that

$$\nabla_\Gamma u\cdot\nabla_\Gamma\varepsilon_\Gamma\cdot\nabla_\Gamma w-\nabla_\Gamma u\cdot\nabla_\Gamma\varepsilon\cdot\nabla_\Gamma w=-(\varepsilon\cdot\nu)\nabla_\Gamma u\cdot\nabla_\Gamma\nu\cdot\nabla_\Gamma w.$$

Since the tensor $\nabla_\Gamma\nu$ is symmetric (see for example [19, theorem 2.5.18]), we also obtain

$$\nabla_\Gamma u\cdot(\nabla_\Gamma\varepsilon_\Gamma)^T\cdot\nabla_\Gamma w-\nabla_\Gamma u\cdot(\nabla_\Gamma\varepsilon)^T\cdot\nabla_\Gamma w=-(\varepsilon\cdot\nu)\nabla_\Gamma u\cdot\nabla_\Gamma\nu\cdot\nabla_\Gamma w.$$

We finally arrive at

$$
\begin{aligned}
&(\varepsilon\cdot\nu)\mathrm{div}\,(\mu_t\nabla_{\Gamma_t}u\cdot\nabla_{\Gamma_t}w(\cdot,x)\nu_t)\,|_{t=0}\\
&=-(\nabla_\Gamma\mu\cdot\varepsilon)(\nabla_\Gamma u\cdot\nabla_\Gamma w)+\mu(\varepsilon\cdot\nu)(\nabla_\Gamma u\cdot\nabla_\Gamma w)(\mathrm{div}_\Gamma\nu)\\
&+\mu(\varepsilon\cdot\nu)\nabla_\Gamma(\nabla u\cdot\nu)\cdot\nabla_\Gamma w+\mu(\nabla u\cdot\nu)\nabla_\Gamma(\varepsilon\cdot\nu)\cdot\nabla_\Gamma w\\
&+\mu(\varepsilon\cdot\nu)\nabla_\Gamma u\cdot\nabla_\Gamma(\nabla w\cdot\nu)+\mu(\nabla w\cdot\nu)\nabla_\Gamma u\cdot\nabla_\Gamma(\varepsilon\cdot\nu)-2\mu(\varepsilon\cdot\nu)\nabla_\Gamma u\cdot\nabla_\Gamma\nu\cdot\nabla_\Gamma w,
\end{aligned}
$$

which completes the proof. □

Gathering propositions 4.6 and 4.7, we establish the main theorem of this section, that is

Theorem 4.8. *The discrepancy between the scattered fields due to obstacle D_ε and the obstacle D is*

$$u^s_\varepsilon(x)-u^s(x)=-\int_{\partial D}B_\varepsilon u(y)w(y,x)\,ds(y)+\mathcal{O}(||\varepsilon||^2),$$

uniformly for x in some compact subset $K\subset\mathbb{R}^d\setminus\overline{D}$, where $w(\cdot,x)$ is the solution of problem (1) associated with $u^i=\Phi(\cdot,x)$, and the surface operator B_ε is defined by

$$
\begin{aligned}
B_\varepsilon u=(\varepsilon\cdot\nu)(k^2-2H\lambda)u+\mathrm{div}_\Gamma\left((Id+2\mu(R-H\,Id))(\varepsilon\cdot\nu)\nabla_\Gamma u\right)+L_{\lambda,\mu}\left((\varepsilon\cdot\nu)L_{\lambda,\mu}u\right)\\
+(\nabla_\Gamma\lambda\cdot\varepsilon_\Gamma)u+\mathrm{div}_\Gamma\left((\nabla_\Gamma\mu\cdot\varepsilon_\Gamma)\nabla_\Gamma u\right),
\end{aligned}
$$

with $2H:=\mathrm{div}_\Gamma\nu$, $R:=\nabla_\Gamma\nu$ and $L_{\lambda,\mu}\cdot:=\mathrm{div}_\Gamma(\mu\nabla_\Gamma\cdot)+\lambda\cdot$.

Proof. From propositions 4.6 and 4.7 it follows that

$$
\begin{aligned}
&(\varepsilon\cdot\nu)\,\mathrm{div}\,(-\mu_t\nabla_{\Gamma_t}u\cdot\nabla_{\Gamma_t}w\nu_t+u\nabla w+\lambda_t uw\nu_t)\,|_{t=0}\\
&=(\varepsilon\cdot\nu)\nabla_\Gamma u\cdot\nabla_\Gamma w+(\varepsilon\cdot\nu)M_\mu u\,M_\mu w-(\varepsilon\cdot\nu)\left(k^2+\lambda^2-\lambda(\mathrm{div}_\Gamma\nu)\right)uw\\
&-(\nabla_\Gamma\lambda\cdot\varepsilon)uw+(\nabla_\Gamma\mu\cdot\varepsilon)\nabla_\Gamma u\cdot\nabla_\Gamma w-\mu(\mathrm{div}_\Gamma\nu)(\varepsilon\cdot\nu)\nabla_\Gamma u\cdot\nabla_\Gamma w\\
&-\mu(\varepsilon\cdot\nu)\nabla_\Gamma(\nabla u\cdot\nu)\cdot\nabla_\Gamma w-\mu(\varepsilon\cdot\nu)\nabla_\Gamma u\cdot\nabla_\Gamma(\nabla w\cdot\nu)\\
&-\mu(\nabla u\cdot\nu)\nabla_\Gamma(\varepsilon\cdot\nu)\cdot\nabla_\Gamma w-\mu(\nabla w\cdot\nu)\nabla_\Gamma(\varepsilon\cdot\nu)\cdot\nabla_\Gamma u\\
&+2\mu(\varepsilon\cdot\nu)(\nabla_\Gamma u\cdot\nabla_\Gamma\nu\cdot\nabla_\Gamma w).
\end{aligned}
$$

