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Steinberg's Conjecture and near-colorings*

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Abstract: Let $\mathcal{F}$ be the family of planar graphs without cycles of length 4 and 5. Steinberg's Conjecture (1976) that says every graph of $\mathcal{F}$ is 3-colorable remains widely open. Motivées par une relaxation proposée par Erdős (1991), plusieurs études ont montré la conjecture pour des sous-classes de $\mathcal{F}$. Par exemple, Borodin *et al.* ont prouvé que tout graphe planaire sans cycles de longueur 4 à 7 est 3-colorable. Dans ce rapport, nous relaxons le problème non pas sur la classe de graphes mais sur le type de coloration en considérant des *quasi-colorations*. Un graphe $G = (V, E)$ est dit (i, j, k)-colorable si son ensemble de sommet peut être partitionner en trois ensembles V_1, V_2, V_3 tels que les graphes $G[V_1], G[V_2], G[V_3]$ induits par ces ensembles soit de degré maximum au plus i, j, k respectivement. Avec cette terminologie, la Conjecture de Steinberg dit que tout graphe de $\mathcal{F}$ est $(0, 0, 0)$-colorable. Un résultat de Xu (2008) implique que tout graphe de $\mathcal{F}$ est $(1, 1, 1)$-colorable. Nous montrons ici que tout graphe de $\mathcal{F}$ est $(2, 1, 0)$-colorable et $(4, 0, 0)$-colorable.

Key-words: graphs, coloring, decomposition, Steinberg's conjecture

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Conjecture de Steinberg et quasi-coloration

Résumé : Soit $\mathcal{F}$ la classe des graphes planaires sans cycles de longueur 4 et 5. La Conjecture de Steinberg (1976) affirmant que tout graphe de $\mathcal{F}$ est 3-colorable, reste largement ouverte.

Mots-clés : graphes, coloration, décomposition, conjecture de Steinberg

1 Introduction

In 1976, Appel and Haken proved that every planar graph is 4-colorable [2, 3], and as early as 1959, Grötzsch [20] showed that every planar graph without 3-cycles is 3-colorable. As proved by Garey, Johnson and Stockmeyer [19], the problem of deciding whether a planar graph is 3-colorable is NP-complete. Therefore, some sufficient conditions for planar graphs to be 3-colorable were stated. In 1976, Steinberg [24] raised the following:

Steinberg's Conjecture '76 *Every planar graph without 4- and 5-cycles is 3-colorable.*

There were then no progress in this direction until Erdős (1991) proposed the following relaxation of Steinberg's Conjecture:

Erdős' relaxation '91 Determine the smallest value of k, if it exists, such that every planar graph without cycles of length from 4 to k is 3-colorable.

Abbott and Zhou [1] proved that such a k does exist, with $k \leq 11$. This result was later on improved to $k \leq 10$ by Borodin [4], to $k \leq 9$ by Borodin [5] and Sanders and Zhao [22], to $k \leq 8$ by Salavatipour [21]. The best known bound for such a k is 7 which was proved by Borodin, Glebov, Raspaud and Salavatipour [10].

This approach was at the origin of sufficient conditions of 3-colorability of subfamilies of planar graphs where some families of cycles are forbidden. See for examples [8, 9, 12, 13, 14, 15, 16, 17, 25].

A graph G is called *improperly* $(d_1, d_2, \ldots, d_k)$*-colorable*, or simply $(d_1, d_2, \ldots, d_k)$*-colorable*, if the vertex set of G can be partitioned into subsets $V_1, V_2, \ldots, V_k$ such that the graph $G[V_i]$ induced by V_i has maximum degree at most d_i for $1 \leq i \leq k$. This notion generalizes those of proper k-coloring (when $d_1 = d_2 = \ldots = d_k = 0$) and d-improper k-coloring (when $d_1 = d_2 = \ldots = d_k = d \geq 0$). Under this terminology, the Four Color Theorem says that every planar graph is $(0, 0, 0, 0)$-colorable. Eaton and Hull [18] and independently Škrekovski [23] proved that every planar graph is 2-improperly 3-colorable (in fact, 2-improperly 3-choosable), i.e. $(2, 2, 2)$-colorable.

In this note we focus on near-colorings and Steinberg's Conjecture. Let $\mathcal{F}$ be the family of planar graphs without cycles of length 4 and 5. We prove:

Theorem 1 *Every graph of $\mathcal{F}$ is $(2, 1, 0)$-colorable and $(4, 0, 0)$-colorable.*

The remaining of the paper is dedicated to the proof of this theorem.

2 General setting for (s_1, s_2, s_3)-colorability of $\mathcal{F}$

The proof of the main theorem is done by reducible configurations and discharging procedure. Suppose the theorem is not true. Let $G = (V, E, F)$ be a counterexample with the minimum order embedded in the plane. We apply a discharging procedure to reach to a contradiction.

