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## *Steinberg's Conjecture and near-colorings*

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## Steinberg's Conjecture and near-colorings\*

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**Abstract:** Let  $\mathcal{F}$  be the family of planar graphs without cycles of length 4 and 5. Steinberg's Conjecture (1976) that says every graph of  $\mathcal{F}$  is 3-colorable remains widely open. Motivées par une relaxation proposée par Erdős (1991), plusieurs études ont montré la conjecture pour des sous-classes de  $\mathcal{F}$ . Par exemple, Borodin *et al.* ont prouvé que tout graphe planaire sans cycles de longueur 4 à 7 est 3-colorable. Dans ce rapport, nous relaxons le problème non pas sur la classe de graphes mais sur le type de coloration en considérant des *quasi-colorations*. Un graphe  $G = (V, E)$  est dit  $(i, j, k)$ -colorable si son ensemble de sommet peut être partitionner en trois ensembles  $V_1, V_2, V_3$  tels que les graphes  $G[V_1], G[V_2], G[V_3]$  induits par ces ensembles soit de degré maximum au plus  $i, j, k$  respectivement. Avec cette terminologie, la Conjecture de Steinberg dit que tout graphe de  $\mathcal{F}$  est  $(0, 0, 0)$ -colorable. Un résultat de Xu (2008) implique que tout graphe de  $\mathcal{F}$  est  $(1, 1, 1)$ -colorable. Nous montrons ici que tout graphe de  $\mathcal{F}$  est  $(2, 1, 0)$ -colorable et  $(4, 0, 0)$ -colorable.

**Key-words:** graphs, coloring, decomposition, Steinberg's conjecture

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## **Conjecture de Steinberg et quasi-coloration**

**Résumé :** Soit  $\mathcal{F}$  la classe des graphes planaires sans cycles de longueur 4 et 5. La Conjecture de Steinberg (1976) affirmant que tout graphe de  $\mathcal{F}$  est 3-colorable, reste largement ouverte.

**Mots-clés :** graphes, coloration, décomposition, conjecture de Steinberg

## 1 Introduction

In 1976, Appel and Haken proved that every planar graph is 4-colorable [2, 3], and as early as 1959, Grötzsch [20] showed that every planar graph without 3-cycles is 3-colorable. As proved by Garey, Johnson and Stockmeyer [19], the problem of deciding whether a planar graph is 3-colorable is NP-complete. Therefore, some sufficient conditions for planar graphs to be 3-colorable were stated. In 1976, Steinberg [24] raised the following:

**Steinberg's Conjecture '76** *Every planar graph without 4- and 5-cycles is 3-colorable.*

There were then no progress in this direction until Erdős (1991) proposed the following relaxation of Steinberg's Conjecture:

**Erdős' relaxation '91** Determine the smallest value of  $k$ , if it exists, such that every planar graph without cycles of length from 4 to  $k$  is 3-colorable.

Abbott and Zhou [1] proved that such a  $k$  does exist, with  $k \leq 11$ . This result was later on improved to  $k \leq 10$  by Borodin [4], to  $k \leq 9$  by Borodin [5] and Sanders and Zhao [22], to  $k \leq 8$  by Salavatipour [21]. The best known bound for such a  $k$  is 7 which was proved by Borodin, Glebov, Raspaud and Salavatipour [10].

This approach was at the origin of sufficient conditions of 3-colorability of subfamilies of planar graphs where some families of cycles are forbidden. See for examples [8, 9, 12, 13, 14, 15, 16, 17, 25].

A graph  $G$  is called *improperly*  $(d_1, d_2, \dots, d_k)$ -colorable, or simply  $(d_1, d_2, \dots, d_k)$ -colorable, if the vertex set of  $G$  can be partitioned into subsets  $V_1, V_2, \dots, V_k$  such that the graph  $G[V_i]$  induced by  $V_i$  has maximum degree at most  $d_i$  for  $1 \leq i \leq k$ . This notion generalizes those of proper  $k$ -coloring (when  $d_1 = d_2 = \dots = d_k = 0$ ) and  $d$ -improper  $k$ -coloring (when  $d_1 = d_2 = \dots = d_k = d \geq 0$ ). Under this terminology, the Four Color Theorem says that every planar graph is  $(0, 0, 0, 0)$ -colorable. Eaton and Hull [18] and independently Škrekovski [23] proved that every planar graph is 2-improperly 3-colorable (in fact, 2-improperly 3-choosable), i.e.  $(2, 2, 2)$ -colorable.

In this note we focus on near-colorings and Steinberg's Conjecture. Let  $\mathcal{F}$  be the family of planar graphs without cycles of length 4 and 5. We prove:

**Theorem 1** *Every graph of  $\mathcal{F}$  is  $(2, 1, 0)$ -colorable and  $(4, 0, 0)$ -colorable.*

The remaining of the paper is dedicated to the proof of this theorem.

## 2 General setting for $(s_1, s_2, s_3)$ -colorability of $\mathcal{F}$

The proof of the main theorem is done by reducible configurations and discharging procedure. Suppose the theorem is not true. Let  $G = (V, E, F)$  be a counterexample with the minimum order embedded in the plane. We apply a discharging procedure to reach to a contradiction.