Using the boundary condition for u and $w(\cdot,x)$ on ∂D, we obtain

$$\begin{aligned}
&(\varepsilon\cdot\nu)\,\mathrm{div}\,(-\mu_t\nabla_{\Gamma_t}u\cdot\nabla_{\Gamma_t}w\nu_t+u\nabla w+\lambda_t uw\nu_t)\,|_{t=0}\\
&=-(\varepsilon\cdot\nu)(k^2-2H\lambda)uw+(\varepsilon\cdot\nu)\nabla_\Gamma u\cdot(Id+2\mu(R-H\,Id))\cdot\nabla_\Gamma w\\
&\quad-(\nabla_\Gamma\lambda\cdot\varepsilon)uw+(\nabla_\Gamma\mu\cdot\varepsilon)\nabla_\Gamma u\cdot\nabla_\Gamma w\\
&\quad+(\varepsilon\cdot\nu)M_\mu u\,M_\mu w-(\varepsilon\cdot\nu)\lambda^2uw\\
&+\mu(\varepsilon\cdot\nu)\nabla_\Gamma(M_\mu u+\lambda u)\cdot\nabla_\Gamma w+\mu(\varepsilon\cdot\nu)\nabla_\Gamma u\cdot\nabla_\Gamma(M_\mu w+\lambda w)\\
&+\mu(M_\mu u+\lambda u)\nabla_\Gamma(\varepsilon\cdot\nu)\cdot\nabla_\Gamma w+\mu(M_\mu w+\lambda w)\nabla_\Gamma(\varepsilon\cdot\nu)\cdot\nabla_\Gamma u.
\end{aligned}$$

The three last lines of the above expression may be written as

$$\begin{aligned}
&(\varepsilon\cdot\nu)(L_{\lambda,\mu}u)(L_{\lambda,\mu}w)-\lambda(\varepsilon\cdot\nu)w(L_{\lambda,\mu}u)-\lambda(\varepsilon\cdot\nu)u(L_{\lambda,\mu}w)\\
&\quad+\mu\nabla_\Gamma((\varepsilon\cdot\nu)L_{\lambda,\mu}u)\cdot\nabla_\Gamma w+\mu\nabla_\Gamma((\varepsilon\cdot\nu)L_{\lambda,\mu}w)\cdot\nabla_\Gamma u.
\end{aligned}$$

The integral over ∂D of the above expression is, after integration by parts and simplification,

$$-\int_{\partial D}(\varepsilon\cdot\nu)(L_{\lambda,\mu}u)(L_{\lambda,\mu}w)\,ds.$$

To complete the proof, we simply use proposition 4.4 and integration by parts. □

Corollary 4.9. *We assume that ∂D, λ and μ are analytic, and (λ,μ) satisfy assumption 2.1. Then the far–field operator $T:\ (\lambda,\mu,\partial D)\to u^\infty$ is differentiable with respect to ∂D according to definition 4.1 and its Fréchet derivative is given by*

$$T'_{\lambda,\mu}(\partial D)\cdot\varepsilon=v_\varepsilon^\infty,$$

where v_ε^∞ is the far–field associated with the outgoing solution v_ε^s of the Helmholtz equation outside D which satisfies the GIBC condition

$$\frac{\partial v_\varepsilon^s}{\partial\nu}+\mathrm{div}(\mu\nabla_\Gamma v_\varepsilon^s)+\lambda v_\varepsilon^s=B_\varepsilon u\quad\text{on }\partial D,$$

where $B_\varepsilon u$ is given by theorem 4.8.

Proof. Proceeding as in [12], we can drop the assumption $\overline{D}\subset D_\varepsilon$ provided we assume that ∂D, λ and μ be analytic. The result then follows from theorem 4.8 and an integral representation for the scattered field $v_\varepsilon^s=u_\varepsilon^s-u^s$. □

Remark 2. *With classical impedance boundary condition, that is $\mu=0$, we retrieve the result of [12, theorem 2.5]. Let us also remark that in this case the surface operator B_ε in theorem 4.8 is a second-order operator, while it becomes a fourth-order differential operator when $\mu\neq0$.*

Remark 3. *Classically, the shape derivative only involves the normal part $(\varepsilon\cdot\nu)$ of field ε (see for example [14, proposition 5.9.1]). In view of theorem 4.8, the expression of B_ε may be split in two parts: one part involves the normal component $(\varepsilon\cdot\nu)$, the second part involves the tangential component ε_Γ. This is due to the fact that the impedances λ and μ are surface functions.*

5 An optimization technique to solve the inverse problem

This section is dedicated to the effective reconstruction of both the obstacle ∂D and the functional impedances (λ,μ) from the observed far–fields $u^\infty_{\mathrm{obs},j}:=T_j(\lambda_0,\mu_0,\partial D_0)\in L^2(S^{d-1})$ associated with N given plane wave directions, where T_j refers to incident direction $\hat d_j$. We shall minimize the cost function

$$F(\lambda,\mu,\partial D)=\frac12\sum_{j=1}^N\|T_j(\lambda,\mu,\partial D)-u^\infty_{\mathrm{obs},j}\|^2_{L^2(S^{d-1})}\qquad(14)$$

with respect to ∂D and (λ, μ) by using a steepest descent method.
To do so, we first compute the Fréchet derivative of T with respect to (λ, μ) for fixed D. We have the following theorem.

