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Abstract: This paper deals with optimal control problems for systems affine in the control variable. We consider nonnegativity constraints on the control, and finitely many equality and inequality constraints on the final state. First, we obtain second order necessary optimality conditions. Secondly, we derive a second order sufficient condition for the scalar control case.

Key-words: Optimal control, second order conditions, control constraints, singular arc, bang-singular solutions, SADCO.

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Conditions quadratiques pour des extrémales bang-singulières

Résumé : Dans ce travail nous étudions le problème de commande optimale avec des contrôles affines dans la dynamique. On considère des contraintes de non-négativité sur la commande et une quantité finie de contraintes d'égalité et d'inégalité sur la valeur finale de l'état. Premièrement on obtient des conditions nécessaires d'optimalité de second ordre. Ensuite, on présente une condition suffisante pour le cas d'une commande scalaire.

Mots-clés : Commande optimale, conditions du second ordre, contraintes sur la commande, arc singulier, solutions bang-singulières, SADCO.

1 Introduction

In this article we obtain second order conditions for an optimal control problem affine in the control. First we consider a pointwise nonnegativity constraint on the control, end-point state constraints and a fixed time interval. Then we extend the result to bound constraints on the control, initial-final state constraints and problems involving parameters. We do not assume that the multipliers are unique. We study weak and Pontryagin minima.

There is already an important literature on this subject. The case without control constraints, i.e. when the extremal is totally singular, has been extensively studied since the mid 1960s. Kelley in [32] treated the scalar control case and presented a necessary condition involving the second order derivative of the switching function. The result was extended by Kopp and Moyer [34] for higher order derivatives, and in [33] it was shown that the order had to be even. Goh in [27] proposed a special change of variables obtained via a linear ODE and in [26] used this transformation to derive a necessary condition for the vector control problem. An extensive survey of these articles can be found in Gabasov and Kirillova [24]. Jacobson and Speyer in [30], and together with Lele in [31] obtained necessary conditions by adding a penalization term to the cost functional. Gabasov and Kirillova [24], Krener [35], Agrachev and Gamkrelidze [1] obtained a countable series of necessary conditions that in fact use the idea behind the Goh transformation. Milyutin in [43] discovered an abstract essence of this approach and obtained even stronger necessary conditions. In [2] Agrachev and Sachkov investigated second order optimality conditions of the minimum time problem of a single-input system. The main feature of this kind of problem, where the control enters linearly, is that the corresponding second variation does not contain the Legendre term, so the methods of the classical calculus of variations are not applicable for obtaining sufficient conditions. This is why the literature was mostly devoted to necessary conditions, which are actually a consequence of the nonnegativity of the second variation. A sufficient condition for time optimality was given by Moyer [45] for a system with a scalar control variable and fixed endpoints. On the other hand, Goh's transformation abovementioned allows one to convert the second variation into another functional that hopefully turns out to be coercive with respect to the L_2-norm of some state variable. Dmitruk in [12] proved that this coercivity is a sufficient condition for the weak optimality, and presented a closely related necessary condition. He used the abstract approach developed by Levitin, Milyutin and Osmolovskii in [38], and considered finitely many inequality and equality constraints on the endpoints and the possible existence of several multipliers. In [13, 15] he also obtained necessary and sufficient conditions for this norm, again closely related, for Pontryagin minimality. More recently, Bonnard et al. in [6] provided second order sufficient conditions for the minimum time problem of a single-input system in terms of the existence of a conjugate time.

On the other hand, the case with linear control constraints and a "purely" bang-bang control without singular subarcs has been extensively investigated over the past 15 years. Milyutin and Osmolovskii in [44] provided necessary and sufficient conditions based on the general theory of [38]. Osmolovskii in [46] completed some of the proofs of the latter article. Sarychev in [53] gave first and second order sufficient condition for Pontryagin solutions. Agrachev, Stefani, Zezza [3] reduced the problem to a finite dimensional problem with the switching instants as variables and obtained a sufficient condition for strong optimality. The result was recently extended by Poggiolini and Spadini in [47]. On the other hand, Maurer and Osmolovskii in [42, 41] gave a second order sufficient condition that is suitable for practical verifications and presented a numerical procedure that allows to verify the positivity of certain quadratic forms. Felgenhauer in [21, 22, 23] studied both second order optimality conditions and sensitivity of the optimal solution.

The mixed case, where the control is partly bang-bang, partly singular was studied in [48] by Poggiolini and Stefani. They obtained a second order sufficient condition with an additional geometrical hypothesis (which is not needed here) and claimed that it is not clear whether this hypothesis is 'almost necessary', in the sense that it is not obtained straightforward from a necessary condition by strengthening an inequality. In [49, 50] they derived a second order sufficient condition for the special case of a time-optimal problem. The main result of the present article is to provide a sufficient condition that is 'almost necessary' for bang-singular extremals in a general Mayer problem.

On the other hand, the single-input time-optimal problem was extensively studied by means of and synthesis-like methods. See, among others, Sussmann [59, 58, 57], Schättler [54] and Schättler-Jankovic [55]. Both bang-bang and bang-singular structures were analysed in these works.

The article is organized as follows. In the second section we present the problem and give basic definitions. In the third section we perform a second order analysis. More precisely, we obtain the second variation of the Lagrangian functions and a necessary condition. Afterwards, in the fourth section, we present the Goh transformation and a new necessary condition in the transformed variables. In the fifth section we show a sufficient condition for scalar control. Finally, we give an example with a scalar control where the second order sufficient condition can be verified. The appendix is devoted to a series of technical properties that are used to prove the main results.

2 Statement of the problem and assumptions

2.1 Statement of the problem

Consider the spaces $\mathcal{U} := L_\infty(0,T;\mathbb{R}^m)$ and $\mathcal{X} := W^1_\infty(0,T;\mathbb{R}^n)$ as control and state spaces, respectively. Denote with u and x their elements, respectively. When needed, put $w = (x,u)$ for a point in $\mathcal{W} := \mathcal{X} \times \mathcal{U}$. In this paper we investigate the optimal control problem

$$J := \varphi_0(x(T)) \to \min, \tag{2.1}$$

$$\dot{x}(t) = \sum_{i=0}^{m} u_i f_i(x), \quad x(0) = x_0, \tag{2.2}$$

$$u(t) \geq 0, \text{ a.e. on } t \in [0,T], \tag{2.3}$$

$$\varphi_i(x(T)) \leq 0, \text{ for } i = 1,\dots,d_\varphi, \quad \eta_j(x(T)) = 0, \text{ for } j = 1\dots,d_\eta. \tag{2.4}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}^n$ for $i = 0,\dots,m$, $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 0,\dots,d_\varphi$, $\eta_j : \mathbb{R}^n \to \mathbb{R}$ for $j = 1,\dots,d_\eta$ and $u_0 \equiv 1$. Assume that data functions f_i are twice continuously differentiable. Functions φ_i and η_j are assumed to be twice differentiable.

A *trajectory* is an element $w \in \mathcal{W}$ that satisfies the state equation (2.2). If, in addition, constraints (2.3) and (2.4) hold, we say that w is a *feasible point* of the problem (2.1)-(2.4). Denote by $\mathcal{A}$ the *set of feasible points.* A *feasible variation* for $\hat{w} \in \mathcal{A}$ is an element $\delta w \in \mathcal{W}$ such that $\hat{w} + \delta w \in \mathcal{A}$.

Definition 2.1. *A pair $w^0 = (x^0,u^0) \in \mathcal{W}$ is said to be a weak minimum of problem* (2.1)-(2.4) *if there exists an $\varepsilon > 0$ such that the cost function attains at w^0 its minimum on the set*

$$\left\{ w = (x,u) \in \mathcal{A} : \|x - x^0\|_\infty < \varepsilon, \|u - u^0\|_\infty < \varepsilon \right\}.$$

We say w^0 is a Pontryagin minimum of problem (2.1)-(2.4) *if, for any positive N, there exists an $\varepsilon_N > 0$ such that w^0 is a minimum point on the set*

$$\left\{ w = (x,u) \in \mathcal{A} : \|x - x^0\|_\infty < \varepsilon_N, \|u - u^0\|_\infty \leq N, \|u - u^0\|_1 < \varepsilon_N \right\}.$$

Consider $\lambda = (\alpha, \beta, \psi) \in \mathbb{R}^{d_\varphi+1,*} \times \mathbb{R}^{d_\eta,*} \times W^1_\infty(0,T;\mathbb{R}^{n,*})$, i.e. ψ is a Lipschitz-continuous function with values in the $n-$dimensional space of row-vectors with real components $\mathbb{R}^{n,*}$. Define the *pre-Hamiltonian* function

$$H[\lambda](x,u,t) := \psi(t) \sum_{i=0}^{m} u_i f_i(x),$$

the *terminal Lagrangian* function

$$\ell[\lambda](q) := \sum_{i=0}^{d_\varphi} \alpha_i \varphi_i(q) + \sum_{j=1}^{d_\eta} \beta_j \eta_j(q),$$

and the *Lagrangian* function

$$\Phi[\lambda](w) := \ell[\lambda](x(T)) + \int_0^T \psi(t) \left(\sum_{i=0}^{m} u_i(t) f_i(x(t)) - \dot{x}(t) \right) \mathrm{d}t. \tag{2.5}$$

In this article the optimality of a given feasible trajectory $\hat{w} = (\hat{x}, \hat{u})$ is studied. Whenever some argument of f_i, H, ℓ, Φ or their derivatives is omitted, assume that they are evaluated over this trajectory. Without loss of generality suppose that

$$\varphi_i(\hat{x}(T)) = 0, \text{ for all } i = 0, 1, \ldots, d_\varphi. \tag{2.6}$$

2.2 First order analysis

Definition 2.2. *Denote by $\Lambda \subset \mathbb{R}^{d_\varphi+1,*} \times \mathbb{R}^{d_\eta,*} \times W^1_\infty(0,T;\mathbb{R}^{n,*})$ the set of Pontryagin multipliers associated with $\hat{w}$ consisting of the elements $\lambda = (\alpha, \beta, \psi)$ satisfying the Pontryagin Maximum Principle, i.e. having the following properties:*

$$|\alpha| + |\beta| = 1, \tag{2.7}$$

$$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{d_\varphi}) \geq 0, \tag{2.8}$$

function ψ is solution of the costate equation and satisfies the transversality condition at the endpoint T, i.e.

$$-\dot{\psi}(t) = H_x[\lambda](\hat{x}(t), \hat{u}(t), t), \ \psi(T) = \ell'[\lambda](\hat{x}(T)), \tag{2.9}$$

and the following minimum condition holds

$$H[\lambda](\hat{x}(t), \hat{u}(t), t) = \min_{v \geq 0} H[\lambda](\hat{x}(t), v, t), \text{ a.e. on } [0,T]. \tag{2.10}$$

Remark 2.3. *For every $\lambda \in \Lambda$, the following two conditions hold.*

(i) $H_{u_i}[\lambda]$ is continuous in time,

(ii) $H_{u_i}[\lambda](t) \geq 0$, a.e. on $[0,T]$.

Recall the following well known result for which a proof can be found e.g. in Alekseev and Tikhomirov [4], Kurcyusz and Zowe [36].

Theorem 2.4. *The set Λ is not empty.*

Remark 2.5. *Since ψ may be expressed as a linear continuous mapping of (α,β) and since (2.7) holds, Λ is a finite-dimensional compact set. Thus, it can be identified with a compact subset of $\mathbb{R}^s$, where $s := d_\varphi + d_\eta + 1$.*

The following expression for the derivative of the Lagrangian function holds

$$\Phi_u[\lambda](\hat{w})v = \int_0^T H_u[\lambda](\hat{x}(t),\hat{u}(t),t)v(t)\mathrm{d}t. \tag{2.11}$$

Consider $v \in \mathcal{U}$ and the *linearized state equation:*

$$\begin{cases} \dot{z}(t) = \sum_{i=0}^m \hat{u}_i(t) f_i'(\hat{x}(t))z(t) + \sum_{i=1}^m v_i(t) f_i(\hat{u}(t)), & \text{a.e. on } [0,T], \\ z(0) = 0. \end{cases} \tag{2.12}$$

Its solution z is called the *linearized state variable.*

With each index $i = 1,\dots,m$, we associate the sets

$$I_0^i := \left\{ t \in [0,T] : \max_{\lambda\in\Lambda} H_{u_i}[\lambda](t) > 0 \right\}, \quad I_+^i := [0,T]\backslash I_0^i, \tag{2.13}$$

and the *active set*

$$\tilde{I}_0^i := \{t \in [0,T] : \hat{u}_i(t) = 0\}. \tag{2.14}$$

Notice that $I_0^i \subset \tilde{I}_0^i$, and that I_0^i is relatively open in $[0,T]$ as each $H_{u_i}[\lambda]$ is continuous.

Assumption 1. Assume *strict complementarity for the control constraint,* i.e. for every $i = 1,\dots,m$,

$$I_0^i = \tilde{I}_0^i, \text{ up to a set of null measure.} \tag{2.15}$$

Observe then that for any index $i = 1,\dots,m$, the control $\hat{u}_i(t) > 0$ a.e. on I_+^i, and given $\lambda \in \Lambda$,

$$H_{u_i}[\lambda](t) = 0, \text{ a.e. on } I_+^i.$$

Assumption 2. For every $i = 1,\dots,m$, the active set I_0^i is a finite union of intervals, i.e.

$$I_0^i = \bigcup_{j=1}^{N_i} I_j^i,$$

for I_j^i subintervals of $[0,T]$ of the form $[0,d)$, $(c,T]$; or (c,d) if $c \neq 0$ and $d \neq T$. Denote by $c_1^i < d_1^i < c_2^i < \ldots < c_{N_i}^i < d_{N_i}^i$ the endpoints of these intervals. Consequently, I_+^i is a finite union of intervals as well.

Remark 2.6 (On the multi-dimensional control case)**.** *We would like to make a comment concerning solutions with more than one control component being singular at the same time. In [9, 10], Chitour et al. proved that generic systems with three or more control variables, or with two controls and drift did not admit singular optimal trajectories (by means of Goh's necessary condition [26]). Consequently, the study of generic properties of control-affine systems is restricted to problems having either one dimensional control or two control variables and no drift. Nevertheless, there are motivations for investigating problems with an arbitrary number of inputs that we point out next. In [37], Ledzewicz and Schättler worked on a model of cancer treatment having two control variables entering linearly in the pre-Hamiltonian and nonzero drift. They provided necessary optimality conditions for solutions with both controls being singular at the same* *time. Even if they were not able to give a proof of optimality they claimed to have strong expectations that this structure is part of the solution. Other examples can be found in the literature. Maurer in [40] analyzed a resource allocation problem (taken from Bryson-Ho [8]). The model had two controls and drift, and numerical computations yielded a candidate solution containing two simultaneous singular arcs. For a system with a similar structure, Gajardo et al. in [25] discussed the optimality of an extremal with two singular control components at the same time. Another motivation that we would like to point out is the technique used in Aronna et al. [5] to study the shooting algorithm for bang-singular solutions. In order to treat this kind of extremals, they perform a transformation that yields a new system and an associated totally singular solution. This new system involves as many control variables as singular arcs of the original solution. Hence, even a one-dimensional problem can lead to a multi-dimensional totally singular solution. These facts give a motivation for the investigation of multi-input control-affine problems.*

2.3 Critical cones

Let $1 \leq p \leq \infty$, and call $\mathcal{U}_p := L_p(0,T;\mathbb{R}^m)$, $\mathcal{U}_p^+ := L_p(0,T;\mathbb{R}^m_+)$ and $\mathcal{X}_p := W^1_p(0,T;\mathbb{R}^n)$. Recall that given a topological vector space E, a subset $D \subset E$ and $x \in E$, a *tangent direction* to D at x is an element $d \in E$ such that there exists sequences $(\sigma_k) \subset \mathbb{R}_+$ and $(x_k) \subset D$ with

$$\frac{x_k - x}{\sigma_k} \to d.$$

It is a well known result, see e.g. [11], that the tangent cone to $\mathcal{U}_2^+$ at $\hat{u}$ is

$$\{v \in \mathcal{U}_2 : \ v_i \geq 0 \text{ on } I_0^i, \ \text{for}\, i = 1, \dots, m\}.$$

Given $v \in \mathcal{U}_p$ and z the solution of (2.12), consider the *linearization of the cost and final constraints*

$$\begin{cases} \varphi_i'(\hat{x}(T))z(T) \leq 0, \ i = 0, \dots, d_\varphi, \\ \eta_j'(\hat{x}(T))z(T) = 0, \ j = 1, \dots, d_\eta. \end{cases} \tag{2.16}$$

For $p \in \{2, \infty\}$, define the L_p-*critical cone* as

$$\mathcal{C}_p := \left\{(z,v) \in \mathcal{X}_p \times \mathcal{U}_p : v \text{ tangent to } \mathcal{U}_p^+, \ (2.12) \text{ and } (2.16) \text{ hold}\right\}.$$

Certain relations of inclusion and density between some approximate critical cones are needed. Given $\varepsilon \geq 0$ and $i = 1, \dots, m$, define the $\varepsilon-$*active sets,* up to a set of null measure

$$I_\varepsilon^i := \{t \in (0,T) : \hat{u}_i(t) \leq \varepsilon\},$$

and the sets

$$\mathcal{W}_{p,\varepsilon} := \{(z,v) \in \mathcal{X}_p \times \mathcal{U}_p : v_i = 0 \text{ on } I_\varepsilon^i, \ (2.12) \text{ holds}\}.$$

By Assumption 1, the following explicit expression for $\mathcal{C}_2$ holds

$$\mathcal{C}_2 = \{(z,v) \in \mathcal{W}_{2,0} : (2.16) \text{ holds}\}. \tag{2.17}$$

Consider the $\varepsilon-$*critical cones*

$$\mathcal{C}_{p,\varepsilon} := \{(z,v) \in \mathcal{W}_{p,\varepsilon} : (2.16) \text{ holds}\}. \tag{2.18}$$

Let $\varepsilon > 0$. Note that by (2.17), $\mathcal{C}_{2,\varepsilon} \subset \mathcal{C}_2$. On the other hand, given $(z,v) \in \mathcal{C}_{\infty,\varepsilon}$, it easily follows that $\hat{u} + \sigma v \in \mathcal{U}^+$ for small positive σ. Thus v is tangent to $\mathcal{U}^+$ at $\hat{u}$, and this yields $\mathcal{C}_{\infty,\varepsilon} \subset \mathcal{C}_\infty$.