We first assign to each vertex  $v$  and face  $f$  of  $G$  a charge  $\omega$  such that  $\omega(v) = 2d(v) - 6$  and  $\omega(f) = r(f) - 6$ , where  $d(v)$  and  $r(f)$  denote the degree of the vertex  $v$  and the length of the face  $f$  respectively. By Euler's Formula  $|V| - |E| + |F| = 2$  and formula  $\sum_{v \in V} d(v) = 2|E| = \sum_{f \in F} r(f)$ , we have:

$$\sum_{v \in V} \omega(v) + \sum_{f \in F} \omega(f) = -12 < 0.$$

We then redistribute the charges according to some discharging rules. During the process, no charges are created or disappear. Let  $\omega^*$  be the new charge on each vertex and face after the procedure. It follows that:

$$\sum_{v \in V} \omega(v) + \sum_{f \in F} \omega(f) = \sum_{v \in V} \omega^*(v) + \sum_{f \in F} \omega^*(f).$$

However, we will show that under some structural properties of  $G$  the new charge on each vertex and face is non-negative. This leads to the following obvious contradiction

$$-12 = \sum_{v \in V} \omega(v) + \sum_{f \in F} \omega(f) = \sum_{v \in V} \omega^*(v) + \sum_{f \in F} \omega^*(f) > 0$$

implying that no counterexample can exist.

Establishing structural properties is essential in the proof of the theorem. Although the properties for  $(2, 1, 0)$ -coloring and for  $(4, 0, 0)$ -coloring are not the same, they share some common part. In this section, we derive lemmas for a general setting. Suppose  $s_1 \geq s_2 \geq s_3 \geq 0$  and  $s = s_1 + s_2 + s_3$ . In this section we assume that  $G$  is a minimum counterexample in  $\mathcal{F}$  that is not  $(s_1, s_2, s_3)$ -colorable.

A vertex of degree  $k$  (resp. at least  $k$ , at most  $k$ ) will be called  $k$ -vertex (resp.  $k^+$ -vertex,  $k^-$ -vertex). A similar notation will be used for cycles and faces. A  $k$ -neighbor (resp.  $k^+$ -neighbor,  $k^-$ -neighbor) of some vertex  $u$  is a neighbor of  $u$  which is a  $k$ -vertex. An  $(a, b, c)$ -face is a 3-face  $uvw$  such that  $d(u) = a$ ,  $d(v) = b$  and  $d(w) = c$ . In addition,  $a^-$  (resp.  $a^+$ ) will mean  $d(u) \leq a$  (resp.  $d(u) \geq a$ ) and  $*$  will mean any degree. For example, a  $(3, 4^-, *)$ -face is a 3-face  $uvw$  such that  $d(u) = 3$ ,  $d(v) \leq 4$  and  $w$  has no restriction on its degree. A *pendent 3-face* of a vertex  $v$  is a 3-face not containing  $v$  but is incident to a 3-vertex adjacent to  $v$ . In the following we will color the vertices of the graphs by partitioning the vertex set into  $V_1, V_2, V_3$  such that each  $V_i$  induces a subgraph of maximum degree at most  $s_i$ . Coloring a vertex with color  $i$  means adding the vertex into  $V_i$ . We will say that we *nicely color* a vertex if we color it by  $i$  and at most  $\max\{0, s_i - 1\}$  of its neighbors are colored by  $i$ . We say that we *properly color* a vertex if we color it by a color not used by its neighbors. Properly colored vertices are nicely colored. When the colored neighbors of an uncolored vertex  $v$  use at most two colors, in particular when  $v$  has at most two colored neighbors, we can always color  $v$  properly by using the third color not used by its neighbors. We will use this frequently. As an easy consequence, every vertex of  $G$  has degree at least 3.

First, since  $G$  has no 4-cycles, we have the following:

**Observation 2** *Two 3-faces may not share an edge. If a  $k$ -vertex  $v$  is incident to  $\alpha$  3-faces and has  $\beta$  pendent 3-faces, then  $2\alpha + \beta \leq k$ .*

Next, three useful lemmas.

**Lemma 3** *Let  $v$  be an  $(s + 2)^-$ -vertex of  $G$ . If  $G - v$  has an  $(s_1, s_2, s_3)$ -coloring such that all neighbors of  $v$  are nicely colored, then  $G$  is  $(s_1, s_2, s_3)$ -colorable.*

PROOF. For  $1 \leq i \leq 3$ , if we cannot assign color  $i$  to  $v$ , then  $v$  has at least  $s_i + 1$  neighbors colored by  $i$ . It follows that  $v$  has degree at least  $\sum_{i=1}^3 (s_i + 1) = s + 3$ , a contradiction.  $\square$

**Lemma 4** *Graph  $G$  contains no  $(s + 2)^-$ -vertex  $v$  adjacent only to  $4^-$ -vertices, each 4-neighbor of which is adjacent some 3-neighbor of  $v$ .*

PROOF. Suppose to the contrary that  $G$  contains such a  $(s + 2)^-$ -vertex  $v$ . By the minimality of  $G$ , the graph  $G'$  obtained from  $G$  by deleting  $v$  and all of its neighbors admits an  $(s_1, s_2, s_3)$ -coloring. We first color all 4-neighbors of  $v$  properly, and then color all 3-neighbors of  $v$  properly. Then all neighbors of  $v$  are nicely colored. Thus, by Lemma 3,  $G$  is  $(s_1, s_2, s_3)$ -colorable, a contradiction.  $\square$