Theorem 5.1. *We assume that D is Lipschitz continuous. Then for $(\lambda, \mu) \in (L^\infty(\partial D))^2$ which satisfy assumption 2.1, the function $T : (\lambda, \mu, \partial D) \to u^\infty$ is Fréchet differentiable with respect to (λ, μ) and its Fréchet derivative is given by*

$$T'_{\partial D}(\lambda, \mu) \cdot (h, l) = v^\infty_{h,l}(\hat{x}) := \langle p(\cdot, \hat{x}), \mathrm{div}_\Gamma(l \nabla_\Gamma u) + h\, u \rangle_{H^1(\Gamma), H^{-1}(\Gamma)}, \quad \forall \hat{x} \in S^{d-1},$$

where

- *u is the solution of the problem (1),*
- *$p(\cdot, \hat{x}) = \Phi^\infty(\cdot, \hat{x}) + p^s(\cdot, \hat{x})$ is the solution of (1) with u^i replaced by $\Phi^\infty(\cdot, \hat{x})$.*

Proof. The proof of this result can be found in [4]. □

The Fréchet derivative of T with respect to ∂D for fixed (λ, μ) is given by theorem 4.8 and its corollary 4.9. With the help of corollary 4.9 and theorem 5.1, and in the case ∂D, λ and μ are analytic, we obtain the following expressions for the partial derivatives of the cost function F with respect to (λ, μ) and ∂D respectively.

$$F'_{\partial D}(\lambda, \mu) \cdot (h, l) = \sum_{j=1}^{N} \Re e \left(\int_{\partial D} G_j(\mathrm{div}_\Gamma(l \nabla_\Gamma u_j) + h u_j)\, dy \right), \tag{15}$$

$$F'_{\lambda,\mu}(\partial D) \cdot \varepsilon = -\sum_{j=1}^{N} \Re e \left(\int_{\partial D} G_j(B_\varepsilon u_j)\, dy \right) \tag{16}$$

where

- u_j is the solution of the problem (1) wich is associated to plane wave direction $\hat{d}_j$,
- $G_j = G^i_j + G^s_j$ is the solution of problem (1) with u^i replaced by

$$G^i_j(y) := \int_{S^{d-1}} \Phi^\infty(y, \hat{x}) \overline{(T_j(\lambda, \mu, \partial D) - u^\infty_{\mathrm{obs},j})}\, d\hat{x}.$$

In the numerical part of the paper we restrict ourselves to the two dimensional setting, that is $d = 2$. The minimization of the cost function F alternatively with respect to D, λ and μ relies on the directions of steepest descent given by (15) and (16). The minimization with respect to (λ, μ) is already exposed in [4], so that we only describe the minimization with respect to D. It is essential to remark from theorem 4.8 that the partial derivative with respect to D depends only on the values of ε on ∂D. With the decomposition $\varepsilon = \varepsilon_\tau \tau + \varepsilon_\nu \nu$, where τ is the tangential unit vector, we formally compute $(\varepsilon_\tau, \varepsilon_\nu)$ on ∂D such that

$$\varepsilon_\tau \tau + \varepsilon_\nu \nu = -\alpha F'_{\lambda,\mu}(\partial D),$$

where $\alpha > 0$ is the descent coefficient. In order to decrease the oscillations of the updated boundary, similarly to [4] we use a H^1-regularization, that is we search ε_τ and ε_ν in $H^1(\partial D)$ such that for all $\phi \in H^1(\partial D)$,

$$\eta_\tau \int_{\partial D} \nabla_\Gamma \varepsilon_\tau \cdot \nabla_\Gamma \phi\, ds + \int_{\partial D} \varepsilon_\tau \phi\, ds = -\alpha F'_{\lambda,\mu}(\partial D) \cdot (\phi\, \tau), \tag{17}$$

$$\eta_\nu \int_{\partial D} \nabla_\Gamma \varepsilon_\nu \cdot \nabla_\Gamma \phi\, ds + \int_{\partial D} \varepsilon_\nu \phi\, ds = -\alpha F'_{\lambda,\mu}(\partial D) \cdot (\phi\, \nu), \tag{18}$$

where η_τ, $\eta_\nu > 0$ are regularization coefficients, while $F'_{\lambda,\mu}(\partial D)\cdot(\phi\,\tau)$ and $F'_{\lambda,\mu}(\partial D)\cdot(\phi\,\nu)$ have from theorem 4.8 the simplified expressions

$$F'_{\lambda,\mu}(\partial D)\cdot(\phi\,\tau) = -\sum_{j=1}^{N}\Re e\left\{\int_{\partial D}\left(\frac{\partial\lambda}{\partial\tau}u_jG_j - \frac{\partial\mu}{\partial\tau}\frac{\partial u_j}{\partial\tau}\frac{\partial G_j}{\partial\tau}\right)\phi\,ds\right\},$$

$$F'_{\lambda,\mu}(\partial D)\cdot(\phi\,\nu) =$$

$$-\sum_{j=1}^{N}\Re e\left\{\int_{\partial D}\left((k^2-2H\lambda)u_jG_j - (1+2H\mu)\frac{\partial u_j}{\partial\tau}\frac{\partial G_j}{\partial\tau} + (L_{\lambda,\mu}u_j)(L_{\lambda,\mu}G_j)\right)\phi\,ds\right\}.$$