We first assign to each vertex v and face f of G a charge ω such that $\omega(v) = 2d(v) - 6$ and $\omega(f) = r(f) - 6$, where $d(v)$ and $r(f)$ denote the degree of the vertex v and the length of the face f respectively. By Euler's Formula $|V| - |E| + |F| = 2$ and formula $\sum_{v \in V} d(v) = 2|E| = \sum_{f \in F} r(f)$, we have:

$$\sum_{v \in V} \omega(v) + \sum_{f \in F} \omega(f) = -12 < 0.$$

We then redistribute the charges according to some discharging rules. During the process, no charges are created or disappear. Let ω^* be the new charge on each vertex and face after the procedure. It follows that:

$$\sum_{v \in V} \omega(v) + \sum_{f \in F} \omega(f) = \sum_{v \in V} \omega^*(v) + \sum_{f \in F} \omega^*(f).$$

However, we will show that under some structural properties of G the new charge on each vertex and face is non-negative. This leads to the following obvious contradiction

$$-12 = \sum_{v\in V}\omega(v) + \sum_{f\in F}\omega(f) = \sum_{v\in V}\omega^*(v) + \sum_{f\in F}\omega^*(f) > 0$$

implying that no counterexample can exist.

Establishing structural properties is essential in the proof of the theorem. Although the properties for $(2,1,0)$-coloring and for $(4,0,0)$-coloring are not the same, they share some common part. In this section, we derive lemmas for a general setting. Suppose $s_1 \geq s_2 \geq s_3 \geq 0$ and $s = s_1+s_2+s_3$. In this section we assume that G is a minimum counterexample in $\mathcal{F}$ that is not (s_1,s_2,s_3)-colorable.

A vertex of degree k (resp. at least k, at most k) will be called *k-vertex* (resp. *k^+-vertex*, *k^--vertex*). A similar notation will be used for cycles and faces. A *k-neighbor* (resp. *k^+-neighbor*, *k^--neighbor*) of some vertex u is a neighbor of u which is a k-vertex. An *(a,b,c)-face* is a 3-face uvw such that $d(u)=a$, $d(v)=b$ and $d(w)=c$. In addition, a^- (resp. a^+) will mean $d(u)\leq a$ (resp. $d(u)\geq a$) and $*$ will mean any degree. For example, a $(3,4^-,*)$-face is a 3-face uvw such that $d(u)=3$, $d(v)\leq 4$ and w has no restriction on its degree. A *pendent 3-face* of a vertex v is a 3-face not containing v but is incident to a 3-vertex adjacent to v. In the following we will color the vertices of the graphs by partitioning the vertex set into V_1,V_2,V_3 such that each V_i induces a subgraph of maximum degree at most s_i. Coloring a vertex with color i means adding the vertex into V_i. We will say that we *nicely color* a vertex if we color it by i and at most $\max\{0,s_i-1\}$ of its neighbors are colored by i. We say that we *properly color* a vertex if we color it by a color not used by its neighbors. Properly colored vertices are nicely colored. When the colored neighbors of an uncolored vertex v use at most two colors, in particular when v has at most two colored neighbors, we can always color v properly by using the third color not used by its neighbors. We will use this frequently. As an easy consequence, every vertex of G has degree at least 3.

First, since G has no 4-cycles, we have the following:

Observation 2 *Two 3-faces may not share an edge. If a k-vertex v is incident to α 3-faces and has β pendent 3-faces, then $2\alpha+\beta\leq k$.*

Next, three useful lemmas.

Lemma 3 *Let v be an $(s+2)^-$-vertex of G. If $G-v$ has an (s_1,s_2,s_3)-coloring such that all neighbors of v are nicely colored, then G is (s_1,s_2,s_3)-colorable.*

PROOF. For $1\leq i\leq 3$, if we cannot assign color i to v, then v has at least s_i+1 neighbors colored by i. It follows that v has degree at least $\sum_{i=1}^{3}(s_i+1) = s+3$, a contradiction. □

Lemma 4 *Graph G contains no $(s+2)^-$-vertex v adjacent only to 4^--vertices, each 4-neighbor of which is adjacent some 3-neighbor of v.*

PROOF. Suppose to the contrary that G contains such a $(s+2)^-$-vertex v. By the minimality of G, the graph G' obtained from G by deleting v and all of its neighbors admits an (s_1,s_2,s_3)-coloring. We first color all 4-neighbors of v properly, and then color all 3-neighbors of v properly. Then all neighbors of v are nicely colored. Thus, by Lemma 3, G is (s_1,s_2,s_3)-colorable, a contradiction. □

Lemma 5 *The three neighbors x_1,x_2,x_3 of a 3-vertex v of G use different colors in an (s_1,s_2,s_3)-coloring of $G-v$. Moreover, assume x_i is colored by i, we have $d(x_i)\geq s_i+3$ for $1\leq i\leq 3$. Furthermore, if $s_i>0$ and x_i is adjacent to x_j, then either $d(x_i)>s_i+3$ or $d(x_j)>s_j+3$.*

PROOF. If x_1,x_2,x_3 do not use three distinct colors, then we can properly color v, a contradiction. Hence w.l.o.g. we can assume that x_i is colored by i for $1\leq i\leq 3$.