Recall the following technical result, see Dmitruk [16].

Lemma 2.7 (on density)**.** *Consider a locally convex topological space X, a finite-faced cone $C \subset X$, and a linear manifold L dense in X. Then the cone $C \cap L$ is dense in C.*

Lemma 2.8. *Given $\varepsilon > 0$ the following properties hold.*

(a) $\mathcal{C}_{\infty,\varepsilon} \subset \mathcal{C}_{2,\varepsilon}$ *with dense inclusion.*

(b) $\bigcup_{\varepsilon>0} \mathcal{C}_{2,\varepsilon} \subset \mathcal{C}_2$ *with dense inclusion.*

Proof. **(a)** The inclusion is immediate. As $\mathcal{U}$ is dense in $\mathcal{U}_2$, $\mathcal{W}_{\infty,\varepsilon}$ is a dense subspace of $\mathcal{W}_{2,\varepsilon}$. By Lemma 2.7, $\mathcal{C}_{2,\varepsilon} \cap \mathcal{W}_{\infty,\varepsilon}$ is dense in $\mathcal{C}_{2,\varepsilon}$, as desired.
(b) The inclusion is immediate. In order to prove density, consider the following dense subspace of $\mathcal{W}_{2,0}$:

$$\mathcal{W}_{2,\bigcup} := \bigcup_{\varepsilon>0} \mathcal{W}_{2,\varepsilon},$$

and the finite-faced cone in $\mathcal{C}_2 \subset \mathcal{W}_{2,0}$. By Lemma 2.7, $\mathcal{C}_2 \cap \mathcal{W}_{2,\bigcup}$ is dense in $\mathcal{C}_2$, which is what we needed to prove. □

3 Second order analysis

3.1 Second variation

Consider the following quadratic mapping on $\mathcal{W}$;

$$\begin{aligned}\Omega[\lambda](\delta x, \delta u) :=& \tfrac{1}{2}\ell''[\lambda](\hat{x}(T))(\delta x(T))^2 \\ &+ \tfrac{1}{2}\int_0^T [(H_{xx}[\lambda]\delta x, \delta x) + 2(H_{ux}[\lambda]\delta x, \delta u)]\,\mathrm{d}t.\end{aligned}$$

The next lemma provides a second order expansion for the Lagrangian function involving operator Ω. Recall the following notation: given two functions $h : \mathbb{R}^n \to \mathbb{R}^{n_h}$ and $k : \mathbb{R}^n \to \mathbb{R}^{n_k}$, we say that h is a *big-O* of k around 0 and denote it by

$$h(x) = \mathcal{O}(k(x)),$$

if there exists positive constants δ and M such that $|h(x)| \le M|k(x)|$ for $|x| < \delta$. It is a *small-o* if M goes to 0 as $|x|$ goes to 0. Denote this by

$$h(x) = o(k(x)).$$

Lemma 3.1. *Let $\delta w = (\delta x, \delta u) \in \mathcal{W}$. Then for every multiplier $\lambda \in \Lambda$, the function Φ has the following expansion (omitting time arguments):*

$$\begin{aligned}\Phi[\lambda](\hat{w} + \delta w) =& \int_0^T H_u[\lambda]\delta u \mathrm{d}t + \Omega[\lambda](\delta x, \delta u) + \tfrac{1}{2}\int_0^T (H_{uxx}[\lambda]\delta x, \delta x, \delta u)\mathrm{d}t \\ &+ \mathcal{O}(|\delta x(T)|^3) + \int_0^T |(\hat{u} + \delta u)(t)|\mathcal{O}(|\delta x(t)|^3)\,\mathrm{d}t.\end{aligned}$$

Proof. Omit the dependence on λ for the sake of simplicity. Use the Taylor expansions

$$\ell(\hat{x}(T) + \delta x(T)) = \ell(\hat{x}(T)) + \ell'(\hat{x}(T))\delta x(T) + \tfrac{1}{2}\ell''(\hat{x}(T))(\delta x(T))^2 + \mathcal{O}(|\delta x(T)|^3),$$

$$f_i(\hat{x}(t) + \delta x(t)) = f_i(\hat{x}(t)) + f_i'(\hat{x}(t))\delta x(t) + \tfrac{1}{2}f_i''(\hat{x}(t))(\delta x(t))^2 + \mathcal{O}(|\delta x(t)|^3),$$

in the expression

$$\Phi(\hat{w}+\delta w)=\ell(\hat{x}+\delta x(T))+\int_0^T\psi\left[\sum_{i=0}^m(\hat{u}_i+\delta u_i)f_i(\hat{x}+\delta x)-\dot{\hat{x}}-\dot{\delta x}\right]\mathrm{d}t.$$

Afterwards, use the identity

$$\int_0^T\psi\sum_{i=0}^m\hat{u}_i f_i'(\hat{x})\delta x\mathrm{d}t=-\ell'(\hat{x}(T))\delta x(T)+\int_0^T\psi\dot{\delta x}\mathrm{d}t,$$

obtained by integration by parts and equation (2.2) to get the desired result. □

The previous lemma yields the following identity for every $(\delta x,\delta u)\in\mathcal{W}$:

$$\Omega[\lambda](\delta x,\delta u)=\tfrac{1}{2}D^2\Phi[\lambda](\hat{w})(\delta x,\delta u)^2.$$

3.2 Necessary condition

This section provides the following second order necessary condition in terms of Ω and the critical cone $\mathcal{C}_2$.

Theorem 3.2. *If $\hat{w}$ is a weak minimum then*

$$\max_{\lambda\in\Lambda}\Omega[\lambda](z,v)\geq 0,\quad \text{for all } (z,v)\in\mathcal{C}_2. \tag{3.1}$$

For the sake of simplicity, define $\bar{\varphi}:\mathcal{U}\to\mathbb{R}^{d_\varphi+1}$, and $\bar{\eta}:\mathcal{U}\to\mathbb{R}^{d_\eta}$ as

$$\begin{aligned}\bar{\varphi}_i(u)&:=\varphi_i(x(T)),\ \text{for } i=0,1,\dots,d_\varphi,\\ \bar{\eta}_j(u)&:=\eta_j(x(T)),\ \text{for } j=1,\dots,d_\eta,\end{aligned} \tag{3.2}$$

where x is the solution of (2.2) corresponding to u.

Definition 3.3. *We say that the equality constraints are nondegenerate if*

$$\bar{\eta}'(\hat{u}) \text{ is onto from } \mathcal{U} \text{ to } \mathbb{R}^{d_\eta}. \tag{3.3}$$

If (3.3) *does not hold, we call them degenerate.*

Write the problem in the following way

$$\bar{\varphi}_0(u)\to\min;\quad \bar{\varphi}_i(u)\leq 0,\ i=1,\dots,d_\varphi,\ \bar{\eta}(u)=0,\ u\in\mathcal{U}_+. \tag{P}$$

Suppose that $\hat{u}$ is a local weak solution of (P)**.** Next we prove Theorem 3.2. Its proof is divided into two cases: degenerate and nondegenerate equality constraints. For the first case the result is immediate and is tackled in the next Lemma. In order to show Theorem 3.2 for the latter case we introduce an auxiliary problem parameterized by certain critical directions (z,v), denoted by (QP$_v$). We prove that val(QP$_v$) ≥ 0 and, by a result on duality, the desired second order condition will be derived.

Lemma 3.4. *If equality constraints are degenerate, then* (3.1) *holds.*

Proof. Notice that there exists $\beta \neq 0$ such that $\sum_{j=1}^{d_\eta} \beta_j \eta_j'(\hat{x}(T)) = 0$, since $\bar{\eta}'(\hat{u})$ is not onto. Consider $\alpha = 0$ and $\psi = 0$. Take $\lambda := (\alpha, \beta, \psi)$ and notice that both λ and $-\lambda$ are in Λ. Observe that

$$\Omega[\lambda](z, v) = \tfrac{1}{2} \sum_{j=1}^{d_\eta} \beta_j \eta_j''(\hat{x}(T))(z(T))^2.$$

Thus $\Omega[\lambda](z, v) \geq 0$ either for λ or $-\lambda$. The required result follows. $\square$

Take $\varepsilon > 0$, $(z, v) \in \mathcal{C}_{\infty,\varepsilon}$, and rewrite (2.18) using the notation in (3.2),

$$\begin{aligned} \mathcal{C}_{\infty,\varepsilon} = \{(z,v) \in \mathcal{X} \times \mathcal{U} :\ & v_i(t) = 0 \text{ on } I_\varepsilon^i, \quad i = 1, \ldots, m, \\ & (2.12) \text{ holds},\ \bar{\varphi}_i'(\hat{u})v \leq 0,\ i = 0, \ldots, d_\varphi,\ \bar{\eta}'(\hat{u})v = 0\}. \end{aligned}$$

Consider the problem

$$\begin{aligned} & \delta\zeta \to \min \\ & \bar{\varphi}_i'(\hat{u})r + \bar{\varphi}_i''(\hat{u})(v,v) \leq \delta\zeta, \text{ for } i = 0, \ldots, d_\varphi, \\ & \bar{\eta}'(\hat{u})r + \bar{\eta}''(\hat{u})(v,v) = 0, \\ & -r_i(t) \leq \delta\zeta, \ \text{on } I_0^i, \ \text{for } i = 1, \ldots, m. \end{aligned} \tag{QP$_v$}$$

Proposition 3.5. *Let* $(z, v) \in \mathcal{C}_{\infty,\varepsilon}$. *If the equality constraints are nondegenerate, problem* (QP$_v$) *is feasible and* $\operatorname{val}(\text{QP}_v) \geq 0$.

Proof. Let us first prove feasibility. As $\bar{\eta}'(\hat{u})$ is onto, there exists $r \in \mathcal{U}$ such that the equality constraint in (QP$_v$) is satisfied. Take

$$\delta\zeta := \max(\|r\|_\infty, \bar{\varphi}_i'(\hat{u})r + \bar{\varphi}''(\hat{u})(v,v)).$$

Thus the pair $(r, \delta\zeta)$ is feasible for (QP$_v$).

Let us now prove that $\operatorname{val}(\text{QP}_v) \geq 0$. On the contrary suppose that there exists a feasible solution $(r, \delta\zeta)$ with $\delta\zeta < 0$. The last constraint in (QP$_v$) implies $\|r\|_\infty \neq 0$. Set, for $\sigma > 0$,

$$\tilde{u}(\sigma) := \hat{u} + \sigma v + \tfrac{1}{2}\sigma^2 r, \quad \tilde{\zeta}(\sigma) := \tfrac{1}{2}\sigma^2 \delta\zeta. \tag{3.4}$$

The goal is finding $u(\sigma)$ feasible for (P) such that for small σ,

$$u(\sigma) \xrightarrow{\mathcal{U}} \hat{u}, \text{ and } \bar{\varphi}_0(u(\sigma)) < \bar{\varphi}_0(\hat{u}),$$

contradicting the weak optimality of $\hat{u}$.

Notice that $\hat{u}_i(t) > \varepsilon$ a.e. on $[0,T] \backslash I_\varepsilon^i$, and then $\tilde{u}(\sigma)_i(t) > -\tilde{\zeta}(\sigma)$ for sufficiently small σ. On I_ε^i, if $\tilde{u}(\sigma)_i(t) < -\tilde{\zeta}(\sigma)$ then necessarily

$$\hat{u}_i(t) < \tfrac{1}{2}\sigma^2(\|r\|_\infty + |\delta\zeta|),$$

as $v_i(t) = 0$. Thus, defining the set

$$J_\sigma^i := \{t : 0 < \hat{u}_i(t) < \tfrac{1}{2}\sigma^2(\|r\|_\infty + |\delta\zeta|)\},$$

we get $\{t \in [0,T] : \tilde{u}(\sigma)_i(t) < -\tilde{\zeta}(\sigma)\} \subset J_\sigma^i$. Observe that on J_σ^i, the function $|\tilde{u}(\sigma)_i(t) + \tilde{\zeta}(\sigma)|/\sigma^2$ is dominated by $\|r\|_\infty + |\delta\zeta|$. Since $\operatorname{meas}(J_\sigma^i)$ goes to 0 by the Dominated Convergence Theorem, we obtain

$$\int_{J_\sigma^i} |\tilde{u}(\sigma)_i(t) + \tilde{\zeta}(\sigma)| \mathrm{d}t = o(\sigma^2).$$

Take

$$\tilde{\tilde{u}}(\sigma) := \begin{cases} \tilde{u}(\sigma) & \text{on } [0,T]\backslash J_\sigma^i, \\ -\tilde{\zeta}(\sigma) & \text{on } J_\sigma^i. \end{cases}$$

Thus, $\tilde{\tilde{u}}$ satisfies

$$\begin{gathered} \tilde{\tilde{u}}(\sigma)(t) \geq -\tilde{\zeta}(\sigma), \quad \text{a.e. on } [0,T], \\ \|\tilde{\tilde{u}}(\sigma) - \hat{u}\|_1 = o(\sigma^2), \quad \|\tilde{\tilde{u}}(\sigma) - \hat{u}\|_\infty = O(\sigma^2), \end{gathered} \tag{3.5}$$

and the following estimates hold

$$\begin{aligned} \bar{\varphi}_i(\tilde{\tilde{u}}(\sigma)) &= \bar{\varphi}_i(\hat{u}) + \sigma\bar{\varphi}_i'(\hat{u})v + \tfrac{1}{2}\sigma^2[\bar{\varphi}_i'(\hat{u})r + \bar{\varphi}_i''(\hat{u})(v,v)] + o(\sigma^2) \\ &< \bar{\varphi}_i(\hat{u}) + \tilde{\zeta}(\sigma) + o(\sigma^2), \end{aligned} \tag{3.6}$$

$$\bar{\eta}(\tilde{\tilde{u}}(\sigma)) = \sigma\bar{\eta}'(\hat{u})v + \tfrac{1}{2}\sigma^2[\bar{\eta}'(\hat{u})r + \bar{\eta}''(\hat{u})(v,v)] + o(\sigma^2) = o(\sigma^2).$$

As $\bar{\eta}'(\hat{u})$ is onto on $\mathcal{U}$ we can find a corrected control $u(\sigma)$ satisfying the equality constraint and such that $\|u(\sigma) - \tilde{\tilde{u}}(\sigma)\|_\infty = o(\sigma^2)$. Deduce by (3.5) that $u(\sigma) \geq 0$ a.e. on $[0,T]$, and by (3.6) that it satisfies the terminal inequality constraints. Thus $u(\sigma)$ is feasible for (P) and it satisfies (3.4). This contradicts the weak optimality of $\hat{u}$. □

Recall that a *Lagrange multiplier* associated with $\hat{w}$ is a pair (λ,μ) in $\mathbb{R}^{d_\varphi+1} \times \mathbb{R}^{d_\eta} \times W^1_\infty(0,T;\mathbb{R}^{n,*}) \times \mathcal{U}^*$ with $\lambda = (\alpha,\beta,\psi)$ satisfying (2.7), (2.8), $\mu \geq 0$ and the *stationarity condition*

$$\int_0^T H_u[\lambda](t)v(t)\mathrm{d}t + \int_0^T v(t)\mathrm{d}\mu(t) = 0, \quad \text{for every } v \in \mathcal{U}.$$

Here $\mathcal{U}^*$ denotes the dual space of $\mathcal{U}$. Simple computations show that (λ,μ) is a Lagrange multiplier if and only if λ is a Pontryagin multiplier and $\mu = H_u[\lambda]$. Thus $\mu \in L_\infty(0,T;\mathbb{R}^{m,*})$.

Let us come back to Theorem 3.2.