**Lemma 5** *The three neighbors  $x_1, x_2, x_3$  of a 3-vertex  $v$  of  $G$  use different colors in an  $(s_1, s_2, s_3)$ -coloring of  $G - v$ . Moreover, assume  $x_i$  is colored by  $i$ , we have  $d(x_i) \geq s_i + 3$  for  $1 \leq i \leq 3$ . Furthermore, if  $s_i > 0$  and  $x_i$  is adjacent to  $x_j$ , then either  $d(x_i) > s_i + 3$  or  $d(x_j) > s_j + 3$ .*

PROOF. If  $x_1, x_2, x_3$  do not use three distinct colors, then we can properly color  $v$ , a contradiction. Hence w.l.o.g. we can assume that  $x_i$  is colored by  $i$  for  $1 \leq i \leq 3$ .

Suppose for a contradiction that some  $d(x_i) \leq s_i + 2$  for some  $i$ . Then  $s_i \geq 1$  as  $d(x_i) \geq 3$ . If  $x_i$  is nicely colored by  $i$ , then we color  $v$  by  $i$  and this extends the coloring to  $G$ , a contradiction.

Hence,  $x_i$  has at least  $s_i$  neighbors colored by  $i$ . Since  $x_i$  has an uncolored neighbor  $v$ , there is at least one color different from  $i$  not used by its neighbors. We then color  $v$  by  $i$  and recolor  $x_i$  by the unused color. This extends the coloring to  $G$ , a contradiction.

Suppose for a contradiction that  $x_i$  is adjacent to  $x_j$ , but  $d(x_i) = s_i + 3$  and  $d(x_j) = s_j + 3$ . Let  $k$  be the color distinct from  $i$  and  $j$ . Since  $G$  has no 4-cycle,  $x_k$  is not adjacent to  $x_i$  and  $x_j$ . As above,  $x_i$  (resp.  $x_j$ ) has  $s_i$  (resp.  $s_j$ ) neighbors colored by  $i$  (resp.  $j$ ) and another colored neighbor  $x'_i$  (resp.  $x'_j$ ) other than  $x_j$  (resp.  $x_i$ ). If  $x'_i$  is colored by  $j$ , then we may color  $v$  by  $i$  and recolor  $x_i$  by  $k$  to get an  $(s_1, s_2, s_3)$ -coloring of  $G$ , a contradiction. Hence,  $x'_i$  is colored by  $k$ . Similarly,  $x'_j$  is also colored by  $k$ . Then we may color  $v$  by  $i$ , recolor  $x_i$  by  $j$  and recolor  $x_j$  by  $i$  to get an  $(s_1, s_2, s_3)$ -coloring of  $G$  (notice that  $s_i > 0$ ), again a contradiction. Hence,  $d(x_i) > s_i + 3$  or  $d(x_j) > s_j + 3$ .  $\square$

### 3 (2, 1, 0)-colorability of $\mathcal{F}$

In this section we prove that every graph in  $\mathcal{F}$  is  $(2, 1, 0)$ -colorable, namely we consider the case  $(s_1, s_2, s_3) = (2, 1, 0)$  for which  $s = s_1 + s_2 + s_3 = 3$ .

#### 3.1 Reducible configurations for $(2, 1, 0)$ -coloring

We first establish structural properties of  $G$ . More precisely, we prove that some ‘configurations’, i.e. subgraphs, are ‘reducible’, i.e. cannot appear in  $G$  because it is a minimum counterexample. Lots of this configurations are depicted in Figure 1.

A *light 5-vertex* is a 5-vertex incident to a  $(3, 5, 5)$ -face  $f$  and adjacent to three 3-vertices not in  $f$ . A *poor  $(3, 5, 5)$ -face* is a  $(3, 5, 5)$ -face incident to a light 5-vertex. If a 3-vertex is incident to a 3-face, then its neighbor not incident to this 3-face is said to be its *outer neighbor*.

As already mentioned we have the following.

(C1)  $G$  contains no  $2^-$ -vertices.

The two following claims come from Lemma 4 with  $s = 3$ .

(C2)  $G$  contains no 5-vertex adjacent to five 3-vertices.

(C3)  $G$  does not contain 5-vertices  $v$  incident to a  $(3, 4, 5)$ -face  $f$  and adjacent to three 3-vertices not in  $f$ .

(C4)  $G$  contains no non-light 5-vertex incident to a poor  $(3, 5, 5)$ -face and a  $(3, 5^-, 5)$ -face, and adjacent to a 3-vertex not in these faces.

*Proof.* Suppose to the contrary that  $G$  contains such a 5-vertex  $v$ . Let  $uvw$  be the poor  $(3, 5, 5)$ -face,  $rvs$  be the  $(3, 5^-, 5)$ -face with  $d(u) = d(r) = 3$ , and  $x$  be the neighbor of  $v$  not in these faces. Vertex  $w$  is light and thus is adjacent to three 3-vertices distinct from  $u$ , say  $w_1, w_2, w_3$ . By the minimality of  $G$ , the graph  $G - \{u, v, w, w_1, w_2, w_3, r, x\}$  admits a  $(2, 1, 0)$ -coloring. Now we extend this coloring as follows. We may assume that, if  $s$  is colored by 1, then it has at most one neighbor colored by 1, otherwise we can properly recolor it. Then we color  $r$  and  $x$  properly. If  $s, r, x$  use different colors, then we color  $v$  with 1; otherwise we color  $v$  properly. We then color  $u, w_1, w_2, w_3$  properly. It follows that all neighbors of  $w$  are nicely colored. By Lemma 3,  $G$  is  $(2, 1, 0)$ -colorable, a contradiction.  $\square$

(C5)  $G$  does not contain a poor  $(3, 5, 5)$ -face incident to two light 5-vertices.