Suppose for a contradiction that some $d(x_i)\leq s_i+2$ for some i. Then $s_i\geq 1$ as $d(x_i)\geq 3$. If x_i is nicely colored by i, then we color v by i and this extends the coloring to G, a contradiction.

Hence, x_i has at least s_i neighbors colored by i. Since x_i has an uncolored neighbor v, there is at least one color different from i not used by its neighbors. We then color v by i and recolor x_i by the unused color. This extends the coloring to G, a contradiction.

Suppose for a contradiction that x_i is adjacent to x_j, but $d(x_i) = s_i + 3$ and $d(x_j) = s_j + 3$. Let k be the color distinct from i and j. Since G has no 4-cycle, x_k is not adjacent to x_i and x_j. As above, x_i (resp. x_j) has s_i (resp. s_j) neighbors colored by i (resp. j) and another colored neighbor x'_i (resp. x'_j) other than x_j (resp. x_i). If x'_i is colored by j, then we may color v by i and recolor x_i by k to get an (s_1, s_2, s_3)-coloring of G, a contradiction. Hence, x'_i is colored by k. Similarly, x'_j is also colored by k. Then we may color v by i, recolor x_i by j and recolor x_j by i to get an (s_1, s_2, s_3)-coloring of G (notice that $s_i > 0$), again a contradiction. Hence, $d(x_i) > s_i + 3$ or $d(x_j) > s_j + 3$. □

3 $(2, 1, 0)$-colorability of $\mathcal{F}$

In this section we prove that every graph in $\mathcal{F}$ is $(2, 1, 0)$-colorable, namely we consider the case $(s_1, s_2, s_3) = (2, 1, 0)$ for which $s = s_1 + s_2 + s_3 = 3$.

3.1 Reducible configurations for $(2, 1, 0)$-coloring

We first establish structural properties of G. More precisely, we prove that some 'configurations', i.e. subgraphs, are 'reducible', i.e cannot appear in G because it is a minimum counterexample. Lots of this configuartions are depicted in Figure 1.

A *light* 5*-vertex* is a 5-vertex incident to a $(3, 5, 5)$-face f and adjacent to three 3-vertices not in f. A *poor* $(3, 5, 5)$*-face* is a $(3, 5, 5)$-face incident to a light 5-vertex. If a 3-vertex is incident to a 3-face, then its neighbor not incident to this 3-face is said to be its *outer neighbor*.

As already mentioned we have the following.

(C1) *G contains no 2^--vertices.*

The two following claims come from Lemma 4 with $s = 3$.

(C2) *G contains no 5-vertex adjacent to five 3-vertices.*

(C3) *G does not contain 5-vertices v incident to a $(3, 4, 5)$-face f and adjacent to three 3-vertices not in f.*

(C4) *G contains no non-light 5-vertex incident to a poor $(3, 5, 5)$-face and a $(3, 5^-, 5)$-face, and adjacent to a 3-vertex not in these faces.*

Proof. Suppose to the contrary that G contains such a 5-vertex v. Let uvw be the poor $(3, 5, 5)$-face, rvs be the $(3, 5^-, 5)$-face with $d(u) = d(r) = 3$, and x be the neighbor of v not in these faces. Vertex w is light and thus is adjacent to three 3-vertices distinct from u, say w_1, w_2, w_3. By the minimality of G, the graph $G - \{u, v, w, w_1, w_2, w_3, r, x\}$ admits a $(2, 1, 0)$-coloring. Now we extend this coloring as follows. We may assume that, if s is colored by 1, then it has at most one neighbor colored by 1, otherwise we can properly recolor it. Then we color r and x properly. If s, r, x use different colors, then we color v with 1; otherwise we color v properly. We then color u, w_1, w_2, w_3 properly. It follows that all neighbors of w are nicely colored. By Lemma 3, G is $(2, 1, 0)$-colorable, a contradiction. □

(C5) *G does not contain a poor $(3, 5, 5)$-face incident to two light 5-vertices.*