The updated obstacle D_ε is then obtained by moving the mesh points x of ∂D to the points x_ε defined by $x_\varepsilon = x + (\varepsilon_\tau\tau + \varepsilon_\nu\nu)(x)$, while the extended impedances on ∂D_ε are defined, following (9), by $\lambda_\varepsilon(x_\varepsilon) = \lambda(x)$ and $\mu_\varepsilon(x_\varepsilon) = \mu(x)$. The points x_ε enable us to define a new domain D_ε, and we have to remesh the complementary domain $\Omega_R^\varepsilon = B_R \setminus \overline{D_\varepsilon}$ to solve the next forward problems. The descent coefficient α and the regularization parameters η_τ, η_ν are determined as follows: α is increased (resp. decreased) and η_τ, η_ν are decreased (resp. increased) as soon as the cost function decreases (resp. increases). The algorithm stops as soon as α is too small. With the help of the relative cost function, namely

$$\text{Error} := \frac{1}{N}\sum_{j=1}^{N}\frac{\|T_j(\lambda,\mu,\partial D) - u^\infty_{\text{obs},j}\|_{L^2(S^1)}}{\|u^\infty_{\text{obs},j}\|_{L^2(S^1)}}, \tag{19}$$

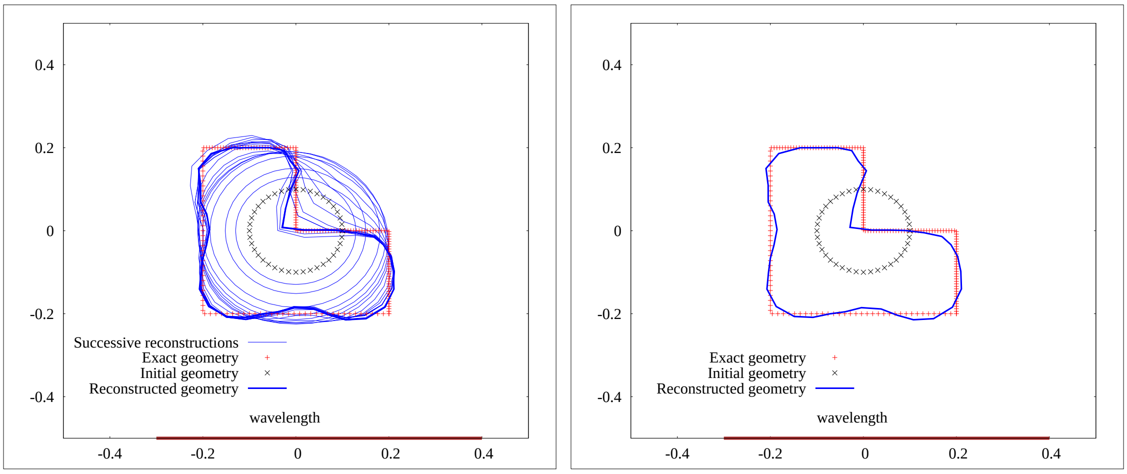
we are able to determine if the computed $(\lambda, \mu, \partial D)$ corresponds to a global or a local minimum: in the first case Error is approximately equal to the amplitude of noise while in the second case it is much larger.

6 Some numerical results

In order to handle dimensionless impedances, we replace λ by $k\lambda$ and μ by μ/k in the boundary condition of problem (2) without changing the notations. Problem (2) is solved by using a finite element method based on the variational formulation associated with problem (2) and which is introduced in [4], more precisely we have used classical Lagrange finite elements. The variational formulations (17) (18) as well as those used to update the impedances λ and μ (see [4]) are solved by using the same finite element basis. All computations were performed with the help of the software FreeFem++ [24]. We obtain some artificial data with forward computations for some given data $(\lambda_0, \mu_0, \partial D_0)$. The resulting far–fields $u^\infty_{\text{obs},j}$, $j = 1, N$ are then contaminated by some Gaussian noise of various amplitude. More precisely, for each Fourier coefficient of the far–field we compute a Gaussian noise with normal distribution. Such a perturbation is multiplied by a constant which is calibrated in order to obtain a global relative L^2 error of prescribed amplitude: 1% or 5%.

6.1 Reconstruction of an obstacle with known impedances

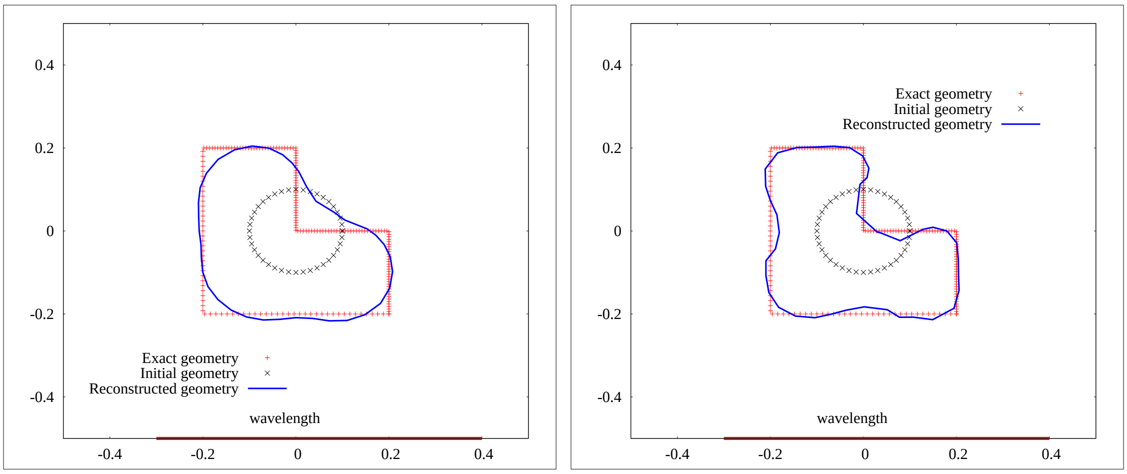
First we try to reconstruct a $L-$ shaped obstacle D_0 with known impedances (λ_0, μ_0). The result is shown on figure 1 in the case of 1% noise by using only two incident waves, namely $N = 2$. The results are shown on figure 2 in the case of 5% noise and $N = 2$, as well as 5% noise and $N = 8$, respectively. This enables us to test the influence of the amplitude of noise as well as the influence of the number of incident waves. In order to evaluate the impact of the initial guess on the quality of the reconstruction, we consider another initial guess which is farther from the true obstacle than the first one, in the presence of 5% noise. The result is very bad with only two incident waves, and becomes much better with eight incident waves, as shown on figure 3. The figure 4 illustrates our numerical approach using a finite element method in a bounded domain and a remeshing process. In the remainder of the numerical section all reconstructions will be based on eight incident waves.