*Proof.* Suppose to the contrary that  $G$  contains a poor  $(3, 5, 5)$ -face  $uvw$  with light vertices  $v$  and  $w$ . For  $x \in \{v, w\}$ , let  $x_1, x_2, x_3$  be the three neighbors of  $x$  not in  $\{u, v, w\}$ . By the minimality of  $G$ , the graph  $G - \{u, v, w, w_1, w_2, w_3, v_1, v_2, v_3\}$  admits a  $(2, 1, 0)$ -coloring. We extend the coloring to  $\{v_1, v_2, v_3\}$  by coloring each of them properly. If  $v_1, v_2, v_3$  use three distinct colors, then



we color  $v$  with 1, and properly otherwise. After this, we color  $u, w_1, w_2, w_3$  properly. It follows that all neighbors of  $w$  are nicely colored. By Lemma 3,  $G$  is  $(2, 1, 0)$ -colorable, a contradiction.  $\square$

Let  $v$  be a 3-vertex adjacent to three vertices  $y_1, y_2, y_3$ . Consider  $G - v$ . By Lemma 5, the colors 1, 2, and 3 appear on the neighbors of  $v$ . Moreover the vertex colored with 1 (resp. 2, 3) has degree at least 5 (resp. 4, 3). Thus (C6) and (C7) follow.

(C6)  $G$  does not contain 3-vertices adjacent to two 3-vertices.

(C7) If  $uvw$  is a  $(3, 4, 4)$ -face with  $d(u) = 3$ , then the outer neighbor of  $u$  has degree at least 5.

Now, if the three vertices  $y_1, y_2, y_3$  satisfy  $d(y_1) = 3, d(y_2) \leq 4$  and  $d(y_2) \leq d(y_3)$ , then  $y_1$  (resp.  $y_2, y_3$ ) is colored with 3 (resp. 2, 1) and has degree 3 (resp. 4, at least 5). By the last sentence of Lemma 5, the vertices  $y_1, y_2$  are non-adjacent; moreover if  $d(y_3) = 5$ , then  $y_3$  is not adjacent to  $y_1$  or  $y_2$ . Thus (C8), (C9), and (C10) follow.

(C8)  $G$  does not contain  $(3, 3, 4^-)$ -faces.

(C9) If  $uvw$  is a  $(3, 3, 5)$ -face with  $d(u) = 3$ , then the outer neighbor of  $u$  has degree at least 5.

(C10) If  $uvw$  is a  $(3, 4, 5)$ -face with  $d(u) = 3, d(v) = 4$  and  $d(w) = 5$ , then the outer neighbor of  $u$  has degree at least 4.

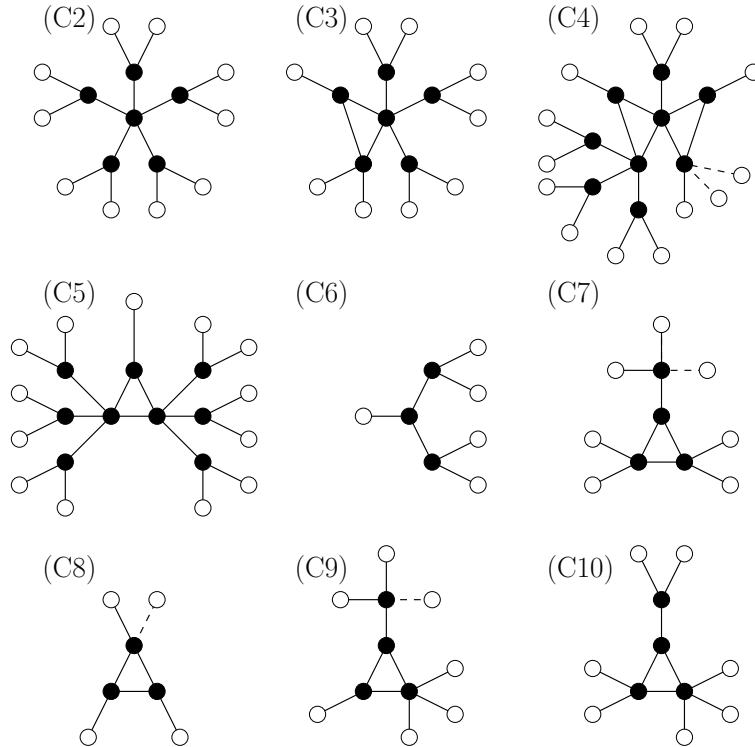


Figure 1: Reducible configurations (C2)-(C10). Black dots represent vertices all neighbours of which are drawn in the figure; the white dots represent vertices that can have nondepicted neighbours. Dashed lines represent edges that may possibly not exist.