Proof. Suppose to the contrary that G contains a poor $(3, 5, 5)$-face uvw with light vertices v and w. For $x \in \{v, w\}$, let x_1, x_2, x_3 be the three neighbors of x not in $\{u, v, w\}$. By the minimality of G, the graph $G - \{u, v, w, w_1, w_2, w_3, v_1, v_2, v_3\}$ admits a $(2, 1, 0)$-coloring. We extend the coloring to $\{v_1, v_2, v_3\}$ by coloring each of them properly. If v_1, v_2, v_3 use three distinct colors, then we color v with 1, and properly otherwise. After this, we color u, w_1, w_2, w_3 properly. It follows that all neighbors of w are nicely colored. By Lemma 3, G is $(2,1,0)$-colorable, a contradiction. $\square$

Let v be a 3-vertex adjacent to three vertices y_1, y_2, y_3. Consider $G - v$. By Lemma 5, the colors 1, 2, and 3 appear on the neighbors of v. Moreover the vertex colored with 1 (resp. 2, 3) has degree at least 5 (resp. 4, 3). Thus (C6) and (C7) follow.

(C6) *G does not contain 3-vertices adjacent to two 3-vertices.*

(C7) *If uvw is a $(3,4,4)$-face with $d(u) = 3$, then the outer neighbor of u has degree at least 5.*

Now, if the three vertices y_1, y_2, y_3 satisfy $d(y_1) = 3, d(y_2) \leq 4$ and $d(y_2) \leq d(y_3)$, then y_1 (resp. y_2, y_3) is colored with 3 (resp. 2, 1) and has degree 3 (resp. 4, at least 5). By the last sentence of Lemma 5, the vertices y_1, y_2 are non-adjacent; moreover if $d(y_3) = 5$, then y_3 is not adjacent to y_1 or y_2. Thus (C8), (C9), and (C10) follow.

(C8) *G does not contain $(3,3,4^-)$-faces.*

(C9) *If uvw is a $(3,3,5)$-face with $d(u) = 3$, then the outer neighbor of u has degree at least 5.*

(C10) *If uvw is a $(3,4,5)$-face with $d(u) = 3, d(v) = 4$ and $d(w) = 5$, then the outer neighbor of u has degree at least 4.*

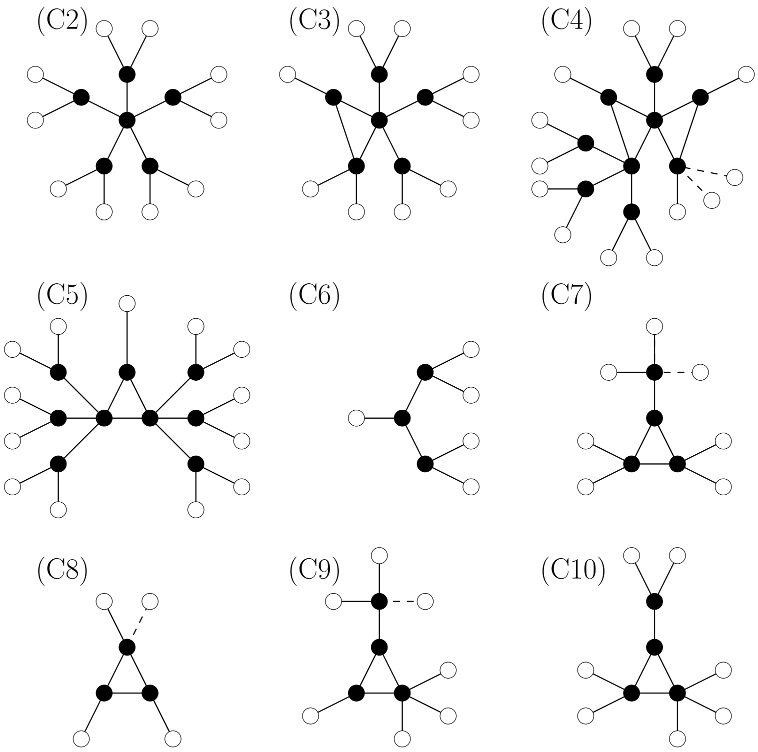

Figure 1: Reducible configurations (C2)-(C10). Black dots represent vertices all neighbours of which are drawn in the figure; the white dots represent vertices that can have nondepicted neighbours. Dashed lines represent edges that may possibly not exist.