Proof. [of Theorem 3.2] Lemma 3.4 covers the degenerate case. Assume thus that $\bar{\eta}'(\hat{u})$ is onto. Take $\varepsilon > 0$ and $(z,v) \in \mathcal{C}_{\infty,\varepsilon}$. Applying Proposition 3.5, we see that there cannot exist r and $\delta\zeta < 0$ such that

$$\begin{aligned} &\bar{\varphi}_i'(\hat{u})r + \bar{\varphi}_i''(\hat{u})(v,v) \leq \delta\zeta, \ i = 0,\dots,d_\varphi, \\ &\bar{\eta}'(\hat{u})r + \bar{\eta}''(\hat{u})(v,v) = 0, \\ &-r_i(t) \leq \delta\zeta, \text{ on } I_0^i, \text{ for } i = 1,\dots,m. \end{aligned}$$

By the Dubovitskii-Milyutin Theorem (see [19]) we obtain the existence of $(\alpha,\beta) \in \mathbb{R}^s$ and $\mu \in \mathcal{U}^*$ with supp $\mu_i \subset I_0^i$, and $(\alpha,\beta,\mu) \neq 0$ such that

$$\sum_{i=0}^{d_\varphi} \alpha_i\bar{\varphi}_i'(\hat{u}) + \sum_{i=1}^{d_\eta} \beta_j\bar{\eta}_j'(\hat{u}) - \mu = 0, \tag{3.7}$$

and denoting $\lambda := (\alpha,\beta,\psi)$, with ψ being solution of (2.9), the following holds:

$$\sum_{i=0}^{d_\varphi} \alpha_i\bar{\varphi}_i''(\hat{u})(v,v) + \sum_{i=1}^{d_\eta} \beta_j\bar{\eta}_j''(\hat{u})(v,v) \geq 0.$$

By Lemma 8.2 we obtain

$$\Omega[\lambda](z,v) \geq 0. \tag{3.8}$$

Observe that (3.7) implies that $\lambda \in \Lambda$. Consider now $(\bar{z},\bar{v}) \in \mathcal{C}_2$, and note that Lemma 2.8 guarantees the existence of a sequence $\{(z_\varepsilon,v_\varepsilon)\} \subset \mathcal{C}_{\infty,\varepsilon}$ converging to $(\bar{z},\bar{v})$ in $\mathcal{X}_2 \times \mathcal{U}_2$. Recall Remark 2.5. Let $\lambda_\varepsilon \in \Lambda$ be such that (3.8) holds for $(\lambda_\varepsilon, z_\varepsilon, v_\varepsilon)$. Since (λ_ε) is bounded, it contains a limit point $\bar{\lambda} \in \Lambda$. Thus (3.8) holds for $(\bar{\lambda},\bar{z},\bar{v})$, as required. □

4 Goh Transformation

Consider an arbitrary linear system:

$$\begin{cases} \dot{z}(t) = A(t)z(t) + B(t)v(t), \quad \text{a.e. on } [0,T], \\ z(0) = 0, \end{cases} \tag{4.1}$$

where $A(t) \in \mathcal{L}(\mathbb{R}^n;\mathbb{R}^n)$ is an essentially bounded function of t, and $B(t) \in \mathcal{L}(\mathbb{R}^m;\mathbb{R}^n)$ is a Lipschitz-continuous function of t. With each $v \in \mathcal{U}$ associate the state variable $z \in \mathcal{X}$ solution of (2.12). Let us present a transformation of the variables $(z,v) \in \mathcal{W}$, first introduced by Goh in [27]. Define two new state variables as follows:

$$\begin{cases} y(t) := \displaystyle\int_0^t v(s)\mathrm{d}s, \\ \xi(t) := z(t) - B(t)y(t). \end{cases} \tag{4.2}$$

Thus $y \in \mathcal{Y} := W^1_\infty(0,T;\mathbb{R}^m)$, $y(0) = 0$ and ξ is an element of space $\mathcal{X}$. It easily follows that ξ is a solution of the linear differential equation

$$\dot{\xi}(t) = A(t)\xi(t) + B_1(t)y(t), \quad \xi(0) = 0, \tag{4.3}$$

where

$$B_1(t) := A(t)B(t) - \dot{B}(t). \tag{4.4}$$

For the purposes of this article take

$$A(t) := \sum_{i=0}^m \hat{u}_i f_i'(\hat{x}(t)), \text{ and } B(t)v(t) := \sum_{i=1}^m v_i(t) f_i(\hat{u}(t)). \tag{4.5}$$

Then (4.1) coincides with the linearized equation (2.12).

4.1 Transformed critical directions

As optimality conditions on the variables obtained by the Goh Transformation will be derived, a new set of critical directions is needed. Take a point (z,v) in $\mathcal{C}_\infty$, and define ξ and y by the transformation (4.2). Let $h := y(T)$ and notice that since (2.16) is satisfied, the following inequalities hold,

$$\begin{aligned} &\varphi_i'(\hat{x}(T))(\xi(T) + B(T)h) \leq 0, \text{ for } i = 0,\dots,d_\varphi, \\ &\eta_j'(\hat{x}(T))(\xi(T) + B(T)h) = 0, \text{ for } j = 1,\dots,d_\eta. \end{aligned} \tag{4.6}$$

Define the set of *transformed critical directions*

$$\mathcal{P} := \left\{ \begin{array}{l} (\xi,y,h) \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}^m : \dot{y}_i = 0 \text{ over } I_0^i,\ y(0) = 0,\ h := y(T), \\ (4.3) \text{ and } (4.6) \text{ hold} \end{array} \right\}.$$

Observe that for every $(\xi,y,h) \in \mathcal{P}$ and $1 \leq i \leq m$,

$$y_i \text{ is constant over each connected component of } I_0^i, \tag{4.7}$$

and at the endpoints the following conditions hold

$$\begin{aligned} &y_i = 0 \text{ on } [0, d_1^i),\ \text{ if } 0 \in I_0^i, \text{ and} \\ &y_i = h_i \text{ on } (c_{N_i}^i, T],\ \text{ if } T \in I_0^i, \end{aligned} \tag{4.8}$$

where c_1^i and d_1^i were introduced in Assumption 2. Define the set

$$\mathcal{P}_2 := \{(\xi, y, h) \in \mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}^m : (4.3), (4.6), (4.7) \text{ and } (4.8) \text{ hold}\}.$$

Lemma 4.1. *$\mathcal{P}$ is a dense subset of $\mathcal{P}_2$ in the $\mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}^m-$topology.*

Proof. The inclusion is immediate. In order to prove the density, consider the following sets.

$$X := \{(\xi, y, h) \in \mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}^m : (4.3), (4.7) \text{ and } (4.8) \text{ hold}\},$$

$$L := \{(\xi, y, y(T)) \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}^m : \ y(0) = 0, \ (4.3) \text{ and } (4.7) \text{ hold}\},$$

$$C := \{(\xi, y, h) \in X : \ (4.6) \text{ holds}\}.$$

By Lemma 8.1, L is a dense subset of X. The conclusion follows with Lemma 2.7. □

4.2 Transformed second variation

We are interested in writing Ω in terms of variables y and ξ defined in (4.2). Introduce the following notation for the sake of simplifying the presentation.

Definition 4.2. *Consider the following matrices of sizes $n \times n, m \times n$ and $m \times n$, respectively.*

$$Q[\lambda] := H_{xx}[\lambda], \quad C[\lambda] := H_{ux}[\lambda], \quad M[\lambda] := B^\top Q[\lambda] - \dot{C}[\lambda] - C[\lambda]A, \tag{4.9}$$

where A and B were defined in (4.5). Notice that M is well-defined as C is Lipschitz-continuous on t. Decompose matrix $C[\lambda]B$ into its symmetric and skew-symmetric parts, i.e. consider

$$S[\lambda] := \tfrac{1}{2}(C[\lambda]B + (C[\lambda]B)^\top), \quad V[\lambda] := \tfrac{1}{2}(C[\lambda]B - (C[\lambda]B)^\top). \tag{4.10}$$

Remark 4.3. *Observe that, since $C[\lambda]$ and B are Lipschitz-continuous, $S[\lambda]$ and $V[\lambda]$ are Lipschitz-continuous as well. In fact, simple computations yield*

$$S_{ij}[\lambda] = \tfrac{1}{2}\psi(f_i' f_j + f_j' f_i), \quad V_{ij}[\lambda] = \tfrac{1}{2}\psi[f_i, f_j], \quad \text{for } i, j = 1, \dots, m, \tag{4.11}$$

where

$$[f_i, f_j] := f_i' f_j - f_j' f_i. \tag{4.12}$$

With this notation, Ω takes the form

$$\Omega[\lambda](\delta x, v) = \tfrac{1}{2}\ell''[\lambda](\hat{x}(T))(\delta x(T))^2 + \tfrac{1}{2}\int_0^T [(Q[\lambda]\delta x, \delta x) + 2(C[\lambda]\delta x, v)]\mathrm{d}t.$$

Define the $m \times m$ matrix

$$R[\lambda] := B^\top Q[\lambda] B - C[\lambda] B_1 - (C[\lambda] B_1)^\top - \dot{S}[\lambda], \tag{4.13}$$

where B_1 was introduced in equation (4.4). Consider the function $g[\lambda]$ from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}$ defined by:

$$g[\lambda](\zeta, h) := \tfrac{1}{2}\ell''[\lambda](\hat{x}(T))(\zeta + B(T)h)^2 + \tfrac{1}{2}(C[\lambda](T)(2\zeta + B(T)h), h). \tag{4.14}$$

Remark 4.4. *(i) We use the same notation for the matrices $Q[\lambda]$, $C[\lambda]$, $M[\lambda]$, $\ell''[\lambda](\hat{x}(T))$ and for the bilinear mapping they define.*

(ii) Observe that when $m = 1$, the function $V[\lambda] \equiv 0$ since it becomes a skew-symmetric scalar.

Definition 4.5. *Define the mapping over $\mathcal{X} \times \mathcal{Y} \times \mathcal{U}$ given by*

$$\begin{aligned}\Omega_{\mathcal{P}}[\lambda](\xi, y, v) &:= g[\lambda](\xi(T), y(T)) \\ &+ \int_0^T \{\tfrac{1}{2}(Q[\lambda]\xi, \xi) + 2(M[\lambda]\xi, y) + \tfrac{1}{2}(R[\lambda]y, y) + (V[\lambda]y, v)\}\mathrm{d}t,\end{aligned} \tag{4.15}$$

with $g[\lambda]$, $Q[\lambda]$, $M[\lambda]$, $R[\lambda]$ and $V[\lambda]$ defined in (4.9)-(4.14).

The following theorem shows that $\Omega_{\mathcal{P}}$ coincides with Ω. See e.g. [15].

Theorem 4.6. *Let $(z, v) \in \mathcal{W}$ satisfying* (2.12) *and (ξ, y) be defined by* (4.2)*. Then*

$$\Omega[\lambda](z, v) = \Omega_{\mathcal{P}}[\lambda](\xi, y, v).$$

Proof. We omit the dependence on λ for the sake of simplicity. Replace z by its expression in (4.2) and obtain

$$\begin{aligned}\Omega(z, v) &= \tfrac{1}{2}\ell''(\hat{x}(T))(\xi(T) + B(T)y(T))^2 \\ &+ \tfrac{1}{2}\int_0^T [(Q(\xi + By), \xi + By) + (C(\xi + By), v) + (C^\top v, \xi + By)]\mathrm{d}t.\end{aligned} \tag{4.16}$$

Integrating by parts yields

$$\int_0^T (C\xi, v)\mathrm{d}t = [(C\xi, y)]_0^T - \int_0^T (\dot{C}\xi + C(A\xi + B_1 y), y)\mathrm{d}t, \tag{4.17}$$

and

$$\begin{aligned}\int_0^T (CBy, v)\mathrm{d}t &= \int_0^T ((S + V)y, v)\mathrm{d}t \\ &= \tfrac{1}{2}[(Sy, y)]_0^T + \int_0^T (-\tfrac{1}{2}(\dot{S}y, y) + (Vy, v))\mathrm{d}t.\end{aligned} \tag{4.18}$$

Combining (4.16), (4.17) and (4.18) we get the desired result. □

Corollary 4.7. *If $V[\lambda] \equiv 0$ then Ω does not involve v explicitly, and it can be expressed in terms of $(\xi, y, y(T))$.*

In view of (4.11), the previous corollary holds in particular if $[f_i, f_j] = 0$ on the reference trajectory for each pair $1 \le i < j \le m$.

Corollary 4.8. *If $\hat{w}$ is a weak minimum, then*

$$\max_{\lambda \in \Lambda} \Omega_{\mathcal{P}}[\lambda](\xi, y, v) \ge 0,$$

for every $(z, v) \in \mathcal{C}_2$ and (ξ, y) defined by (4.2).

4.3 New second order condition

In this section we present a necessary condition involving the variable (ξ, y, h) in $\mathcal{P}_2$. To achieve this we remove the explicit dependence on v from the second variation, for certain subset of multipliers. Recall that we consider $\lambda = (\alpha, \beta)$ as elements of $\mathbb{R}^s$.

Definition 4.9. *Given $M \subset \mathbb{R}^s$, define*

$$G(M) := \{\lambda \in M : V_{ij}[\lambda](t) = 0 \text{ on } I_+^i \cap I_+^j, \text{ for any pair } 1 \leq i < j \leq m\}.$$

Theorem 4.10. *Let $M \subset \mathbb{R}^s$ be convex and compact, and assume that*

$$\max_{\lambda \in M} \Omega_{\mathcal{P}}[\lambda](\xi, y, \dot{y}) \geq 0, \quad \text{for all} \quad (\xi, y, h) \in \mathcal{P}. \tag{4.19}$$

Then

$$\max_{\lambda \in G(M)} \Omega_{\mathcal{P}}[\lambda](\xi, y, \dot{y}) \geq 0, \quad \text{for all} \quad (\xi, y, h) \in \mathcal{P}.$$

The proof is based on some techniques introduced in Dmitruk [12, 15] for the proof of similar theorems.

Let $1 \leq i < j \leq m$ and $t^* \in \operatorname{int} I_+^i \cap I_+^j$. Take $y \in \mathcal{Y}$ satisfying

$$y(0) = y(T) = 0, \quad y_k = 0, \text{ for } k \neq i, k \neq j. \tag{4.20}$$

Such functions define a linear continuous mapping $r : \mathbb{R}^{s,*} \to \mathbb{R}$ by

$$\lambda \mapsto r[\lambda] := \int_0^T (V[\lambda](t^*)y, \dot{y})\mathrm{d}t. \tag{4.21}$$

By condition (4.20), and since $V[\lambda]$ is skew-symmetric,

$$\int_0^T (V[\lambda](t^*)y, \dot{y})\mathrm{d}t = V_{ij}[\lambda](t^*) \int_0^T (y_i \dot{y}_j - y_j \dot{y}_i)\mathrm{d}t.$$

Each r is an element of the dual space of $\mathbb{R}^{s,*}$, and it can thus be identified with an element of $\mathbb{R}^s$. Consequently, the subset of $\mathbb{R}^s$ defined by

$$R_{ij}(t^*) := \{r \in \mathbb{R}^s : y \in \mathcal{Y} \text{ satisfies (4.20)}, \ r \text{ is defined by (4.21)}\},$$

is a linear subspace of $\mathbb{R}^s$. Now, consider all the finite collections

$$\Theta_{ij} := \left\{\theta = \{t^1 < \cdots < t^{N_\theta}\} : t^k \in \operatorname{int} I_+^i \cap I_+^j \text{ for } k = 1, \ldots, N_\theta\right\}.$$

Define

$$\mathcal{R} := \sum_{i<j} \bigcup_{\theta \in \Theta_{ij}} \sum_{k=1}^{N_\theta} R_{ij}(t^k).$$

Note that $\mathcal{R}$ is a linear subspace of $\mathbb{R}^s$. Given $(\xi, y, y(T)) \in \mathcal{P}$, let the mapping $p_y : \mathbb{R}^{s,*} \to \mathbb{R}$ be given by

$$\lambda \mapsto p_y[\lambda] := \Omega_{\mathcal{P}}[\lambda](\xi, y, \dot{y}). \tag{4.22}$$

Thus, p_y is an element of $\mathbb{R}^s$.