(a) Intermediate steps

(b) 1% noise, $N = 2$, Final Error = 2.1%

Figure 1: Case of known impedances and good initial guess



(a) 5% noise, $N = 2$, Final Error = 6.3%

(b) 5% noise, $N = 8$, Final Error = 5.0%

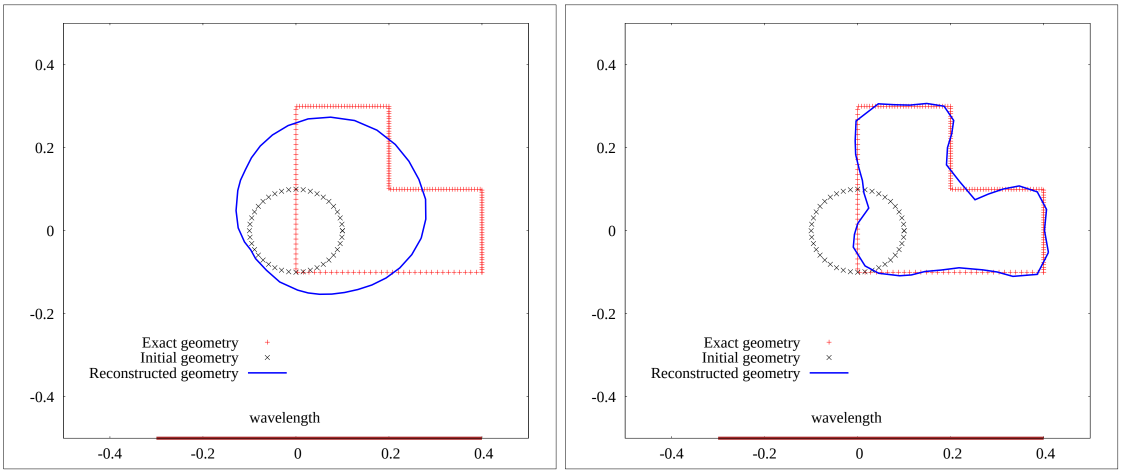
Figure 2: Case of known impedances and good initial guess, influence of noise and of N

6.2 Reconstruction of the geometry and constant impedances

Secondly we assume that both the obstacle D_0 and the impedances $(\lambda_0, \mu_0) = (0.5i, 2)$ are unknown, but these impedances are constants. Starting from $(i, 1.5)$ as initial guess for (λ, μ), the retrieved impedances are $(\lambda, \mu) = (0.49i, 1.99)$ for 1% noise and $(\lambda, \mu) = (0.51i, 1.93)$ for 5% noise, while the corresponding retrieved obstacles are shown on figure 5.

6.3 Reconstruction of the geometry and functional impedances

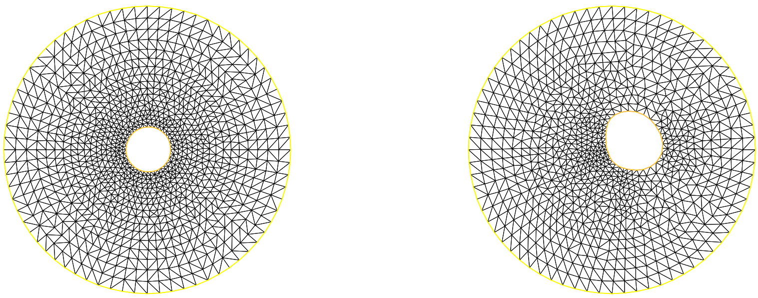
In order to emphasize the role played by the tangential part of the mapping ε in the optimization of the cost function F for functional impedances (see remark 3), we first consider a very academic case. We try to reconstruct a circle D_0 of radius $R_0 = 0.3$ and an impedance $\lambda_0(\theta) = 0.5(1+\sin^2(\theta+\frac{\pi}{6}))$, where θ is the polar angle, starting from an initial circle of same center and radius 0.2 and from the initial impedance $\lambda(\theta) = 0.5(1 + \sin^2(\theta))$. Compared to the true obstacle, the initial guess is hence a smaller and rotated circle. Here $\mu = 0$ for sake of simplicity. Amplitude of noise is 5% and we use eight incident waves. As can be seen on figure 6, the obstacle D_0 and the impedance



(a) 5% noise, $N = 2$, Final Error = 25.2%

(b) 5% noise, $N = 8$, Final Error = 5.8%

Figure 3: Case of known impedances and bad initial guess, increasing number of incident waves