3.2 Discharging procedure for $(2, 1, 0)$-coloring

We now apply a discharging procedure to reach to a contradiction. The discharging rules are as follows:

R1. Every 4-vertex gives $\frac{1}{2}$ to each pendent 3-face.
R2. Every 5^+-vertex gives 1 to each pendent 3-face.
R3. Every 4-vertex gives 1 to each incident 3-face.
R4. Every non-light 5-vertex gives 2 to each incident poor $(3, 5, 5)$-face.
R5. Every 5-vertex gives $\frac{3}{2}$ to each incident non-poor $(3, 5, 5)$-face or $(3, 4, 5)$-face.
R6. Every 5-vertex gives 1 to each other incident 3-face.
R7. Every 6^+-vertex gives 2 to each incident 3-face.

Let v be a k-vertex with $k \geq 3$ by (C1).

Case $k = 3$**.** The discharging procedure does not involves 3-vertices. Hence $\omega^*(v) = \omega(v) = 0$.

Case $k = 4$**.** Initially $\omega(v) = 2$. Vertex v gives 1 to each of the α incident 3-faces by R3 and $\frac{1}{2}$ to each of the β pendent 3-faces by R1. By Observation 2, $\omega^*(v) \geq 2 - (\alpha + \frac{1}{2}\beta) \geq 2 - \frac{1}{2} \cdot 4 = 0$.

Case $k = 5$**.** Initially $\omega(v) = 4$. Assume v is not incident to any 3-face. By (C2), v is adjacent to at most four 3-vertices and so has at most four pendent 3-faces. By R2, $\omega^*(v) \geq 4 - 4 \cdot 1 = 0$.

Assume v is incident to exactly one 3-face f. If v is a non-light 5-vertex and f is a poor $(3, 5, 5)$-face, then v has at most two pendent 3-faces by definition. By R4 and R2, $\omega^*(v) \geq 4 - 2 - 2 \cdot 1 = 0$. If f is a non-poor $(3, 5, 5)$-face, then v has at most two pendent 3-faces by definition. By R5 and R2, $\omega^*(v) \geq 4 - \frac{3}{2} - 2 \cdot 1 > 0$. If f is a $(3, 4, 5)$-face, then v has at most two pendent 3-faces by (C3). By R5 and R2, $\omega^*(v) \geq 4 - \frac{3}{2} - 2 \cdot 1 > 0$. If f is a 3-face of other type, then by R6 and R2 $\omega^*(v) \geq 4 - 1 - 3 \cdot 1 = 0$.

Assume v is incident to exactly two 3-faces f_1 and f_2. If v gives twice at most $\frac{3}{2}$ to the 3-faces, then $\omega^*(v) \geq 4 - 2 \cdot \frac{3}{2} - 1 = 0$. So we may assume that f_1 or f_2, say f_1, is a poor $(3, 5, 5)$-face. If f_2 is a $(3, 5^-, 5)$-face, then v has no pendent 3-faces by (C4) and $\omega^*(v) \geq 4 - 2 - 2 = 0$. If f_2 is a 3-face of other type, then v may have a pendent 3-face and $\omega^*(v) \geq 4 - 2 - 1 - 1 = 0$ by R6.

Case $k \geq 6$**.** Initially $\omega(v) = 2k - 6$. Vertex v gives 2 to each of the α incident 3-faces by R7 and 1 to each of the β pendent 3-faces by R2. By Observation 2, $\omega^*(v) \geq 2k - 6 - 2\alpha - \beta \geq 2k - 6 - k = k - 6 \geq 0$.

Let f be a k-face.

Case $k = 3$**.** Initially $\omega(f) = -3$. By (C8), f is not a $(3, 3, 4^-)$-face.

Let $f = uvw$ be a $(3, 3, 5)$-face so that $d(u) = d(v) = 3$ and $d(w) = 5$. By (C9) the outer neighbor of u (resp. v) has degree at least 5 and so gives at least 1 to f by R2. By R6, w gives 1 to f. It follows that $\omega^*(f) = -3 + 2 \cdot 1 + 1 = 0$.

Let $f = uvw$ be a $(3, 3, 6^+)$-face so that $d(u) = d(v) = 3$ and $d(w) \geq 6$. By (C6), the outer neighbor of u (resp. v) has degree at least 4 and so gives at least $\frac{1}{2}$ to f by R1. By R7, w gives 2 to f. It follows that $\omega^*(f) = -3 + 2 \cdot \frac{1}{2} + 2 = 0$.

Let $f = uvw$ be a $(3, 4, 4)$-face so that $d(u) = 3$ and $d(v) = d(w) = 4$. By (C7) the outer neighbor of u has degree at least 5 and so gives 1 to f by R2. Vertices v (resp. w) give 1 to f by R3. Hence $\omega^*(f) = -3 + 1 + 2 \cdot 1 = 0$.

Let $f = uvw$ be a $(3, 4, 5)$-face so that $d(u) = 3, d(v) = 4$ and $d(w) = 5$. By (C10), the outer neighbor of u has degree at least 4 and so gives at least $\frac{1}{2}$ to f by R1. Vertices v and w give each 1 and $\frac{3}{2}$ to f respectively by R3 and R5. Hence $\omega^*(f) = -3 + \frac{1}{2} + 1 + \frac{3}{2} = 0$.