### 3.2 Discharging procedure for $(2, 1, 0)$ -coloring

We now apply a discharging procedure to reach to a contradiction. The discharging rules are as follows:

- R1.** Every 4-vertex gives  $\frac{1}{2}$  to each pendent 3-face.
- R2.** Every  $5^+$ -vertex gives 1 to each pendent 3-face.
- R3.** Every 4-vertex gives 1 to each incident 3-face.
- R4.** Every non-light 5-vertex gives 2 to each incident poor  $(3, 5, 5)$ -face.
- R5.** Every 5-vertex gives  $\frac{3}{2}$  to each incident non-poor  $(3, 5, 5)$ -face or  $(3, 4, 5)$ -face.
- R6.** Every 5-vertex gives 1 to each other incident 3-face.
- R7.** Every  $6^+$ -vertex gives 2 to each incident 3-face.

Let  $v$  be a  $k$ -vertex with  $k \geq 3$  by (C1).

**Case  $k = 3$ .** The discharging procedure does not involves 3-vertices. Hence  $\omega^*(v) = \omega(v) = 0$ .

**Case  $k = 4$ .** Initially  $\omega(v) = 2$ . Vertex  $v$  gives 1 to each of the  $\alpha$  incident 3-faces by R3 and  $\frac{1}{2}$  to each of the  $\beta$  pendent 3-faces by R1. By Observation 2,  $\omega^*(v) \geq 2 - (\alpha + \frac{1}{2}\beta) \geq 2 - \frac{1}{2} \cdot 4 = 0$ .

**Case  $k = 5$ .** Initially  $\omega(v) = 4$ . Assume  $v$  is not incident to any 3-face. By (C2),  $v$  is adjacent to at most four 3-vertices and so has at most four pendent 3-faces. By R2,  $\omega^*(v) \geq 4 - 4 \cdot 1 = 0$ .

Assume  $v$  is incident to exactly one 3-face  $f$ . If  $v$  is a non-light 5-vertex and  $f$  is a poor  $(3, 5, 5)$ -face, then  $v$  has at most two pendent 3-faces by definition. By R4 and R2,  $\omega^*(v) \geq 4 - 2 - 2 \cdot 1 = 0$ . If  $f$  is a non-poor  $(3, 5, 5)$ -face, then  $v$  has at most two pendent 3-faces by definition. By R5 and R2,  $\omega^*(v) \geq 4 - \frac{3}{2} - 2 \cdot 1 > 0$ . If  $f$  is a  $(3, 4, 5)$ -face, then  $v$  has at most two pendent 3-faces by (C3). By R5 and R2,  $\omega^*(v) \geq 4 - \frac{3}{2} - 2 \cdot 1 > 0$ . If  $f$  is a 3-face of other type, then by R6 and R2  $\omega^*(v) \geq 4 - 1 - 3 \cdot 1 = 0$ .

Assume  $v$  is incident to exactly two 3-faces  $f_1$  and  $f_2$ . If  $v$  gives twice at most  $\frac{3}{2}$  to the 3-faces, then  $\omega^*(v) \geq 4 - 2 \cdot \frac{3}{2} - 1 = 0$ . So we may assume that  $f_1$  or  $f_2$ , say  $f_1$ , is a poor  $(3, 5, 5)$ -face. If  $f_2$  is a  $(3, 5^-, 5)$ -face, then  $v$  has no pendent 3-faces by (C4) and  $\omega^*(v) \geq 4 - 2 - 2 = 0$ . If  $f_2$  is a 3-face of other type, then  $v$  may have a pendent 3-face and  $\omega^*(v) \geq 4 - 2 - 1 - 1 = 0$  by R6.

**Case  $k \geq 6$ .** Initially  $\omega(v) = 2k - 6$ . Vertex  $v$  gives 2 to each of the  $\alpha$  incident 3-faces by R7 and 1 to each of the  $\beta$  pendent 3-faces by R2. By Observation 2,  $\omega^*(v) \geq 2k - 6 - 2\alpha - \beta \geq 2k - 6 - k = k - 6 \geq 0$ .

Let  $f$  be a  $k$ -face.

**Case  $k = 3$ .** Initially  $\omega(f) = -3$ . By (C8),  $f$  is not a  $(3, 3, 4^-)$ -face.

Let  $f = uvw$  be a  $(3, 3, 5)$ -face so that  $d(u) = d(v) = 3$  and  $d(w) = 5$ . By (C9) the outer neighbor of  $u$  (resp.  $v$ ) has degree at least 5 and so gives at least 1 to  $f$  by R2. By R6,  $w$  gives 1 to  $f$ . It follows that  $\omega^*(f) = -3 + 2 \cdot 1 + 1 = 0$ .

Let  $f = uvw$  be a  $(3, 3, 6^+)$ -face so that  $d(u) = d(v) = 3$  and  $d(w) \geq 6$ . By (C6), the outer neighbor of  $u$  (resp.  $v$ ) has degree at least 4 and so gives at least  $\frac{1}{2}$  to  $f$  by R1. By R7,  $w$  gives 2 to  $f$ . It follows that  $\omega^*(f) = -3 + 2 \cdot \frac{1}{2} + 2 = 0$ .

Let  $f = uvw$  be a  $(3, 4, 4)$ -face so that  $d(u) = 3$  and  $d(v) = d(w) = 4$ . By (C7) the outer neighbor of  $u$  has degree at least 5 and so gives 1 to  $f$  by R2. Vertices  $v$  (resp.  $w$ ) give 1 to  $f$  by R3. Hence  $\omega^*(f) = -3 + 1 + 2 \cdot 1 = 0$ .