(a) Initial obstacle

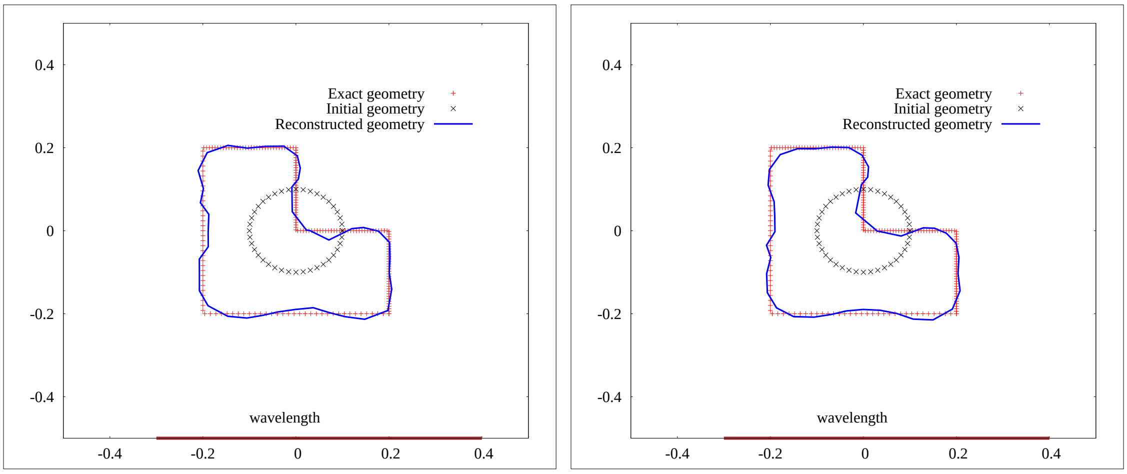
(b) After 5 iterations

Figure 4: Finite element method and remeshing (final reconstruction given in figure 3)

λ_0 are quite well reconstructed even if we use only the gradient iterations on the geometry (we do not use the gradient iterations on the impedance).
We complete the numerical section with a more complicated example. The aim is to retrieve the obstacle D_0 defined with polar coordinates (r, θ) by $r = 0.3 + 0.08\cos(3\theta)$, as well as the impedances $\lambda_0 = 0.5(1 + \sin^2\theta)i$ and $\mu_0 = 0.5(1 + \cos^2\theta)$, assuming that both the real part of λ and the imaginary part of μ are 0, in the presence of 5% noise with eight incident waves. Note that in this case the obstacle is star-shaped, which is not necessary to apply our optimization process, but it enables us to compare the retrieved and the exact impedances in a simple way. The result are presented on figure 7 and show a good accuracy.

Appendix

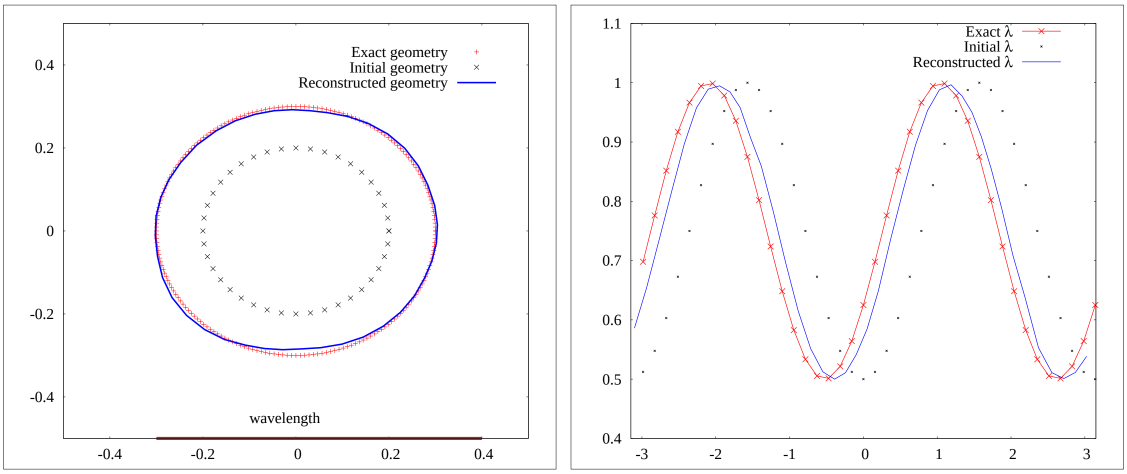
We give below the proof of lemma 4.5. In order to obtain this lemma, we consider the local basis (e^t_j, e^t_d), $j = 1, d-1$, where vectors e^t_j are defined by (11), while $e^t_d = \varepsilon$. We can hence define the associated covariant basis (f^i_t), $i = 1, d$. Note that $f^i_t \neq e^i_t$ $(i = 1, d-1)$, where covariant vectors e^i_t are defined by (12). We begin with the proof of the first part of lemma 4.5. We have, denoting



(a) 1% of noise, Final Error = 1.4%

(b) 5% of noise, Final Error = 5.8%

Figure 5: Case of constant but unknown impedances



(a) Reconstruction of D

(b) Reconstruction of $\mathcal{I}m\lambda$

Figure 6: On the use of the tangential component of mapping ε

$\tilde{\lambda} = \lambda \circ \phi$

$$\nabla \lambda_t = \sum_{i=1}^{d-1} \frac{\partial \tilde{\lambda}}{\partial \xi_i} f_t^i + \frac{\partial \tilde{\lambda}}{\partial t} f_t^d.$$

By the definition of $\tilde{\lambda}$, we have $\partial \tilde{\lambda}/\partial t = 0$. We hence have, with $f^i := f_0^i$,