Let $f = uvw$ be a $(3, 4, 6^+)$-face so that $d(u) = 3, d(v) = 4$ and $d(w) \geq 6$. By R3 and R7, vertices v and w give each 1 and 2 to f respectively. Hence $\omega^*(f) = -3 + 1 + 2 = 0$.

Let $f = uvw$ be a $(3, 5, 5)$-face so that $d(u) = 3, d(v) = d(w) = 5$. Assume f is poor and v is light. By (C5) w cannot be light. Hence $\omega^*(f) = -3 + 1 + 2 = 0$ by R4 and R6. Assume f is not poor. Then $\omega^*(f) = -3 + 2 \cdot \frac{3}{2} = 0$ by R5.

Let $f = uvw$ be a $(3, 5^+, 6^+)$-face so that $d(u) = 3, d(v) \geq 5, d(w) \geq 6$. Vertices v and w give each at least 1 and 2 respectively by R6-7. Hence $\omega^*(f) \geq -3 + 1 + 2 = 0$.

Let $f = uvw$ be a $(4^+, 4^+, 4^+)$-face. Each incident vertex gives at least 1 to f by R3-7. Hence $\omega^*(f) \geq -3 + 3 \cdot 1 = 0$.

Case $k \geq 4$**.** Faces of length 4 and 5 do not exist by hypothesis. Faces of length at least 6 are not involved in the discharging procedure. Hence $\omega^*(f) = \omega(f) = r(f) - 6 \geq 0$.

It follows that every vertex and face has a non-negative charge as required. This completes the proof.

4 $(4, 0, 0)$-colorability of $\mathcal{F}$

In this section we prove that every graph of $\mathcal{F}$ is $(4, 0, 0)$-colorable, namely we consider the case of $(s_1, s_2, s_3) = (4, 0, 0)$ for which $s = s_1 + s_2 + s_3 = 4$.

4.1 Reducible configurations for $(4, 0, 0)$-coloring

In this section we study structural properties of G and establish a number of reducible configuarions. See Figure 3.

A *bad* 8*-vertex* is a 8-vertex v incident to three $(3, 3, 8)$-faces and to a $(3, 8, *)$-face $f = uvw$ with $d(u) = 3, d(v) = 8$, where the vertex w is called the *sponsor* of f and f is a *bad face* of v. See Figure 2.

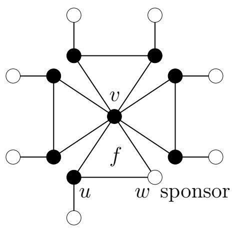

Figure 2: A bad 8-vertex v whose bad face is uvw with sponsor w. (Drawing conventions are the same as in Figure 1.)

(C1') *G contains no 2^--vertices.*

(C2') *For $8 \leq k \leq 10$, a k-vertex cannot be incident to exactly $k - 5$ $(3, 3, k)$-faces and adjacent to k 3-vertices.*

Proof. Suppose v is a k-vertex incident to exactly $k - 5$ $(3, 3, k)$-faces and adjacent to $10 - k$ other 3-vertices not in these $(3, 3, k)$-faces. By the minimality of G, the graph G' obtained from G by deleting v and all its neighbors admits a $(4, 0, 0)$-coloring. We color properly and sequentially all neighbors of v. Since each $(3, 3, k)$-face contains at most one vertex colored by 1, color 1 appears at most $k - 5 + 10 - k = 5$ times on the neighbors of v. If it appears less than 5 times, we can color v with 1, a contradiction. Hence color 1 appears exactly 5 times, once in each $(3,3,k)$-face and on all the $10-k$ other 3-vertices. For each $(3,3,k)$-face vxy with $d(x)=d(y)=3$, where x is colored by 1, y is colored by 2 or 3. In the case of y is colored by 3, if the outer neighbor of y is colored by 1 (resp. 2), then we can recolor y by 2 (resp. 1). Then we can color v with 3 to obtain a $(4,0,0)$-coloring of G, a contradiction. □

(C3') *Every* 3-*vertex of* G *is adjacent to at least one* 7^+-*vertex.*

Proof. This follows from the fact that the degree sequence for the three neighbors of a 3-vertex is lexicographically at least $(7,3,3)$ by Lemma 5. □

(C4') *If* uvw *is a* $(3,3,7)$-*face with* $d(u)=3$*, then the outer neighbor of* u *has degree at least* 4.