Lemma 4.11. *Let $(\bar{\xi}, \bar{y}, \bar{y}(T)) \in \mathcal{P}$ and $r \in \mathcal{R}$. Then there exists a sequence $\{(\xi^\nu, y^\nu, y^\nu(T))\}$ in $\mathcal{P}$ such that*

$$\Omega_{\mathcal{P}}[\lambda](\xi^\nu, y^\nu, \dot{y}^\nu) \longrightarrow p_{\bar{y}}[\lambda] + r[\lambda]. \tag{4.23}$$

Proof. Take $(\bar\xi,\bar y,\bar y(T))\in\mathcal{P}$, its corresponding critical direction $(\bar z,\bar v)\in\mathcal{C}$ related via (4.2) and $p_{\bar y}$ defined in (4.22). Assume that $r\in R_{ij}(t^*)$ for some $1\le i<j\le m$ and $t^*\in \text{int } I_+^i\cap I_+^j$, i.e. r is associated via (4.21) to some function $\tilde y$ verifying (4.20). Take $\tilde y(t)=0$ when $t\notin[0,T]$. Consider

$$\tilde y^\nu(t):=\tilde y(\nu(t-t^*)),\quad \breve y^\nu:=\bar y+\tilde y^\nu. \tag{4.24}$$

Let $\breve\xi^\nu$ be the solution of (4.3) corresponding to $\breve y^\nu$. Observe that for large enough ν, as $t^*\in$ int $I_+^i\cap I_+^j$,

$$\dot{\breve y}_k^\nu=0,\ \text{a.e. on } I_0^k,\ \text{for } k=1,\dots,m. \tag{4.25}$$

Let $(\tilde z^\nu,\tilde v^\nu)$ and $(\breve z^\nu,\breve v^\nu)$ be the points associated by transformation (4.2) with $(\tilde\xi^\nu,\tilde y^\nu,\tilde y^\nu(T))$ and $(\breve\xi^\nu,\breve y^\nu,\breve y^\nu(T))$, respectively. By (4.25), we get

$$\breve v_k^\nu=0,\ \text{a.e. on } I_0^k,\ \text{for } k=1,\dots,m.$$

Note, however, that $(\breve z^\nu,\breve v^\nu)$ can violate the terminal constraints defining $\mathcal{C}_\infty$, i.e. the constraints defined in (2.16). Let us look for an estimate of the magnitude of this violation. Since

$$\|\tilde y^\nu\|_1=\mathcal{O}(1/\nu), \tag{4.26}$$

and $(\tilde\xi^\nu,\tilde y^\nu)$ is solution of (4.3), Gronwall's Lemma implies

$$|\tilde\xi^\nu(T)|=\mathcal{O}(1/\nu).$$

On the other hand, notice that $\breve z^\nu(T)=\bar z(T)+\tilde\xi^\nu(T)$, and thus

$$|\breve z^\nu(T)-\bar z(T)|=\mathcal{O}(1/\nu).$$

By Hoffman's Lemma (see [29]), there exists $(\Delta z^\nu,\Delta v^\nu)\in\mathcal{W}$ satisfying $\|\Delta v^\nu\|_\infty+\|\Delta z^\nu\|_\infty=\mathcal{O}(1/\nu)$, and such that $(z^\nu,v^\nu):=(\breve z^\nu,\breve v^\nu)+(\Delta z^\nu,\Delta v^\nu)$ belongs to $\mathcal{C}_\infty$. Let $(\xi^\nu,y^\nu,y^\nu(T))\in\mathcal{P}$ be defined by (4.2). Let us show that for each $\lambda\in M$,

$$\lim_{\nu\to\infty}\Omega_{\mathcal{P}}[\lambda](\xi^\nu,y^\nu,\dot y^\nu)=p_{\bar y}[\lambda]+r[\lambda].$$

Observe that

$$\lim_{\nu\to\infty}\Omega_{\mathcal{P}}[\lambda](\xi^\nu,y^\nu,\dot y^\nu)-p_{\bar y}[\lambda]=\lim_{\nu\to\infty}\int_0^T\{(V[\lambda]\bar y,\dot{\tilde y}^\nu)+(V[\lambda]\tilde y^\nu,\dot{\tilde y}^\nu)\}\mathrm{d}t, \tag{4.27}$$

since the terms involving $\xi^\nu-\bar\xi$, $y^\nu-\bar y$ or Δv^ν vanish as $\|\xi^\nu-\bar\xi\|_\infty\to0$ and $\|y^\nu-\bar y\|_1\to0$. Integrating by parts the first term in the right hand-side of (4.27). we obtain

$$\int_0^T(V[\lambda]\bar y,\dot{\tilde y}^\nu)\mathrm{d}t=[(V[\lambda]\bar y,\tilde y^\nu)]_0^T-\int_0^T\{(\dot V[\lambda]\bar y,\tilde y^\nu)+(V[\lambda]\dot{\bar y},\tilde y^\nu)\}\mathrm{d}t\overset{\nu\to\infty}{\to}0,$$

by (4.26) and since $\tilde y^\nu(0)=\tilde y^\nu(T)=0$. Coming back to (4.27) we have

$$\begin{aligned}\lim_{\nu\to\infty}\Omega_{\mathcal{P}}[\lambda](\xi^\nu,y^\nu,\dot y^\nu)-p_{\bar y}[\lambda]&=\lim_{\nu\to\infty}\int_0^T(V[\lambda]\tilde y^\nu,\dot{\tilde y}^\nu)\mathrm{d}t\\&=\lim_{\nu\to\infty}\int_0^T(V[\lambda](t)\tilde y(\nu(t-t^*)),\dot{\tilde y}(\nu(t-t^*)))\mathrm{d}\nu t\\&=\lim_{\nu\to\infty}\int_{-\nu t^*}^{\nu(T-t^*)}(V[\lambda](t^*+s/\nu)\tilde y(s),\dot{\tilde y}(s))\mathrm{d}s=r[\lambda],\end{aligned}$$

and thus (4.23) holds when $r \in R_{ij}(t^*)$.

Consider the general case when $r \in \mathcal{R}$, i.e. $r = \sum_{i<j} \sum_{k=1}^{N_{ij}} r_{ij}^k$, with each r_{ij}^k in $R_{ij}(t_{ij}^k)$. Let $\tilde{y}_{ij}^k$ be associated with r_{ij}^k by (4.21). Define $\tilde{y}_{ij}^{k,\nu}$ as in (4.24), and follow the previous procedure for $\bar{y} + \sum_{i<j} \sum_{k=1}^{N_{ij}} \tilde{y}_{ij}^{k,\nu}$ to get the desired result.

□

Proof. **[of Theorem 4.10]** Take $(\bar{\xi}, \bar{y}, \bar{y}(T)) \in \mathcal{P}$ and $r \in \mathcal{R}$. By Lemma 4.11 there exists a sequence $\{(\xi^\nu, y^\nu, y^\nu(T))\}$ in $\mathcal{P}$ such that for each $\lambda \in M$,

$$\Omega_{\mathcal{P}}[\lambda](\xi^\nu, y^\nu, \dot{y}^\nu) \to \Omega_{\mathcal{P}}[\lambda](\bar{\xi}, \bar{y}, \dot{\bar{y}}) + r[\lambda].$$

Since this convergence is uniform over M, from (4.19) we get that

$$\max_{\lambda \in M} (\Omega_{\mathcal{P}}[\lambda](\bar{\xi}, \bar{y}, \dot{\bar{y}}) + r[\lambda]) \geq 0, \quad \text{for all } r \in \mathcal{R}.$$

Hence

$$\inf_{r \in \mathcal{R}} \max_{\lambda \in M} (\Omega_{\mathcal{P}}[\lambda](\bar{\xi}, \bar{y}, \dot{\bar{y}}) + r[\lambda]) \geq 0, \tag{4.28}$$

where the expression in brackets is linear both in λ and r. Furthermore, note that M and $\mathcal{R}$ are convex, and M is compact. In light of MinMax Theorem [51, Corollary 37.3.2, page 39] we can invert the order of inf and max in (4.28) and obtain

$$\max_{\lambda \in M} \inf_{r \in \mathcal{R}} (\Omega_{\mathcal{P}}[\lambda](\bar{\xi}, \bar{y}, \dot{\bar{y}}) + r[\lambda]) \geq 0. \tag{4.29}$$

Suppose that, for certain $\lambda \in M$, there exists $r \in \mathcal{R}$ with $r[\lambda] \neq 0$. Then the infimum in (4.29) is $-\infty$ since $\mathcal{R}$ is a linear subspace. Hence, this λ does not provide the maximal value of the infima, and so, we can restrict the maximization to the set of $\lambda \in M$ for which $r[\lambda] = 0$ for every $r \in \mathcal{R}$. Note that this set is $G(M)$, and thus the conclusion follows. □

Consider for $i, j = 1, \ldots, m$:

$$I_{ij} := \{t \in (0, T) : \hat{u}_i(t) = 0, \ \hat{u}_j(t) > 0\}.$$

By Assumption 2, I_{ij} can be expressed as a finite union of intervals, i.e.

$$I_{ij} = \bigcup_{k=1}^{K_{ij}} I_{ij}^k, \text{ where } I_{ij}^k := (c_{ij}^k, d_{ij}^k).$$

Let $(z, v) \in \mathcal{C}_\infty$, $i \neq j$, and y be defined by (4.2). Notice that y_i is constant on each (c_{ij}^k, d_{ij}^k). Denote with $y_{i,j}^k$ its value on this interval.

Proposition 4.12. *Let* $(z, v) \in \mathcal{C}_\infty$, y *be defined by* (4.2) *and* $\lambda \in G(\Lambda)$. *Then*

$$\int_0^T (V[\lambda]y, v)\mathrm{d}t = \sum_{\substack{i \neq j \\ i,j=1}}^{m} \sum_{k=1}^{K_{ij}} y_{i,j}^k \left\{ [V_{ij}[\lambda]y_j]_{c_{ij}^k}^{d_{ij}^k} - \int_{c_{ij}^k}^{d_{ij}^k} \dot{V}_{ij}[\lambda]y_j \mathrm{d}t \right\}.$$

Proof. Observe that

$$\int_0^T (V[\lambda]y,v)\mathrm{d}t = \sum_{\substack{i\neq j\\ i,j=1}}^{m} \int_0^T V_{ij}[\lambda]y_i v_j \mathrm{d}t, \tag{4.30}$$

since $V_{ii}[\lambda] \equiv 0$. Fix $i \neq j$, and recall that that $V_{ij}[\lambda]$ is differentiable in time (see expression (4.11)). Since $(z,v) \in \mathcal{C}_\infty$ and $\lambda \in G(\Lambda)$,

$$\begin{aligned}\int_0^T V_{ij}[\lambda]y_i v_j \mathrm{d}t = \int_{I_{ij}} V_{ij}[\lambda]y_i v_j \mathrm{d}t &= \sum_{k=1}^{K_{ij}} \int_{c_{ij}^k}^{d_{ij}^k} V_{ij}[\lambda]y_i v_j \mathrm{d}t \\ &= \sum_{k=1}^{K_{ij}} y_{i,j}^k \left\{ [V_{ij}[\lambda]y_j]_{c_{ij}^k}^{d_{ij}^k} - \int_{c_{ij}^k}^{d_{ij}^k} \dot{V}_{ij}[\lambda]y_j \mathrm{d}t \right\},\end{aligned} \tag{4.31}$$

where the last equality was obtained by integrating by parts and knowing that y_i is constant on I_{ij}. The desired result follows from (4.30) and (4.31). □

Given a real function h and $c \in \mathbb{R}$, define

$$h(c+) := \lim_{t\to c+} h(t), \text{ and } h(c-) := \lim_{t\to c-} h(t).$$

Definition 4.13. *Let $(\xi,y,h) \in \mathcal{P}_2$ and $\lambda \in G(\Lambda)$. Define*

$$\Xi[\lambda](\xi,y,h) :=$$
$$2\sum_{\substack{i\neq j\\ i,j=1}}^{m} \sum_{\substack{k=1\\ c_{ij}^k\neq 0}}^{K_{ij}} y_{i,j}^k \left\{ V_{ij}[\lambda](d_{ij}^k)y_j(d_{ij}^k+) - V_{ij}[\lambda](c_{ij}^k)y_j(c_{ij}^k-) - \int_{c_{ij}^k}^{d_{ij}^k} \dot{V}_{ij}[\lambda]y_j \mathrm{d}t \right\},$$

where the above expression is interpreted as follows:

(i) $y_j(d_{ij}^k+) := h_j$, *if* $d_{ij}^k = T$,

(ii) $V_{ij}[\lambda](c_{ij}^k)y_j(c_{ij}^k-) := 0$, *if* $\hat{u}_i > 0$ *and* $\hat{u}_j > 0$ *for* $t < c_{ij}^k$,

(iii) $V_{ij}[\lambda](d_{ij}^k)y_j(d_{ij}^k+) := 0$, *if* $\hat{u}_i > 0$ *and* $\hat{u}_j > 0$ *for* $t > d_{ij}^k$.

Proposition 4.14. *The following properties for Ξ hold.*

(i) $\Xi[\lambda](\xi,y,h)$ *is well-defined for each* $(\xi,y,h) \in \mathcal{P}_2$, *and* $\lambda \in G(\Lambda)$.

(ii) If $\{(\xi^\nu, y^\nu, y^\nu(T))\} \subset \mathcal{P}$ *converges in the* $\mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}^m-$ *topology to* $(\xi,y,h) \in \mathcal{P}_2$ *, then*

$$\int_0^T (V[\lambda]y^\nu, \dot{y}^\nu)\mathrm{d}t \overset{\nu\to\infty}{\longrightarrow} \Xi[\lambda](\xi,y,h).$$

Proof. **(i)** Take $(\xi,y,h) \in \mathcal{P}_2$. First observe that $y_i \equiv y_{i,j}^k$ over (c_{ij}^k, d_{ij}^k). As $c_{ij}^k \neq 0$, two possible situations can arise,

(a) for $t < c_{ij}^k : \hat{u}_j = 0$, thus y_j is constant, and consequently $y_j(c_{ij}^k-)$ is well-defined,

(b) for $t < c_{ij}^k : \hat{u}_i > 0$ and $\hat{u}_j > 0$, thus $V_{ij}[\lambda](c_{ij}^k) = 0$ since $\lambda \in G(\Lambda)$.

The same analysis can be done for $t > d_{ij}^k$ when $d_{ij}^k \neq T$. We conclude that Ξ is correctly defined.
(ii) Observe that since y^ν converges to y in the $\mathcal{U}_2-$topology and since y_i^ν is constant over I_{ij}, then y_i is constant as well, and y_i^ν goes to y_i pointwise on I_{ij}. Thus, $y_i^\nu(c_{ij}^k) \longrightarrow y_{i,j}^k$, and $y_i^\nu(d_{ij}^k) \longrightarrow y_{i,j}^k$. Now, for the terms on y_j, the same analysis can be made, which yields either $y_j^\nu(c_{ij}^k) \longrightarrow y_j(c_{ij}^k-)$ or $V_{ij}[\lambda](c_{ij}^k) = 0$; and, either $y_j^\nu(d_{ij}^k) \longrightarrow y_j(d_{ij}^k+)$ or $V_{ij}[\lambda](d_{ij}^k) = 0$, when $d_{ij}^k < T$. For $d_{ij}^k = T$, $y_j^\nu(T) \longrightarrow h_j$ holds. □

Definition 4.15. *For $(\xi, y, h) \in \mathcal{P}_2$ and $\lambda \in G(\Lambda)$ define*

$$\begin{aligned}\Omega_{\mathcal{P}_2}[\lambda](\xi, y, h) :=& g[\lambda](\xi(T), h) + \Xi[\lambda](\xi, y, h)\\ &+ \int_0^T ((Q[\lambda]\xi, \xi) + 2(M[\lambda]\xi, y) + (R[\lambda]y, y))\mathrm{d}t.\end{aligned}$$

Remark 4.16. *Observe that when $m = 1$, the mapping $\Xi \equiv 0$ since $V \equiv 0$. Thus, in this case, $\Omega_{\mathcal{P}_2}$ can be defined for any element $(\xi, y, h) \in \mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}$ and any $\lambda \in \Lambda$. If we take $(z, v) \in \mathcal{W}$ satisfying* (2.12)*, and define (ξ, y) by* (4.2)*, then*

$$\Omega[\lambda](z, v) = \Omega_{\mathcal{P}}[\lambda](\xi, y, \dot{y}) = \Omega_{\mathcal{P}_2}[\lambda](\xi, y, y(T)).$$

For $m > 1$, the previous equality holds for $(z, v) \in \mathcal{C}_\infty$.

Lemma 4.17. *Let $\{(\xi^\nu, y^\nu, y^\nu(T)\} \subset \mathcal{P}$ be a sequence converging to $(\xi, y, h) \in \mathcal{P}_2$ in the $\mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}^m-$topology. Then*

$$\lim_{\nu\to\infty} \Omega_{\mathcal{P}}[\lambda](\xi^\nu, y^\nu, \dot{y}^\nu) = \Omega_{\mathcal{P}_2}[\lambda](\xi, y, h).$$

Denote with $\operatorname{co}\Lambda$ the convex hull of Λ.

Theorem 4.18. *Let $\hat{w}$ be a weak minimum, then*

$$\max_{\lambda\in G(\operatorname{co}\Lambda)} \Omega_{\mathcal{P}_2}[\lambda](\xi, y, h) \geq 0, \quad \text{for all } (\xi, y, h) \in \mathcal{P}_2. \tag{4.32}$$

Proof. Corollary 4.8 together with Theorem 4.10 applied to $M := \operatorname{co}\Lambda$ yield

$$\max_{\lambda\in G(\operatorname{co}\Lambda)} \Omega_{\mathcal{P}}[\lambda](\xi, y, \dot{y}) \geq 0, \quad \text{for all } (\xi, y, y(T)) \in \mathcal{P}.$$

The result follows from Lemma 4.1 and Lemma 4.17. □

Remark 4.19. *Notice that in case* (3.3) *is not satisfied, condition* (4.32) *does not provide any useful information as $0 \in \operatorname{co}\Lambda$. On the other hand, if* (3.3) *holds, every $\lambda = (\alpha, \beta, \psi) \in \Lambda$ necessarily has $\alpha \neq 0$, and thus $0 \notin \operatorname{co}\Lambda$.*

5 Sufficient condition

Consider the problem for a scalar control, i.e. let $m = 1$. This section provides a sufficient condition for Pontryagin optimality.