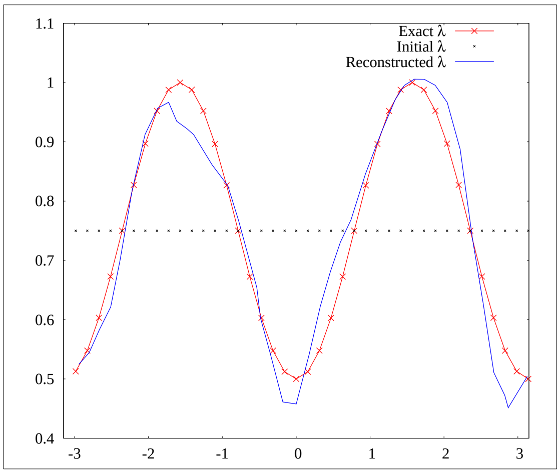
$$(\nabla \lambda_t)|_{t=0} = \sum_{i=1}^{d-1} \frac{\partial \tilde{\lambda}}{\partial \xi_i} f^i. \tag{20}$$

It remains to compute the covariant vectors f^i for $i = 1, d-1$. In this view we search f^1 in the form

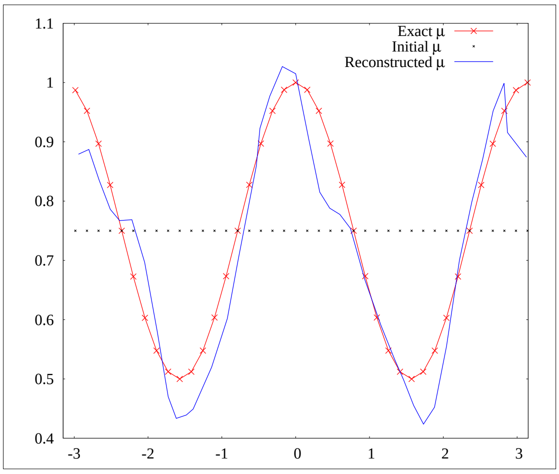
$$f^1 = \sum_{i=1}^{d-1} \beta_i e^i + \alpha \nu.$$

The coefficients α, β_i are uniquely defined by

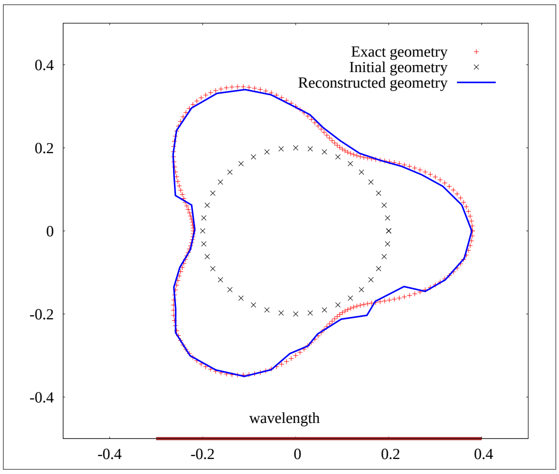
$$f^1 \cdot e_1 = 1, \quad f^1 \cdot e_j = 0, \quad j = 2, d-1, \quad f^1 \cdot \varepsilon = 0.$$



(a) Reconstruction of $\mathcal{I}m\lambda$



(b) Reconstruction of $\mathcal{R}e\mu$



(c) Reconstruction of D

Figure 7: Case of unknown geometry and unknown functional impedances

This implies that

$$\beta_1 = 1, \quad \beta_j = 0, \quad j = 2, d-1, \quad \alpha(\nu \cdot \varepsilon) = -e^1 \cdot \varepsilon.$$

As a conclusion, we have

$$f^1 = e^1 - \frac{1}{\nu \cdot \varepsilon}(e^1 \cdot \varepsilon)\nu.$$

We obtain a symmetric expression for f^i, $i = 2, d-1$ and coming back to (20), we obtain

$$(\nabla \lambda_t \cdot \nu_t)|_{t=0} = (\nabla \lambda_t)|_{t=0} \cdot \nu = -\sum_{i=1}^{d-1} \frac{1}{\nu \cdot \varepsilon}(e^i \cdot \varepsilon)\frac{\partial \tilde{\lambda}}{\partial \xi_i},$$

and lastly,

$$(\nu \cdot \varepsilon)(\nabla \lambda_t \cdot \nu_t)|_{t=0} = -(\nabla_\Gamma \lambda \cdot \varepsilon),$$

which completes the proof of the first statement of lemma 4.5.
Now let us give the proof of the second statement of lemma 4.5. In this view we also need an

expression of the covariant vector f^d. We again search f^d in the form

$$f^d = \sum_{i=1}^{d-1} \beta_i e^i + \alpha\nu.$$

The coefficients α, β_i are now uniquely defined by

$$f^d \cdot e_i = 0, \quad i = 1, d-1, \quad f^d \cdot \varepsilon = 1.$$

After simple calculations, we obtain

$$f^d = \frac{1}{\nu \cdot \varepsilon}\nu.$$

We have

$$\mathrm{div}\nu_t = \sum_{i=1}^{d-1} \frac{\partial \nu_t}{\partial \xi_i} \cdot f_t^i + \frac{\partial \nu_t}{\partial t} \cdot f_t^d.$$