Proof. Suppose to the contrary that G has a $(3,3,7)$-face uvw with $d(u)=d(v)=3$ and $d(w)=7$, but the outer vertex x of u has $d(x)=3$. By Lemma 5, the degree sequence for the three neighbors of u is lex-graphically at least $(7,3,3)$. Hence w is colored by 1 and v is colored by 2 or 3. This contradicts the last sentence of Lemma 5 as w is adjacent to v. □

(C5') *The sponsor* w *of a bad* 8-*vertex* v *has degree at least* 8 *and is not a bad* 8-*vertex.*

Proof. Suppose to the contrary that the bad 8-vertex v is incident to three $(3,3,8)$-faces x_1x_2v, y_1y_2v and z_1z_2v and to a $(3,8,*)$-face uvw with $d(u)=3$ and $3\le d(w)\le 7$ or w a bad 8-vertex. By the minimality of G, the graph $G'=G-\{v,x_1,x_2,y_1,y_2,z_1,z_2,u\}$ admits a $(4,0,0)$-coloring. We can assume that w is nicely colored; otherwise, if $d(w)\le 7$, then we can recolor it properly, and if w is a bad 8-vertex, then we can recolor properly all its colored neighborhood and then color w nicely. Now we color properly and sequentially $x_1,x_2,y_1,y_2,z_1,z_2,u$, and we assign color 1 to v (color 1 appears at most 4 times on the neighbors of v). This extends the $(4,0,0)$-coloring to G, a contradiction. □

4.2 Discharging procedure for $(4,0,0)$-coloring

We now apply a discharging procedure to reach a contradiction. The discharging rules are as follows:

R1'. For $4\le k\le 6$, every k-vertex gives $\frac{1}{2}$ to each pendent 3-face.

R2'. Every 7^+-vertex gives 1 to each pendent 3-face.

R3'. For $4\le k\le 6$, every k-vertex gives 1 to each incident 3-face.

R4'. Every 7^+-vertex gives 1 to each incident $(4^+,4^+,4^+)$-face.

R5'. Every non-bad7^+-vertex gives 2 to each incident $(3,4^+,4^+)$-face; every bad 8-vertex gives 1 to its bad 3-face.

R6'. Every 7-vertex gives 2 to each incident $(3,3,7)$-face.

R7'. For $k\ge 8$, every k-vertex gives 3 to each incident $(3,3,k)$-face.

Let v be a k-vertex with $k\ge 3$ by (C1'). Initially $\omega(v)=2k-6$.

Case $k=3$. The discharging procedure does not involves 3-vertices. Hence $\omega^*(v)=\omega(v)=0$.

Case $4\le k\le 6$. Vertex v gives 1 to each of the α incident 3-faces by R3' and $\frac{1}{2}$ to each of the β pendent 3-faces by R1'. By Observation 2, $\omega^*(v)\ge 2k-6-(\alpha+\frac{1}{2}\beta)\ge 2k-6-\frac{1}{2}k=\frac{3}{2}k-6\ge 0$.

Case $k=7$. Vertex v gives 2 to each of the α' incident $(3,3^+,4^+)$-faces by R5'-6', 1 to each of the α'' incident $(4^+,4^+,4^+)$-faces by R4', and 1 to each of the β pendent 3-faces by R2'. By Observation 2, $\omega^*(v)\ge 2k-6-(2\alpha'+\alpha''+\beta)\ge 2k-6-k=k-6>0$.

Case $k\ge 8$. For the case when v is a bad 8-vertex, v gives 3 to each incident $(3,3,8)$-face by R7' and 1 to the bad 3-face by R5'. Hence $\omega^*(v)=2\cdot 8-6-3\cdot 3-1=0$.