Definition 5.1. *Given $(y, h) \in \mathcal{U}_2 \times \mathbb{R}$, let*

$$\gamma(y, h) := \int_0^T y(t)^2\mathrm{d}t + |h|^2.$$

Definition 5.2. *A sequence $\{v_k\} \subset \mathcal{U}$ converges to 0 in the Pontryagin sense if $\|v_k\|_1 \to 0$ and there exists N such that $\|v_k\|_\infty < N$.*

Definition 5.3. *We say that $\hat{w}$ satisfies $\gamma-$quadratic growth condition in the Pontryagin sense if there exists $\rho > 0$ such that, for every sequence of feasible variations $\{(\delta x_k, v_k)\}$ with $\{v_k\}$ converging to 0 in the Pontryagin sense,*

$$J(\hat{u} + v_k) - J(\hat{u}) \geq \rho\gamma(y_k, y_k(T)), \tag{5.1}$$

holds for a large enough k, where y_k is defined by (4.2). Equivalently, for all $N > 0$, there exists $\varepsilon > 0$ such that if $\|v\|_\infty < N$ and $\|v\|_1 < \varepsilon$, then (5.1) holds.

Definition 5.4. *We say that $\hat{w}$ is normal if $\alpha_0 > 0$ for every $\lambda \in \Lambda$.*

Theorem 5.5. *Suppose that there exists $\rho > 0$ such that*

$$\max_{\lambda\in\Lambda} \Omega_{\mathcal{P}_2}[\lambda](\xi, y, h) \geq \rho\gamma(y, h), \quad \text{for all } (\xi, y, h) \in \mathcal{P}_2. \tag{5.2}$$

Then $\hat{w}$ is a Pontryagin minimum satisfying $\gamma-$ quadratic growth. Furthermore, if $\hat{w}$ is normal, the converse holds.

Remark 5.6. *In case the bang arcs are absent, i.e. the control is totally singular, this theorem reduces to one proved in Dmitruk [13, 15].*

Recall that Φ is defined in (2.5). We will use the following technical result.

Lemma 5.7. *Consider $\{v_k\} \subset \mathcal{U}$ converging to 0 in the Pontryagin sense. Let $u_k := \hat{u} + v_k$ and let x_k be the corresponding solution of equation (2.2). Then for every $\lambda \in \Lambda$,*

$$\Phi[\lambda](x_k, u_k) = \Phi[\lambda](\hat{x}, \hat{u}) + \int_0^T H_u[\lambda](t)v_k(t)\mathrm{d}t + \Omega[\lambda](z_k, v_k) + o(\gamma_k), \tag{5.3}$$

where z_k is defined by (2.12), $\gamma_k := \gamma(y_k, y_k(T))$, and y_k is defined by (4.2).

Proof. By Lemma 3.1 we can write

$$\Phi[\lambda](x_k, u_k) = \Phi[\lambda](\hat{x}, \hat{u}) + \int_0^T H_u[\lambda](t)v_k(t)\mathrm{d}t + \Omega[\lambda](z_k, v_k) + R_k,$$

where, in view of Lemma 8.5,

$$R_k := \Delta_k\Omega[\lambda] + \int_0^T (H_{uxx}[\lambda](t)\delta x_k(t), \delta x_k(t), v_k(t))\mathrm{d}t + o(\gamma_k), \tag{5.4}$$

with $\delta x_k := x_k - \hat{x}$, and

$$\Delta_k\Omega[\lambda] := \Omega[\lambda](\delta x_k, v_k) - \Omega[\lambda](z_k, v_k). \tag{5.5}$$

Next, we prove that

$$R_k = o(\gamma_k). \tag{5.6}$$

Note that $\mathcal{Q}(a, a) - \mathcal{Q}(b, b) = \mathcal{Q}(a + b, a - b)$, for any bilinear mapping $\mathcal{Q}$, and any pair a, b. Put $\eta_k := \delta x_k - z_k$. Hence, from (5.5), we get

$$\begin{aligned}\Delta_k\Omega[\lambda] =& \tfrac{1}{2}\ell''[\lambda](\hat{x}(T))(\delta x_k(T) + z_k(T), \eta_k(T)) \\ &+ \tfrac{1}{2}\int_0^T (H_{xx}[\lambda](\delta x_k + z_k), \eta_k)\mathrm{d}t + \int_0^T (H_{ux}[\lambda]\eta_k, v_k)\mathrm{d}t.\end{aligned}$$

By Lemmas 8.5 and 8.13 in the Appendix, the first and the second terms are of order $o(\gamma_k)$. Integrate by parts the last term to obtain

$$\int_0^T (H_{ux}[\lambda]\eta_k, v_k)\mathrm{d}t \tag{5.7}$$

$$= [(H_{ux}[\lambda]\eta_k, y_k)]_0^T - \int_0^T \{(\dot{H}_{ux}[\lambda]\eta_k, y_k) + (H_{ux}[\lambda]\dot{\eta}_k, y_k)\}\mathrm{d}t. \tag{5.8}$$

Thus, by Lemma 8.13 we deduce that the first two terms in (5.8) are of order $o(\gamma_k)$. It remains to deal with last term in the integral. Replace $\dot{\eta}_k$ by its expression in equation (8.13) of Lemma 8.13:

$$\begin{aligned}\int_0^T (H_{ux}[\lambda]\dot{\eta}_k, y_k)\mathrm{d}t &= \int_0^T (H_{ux}[\lambda]\left(\sum_{i=0}^1 \hat{u}_i f_i'(\hat{x})\eta_k + v_k f_1'(\hat{x})\delta x_k + \zeta_k\right), y_k)\mathrm{d}t \\ &= o(\gamma_k) + \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y_k^2}{2}\right) H_{ux}[\lambda] f_1'(\hat{x})\delta x_k \mathrm{d}t,\end{aligned} \tag{5.9}$$

where the second equality follows from Lemmas 8.5 and 8.13. Integrating the last term by parts, we obtain

$$\begin{aligned}&\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y_k^2}{2}\right) H_{ux}[\lambda] f_1'(\hat{x})\delta x_k \mathrm{d}t = \left[\frac{y_k^2}{2} H_{ux}[\lambda] f_1'(\hat{x})\delta x_k\right]_0^T \\ &- \int_0^T \frac{y_k^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(H_{ux}[\lambda] f_1'(\hat{x})\right)\delta x_k \mathrm{d}t - \int_0^T \frac{y_k^2}{2} H_{ux}[\lambda] f_1'(\hat{x})\dot{\delta x}_k \mathrm{d}t \\ &= o(\gamma_k) - \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y_k^3}{6}\right) H_{ux}[\lambda] f_1'(\hat{x}) f_1(\hat{x})\mathrm{d}t \\ &= o(\gamma_k) - \left[\frac{y_k^3}{6} H_{ux}[\lambda] f_1'(\hat{x}) f_1(\hat{x})\right]_0^T + \int_0^T \frac{y_k^3}{6}\frac{\mathrm{d}}{\mathrm{d}t}\left(H_{ux}[\lambda] f_1'(\hat{x}) f_1(\hat{x})\right)\mathrm{d}t \\ &= o(\gamma_k),\end{aligned} \tag{5.10}$$

where we used Lemma 8.13 and, in particular, equation (8.14). From (5.9) and (5.10), it follows that the term in (5.7) is of order $o(\gamma_k)$. Thus,

$$\Delta_k \Omega[\lambda] \leq o(\gamma_k). \tag{5.11}$$

Consider now the third order term in (5.4):

$$\begin{aligned}&\int_0^T (H_{uxx}[\lambda]\delta x_k, \delta x_k, v_k)\mathrm{d}t = [y_k \delta x_k^\top H_{uxx}[\lambda]\delta x_k]_0^T \\ &- \int_0^T y_k \delta x_k^\top \dot{H}_{uxx}[\lambda]\delta x_k \mathrm{d}t - 2\int_0^T y_k \delta x_k^\top H_{uxx}[\lambda]\dot{\delta x}_k \mathrm{d}t \\ &= o(\gamma_k) - \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}(y_k^2)\delta x_k^\top H_{uxx}[\lambda] f_1(\hat{x})\mathrm{d}t \\ &= o(\gamma_k) - \left[y_k^2 \delta x_k^\top H_{uxx}[\lambda] f_1(\hat{x})\right]_0^T - \int_0^T y_k^2 v_k f_1(\hat{x})^\top H_{uxx}[\lambda] f_1(\hat{x})\mathrm{d}t = o(\gamma_k),\end{aligned} \tag{5.12}$$

by Lemmas 8.5 and 8.13. The last inequality follows from integrating by parts one more time as it was done in (5.10). Consider expression (5.4). By inequality (5.11) and equation (5.12), equality (5.6) is obtained and thus, the desired result follows.

□

Proof. [of Theorem 5.5] *Part 1.* First we prove that if $\hat{w}$ is a normal Pontryagin minimum satisfying the $\gamma-$quadratic growth condition in the Pontryagin sense then (5.2) holds for some $\rho > 0$. Here the necessary condition of Theorem 3.2 is used. Define $\hat{y}(t) := \int_0^t \hat{u}(s)\mathrm{d}s$, and note that $(\hat{w}, \hat{y})$ is, for some $\rho' > 0$, a Pontryagin minimum of

$$\begin{aligned} &\tilde{J} := J - \rho'\gamma(y - \hat{y}, y(T) - \hat{y}(T)) \to \min, \\ &(2.2)\text{-}(2.4), \dot{y} = u,\ y(0) = 0. \end{aligned} \tag{5.13}$$

Observe that the critical cone $\tilde{\mathcal{C}}_2$ for (5.13) consists of the points $(z, v, \delta y)$ in $\mathcal{X}_2 \times \mathcal{U}_2 \times W_2^1(0, T; \mathbb{R})$ verifying $(z, v) \in \mathcal{C}_2$, $\dot{\delta y} = v$ and $\delta y(0) = 0$. Since the pre-Hamiltonian at point $(\hat{w}, \hat{y})$ coincides with the original pre-Hamiltonian, the set of multipliers for (5.13) consists of the points (λ, ψ_y) with $\lambda \in \Lambda$.

Applying the second order necessary condition of Theorem 3.2 at the point $(\hat{w}, \hat{y})$ we see that, for every $(z, v) \in \mathcal{C}_2$ and $\delta y(t) := \int_0^t v(s)\mathrm{d}s$, there exists $\lambda \in \Lambda$ such that

$$\Omega[\lambda](z, v) - \alpha_0\rho'(\|\delta y\|_2^2 + \delta y^2(T)) \geq 0, \tag{5.14}$$

where $\alpha_0 > 0$ since $\hat{w}$ is normal. Take $\rho := \min_{\lambda\in\Lambda} \alpha_0\rho' > 0$. Applying the Goh transformation in (5.14), condition (5.2) for the constant ρ follows.

Part 2. We shall prove that if (5.2) holds for some $\rho > 0$, then $\hat{w}$ satisfies $\gamma-$quadratic growth in the Pontryagin sense. On the contrary, assume that the quadratic growth condition (5.1) is not valid. Consequently, there exists a sequence $\{v_k\} \subset \mathcal{U}$ converging to 0 in the Pontryagin sense such that, denoting $u_k := \hat{u} + v_k$,

$$J(\hat{u} + v_k) \leq J(\hat{u}) + o(\gamma_k), \tag{5.15}$$

where $y_k(t) := \int_0^t v_k(s)ds$ and $\gamma_k := \gamma(y_k, y_k(T))$. Denote by x_k the solution of equation (2.2) corresponding to u_k, define $w_k := (x_k, u_k)$ and let z_k be the solution of (2.12) associated with v_k. Take any $\lambda \in \Lambda$. Multiply inequality (5.15) by α_0, add the nonpositive term $\sum_{i=0}^{d_\varphi} \alpha_i\varphi_i(x_k(T)) + \sum_{j=1}^{d_\eta} \beta_j\eta_j(x_k(T))$ to its left-hand side, and obtain the inequality

$$\Phi[\lambda](x_k, u_k) \leq \Phi[\lambda](\hat{x}, \hat{u}) + o(\gamma_k). \tag{5.16}$$

Recall expansion (5.3). Let $(\bar{y}_k, \bar{h}_k) := (y_k, y_k(T))/\sqrt{\gamma_k}$. Note that the elements of this sequence have unit norm in $\mathcal{U}_2 \times \mathbb{R}$. By the Banach-Alaoglu Theorem, extracting if necessary a sequence, we may assume that there exists $(\bar{y}, \bar{h}) \in \mathcal{U}_2 \times \mathbb{R}$ such that

$$\bar{y}_k \rightharpoonup \bar{y}, \text{ and } \bar{h}_k \to \bar{h}, \tag{5.17}$$

where the first limit is taken in the weak topology of $\mathcal{U}_2$. The remainder of the proof is split into two parts.

- **(a)** Using equations (5.3) and (5.16) we prove that $(\bar{\xi}, \bar{y}, \bar{h}) \in \mathcal{P}_2$, where $\bar{\xi}$ is a solution of (4.3).
- **(b)** We prove that $(\bar{y}, \bar{h}) = 0$ and that it is the limit of $\{(\bar{y}_k, \bar{h}_k)\}$ in the strong sense. This leads to a contradiction since each $(\bar{y}_k, \bar{h}_k)$ has unit norm.

(a) We shall prove that $(\bar{\xi}, \bar{y}, \bar{h}) \in \mathcal{P}_2$. From (5.3) and (5.16) it follows that

$$0 \leq \int_0^T H_u[\lambda](t)v_k(t)\mathrm{d}t \leq -\Omega_{\mathcal{P}_2}[\lambda](\xi_k, y_k, h_k) + o(\gamma_k),$$

where ξ_k is solution of (4.3) corresponding to y_k. The first inequality holds as $H_u[\lambda]v_k \geq 0$ almost everywhere on $[0,T]$ and we replaced $\Omega_{\mathcal{P}}$ by $\Omega_{\mathcal{P}_2}$ in view of Remark 4.16. By the continuity of mapping $\Omega_{\mathcal{P}_2}[\lambda]$ over $\mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}$ deduce that

$$0 \leq \int_0^T H_u[\lambda](t)v_k(t)\mathrm{d}t \leq O(\gamma_k),$$

and thus, for each composing interval (c,d) of I_0,

$$\lim_{k\to\infty} \int_c^d H_u[\lambda](t)\varphi(t)\frac{v_k(t)}{\sqrt{\gamma_k}}dt = 0, \tag{5.18}$$

for every nonnegative Lipschitz continuous function φ with $\operatorname{supp}\varphi \subset (c,d)$. The latter expression means that the support of φ is included in (c,d). Integrating by parts in (5.18) and by (5.17) we obtain

$$0 = \lim_{k\to\infty} \int_c^d \frac{\mathrm{d}}{\mathrm{d}t}\left(H_u[\lambda](t)\varphi(t)\right)\bar{y}_k(t)\mathrm{d}t = \int_c^d \frac{\mathrm{d}}{\mathrm{d}t}\left(H_u[\lambda](t)\varphi(t)\right)\bar{y}(t)\mathrm{d}t.$$

By Lemma 8.6, $\bar{y}$ is nondecreasing over (c,d). Hence, in view of Lemma 8.8, we can integrate by parts in the previous equation to get

$$\int_c^d H_u[\lambda](t)\varphi(t)\mathrm{d}\bar{y}(t) = 0. \tag{5.19}$$

Take $t_0 \in (c,d)$. By the strict complementary in Assumption 1, there exists $\lambda_0 \in \Lambda$ such that $H_u[\lambda_0](t_0) > 0$. Hence, in view of the continuity of $H_u[\lambda_0]$, there exists $\varepsilon > 0$ such that $H_u[\lambda_0] > 0$ on $(t_0-2\varepsilon, t_0+2\varepsilon) \subset (c,d)$. Choose φ such that $\operatorname{supp}\varphi \subset (t_0-2\varepsilon, t_0+2\varepsilon)$, and $H_u[\lambda_0](t)\varphi(t) = 1$ on $(t_0-\varepsilon, t_0+\varepsilon)$. Since $\mathrm{d}\bar{y} \geq 0$, equation (5.19) yields

$$\begin{aligned} 0 &= \int_c^d H_u[\lambda](t)\varphi(t)\mathrm{d}\bar{y}(t) \geq \int_{t_0-\varepsilon}^{t_0+\varepsilon} H_u[\lambda](t)\varphi(t)\mathrm{d}\bar{y}(t) \\ &= \int_{t_0-\varepsilon}^{t_0+\varepsilon} \mathrm{d}\bar{y}(t) = \bar{y}(t_0+\varepsilon) - \bar{y}(t_0-\varepsilon). \end{aligned}$$

As ε and $t_0 \in (c,d)$ are arbitrary we find that

$$\mathrm{d}\bar{y}(t) = 0, \quad \text{on } I_0, \tag{5.20}$$

and thus (4.7) holds. Let us prove condition (4.8) for $(\bar{\xi}, \bar{y}, \bar{h})$. Suppose that $0 \in I_0$. Take $\varepsilon > 0$, and notice that by Assumption 1 there exists $\lambda' \in \Lambda$ and $\delta > 0$ such that $H_u[\lambda'](t) > \delta$ for $t \in [0, d_1-\varepsilon]$, and thus by (5.18) we obtain $\int_0^{d_1-\varepsilon} v_k(t)/\sqrt{\gamma_k}\mathrm{d}t \to 0$, as $v_k \geq 0$. Then for all $s \in [0, d_1)$, we have

$$\bar{y}_k(s) \to 0,$$

and thus

$$\bar{y} = 0, \text{ on } [0, d_1), \text{ if } 0 \in I_0. \tag{5.21}$$

Suppose that $T \in I_0$. Then, we can derive $\int_{a_N+\varepsilon}^T \bar{v}_k(t)\mathrm{d}t \to 0$ by an analogous argument. Thus, the pointwise convergence

$$\bar{h}_k - \bar{y}_k(s) \to 0,$$

holds for every $s \in (a_N, T]$, and then,

$$\bar{y} = \bar{h}, \text{ on } (a_N, T], \text{ if } T \in I_0. \tag{5.22}$$

It remains to check the final conditions (4.6) for $\bar{h}$. Let $0 \leq i \leq d_\varphi$,

$$\begin{aligned} \varphi_i'(\hat{x}(T))(\bar{\xi}(T) + B(T)\bar{h}) &= \lim_{k\to\infty} \varphi_i'(\hat{x}(T)) \left(\frac{\xi_k(T) + B(T)h_k}{\sqrt{\gamma_k}} \right) \\ &= \lim_{k\to\infty} \varphi_i'(\hat{x}(T)) \frac{z_k(T)}{\sqrt{\gamma_k}}. \end{aligned} \tag{5.23}$$