Let  $f = uvw$  be a  $(3, 4, 5)$ -face so that  $d(u) = 3$ ,  $d(v) = 4$  and  $d(w) = 5$ . By (C10), the outer neighbor of  $u$  has degree at least 4 and so gives at least  $\frac{1}{2}$  to  $f$  by R1. Vertices  $v$  and  $w$  give each 1 and  $\frac{3}{2}$  to  $f$  respectively by R3 and R5. Hence  $\omega^*(f) = -3 + \frac{1}{2} + 1 + \frac{3}{2} = 0$ .

Let  $f = uvw$  be a  $(3, 4, 6^+)$ -face so that  $d(u) = 3, d(v) = 4$  and  $d(w) \geq 6$ . By R3 and R7, vertices  $v$  and  $w$  give each 1 and 2 to  $f$  respectively. Hence  $\omega^*(f) = -3 + 1 + 2 = 0$ .

Let  $f = uvw$  be a  $(3, 5, 5)$ -face so that  $d(u) = 3, d(v) = d(w) = 5$ . Assume  $f$  is poor and  $v$  is light. By (C5)  $w$  cannot be light. Hence  $\omega^*(f) = -3 + 1 + 2 = 0$  by R4 and R6. Assume  $f$  is not poor. Then  $\omega^*(f) = -3 + 2 \cdot \frac{3}{2} = 0$  by R5.

Let  $f = uvw$  be a  $(3, 5^+, 6^+)$ -face so that  $d(u) = 3, d(v) \geq 5, d(w) \geq 6$ . Vertices  $v$  and  $w$  give each at least 1 and 2 respectively by R6-7. Hence  $\omega^*(f) \geq -3 + 1 + 2 = 0$ .

Let  $f = uvw$  be a  $(4^+, 4^+, 4^+)$ -face. Each incident vertex gives at least 1 to  $f$  by R3-7. Hence  $\omega^*(f) \geq -3 + 3 \cdot 1 = 0$ .

**Case  $k \geq 4$ .** Faces of length 4 and 5 do not exist by hypothesis. Faces of length at least 6 are not involved in the discharging procedure. Hence  $\omega^*(f) = \omega(f) = r(f) - 6 \geq 0$ .

It follows that every vertex and face has a non-negative charge as required. This completes the proof.

## 4 $(4, 0, 0)$ -colorability of $\mathcal{F}$

In this section we prove that every graph of  $\mathcal{F}$  is  $(4, 0, 0)$ -colorable, namely we consider the case of  $(s_1, s_2, s_3) = (4, 0, 0)$  for which  $s = s_1 + s_2 + s_3 = 4$ .

### 4.1 Reducible configurations for $(4, 0, 0)$ -coloring

In this section we study structural properties of  $G$  and establish a number of reducible configurations. See Figure 3.

A *bad 8-vertex* is a 8-vertex  $v$  incident to three  $(3, 3, 8)$ -faces and to a  $(3, 8, *)$ -face  $f = uvw$  with  $d(u) = 3, d(v) = 8$ , where the vertex  $w$  is called the *sponsor* of  $f$  and  $f$  is a *bad face* of  $v$ . See Figure 2.

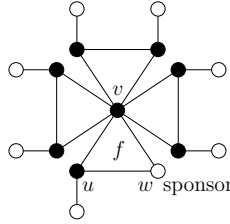


Figure 2: A bad 8-vertex  $v$  whose bad face is  $uvw$  with sponsor  $w$ . (Drawing conventions are the same as in Figure 1.)

(C1')  $G$  contains no  $2^-$ -vertices.

(C2') For  $8 \leq k \leq 10$ , a  $k$ -vertex cannot be incident to exactly  $k - 5$   $(3, 3, k)$ -faces and adjacent to  $k$  3-vertices.

*Proof.* Suppose  $v$  is a  $k$ -vertex incident to exactly  $k - 5$   $(3, 3, k)$ -faces and adjacent to  $10 - k$  other 3-vertices not in these  $(3, 3, k)$ -faces. By the minimality of  $G$ , the graph  $G'$  obtained from  $G$  by deleting  $v$  and all its neighbors admits a  $(4, 0, 0)$ -coloring. We color properly and sequentially all neighbors of  $v$ . Since each  $(3, 3, k)$ -face contains at most one vertex colored by 1, color 1 appears at most  $k - 5 + 10 - k = 5$  times on the neighbors of  $v$ . If it appears less than 5 times, we can

color  $v$  with 1, a contradiction. Hence color 1 appears exactly 5 times, once in each  $(3, 3, k)$ -face and on all the  $10 - k$  other 3-vertices. For each  $(3, 3, k)$ -face  $vxy$  with  $d(x) = d(y) = 3$ , where  $x$  is colored by 1,  $y$  is colored by 2 or 3. In the case of  $y$  is colored by 3, if the outer neighbor of  $y$  is colored by 1 (resp. 2), then we can recolor  $y$  by 2 (resp. 1). Then we can color  $v$  with 3 to obtain a  $(4, 0, 0)$ -coloring of  $G$ , a contradiction.  $\square$

(C3') Every 3-vertex of  $G$  is adjacent to at least one  $7^+$ -vertex.