By differentiation of $|\nu_t|^2 = 1$ with respect to ξ_i and t, we obtain

$$\frac{\partial \nu_t}{\partial \xi_i}|_{t=0}.\nu = 0, \quad i = 1, d-1, \quad \frac{\partial \nu_t}{\partial t}|_{t=0}.\nu = 0,$$

hence

$$\left(\sum_{i=1}^{d-1} \frac{\partial \nu_t}{\partial \xi_i} \cdot f_t^i\right)|_{t=0} = \sum_{i=1}^{d-1} \frac{\partial \nu}{\partial \xi_i} \cdot e^i = \mathrm{div}_\Gamma \nu,$$

and we obtain the second thesis of lemma 4.5.
Lastly, let us give the proof of the third statement of lemma 4.5. Let us denote

$$G = \nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w$$

and $\tilde{G}_t = G \circ \phi_t$. Given the definition of surface gradient 13, we have

$$\nabla G = \sum_{i=1}^{d-1} \frac{\partial \tilde{G}_t}{\partial \xi_i} f_t^i + \frac{\partial \tilde{G}_t}{\partial t} f_t^d.$$

By using the expressions obtained above for the covariant vectors f_t^i, $i = 1, d$, we obtain

$$(\nu \cdot \varepsilon)(\nabla G \cdot \nu_t)|_{t=0} = -\sum_{i=1}^{d-1} (e^i \cdot \varepsilon) \frac{\partial \tilde{G}_t}{\partial \xi_i}|_{t=0} + \frac{\partial \tilde{G}_t}{\partial t}|_{t=0},$$

that is

$$(\nu \cdot \varepsilon)(\nabla G \cdot \nu_t)|_{t=0} = -\varepsilon_\Gamma \cdot \nabla_\Gamma (\nabla_\Gamma u \cdot \nabla_\Gamma w) + \frac{\partial \tilde{G}_t}{\partial t}|_{t=0}. \tag{21}$$

We now have to compute $\partial \tilde{G}_t / \partial t$ at $t = 0$. We have

$$\frac{\partial \tilde{G}_t}{\partial t} = \sum_{i,j=1}^{d-1} \frac{\partial .}{\partial t} \left(\frac{\partial \tilde{u}_t}{\partial \xi_i} e_t^i \cdot \frac{\partial \tilde{w}_t}{\partial \xi_j} e_t^j \right),$$

with

$$\frac{\partial .}{\partial t} \left(\frac{\partial \tilde{u}_t}{\partial \xi_i} e_t^i \cdot \frac{\partial \tilde{w}_t}{\partial \xi_j} e_t^j \right) = \frac{\partial^2 \tilde{u}_t}{\partial \xi_i \partial t} \frac{\partial \tilde{w}_t}{\partial \xi_j} e_t^i \cdot e_t^j + \frac{\partial \tilde{u}_t}{\partial \xi_i} \frac{\partial^2 \tilde{w}_t}{\partial \xi_j \partial t} e_t^i \cdot e_t^j + \frac{\partial \tilde{u}_t}{\partial \xi_i} \frac{\partial \tilde{w}_t}{\partial \xi_j} \left(\frac{\partial e_t^i}{\partial t} \cdot e_t^j + e_t^i \cdot \frac{\partial e_t^j}{\partial t} \right).$$

From differentiation with respect to t of

$$(Id + t(\nabla\varepsilon)^T) e_t^i = e^i,$$

we obtain

$$(\nabla\varepsilon)^T e_t^i + (Id + t(\nabla\varepsilon)^T))\frac{\partial e_t^i}{\partial t} = 0,$$

hence

$$\frac{\partial e_t^i}{\partial t} = -(Id + t(\nabla\varepsilon)^T)^{-1}(\nabla\varepsilon)^T e_t^i,$$

and in particular

$$\frac{\partial e_t^i}{\partial t}|_{t=0} = -(\nabla\varepsilon)^T \cdot e^i.$$

We arrive at

$$\begin{aligned}\frac{\partial \tilde{G}_t}{\partial t}|_{t=0} =& \nabla_\Gamma\left(\frac{\partial \tilde{u}_t}{\partial t}|_{t=0}\right)\cdot\nabla_\Gamma w + \nabla_\Gamma u\cdot\nabla_\Gamma\left(\frac{\partial \tilde{w}_t}{\partial t}|_{t=0}\right)\\ &- \nabla_\Gamma u\cdot\nabla\varepsilon\cdot\nabla_\Gamma w - \nabla_\Gamma u\cdot(\nabla\varepsilon)^T\cdot\nabla_\Gamma w.\end{aligned}$$

By using

$$\frac{\partial \tilde{u}_t}{\partial t}|_{t=0} = \nabla u\cdot\varepsilon, \quad \frac{\partial \tilde{w}_t}{\partial t}|_{t=0} = \nabla w\cdot\varepsilon$$

as well as the decomposition $\varepsilon = \varepsilon_\Gamma + (\varepsilon\cdot\nu)\nu$, we obtain

$$\begin{aligned}\frac{\partial \tilde{G}_t}{\partial t}|_{t=0} = & \nabla_\Gamma(\nabla_\Gamma u\cdot\varepsilon_\Gamma + (\nabla u\cdot\nu)(\varepsilon\cdot\nu))\cdot\nabla_\Gamma w + \nabla_\Gamma u\cdot\nabla_\Gamma(\nabla_\Gamma w\cdot\varepsilon_\Gamma + (\nabla w\cdot\nu)(\varepsilon\cdot\nu))\\ &- \nabla_\Gamma u\cdot(\nabla\varepsilon + (\nabla\varepsilon)^T))\cdot\nabla_\Gamma w.\end{aligned}$$

We complete the proof of lemma 4.5 by using equation (21).

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