A first order Taylor expansion of the function φ_i around $\hat{x}(T)$ gives

$$\varphi_i(x_k(T)) = \varphi_i(\hat{x}(T)) + \varphi_i'(\hat{x}(T))\delta x_k(T) + O(|\delta x_k(T)|^2).$$

By Lemmas 8.5 and 8.13 in the Appendix, we can write

$$\varphi_i(x_k(T)) = \varphi_i(\hat{x}(T)) + \varphi_i'(\hat{x}(T))z_k(T) + o(\sqrt{\gamma_k}).$$

Thus

$$\varphi_i'(\hat{x}(T))\frac{z_k(T)}{\sqrt{\gamma_k}} = \frac{\varphi_i(x_k(T)) - \varphi_i(\hat{x}(T))}{\sqrt{\gamma_k}} + o(1). \tag{5.24}$$

Since x_k satisfies (2.4), equations (5.23) and (5.24) yield, for $1 \leq i \leq d_\varphi$: $\varphi_i'(\hat{x}(T))(\bar{\xi}(T) + B(T)\bar{h}) \leq 0$. For $i = 0$ use inequality (5.15). Analogously,

$$\eta_j'(\hat{x}(T))(\bar{\xi}(T) + B(T)\bar{h}) = 0, \quad \text{for } j = 1, \dots, d_\eta.$$

Thus $(\bar{\xi}, \bar{y}, \bar{h})$ satisfies (4.6), and by (5.20), (5.21) and (5.22), we obtain

$$(\bar{\xi}, \bar{y}, \bar{h}) \in \mathcal{P}_2.$$

(b) Return to the expansion (5.3). Equation (5.16) and $H_u[\lambda] \geq 0$ imply

$$\begin{aligned} &\Omega_{\mathcal{P}_2}[\lambda](\xi_k, y_k, y_k(T)) = \\ &\qquad \Phi[\lambda](x_k, u_k) - \Phi[\lambda](\hat{x}, \hat{u}) - \int_0^T H_u[\lambda] v_k \mathrm{d}t - o(\gamma_k) \leq o(\gamma_k). \end{aligned}$$

Thus

$$\liminf_{k\to\infty} \Omega_{\mathcal{P}_2}[\lambda](\bar{\xi}_k, \bar{y}_k, \bar{h}_k) \leq \limsup_{k\to\infty} \Omega_{\mathcal{P}_2}[\lambda](\bar{\xi}_k, \bar{y}_k, \bar{h}_k) \leq 0. \tag{5.25}$$

Split $\Omega_{\mathcal{P}_2}$ as follows,

$$\Omega_{\mathcal{P}_2,w}[\lambda](\xi, y, h) := \int_0^T \{(Q[\lambda]\xi, \xi) + (M[\lambda]\xi, y)\}\mathrm{d}t + g[\lambda](\xi(T), h),$$

$$\Omega_{\mathcal{P}_2,0}[\lambda](y) := \int_{I_0} (R[\lambda]y, y)\mathrm{d}t,$$

and

$$\Omega_{\mathcal{P}_2,+}[\lambda](y) := \int_{I_+} (R[\lambda]y, y)\mathrm{d}t.$$

Notice that $\Omega_{\mathcal{P}_2,w}[\lambda]$ is weakly continuous in the space $\mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}$. Consider now the subspace

$$\Gamma_2 := \{(\xi, y, h) \in \mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R} : (4.3),\ (4.7) \text{ and } (4.8) \text{ hold}\}.$$

Notice that Γ_2 is itself a Hilbert space. Let $\rho > 0$ be the constant in the positivity condition (5.2) and define

$$\Lambda^\rho := \{\lambda \in \operatorname{co}\Lambda : \Omega_{\mathcal{P}_2}[\lambda] - \rho\gamma \text{ is weakly l.s.c. on } \Gamma_2\}.$$

Equation (5.2) and Lemma 8.12 in the Appendix imply that

$$\max_{\lambda\in\Lambda^\rho} \Omega_{\mathcal{P}_2}[\lambda](\bar{\xi},\bar{y},\bar{h}) \geq \rho\gamma(\bar{y},\bar{h}). \tag{5.26}$$

Denote by $\bar{\lambda}$ the element in Λ^ρ that reaches the maximum in (5.26). Next we show that $R[\bar{\lambda}](t) \geq \rho$ on I_+.

Observe that $\Omega_{\mathcal{P}_2,0}[\bar{\lambda}] - \rho\int_{I_0}|y(t)|^2\mathrm{d}t$ is weakly continuous in the space Γ_2. In fact, consider a sequence $\{(\tilde{\xi}_k,\tilde{y}_k,\tilde{h}_k)\} \subset \Gamma_2$ converging weakly to some $(\tilde{\xi},\tilde{y},\tilde{h}) \in \Gamma_2$. Since $\tilde{y}_k$ and $\tilde{y}$ are constant on I_0, necessarily $\tilde{y}_k \to \tilde{y}$ uniformly in every compact subset of I_0. Easily follows that

$$\lim_{k\to\infty} \Omega_{\mathcal{P}_2,0}[\bar{\lambda}](\tilde{y}_k) - \rho\int_{I_0}|\tilde{y}_k(t)|^2\mathrm{d}t = \Omega_{\mathcal{P}_2,0}[\bar{\lambda}](\tilde{y}) - \rho\int_{I_0}|\tilde{y}(t)|^2\mathrm{d}t, \tag{5.27}$$

and therefore, the weak continuity of $\Omega_{\mathcal{P}_2,0}[\bar{\lambda}] - \rho\int_{I_0}|y(t)|^2\mathrm{d}t$ in Γ_2 holds. Since $\Omega_{\mathcal{P}_2}[\bar{\lambda}] - \rho\gamma$ is weakly l.s.c. in Γ_2, we get that the (remainder) quadratic mapping

$$y \mapsto \Omega_{\mathcal{P}_2,+}[\bar{\lambda}](y) - \rho\int_{I_+}|y(t)|^2\mathrm{d}t, \tag{5.28}$$

is weakly l.s.c. on Γ_2. In particular, it is weakly l.s.c. in the subspace of Γ_2 consisting of the elements for which $y = 0$ on I_0. Hence, in view of Lemma 8.11 in the Appendix, we get

$$R[\bar{\lambda}](t) \geq \rho, \quad \text{on } I_+. \tag{5.29}$$

The following step is proving the strong convergence of $\bar{y}_k$ to $\bar{y}$. With this aim we make use of the uniform convergence on compact subsets of I_0, which is pointed out in Lemma 8.7.

Recall now Assumption 2, and let N be the number of connected components of I_0. Set $\varepsilon > 0$, and for each composing interval (c,d) of I_0, consider a smaller interval of the form $(c+\varepsilon/2N, d-\varepsilon/2N)$. Denote their union as I_0^ε. Notice that $I_0\backslash I_0^\varepsilon$ is of measure ε. Put $I_+^\varepsilon := [0,T]\backslash I_0^\varepsilon$. By the Lemma 8.9 in the Appendix, $R[\bar{\lambda}](t)$ is a continuous function of time, and thus from (5.29) we can assure that $R[\bar{\lambda}](t) \geq \rho/2$ on I_+^ε for ε sufficiently small. Consequently,

$$\Omega^\varepsilon_{\mathcal{P}_2,+}[\bar{\lambda}](y) := \int_{I_+^\varepsilon}(R[\bar{\lambda}]y,y)\mathrm{d}t,$$

is a Legendre form on $L_2(I_+^\varepsilon)$, and thus the following inequality holds for the approximating directions $\bar{y}_k$,

$$\Omega^\varepsilon_{\mathcal{P}_2,+}[\bar{\lambda}](\bar{y}) \leq \liminf_{k\to\infty} \Omega^\varepsilon_{\mathcal{P}_2,+}[\bar{\lambda}](\bar{y}_k). \tag{5.30}$$

Since the sequence $\bar{y}_k$ converges uniformly to $\bar{y}$ on every compact subset of I_0, defining

$$\Omega^\varepsilon_{\mathcal{P}_2,0}[\bar{\lambda}](y) := \int_{I_0^\varepsilon}(R[\bar{\lambda}]y,y)\mathrm{d}t,$$

we get

$$\lim_{k\to\infty} \Omega^\varepsilon_{\mathcal{P}_2,0}[\bar{\lambda}](\bar{\xi}_k,\bar{y}_k,\bar{h}_k) = \Omega^\varepsilon_{\mathcal{P}_2,0}[\bar{\lambda}](\bar{\xi},\bar{y},\bar{h}). \tag{5.31}$$

Notice that the weak continuity of $\Omega^{\varepsilon}_{\mathcal{P}_2,0}[\bar{\lambda}]$ in Γ_2 cannot be applied since $(\bar{\xi}_k, \bar{y}_k, \bar{h}_k) \notin \Gamma_2$. From positivity condition (5.2), equations (5.30), (5.31), and the weak continuity of $\Omega_{\mathcal{P}_2,w}[\bar{\lambda}]$ (in $\mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}$) we get

$$
\begin{aligned}
\rho\gamma(\bar{y},\bar{h}) \leq \Omega_{\mathcal{P}_2}[\bar{\lambda}](\bar{\xi},\bar{y},\bar{h}) &\leq \lim_{k\to\infty} \Omega_{\mathcal{P}_2,w}[\bar{\lambda}](\bar{\xi}_k,\bar{y}_k,\bar{h}_k) + \lim_{k\to\infty} \Omega^{\varepsilon}_{\mathcal{P}_2,0}[\bar{\lambda}](\bar{y}_k) \\
&+ \liminf_{k\to\infty} \Omega^{\varepsilon}_{\mathcal{P}_2,+}[\bar{\lambda}](\bar{y}_k) = \liminf_{k\to\infty} \Omega_{\mathcal{P}_2}[\bar{\lambda}](\bar{\xi}_k,\bar{y}_k,\bar{h}_k).
\end{aligned}
$$

On the other hand, inequality (5.25) implies that the right-hand side of the last expression is nonpositive. Therefore,

$$
(\bar{y},\bar{h}) = 0, \text{ and } \lim_{k\to\infty} \Omega_{\mathcal{P}_2}[\bar{\lambda}](\bar{\xi}_k,\bar{y}_k,\bar{h}_k) = 0.
$$

Equation (5.31) yields $\lim_{k\to\infty} \Omega^{\varepsilon}_{\mathcal{P}_2,0}[\bar{\lambda}](\bar{\xi}_k,\bar{y}_k,\bar{h}_k) = 0$ and thus

$$
\lim_{k\to\infty} \Omega^{\varepsilon}_{\mathcal{P}_2,+}[\bar{\lambda}](\bar{y}_k) = 0. \tag{5.32}
$$

We have: $\Omega^{\varepsilon}_{\mathcal{P}_2,+}[\bar{\lambda}]$ is a Legendre form on $L_2(I^{\varepsilon}_+)$ and $\bar{y}_k \rightharpoonup 0$ on I^{ε}_+. Thus, by (5.32),

$$
\bar{y}_k \to 0, \quad \text{on } L_2(I^{\varepsilon}_+).
$$

As we already noticed, $\{\bar{y}_k\}$ converges uniformly on I^{ε}_0, thus the strong convergence holds on $[0,T]$. Therefore

$$
(\bar{y}_k,\bar{h}_k) \longrightarrow (0,0), \quad \text{on } \mathcal{U}_2 \times \mathbb{R}. \tag{5.33}
$$

This leads to a contradiction since $(\bar{y}_k,\bar{h}_k)$ has unit norm for every $k \in I\!N$. Thus, $\hat{w}$ is a Pontryagin minimum satisfying quadratic growth.

□

6 Extensions and an example

6.1 Including parameters

Consider the following optimal control problem where the initial state is not determined, some parameters are included and a more general control constraint is considered.

$$
J := \varphi_0(x(0), x(T), r(0)) \to \min, \tag{6.1}
$$

$$
\dot{x}(t) = \sum_{i=0}^{m} u_i(t) f_i(x(t), r(t)), \tag{6.2}
$$

$$
\dot{r}(t) = 0, \tag{6.3}
$$

$$
a_i \leq u_i(t) \leq b_i, \text{ for a.a. } t \in (0,T), \ i = 1,\dots,m \tag{6.4}
$$

$$
\varphi_i(x(0), x(T), r(0)) \leq 0, \text{ for } i = 1,\dots,d_\varphi, \tag{6.5}
$$

$$
\eta_j(x(0), x(T), r(0)) = 0, \text{ for } j = 1\dots,d_\eta, \tag{6.6}
$$

where $u \in \mathcal{U}$, $x \in \mathcal{X}$, $r \in \mathbb{R}^{n_r}$ is a parameter considered as a state variable with zero-dynamics, $a, b \in \mathbb{R}^m$, functions $f_i : \mathbb{R}^{n+n_r} \to \mathbb{R}^n$, $\varphi_i : \mathbb{R}^{2n+n_r} \to \mathbb{R}$, and $\eta : \mathbb{R}^{2n+n_r} \to \mathbb{R}^{d_\eta}$ are twice continuously differentiable. As r has zero dynamics, the costate variable ψ_r corresponding to equation (6.3) does not appear in the pre-Hamiltonian. Denote with ψ the costate variable associated with (6.2). The pre-Hamiltonian function for problem (6.1)-(6.6) is given by

$$H[\lambda](x,r,u,t) = \psi(t)\sum_{i=0}^{m} u_i f_i(x,r).$$

Let $(\hat{x},\hat{r},\hat{u})$ be a feasible solution for (6.2)-(6.6). Since $\hat{r}(\cdot)$ is constant, we can denote it by $\hat{r}$. Assume that

$$\varphi_i(\hat{x}(0),\hat{x}(T),\hat{r}) = 0, \quad \text{for } i = 0,\dots,d_\varphi.$$

An element $\lambda = (\alpha,\beta,\psi_x,\psi_r) \in \mathbb{R}^{d_\varphi+d_\eta+1} \times W^1_\infty(0,T;\mathbb{R}^{n,*}) \times W^1_\infty(0,T;\mathbb{R}^{n_r,*})$ is a Pontryagin multiplier for $(\hat{x},\hat{r},\hat{u})$ if it satisfies (2.7), (2.8), the costate equation for ψ

$$\begin{cases} -\dot{\psi}_x(t) = H_x[\lambda](\hat{x}(t),\hat{r},\hat{u}(t),t), \text{ a.e. on } [0,T] \\ \psi_x(0) = -\ell_{x_0}[\lambda](\hat{x}(0),\hat{x}(T),\hat{r}), \\ \psi_x(T) = \ell_{x_T}[\lambda](\hat{x}(0),\hat{x}(T),\hat{r}), \end{cases}$$

and for ψ_r

$$\begin{cases} -\dot{\psi}_r(t) = H_r[\lambda](\hat{x}(t),\hat{r},\hat{u}(t),t), \text{ a.e. on } [0,T] \\ \psi_r(0) = -\ell_r[\lambda](\hat{x}(0),\hat{x}(T),\hat{r}), \ \psi_r(T) = 0. \end{cases} \tag{6.7}$$

Observe that (6.7) implies the stationarity condition

$$\ell_r(\hat{x}(0),\hat{x}(T),\hat{r}) + \int_0^T H_r[\lambda](t)\mathrm{d}t = 0.$$

Take $v \in \mathcal{U}$ and consider the linearized state equation

$$\begin{cases} \dot{z}(t) = \displaystyle\sum_{i=0}^{m} \hat{u}_i(t)[f_{i,x}(\hat{x}(t),\hat{r})z(t) + f_{i,r}(\hat{x}(t),\hat{r})\delta r(t)] + \sum_{i=1}^{m} v_i(t) f_i(\hat{x}(t),\hat{r}), \\ \dot{\delta r}(t) = 0, \end{cases} \tag{6.8}$$

where we can see that $\delta r(\cdot)$ is constant and thus we denote it by δr. Let the linearized initial-final constraints be

$$\begin{aligned} &\varphi_i'(\hat{x}(0),\hat{x}(T),\hat{r})(z(0),z(T),\delta r) \leq 0, \quad \text{for } i = 1,\dots,d_\varphi, \\ &\eta_j'(\hat{x}(0),\hat{x}(T),\hat{r})(z(0),z(T),\delta r) = 0, \quad \text{for } j = 1,\dots,d_\eta. \end{aligned} \tag{6.9}$$

Define for each $i = 1,\dots,m$ the sets

$$\begin{aligned} I_a^i &:= \{t \in [0,T] : \max_{\lambda\in\Lambda} H_{u_i}[\lambda](t) > 0\}, \\ I_b^i &:= \{t \in [0,T] : \max_{\lambda\in\Lambda} H_{u_i}[\lambda](t) < 0\}, \\ I_{\text{sing}}^i &:= [0,T]\backslash(I_a^i \cup I_b^i). \end{aligned}$$

Assumption 3. Consider the natural extension of Assumption 2, i.e. for each $i = 1,\dots,m$, the sets I_a^i and I_b^i are finite unions of intervals, i.e.

$$I_a^i = \bigcup_{j=1}^{N_a^i} I_{j,a}^i, \quad I_b^i = \bigcup_{j=1}^{N_b^i} I_{j,b}^i,$$

for $I^i_{j,a}$ and $I^i_{j,b}$ being subintervals of $[0,T]$ of the form $[0,c)$, $(d,T]$; or (c,d) if $c\neq 0$ and $d\neq T$. Notice that $I^i_a\cap I^i_b=\emptyset$. Call $c^i_{1,a}<d^i_{1,a}<c^i_{2,a}<\ldots<c^i_{N^i_a,a}<d^i_{N^i_a,a}$ the endpoints of these intervals corresponding to bound a, and define them analogously for b. Consequently, $I^i_{\rm sing}$ is a finite union of intervals as well. Assume that a concatenation of a bang arc followed by another bang arc is forbidden.