*Proof.* This follows from the fact that the degree sequence for the three neighbors of a 3-vertex is lexicographically at least  $(7, 3, 3)$  by Lemma 5.  $\square$

(C4') If  $uvw$  is a  $(3, 3, 7)$ -face with  $d(u) = 3$ , then the outer neighbor of  $u$  has degree at least 4.

*Proof.* Suppose to the contrary that  $G$  has a  $(3, 3, 7)$ -face  $uvw$  with  $d(u) = d(v) = 3$  and  $d(w) = 7$ , but the outer vertex  $x$  of  $u$  has  $d(x) = 3$ . By Lemma 5, the degree sequence for the three neighbors of  $u$  is lex-graphically at least  $(7, 3, 3)$ . Hence  $w$  is colored by 1 and  $v$  is colored by 2 or 3. This contradicts the last sentence of Lemma 5 as  $w$  is adjacent to  $v$ .  $\square$

(C5') The sponsor  $w$  of a bad 8-vertex  $v$  has degree at least 8 and is not a bad 8-vertex.

*Proof.* Suppose to the contrary that the bad 8-vertex  $v$  is incident to three  $(3, 3, 8)$ -faces  $x_1x_2v$ ,  $y_1y_2v$  and  $z_1z_2v$  and to a  $(3, 8, *)$ -face  $uvw$  with  $d(u) = 3$  and  $3 \leq d(w) \leq 7$  or  $w$  a bad 8-vertex. By the minimality of  $G$ , the graph  $G' = G - \{v, x_1, x_2, y_1, y_2, z_1, z_2, u\}$  admits a  $(4, 0, 0)$ -coloring. We can assume that  $w$  is nicely colored; otherwise, if  $d(w) \leq 7$ , then we can recolor it properly, and if  $w$  is a bad 8-vertex, then we can recolor properly all its colored neighborhood and then color  $w$  nicely. Now we color properly and sequentially  $x_1, x_2, y_1, y_2, z_1, z_2, u$ , and we assign color 1 to  $v$  (color 1 appears at most 4 times on the neighbors of  $v$ ). This extends the  $(4, 0, 0)$ -coloring to  $G$ , a contradiction.  $\square$

## 4.2 Discharging procedure for $(4, 0, 0)$ -coloring

We now apply a discharging procedure to reach a contradiction. The discharging rules are as follows:

- R1'**. For  $4 \leq k \leq 6$ , every  $k$ -vertex gives  $\frac{1}{2}$  to each pendent 3-face.
- R2'**. Every  $7^+$ -vertex gives 1 to each pendent 3-face.
- R3'**. For  $4 \leq k \leq 6$ , every  $k$ -vertex gives 1 to each incident 3-face.
- R4'**. Every  $7^+$ -vertex gives 1 to each incident  $(4^+, 4^+, 4^+)$ -face.
- R5'**. Every non-bad  $7^+$ -vertex gives 2 to each incident  $(3, 4^+, 4^+)$ -face; every bad 8-vertex gives 1 to its bad 3-face.
- R6'**. Every 7-vertex gives 2 to each incident  $(3, 3, 7)$ -face.
- R7'**. For  $k \geq 8$ , every  $k$ -vertex gives 3 to each incident  $(3, 3, k)$ -face.

Let  $v$  be a  $k$ -vertex with  $k \geq 3$  by (C1'). Initially  $\omega(v) = 2k - 6$ .

**Case  $k = 3$ .** The discharging procedure does not involves 3-vertices. Hence  $\omega^*(v) = \omega(v) = 0$ .

**Case  $4 \leq k \leq 6$ .** Vertex  $v$  gives 1 to each of the  $\alpha$  incident 3-faces by R3' and  $\frac{1}{2}$  to each of the  $\beta$  pendent 3-faces by R1'. By Observation 2,  $\omega^*(v) \geq 2k - 6 - (\alpha + \frac{1}{2}\beta) \geq 2k - 6 - \frac{1}{2}k = \frac{3}{2}k - 6 \geq 0$ .

**Case  $k = 7$ .** Vertex  $v$  gives 2 to each of the  $\alpha'$  incident  $(3, 3^+, 4^+)$ -faces by R5'-6', 1 to each of the  $\alpha''$  incident  $(4^+, 4^+, 4^+)$ -faces by R4', and 1 to each of the  $\beta$  pendent 3-faces by R2'. By Observation 2,  $\omega^*(v) \geq 2k - 6 - (2\alpha' + \alpha'' + \beta) \geq 2k - 6 - k = k - 6 > 0$ .

**Case  $k \geq 8$ .** For the case when  $v$  is a bad 8-vertex,  $v$  gives 3 to each incident  $(3, 3, 8)$ -face by R7' and 1 to the bad 3-face by R5'. Hence  $\omega^*(v) = 2 \cdot 8 - 6 - 3 \cdot 3 - 1 = 0$ .