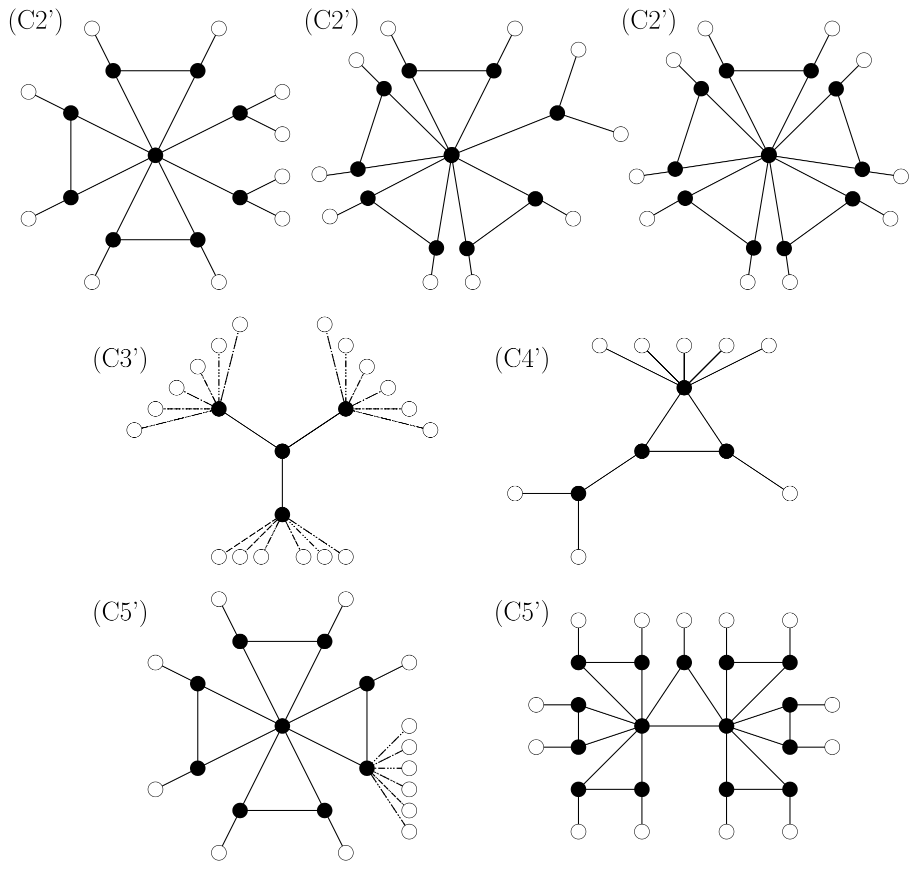

Figure 3: The reducible configurations (C2')-(C5'). (Drawing conventions are the same as in Figure 1.)

Now assume that v is not a bad 8-vertex. By R7', R5', R4' and R2', v gives 3 to each of the α' incident $(3,3,k)$-faces, 2 to each of the α'' incident $(3,4^+,4^+)$-faces, 1 to each of the α''' incident $(4^+,4^+,4^+)$-faces, and 1 to each of the β pendent 3-faces. By Observation 2, $\omega^*(v) = 2k-6-(3\alpha'+2\alpha''+\alpha'''+\beta) \geq 2k-6-\lfloor\frac{3k}{2}\rfloor = \lceil\frac{k}{2}\rceil - 6 \geq 0$ except for the cases (1) $k=10$ with $\alpha'=5$, (2) $k=9$ with $\alpha'=4$ and $\beta=1$, (3) $k=8$ with $\alpha'=3$ and $\beta=2$ (note that the bad 8-vertex case, i.e. $\alpha'=4$ or $\alpha'=3$ with $\alpha''=1$, is excluded). The exceptional cases give a k-vertex, $8 \leq k \leq 10$, with exactly $k-5$ $(3,3,k)$-faces and adjacent only to 3-vertices, a contradiction to (C2').

Let f be a k-face.

Case $k=3$**.** Initially $\omega(f)=-3$.

Let $f=uvw$ be a (a_1,a_2,a_3)-face with $3 \leq a_1 \leq 6, 3 \leq a_2 \leq 6$ and $3 \leq a_3 \leq 6$. By (C3'), the outer neighbor of each 3-vertex incident to f has degree at least 7 and gives each at least 1 to f by R2'. By R3', each d-vertex with $4 \leq d \leq 6$ incident to f gives 1 to f. It follows that $\omega^*(f) = -3+3 = 0$.

Let $f=uvw$ be a $(3,3,7)$-face so that $d(u)=d(v)=3$ and $d(w)=7$. By (C4') the outer neighbor of u (resp. v) has degree at least 4 and so gives at least $\frac{1}{2}$ to f by R1'. By R6', w gives 2 to f. It follows that $\omega^*(f) = -3+2\cdot\frac{1}{2}+2 = 0$.

Let $f=uvw$ be a $(3,3,8^+)$-face so that $d(u)=d(v)=3$ and $d(w)\geq 8$. By R7', w gives 3 to f. It follows that $\omega^*(f)=-3+3=0$.

Let $f=uvw$ be a $(3,4^+,7^+)$-face so that $d(u)\geq 3$, $d(v)\geq 4$ and $d(w)\geq 7$. By R3'-5', vertices v and w gives at least 3 to f and so $\omega^*(f)=-3+3=0$, except for the case when f is a bad 3-face with the pair v,w being either two bad 8-vertices or a bad 8-vertex and a 6^--vertex. But these two exceptional cases are impossible by (C5').

Finally, let $f=uvw$ be a $(4^+,4^+,4^+)$-face. Every incident vertex gives at at least 1 to f by R3'-4'. Hence $\omega^*(f)\geq 0$.

It follows that every vertex and face has a non-negative charge as required. This completes the proof.

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