Assumption 4. Strict complementarity assumption for control constraints:

$$\begin{cases} I^i_a=\{t\in[0,T]:\hat u_i(t)=a_i\}, \text{ up to a set of null measure,}\\ I^i_b=\{t\in[0,T]:\hat u_i(t)=b_i\}, \text{ up to a set of null measure.}\end{cases}$$

Consider

$$\mathcal{C}_2:=\left\{\begin{array}{l}(z,\delta r,v)\in\mathcal{X}_2\times\mathbb{R}^{n_r}\times\mathcal{U}_2: \text{(6.8)-(6.9) hold,}\\ v_i=0 \text{ on } I^i_{\rm a}\cup I^i_{\rm b}, \text{ for } i=1,\ldots,m\end{array}\right\}.$$

The Goh transformation allows us to obtain variables (ξ,y) defined by

$$y(t):=\int_0^t v(s)\mathrm{d}s,\ \xi:=z-\sum_{i=1}^m y_if_i.$$

Notice that ξ satisfies the equation

$$\begin{aligned}\dot\xi&=A^x\xi+A^r\delta r+B_1^xy,\\ \xi(0)&=z(0),\end{aligned} \tag{6.10}$$

where, denoting $[f_i,f_j]^x:=f_{i,x}f_j-f_{j,x}f_i$,

$$A^x:=\sum_{i=0}^m\hat u_if_{i,x},\quad A^r:=\sum_{i=0}^m\hat u_if_{i,r},\quad B_1^xy:=\sum_{j=1}^m y_j\sum_{i=0}^m\hat u_i[f_i,f_j]^x.$$

Consider the transformed version of (6.9),

$$\begin{aligned}&\varphi_i'(\hat x(0),\hat x(T),\hat r)(\xi(0),\xi(T)+B(T)h,\delta r)\leq 0,\ i=1,\ldots,d_\varphi,\\ &\eta_j'(\hat x(0),\hat x(T),\hat r)(\xi(0),\xi(T)+B(T)h,\delta r)=0,\ j=1,\ldots,d_\eta,\end{aligned} \tag{6.11}$$

and let the cone $\mathcal{P}$ be given by

$$\mathcal{P}:=\left\{\begin{array}{l}(\xi,\delta r,y,h)\in\mathcal{X}\times\mathbb{R}^{n_r}\times\mathcal{Y}\times\mathbb{R}^m: y(0)=0,\ h=y(T),\\ \text{(6.10) and (6.11) hold},\ y_i'=0 \text{ on } I^i_{\rm a}\cup I^i_{\rm b}, \text{ for } i=1,\ldots,m\end{array}\right\}.$$

Observe that each $(\xi,\delta r,y,h)\in\mathcal{P}$ satisfies

$$y_i \text{ constant over each composing interval of } I^i_{\rm a}\cup I^i_{\rm b}, \tag{6.12}$$

and at the endpoints,

$$\begin{cases} y_i=0 \text{ on } [0,d],\ \text{if } 0\in I^i_{\rm a}\cup I^i_{\rm b}, \text{ and,}\\ y_i=h_i \text{ on } [c,T],\ \text{if } T\in I^i_{\rm a}\cup I^i_{\rm b},\end{cases} \tag{6.13}$$

where $[0,d)$ is the first maximal composing interval of $I^i_{\rm a}\cup I^d_{\rm b}$ when $0\in I^d_{\rm a}\cup I^d_{\rm b}$, and $(c,T]$ is its last composing interval when $T\in I^i_{\rm a}\cup I^i_{\rm b}$. Define

$$\mathcal{P}_2:=\left\{\begin{array}{l}(\xi,\delta r,y,h)\in\mathcal{X}_2\times\mathbb{R}^{n_r}\times\mathcal{U}_2\times\mathbb{R}^m:\\ \text{(6.10), (6.11), (6.12) and (6.13) hold for } i=1,\ldots,m\end{array}\right\}.$$

Recall definitions in equations (4.9), (4.10), (4.13), (4.14), (4.15). Minor simplifications appear in the computations of these functions as the dynamics of r are null and δr is constant. We outline these calculations in an example.

Consider $M \subset \mathbb{R}^s$ and the subset of $M \subset \mathbb{R}^s$ defined by

$$G(M) := \{\lambda \in M : V_{ij}[\lambda] = 0 \text{ on } I^i_{\text{sing}} \cap I^j_{\text{sing}}, \text{ for every pair } 1 < i \neq j \leq m\}.$$

Using the same techniques, we obtain the equivalent of Theorem 4.18:

Corollary 6.1. *Suppose that* $(\hat{x}, \hat{r}, \hat{u})$ *is a weak minimum for problem* (6.1)-(6.6). *Then*

$$\max_{\lambda \in G(\text{co}\,\Lambda)} \Omega_{\mathcal{P}_2}[\lambda](\xi, \delta r, y, h) \geq 0, \quad \text{for all } (\xi, \delta r, y, h) \in \mathcal{P}_2.$$

By a simple adaptation of the proof of Theorem 5.5 we get the equivalent result.

Corollary 6.2. *Let* $m = 1$. *Suppose that there exists* $\rho > 0$ *such that*

$$\max_{\lambda \in \Lambda} \Omega_{\mathcal{P}_2}[\lambda](\xi, \delta r, y, h) \geq \rho\gamma(y, h), \quad \text{for all } (\xi, \delta r, y, h) \in \mathcal{P}_2. \tag{6.14}$$

Then $(\hat{x}, \hat{r}, \hat{u})$ *is a Pontryagin minimum that satisfies* $\gamma-$*quadratic growth.*

6.2 Application to minimum-time problems

Consider the problem

$$\begin{aligned} &J := T \to \min, \\ &\text{s.t. } (6.2) - (6.6). \end{aligned}$$

Observe that by the change of variables:

$$x(s) \leftarrow x(Ts), \quad u(s) \leftarrow u(Ts), \tag{6.15}$$

we can transform the problem into the following formulation.

$$\begin{aligned} &J := T(0) \to \min, \\ &\dot{x}(s) = T(s) \sum_{i=0}^m u_i(s) f_i(x(s), r(s)), \quad \text{a.e. on } [0,1], \\ &\dot{r}(s) = 0, \quad \text{a.e. on } [0,1], \\ &\dot{T}(s) = 0, \quad \text{a.e. on } [0,1], \\ &a_i \leq u_i(s) \leq b_i, \quad \text{a.e. on } [0,1],\ i = 1, \dots, m, \\ &\varphi_i(x(0), x(1), r(0)) \leq 0, \text{ for } i = 1, \dots, d_\varphi, \\ &\eta_j(x(0), x(T), r(0)) = 0, \text{ for } j = 1 \dots, d_\eta. \end{aligned}$$

We can apply Corollaries 6.1 and 6.2 to the problem written in this form. We outline the calculations in the following example.

6.2.1 Example: Markov-Dubins problem

Consider a problem over the interval $[0,T]$ with free final time T :

$$
\begin{aligned}
&J := T \to \min,\\
&\dot{x}_1 = -\sin x_3,\ x_1(0) = 0,\ x_1(T) = b_1,\\
&\dot{x}_2 = \cos x_3,\ x_2(0) = 0,\ x_2(T) = b_2,\\
&\dot{x}_3 = u,\ x_3(0) = 0,\ x_3(T) = \theta,\\
&-1 \leq u \leq 1,
\end{aligned}
\tag{6.16}
$$

with $0 < \theta < \pi$, b_1 and b_2 fixed.

This problem was originally introduced by Markov in [39] and studied by Dubins in [18]. More recently, the problem was investigated by Sussmann and Tang [60], Soueres and Laumond [56], Boscain and Piccoli [7], among others.

Here we will study the optimality of the extremal

$$
\hat{u}(t) := \begin{cases} 1 & \text{on } [0,\theta], \\ 0 & \text{on } (\theta, \hat{T}]. \end{cases}
\tag{6.17}
$$

Observe that by the change of variables (6.15) we can transform (6.16) into the following problem on the interval $[0,1]$.

$$
\begin{aligned}
&J := T(0) \to \min,\\
&\dot{x}_1(s) = -T(s)\sin x_3(s),\ x_1(0) = 0,\ x_1(1) = b_1,\\
&\dot{x}_2(s) = T(s)\cos x_3(s),\ x_2(0) = 0,\ x_2(1) = b_2,\\
&\dot{x}_3(s) = T(s)u(s),\ x_3(0) = 0,\ x_3(1) = \theta,\\
&\dot{T}(s) = 0,\\
&-1 \leq u(s) \leq 1.
\end{aligned}
\tag{6.18}
$$

We obtain for state variables:

$$
\hat{x}_3(s) = \begin{cases} \hat{T}s & \text{on } [0,\theta/\hat{T}], \\ \theta & \text{on } (\theta/\hat{T}, 1], \end{cases}
\tag{6.19}
$$

$$
\hat{x}_1(s) = \begin{cases} \cos(\hat{T}s) - 1 & \text{on } [0,\theta/\hat{T}], \\ \hat{T}\sin\theta(\theta/\hat{T} - s) + \cos\theta - 1 & \text{on } (\theta/\hat{T}, 1], \end{cases}
$$

$$
\hat{x}_2(s) = \begin{cases} \sin\hat{T}s & \text{on } [0,\theta/\hat{T}], \\ \hat{T}\cos\theta(s - \theta/\hat{T}) + \sin\theta & \text{on } (\theta, \hat{T}]. \end{cases}
$$

Since the terminal values for x_1 and x_2 are fixed, the final time $\hat{T}$ is determined by the previous equalities. The pre-Hamiltonian for problem (6.18) is

$$
H[\lambda](s) := T(s)(-\psi_1(s)\sin x_3(s) + \psi_2(s)\cos x_3(s) + \psi_3(s)u(s)).
\tag{6.20}
$$

The final Lagrangian is

$$
\ell := \alpha_0 T(1) + \sum_{j=1}^{3}(\beta^j x_j(0) + \beta_j x_j(1)).
$$

As $\dot\psi_1 \equiv 0$, and $\dot\psi_2 \equiv 0$, we get

$$\psi_1 \equiv \beta_1, \quad \psi_2 \equiv \beta_2, \quad \text{on } [0,1].$$

Since the candidate control $\hat u$ is singular on $[\theta/\hat T, 1]$, we have $H_u[\lambda] \equiv 0$. By (6.20), we obtain

$$\psi_3(s) = 0, \quad \text{on } [\theta/\hat T, 1]. \tag{6.21}$$

Thus $\beta_3 = 0$. In addition, as the costate equation for ψ_3 is

$$-\dot\psi_3 = \hat T(-\beta_1 \cos \hat x_3 - \beta_2 \sin \hat x_3),$$

by (6.19) and (6.21), we get

$$\beta_1 \cos\theta + \beta_2 \sin\theta = 0. \tag{6.22}$$

From (6.19) and (6.21) and since H is constant and equal to $-\alpha_0$, we get

$$H = \hat T(-\beta_1 \sin\theta + \beta_2 \cos\theta) \equiv -\alpha_0. \tag{6.23}$$

Proposition 6.3. *The following properties hold*

(i) $\alpha_0 > 0$,

(ii) $H_u[\lambda](s) < 0$ *on* $[0, \theta/\hat T)$ *for all* $\lambda \in \Lambda$.

Proof. **Item (i)** Suppose that $\alpha_0 = 0$. By (6.22) and (6.23), we obtain

$$\beta_1 \cos\theta + \beta_2 \sin\theta = 0, \text{ and } -\beta_1 \sin\theta + \beta_2 \cos\theta = 0.$$

Suppose, w.l.g., that $\cos\theta \neq 0$. Then $\beta_1 = -\beta_2 \dfrac{\sin\theta}{\cos\theta}$ and thus

$$\beta_2 \frac{\sin^2\theta}{\cos\theta} + \beta_2 \cos\theta = 0.$$

We conclude that $\beta_2 = 0$ as well. This implies $(\alpha_0, \beta_1, \beta_2, \beta_3) = 0$, which contradicts the non-triviality condition (2.7). So, $\alpha_0 > 0$, as required.
Item (ii) Observe that

$$H_u[\lambda](s) \leq 0, \quad \text{on } [0, \theta/\hat T),$$

and $H_u[\lambda] = \psi_3$. Let us prove that ψ_3 is never 0 on $[0, \theta/\hat T)$. Suppose there exists $s_1 \in [0, \theta/\hat T)$ such that $\psi_3(s_1) = 0$. Thus, since $\psi_3(\theta/\hat T) = 0$ as indicated in (6.21), there exists $s_2 \in (s_1, \theta/\hat T)$ such that $\dot\psi_3(s_2) = 0$, i.e.

$$\beta_1 \cos(\hat T s_2) + \beta_2 \sin(\hat T s_2) = 0. \tag{6.24}$$

Equations (6.22) and (6.24) imply that $\tan(\theta/\hat T) = \tan(s_2/\hat T)$. This contradicts $\theta < \pi$. Thus $\psi_3(s) \neq 0$ for every $s \in [0, \theta/\hat T)$, and consequently,

$$H_u[\lambda](s) < 0, \quad \text{for } s \in [0, \theta/\hat T).$$

□

Since $\alpha_0 > 0$, then $\delta T = 0$ for each element of the critical cone, where δT is the linearized state variable T. Observe that as $\hat{u} = 1$ on $[0, \theta/\hat{T}]$, then

$$y = 0 \text{ and } \xi = 0, \text{ on } [0, \theta/\hat{T}], \text{ for all } (\xi, \delta T, y, h) \in \mathcal{P}_2.$$

We look for the second variation in the interval $[\theta/\hat{T}, 1]$. The Goh transformation gives

$$\xi_3 = z_3 - \hat{T} y,$$

and since $\dot{z}_3 = \hat{T} v$, we get $z_3 = \hat{T} y$ and thus $\xi_3 = 0$. Then, as $H_{ux} = 0$ and $\ell'' = 0$, we get

$$\Omega[\lambda] = \int_{\theta/\hat{T}}^{1} (\beta_1 \sin\theta - \beta_2 \cos\theta) y^2 dt = \alpha_0 \int_0^1 y^2 dt.$$

Notice that if $(\xi, \delta T, y, h) \in \mathcal{P}_2$, then h satisfies $\xi_3(T) + \hat{T} h = 0$, and, as $\xi_3(T) = 0$, we get $h = 0$. Thus

$$\Omega[\lambda](\xi, y, h) = \alpha_0 \int_0^T y^2 dt = \alpha_0 \gamma(y, h), \quad \text{on } \mathcal{P}_2.$$

Since Assumptions 3 and 4 hold, we conclude by Corollary 6.2 that $(\hat{x}, \hat{T}, \hat{u})$ is a Pontryagin minimum satisfying quadratic growth.

7 Conclusion

We provided a set of necessary and sufficient conditions for a bang-singular extremal. The sufficient condition is restricted to the scalar control case. These necessary and sufficient conditions are close in the sense that, to pass from one to the other, one has to strengthen a non-negativity inequality transforming it into a coercivity condition.

This is the first time that a sufficient condition that is 'almost necessary' is established for a bang-singular extremal for the general Mayer problem. In some cases the condition can be easily checked as it can be seen in the example.

8 Appendix

Lemma 8.1. *Let*

$$X := \{(\xi, y, h) \in \mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}^m : \; (4.3), (4.7)\text{-}(4.8) \text{ hold}\},$$

$$L := \{(\xi, y, y(T)) \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}^m : y(0) = 0, \; (4.3) \text{ and } (4.7)\}.$$

Then L is a dense subset of X in the $\mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}^m-$topology.

Proof. (See Lemma 6 in [17].) Let us prove the result for $m = 1$. The general case is a trivial extension. Let $(\bar{\xi}, \bar{y}, \bar{h}) \in X$ and $\varepsilon, \delta > 0$. Consider $\phi \in \mathcal{Y}$ such that $\|\bar{y} - \phi\|_2 < \varepsilon/2$. In order to satisfy condition (4.8) take

$$\begin{cases} y_\delta(t) := 0, & \text{for } t \in [0, d_1], \quad \text{if } c_1 = 0, \\ y_\delta(t) := h, & \text{for } t \in [c_N, T], \quad \text{if } d_N = T, \end{cases}$$

where c_j, d_j were introduced in Assumption 2. Since $\bar{y}$ is constant on each I_j, define y_δ constant over these intervals with the same constant value as $\bar{y}$. It remains to define y_δ over I_+. Over each maximal composing interval (a,b) of I_+, define y_δ as described below. Take $c := \bar{y}(a-)$ if $a>0$, or $c:=0$ if $a=0$; and let $d:=\bar{y}(b+)$ if $b<T$, or $d:=h$ when $b=T$. Define two affine functions $\ell_{1,\delta}$ and $\ell_{2,\delta}$ satisfying

$$\begin{aligned} &\ell_{1,\delta}(a)=c,\ \ \ell_{1,\delta}(a+\delta)=\phi(a+\delta),\\ &\ell_{2,\delta}(b)=d,\ \ \ell_{2,\delta}(b-\delta)=\phi(b-\delta). \end{aligned} \tag{8.1}$$

Take

$$y_\delta(t):=\begin{cases} \ell_{1,\delta}(t), & \text{for } t\in[a,a+\delta],\\ \phi(t), & \text{for } t\in(a+\delta,b-\delta),\\ \ell_{2,\delta}(t), & \text{for } t\in[b-\delta,b], \end{cases} \tag{8.2}$$

and notice that $\|\phi-y_\delta\|_{2,[a,b]}\leq \frac{1}{k}\max(|c|,|d|,M)$, where $M:=\sup_{t\in[a,b]}|\phi(t)|$. Finally, observe that $y_\delta(T)=h$, and, for sufficiently small δ,

$$\|\bar{y}-y_\delta\|_2\leq\|\bar{y}-\phi\|_2+\|\phi-y_\delta\|_2<\varepsilon.$$

Thus, the result follows. □

Lemma 8.2. *Let $\lambda\in\Lambda$ and $(z,v)\in\mathcal{C}_2$. Then*

$$\sum_{i=0}^{d_\varphi}\alpha_i\bar{\varphi}_i''(\hat{u})(v,v)+\sum_{i=1}^{d_\eta}\beta_j\bar{\eta}_j''(\hat{u})(v,v)=\Omega[\lambda](z,v). \tag{8.3}$$

Proof. Let us compute the left-hand side of (8.3). Notice that

$$\sum_{i=0}^{d_\varphi}\alpha_i\bar{\varphi}_i(\hat{u})+\sum_{i=1}^{d_\eta}\beta_j\bar{\eta}_j(\hat{u})=\ell[\lambda](\hat{x}(T)). \tag{8.4}$$

Let us look for a second order expansion for ℓ. Consider first a second order expansion of the state variable:

$$x=\hat{x}+z+\tfrac{1}{2}z_{vv}+o(\|v\|_\infty^2),$$

where z_{vv} satisfies

$$\dot{z}_{vv}=Az_{vv}+D^2_{(x,u)^2}F(\hat{x},\hat{u})(z,v)^2,\quad z_{vv}(0)=0, \tag{8.5}$$

with $F(x,u):=\sum_{i=0}^m u_if_i(x)$. Consider the second order expansion for ℓ :

$$\begin{aligned} \ell[\lambda](x(T)) =&\ell[\lambda]((\hat{x}+z+\tfrac{1}{2}z_{vv})(T))+o(\|v\|_1^2)\\ =&\ell[\lambda](\hat{x}(T))+\ell'[\lambda](\hat{x}(T))(z(T)+\tfrac{1}{2}z_{vv}(T))\\ &+\tfrac{1}{2}\ell''[\lambda](\hat{x}(T))(z(T)+\tfrac{1}{2}z_{vv}(T))^2+o(\|v\|_1^2). \end{aligned} \tag{8.6}$$

Step 1. Compute

$$\begin{aligned} &\ell'[\lambda](\hat{x}(T))z_{vv}(T)=\psi(T)z_{vv}(T)-\psi(0)z_{vv}(0)\\ &=\int_0^T[\dot{\psi}z_{vv}+\psi\dot{z}_{vv}]\mathrm{d}t=\int_0^T\{-\psi Az_{vv}+\psi(Az_{vv}+D^2F_{(x,u)^2}(z,v)^2)\}\mathrm{d}t\\ &=\int_0^T D^2H[\lambda](z,v)^2\mathrm{d}t. \end{aligned}$$

Step 2. Compute $\ell''[\lambda](\hat{x}(T))(z(T), z_{vv}(T))$. Applying Gronwall's Lemma, we obtain $\|z\|_\infty = O(\|v\|_1)$, and $\|z_{vv}\|_\infty = O(\|v^2\|_1)$. Thus

$$|(z(T), z_{vv}(T))| = O(\|v\|_1^3),$$

and we conclude that

$$|\ell''[\lambda](\hat{x}(T))(z(T), z_{vv}(T))| = O(\|v\|_1^3).$$

Step 3. See that $\ell''[\lambda](\hat{x}(T))(z_{vv}(T))^2 = O(\|v\|_1^4)$. Then by (8.6) we get,

$$\begin{aligned} \ell[\lambda](x(T)) =& \ell[\lambda](\hat{x}(T)) + \ell'[\lambda](\hat{x}(T))z(T) \\ &+ \tfrac{1}{2}\ell''[\lambda](\hat{x}(T))z^2(T) + \tfrac{1}{2}\int_0^T D^2_{(x,u)^2}H[\lambda](z,v)^2 \mathrm{d}t + o(\|v\|_1^2) \\ =& \ell[\lambda](\hat{x}(T)) + \ell'[\lambda](\hat{x}(T))z(T) + \Omega[\lambda](z,v) + o(\|v\|_1^2). \end{aligned}$$

The conclusion follows by (8.4). □

Lemma 8.3. *Given* $(z,v) \in \mathcal{W}$ *satisfying* (2.12)*, the following estimation holds for some* $\rho > 0$:

$$\|z\|_2^2 + |z(T)|^2 \leq \rho\gamma(y, y(T)),$$

where y *is defined by* (4.2).

Remark 8.4. ρ *depends on* $\hat{w}$*, i.e. it does not vary with* (z,v).

Proof. Every time we mention ρ_i we are referring to a constant depending on $\|A\|_\infty$, $\|B\|_\infty$ or both. Consider ξ, the solution of equation (4.3) corresponding to y. Gronwall's Lemma and the Cauchy-Schwartz inequality imply

$$\|\xi\|_\infty \leq \rho_1 \|y\|_2. \tag{8.7}$$

This last inequality, together with expression (4.2), implies

$$\|z\|_2 \leq \|\xi\|_2 + \|B\|_\infty \|y\|_2 \leq \rho_2 \|y\|_2. \tag{8.8}$$

On the other hand, equations (4.2) and (8.7) lead to

$$|z(T)| \leq |\xi(T)| + \|B\|_\infty |y(T)| \leq \rho_1 \|y\|_2 + \|B\|_\infty |y(T)|.$$

Then, by the inequality $ab \leq \frac{a^2+b^2}{2}$, we get

$$|z(T)|^2 \leq \rho_3(\|y\|_2^2 + |y(T)|^2). \tag{8.9}$$

The conclusion follows from equations (8.8) and (8.9). □

The next lemma is a generalization of the previous result to the nonlinear case. See Lemma 6.1 in Dmitruk [15].

Lemma 8.5. *Let* $w = (x,u)$ *be the solution of* (2.2) *with* $\|u\|_2 \leq c$ *for some constant* c*. Put* $(\delta x, v) := w - \hat{w}$*. Then*

$$|\delta x(T)|^2 + \|\delta x\|_2^2 \leq \rho\gamma(y, y(T)),$$

where y *is defined by* (4.2) *and* ρ *depends on* c.

Lemma 8.6. *Let* $\{y_k\} \subset L_2(a,b)$ *be a sequence of continuous non-decreasing functions that converges weakly to* $y \in L_2(a,b)$*. Then* y *is non-decreasing.*

Proof. Let $s, t \in (a, b)$ be such that $s < t$, and $\varepsilon_0 > 0$ such that $s + \varepsilon_0 < t - \varepsilon_0$. For every $k \in I\!N$, and every $0 < \varepsilon < \varepsilon_0$, the following inequality holds

$$\int_{s-\varepsilon}^{s+\varepsilon} y_k(\nu)d\nu \leq \int_{t-\varepsilon}^{t+\varepsilon} y_k(\nu)d\nu.$$

Taking the limit as k goes to infinity and multiplying by $\frac{1}{2\varepsilon}$, we deduce that

$$\frac{1}{2\varepsilon}\int_{s-\varepsilon}^{s+\varepsilon} y(\nu)d\nu \leq \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} y(\nu)d\nu.$$

As (a, b) is a finite measure space, y is a function of $L_1(a, b)$ and almost all points in (a, b) are Lebesgue points (see Rudin [52, Theorem 7.7]). Thus, by taking ε to 0, it follows from the previous inequality that

$$y(s) \leq y(t),$$

which is what we wanted to prove. □

Lemma 8.7. *Consider a sequence $\{y_k\}$ of non-decreasing continuous functions in a compact real interval I and assume that $\{y_k\}$ converges weakly to 0 in $L_2(I)$. Then it converges uniformly to 0 on any interval $(a, b) \subset I$.*

Proof. Take an arbitrary interval $(a, b) \subset I$. First prove the pointwise convergence of $\{y_k\}$ to 0. On the contrary, suppose that there exists $c \in (a, b)$ such that $\{y_k(c)\}$ does not converge to 0. Thus there exist $\varepsilon > 0$ and a subsequence $\{y_{k_j}\}$ such that $y_{k_j}(c) > \varepsilon$ for each $j \in I\!N$, or $y_{k_j}(c) < -\varepsilon$ for each $j \in I\!N$. Suppose, without loss of generality, that the first statement is true. Thus

$$0 < \varepsilon(b - c) < y_{k_j}(c)(b - c) \leq \int_c^b y_{k_j}(t)\mathrm{d}t, \tag{8.10}$$

where the last inequality holds since y_{k_j} is nondecreasing. But the right-hand side of (8.10) goes to 0 as j goes to infinity. This contradicts the hypothesis and thus the pointwise convergence of $\{y_k\}$ to 0 follows. The uniform convergence is a direct consequence of the monotonicity of the functions y_k. □

Lemma 8.8. *[20, Theorem 22, Page 154 - Volume I] Let a and b be two functions of bounded variation in $[0, T]$. Suppose that one is continuous and the other is right-continuous. Then*

$$\int_0^T a(t)db(t) + \int_0^T b(t)da(t) = [ab]_{0-}^{T+}.$$

Lemma 8.9. *Let $m = 1$, i.e. consider a scalar control variable. Then, for any $\lambda \in \Lambda$, the function $R[\lambda](t)$ defined in (4.13) is continuous in t.*

Proof. Consider definition (4.10). Condition $V[\lambda] \equiv 0$ yields $S[\lambda] = C[\lambda]B$, and since $R[\lambda]$ is scalar, we can write

$$R[\lambda] = B^\top Q[\lambda]B - 2C[\lambda]B_1 - \dot{C}[\lambda]B - C[\lambda]\dot{B}.$$

Note that $B = f_1$, $B_1 = [f_0, f_1]$, $C[\lambda] = -\psi f_1'$, and $Q[\lambda] = -\psi(f_0'' + \hat{u}f_1''$. Thus

$$\begin{aligned} R[\lambda] =& \psi(f_0'' + \hat{u}f_1'')(f_1, f_1) - 2\psi f_1'(f_0' f_1 - f_1' f_0) \\ &+ \psi(f_0' + \hat{u}f_1')f_1' f_1 - \psi f_1''(f_0 + \hat{u}f_1)f_1 - \psi f_1' f_1'(f_0 + \hat{u}f_1) \\ =& \psi[f_1, [f_1, f_0]]. \end{aligned}$$

Since f_0 and f_1 are twice continuously differentiable, we conclude that $R[\lambda]$ is continuous in time. □

Lemma 8.10. *[28] Consider a quadratic form $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$ where $\mathcal{Q}_1$ is a Legendre form and $\mathcal{Q}_2$ is weakly continuous over some Hilbert space. Then $\mathcal{Q}$ is a Legendre form.*

Lemma 8.11. *[28, Theorem 3.2] Consider a real interval I and a quadratic form $\mathcal{Q}$ over the Hilbert space $L_2(I)$, given by*

$$\mathcal{Q}(y) := \int_I y^\top(t) R(t) y(t) \mathrm{d}t.$$

Then $\mathcal{Q}$ is weakly l.s.c. over $L_2(I)$ iff

$$R(t) \succeq 0, \quad \text{a.e. on } I. \tag{8.11}$$

Lemma 8.12. *[14, Theorem 5] Given a Hilbert space H, and $a_1, a_2, \ldots, a_p \in H$, set*

$$K := \{x \in H : (a_i, x) \leq 0, \text{ for } i = 1, \ldots, p\}.$$

Let M be a convex and compact subset of $\mathbb{R}^s$, and let $\{Q^\psi : \psi \in M\}$ be a family of continuous quadratic forms over H with the mapping $\psi \to Q^\psi$ being affine. Set $M_\# := \{\psi \in M : Q^\psi \text{ is weakly l.s.c.}\}$ and assume that

$$\max_{\psi \in M} Q^\psi(x) \geq 0, \text{ for all } x \in K.$$

Then

$$\max_{\psi \in M_\#} Q^\psi(x) \geq 0, \text{ for all } x \in K.$$

The following result is an adaptation of Lemma 6.5 in [15].

Lemma 8.13. *Consider a sequence $\{v_k\} \subset \mathcal{U}$ and $\{y_k\}$ their primitives defined by (4.2). Call $u_k := \hat{u} + v_k$, x_k its corresponding solution of (2.2), and let z_k denote the linearized state corresponding to v_k, i.e. the solution of (2.12). Define, for each $k \in I\!N$,*

$$\delta x_k := x_k - \hat{x}, \quad \eta_k := \delta x_k - z_k, \quad \gamma_k := \gamma(y_k, y_k(T)). \tag{8.12}$$

Suppose that $\{v_k\}$ converges to 0 in the Pontryagin sense. Then

(i)

$$\dot{\eta}_k = \sum_{i=0}^m \hat{u}_i f_i'(\hat{x}) \eta_k + \sum_{i=1}^m v_{i,k} f_i'(\hat{x}) \delta x_k + \zeta_k, \tag{8.13}$$

$$\dot{\delta x}_k = \sum_{i=0}^m u_{i,k} f_i'(\hat{x}) \delta x_k + \sum_{i=1}^m v_{i,k} f_i(\hat{x}) + \zeta_k, \tag{8.14}$$

where $\|\zeta_k\|_2 \leq o(\sqrt{\gamma_k})$ and $\|\zeta_k\|_\infty \to 0$,

(ii) $\|\eta_k\|_\infty \leq o(\sqrt{\gamma_k})$.

Proof. **(i,ii)** Consider the second order Taylor expansions of f_i,

$$f_i(x_k) = f_i(\hat{x}) + f_i'(\hat{x})\delta x_k + \tfrac{1}{2} f_i''(\hat{x})(\delta x_k, \delta x_k) + o(|\delta x_k(t)|^2).$$

We can write

$$\dot{\delta x}_k = \sum_{i=0}^m u_{i,k} f_i'(\hat{x}) \delta x_k + \sum_{i=1}^m v_{i,k} f_i(\hat{x}) + \zeta_k, \tag{8.15}$$

with

$$\zeta_k := \tfrac{1}{2}\sum_{i=0}^{m} u_{i,k} f_i''(\hat{x})(\delta x_k, \delta x_k) + o(|\delta x_k(t)|^2)\sum_{i=0}^{m} u_{i,k}. \tag{8.16}$$

As $\{u_k\}$ is bounded in L_∞ and $\|\delta x_k\|_\infty \to 0$, we get $\|\zeta_k\|_\infty \to 0$ and the following L_2-norm bound:

$$\begin{aligned}\|\zeta_k\|_2 &\leq const. \sum_{i=0}^{m} \|u_{i,k}(\delta x_k, \delta x_k)\|_2 + o(\gamma_k)\|\sum_{i=0}^{m} u_{i,k}\|_1 \\ &\leq const.\|u_k\|_\infty \|\delta x_k\|_2^2 = O(\gamma_k) \leq o(\sqrt{\gamma_k}).\end{aligned} \tag{8.17}$$

Let us look for the differential equation of η_k defined in (8.12). By (8.15), and adding and substracting the term $\sum_{i=1}^{m} \hat{u}_i f_i'(\hat{x})\delta x_k$ we obtain

$$\dot{\eta}_k = \sum_{i=0}^{m} \hat{u}_i f_i'(\hat{x})\eta_k + \sum_{i=1}^{m} v_{i,k} f_i'(\hat{x})\delta x_k + \zeta_k.$$

Thus we obtain **(i).** Applying Gronwall's Lemma to this last differential equation we get

$$\|\eta_k\|_\infty \leq \|\sum_{i=1}^{m} v_{i,k} f_i'(\hat{x})\delta x_k + \zeta_k\|_1. \tag{8.18}$$

Since $\|v_k\|_\infty < N$ and $\|v_k\|_1 \to 0$, we also find that $\|v_k\|_2 \to 0$. Applying the Cauchy-Schwartz inequality to (8.18), from (8.17) we get **(ii).**

□

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