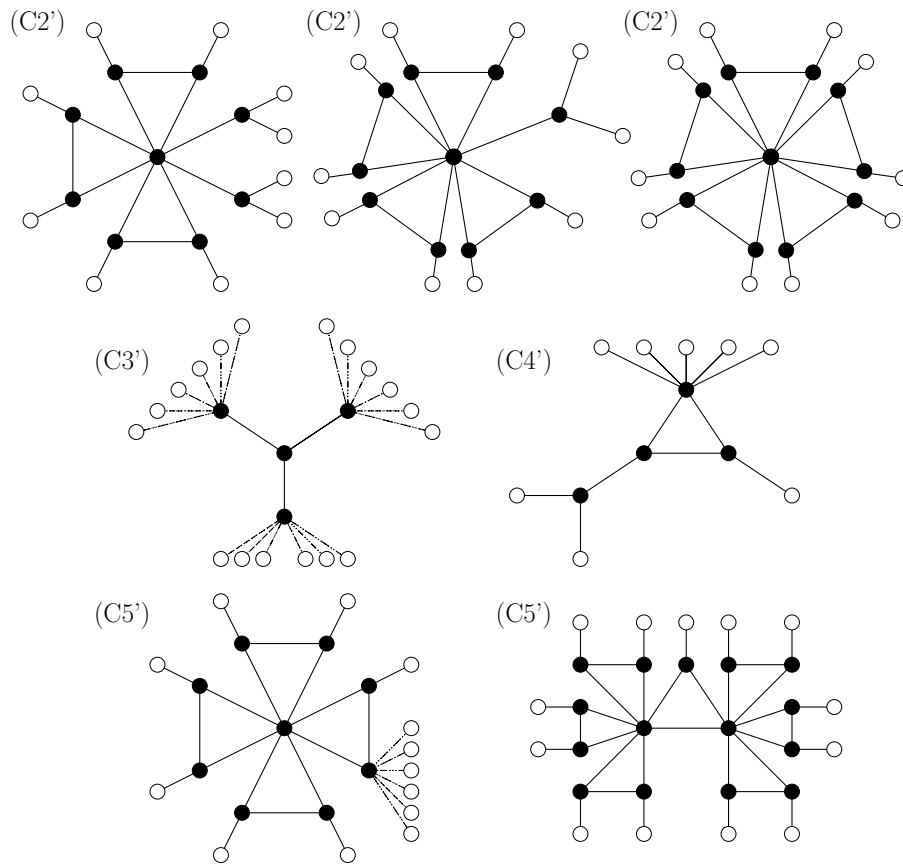


Figure 3: The reducible configurations (C2')-(C5'). (Drawing conventions are the same as in Figure 1.)

Now assume that  $v$  is not a bad 8-vertex. By R7', R5', R4' and R2',  $v$  gives 3 to each of the  $\alpha'$  incident  $(3, 3, k)$ -faces, 2 to each of the  $\alpha''$  incident  $(3, 4^+, 4^+)$ -faces, 1 to each of the  $\alpha'''$  incident  $(4^+, 4^+, 4^+)$ -faces, and 1 to each of the  $\beta$  pendent 3-faces. By Observation 2,  $\omega^*(v) = 2k - 6 - (3\alpha' + 2\alpha'' + \alpha''' + \beta) \geq 2k - 6 - \lfloor \frac{3k}{2} \rfloor = \lceil \frac{k}{2} \rceil - 6 \geq 0$  except for the cases (1)  $k = 10$  with  $\alpha' = 5$ , (2)  $k = 9$  with  $\alpha' = 4$  and  $\beta = 1$ , (3)  $k = 8$  with  $\alpha' = 3$  and  $\beta = 2$  (note that the bad 8-vertex case, i.e.  $\alpha' = 4$  or  $\alpha' = 3$  with  $\alpha'' = 1$ , is excluded). The exceptional cases give a  $k$ -vertex,  $8 \leq k \leq 10$ , with exactly  $k - 5$   $(3, 3, k)$ -faces and adjacent only to 3-vertices, a contradiction to (C2').

Let  $f$  be a  $k$ -face.

**Case  $k = 3$ .** Initially  $\omega(f) = -3$ .

Let  $f = uvw$  be a  $(a_1, a_2, a_3)$ -face with  $3 \leq a_1 \leq 6, 3 \leq a_2 \leq 6$  and  $3 \leq a_3 \leq 6$ . By (C3'), the outer neighbor of each 3-vertex incident to  $f$  has degree at least 7 and gives each at least 1 to  $f$  by R2'. By R3', each  $d$ -vertex with  $4 \leq d \leq 6$  incident to  $f$  gives 1 to  $f$ . It follows that  $\omega^*(f) = -3 + 3 = 0$ .

Let  $f = uvw$  be a  $(3, 3, 7)$ -face so that  $d(u) = d(v) = 3$  and  $d(w) = 7$ . By (C4') the outer neighbor of  $u$  (resp.  $v$ ) has degree at least 4 and so gives at least  $\frac{1}{2}$  to  $f$  by R1'. By R6',  $w$  gives 2 to  $f$ . It follows that  $\omega^*(f) = -3 + 2 \cdot \frac{1}{2} + 2 = 0$ .

Let  $f = uvw$  be a  $(3, 3, 8^+)$ -face so that  $d(u) = d(v) = 3$  and  $d(w) \geq 8$ . By R7',  $w$  gives 3 to  $f$ . It follows that  $\omega^*(f) = -3 + 3 = 0$ .

Let  $f = uvw$  be a  $(3, 4^+, 7^+)$ -face so that  $d(u) \geq 3, d(v) \geq 4$  and  $d(w) \geq 7$ . By R3'-5', vertices  $v$  and  $w$  gives at least 3 to  $f$  and so  $\omega^*(f) = -3 + 3 = 0$ , except for the case when  $f$  is a bad 3-face with the pair  $v, w$  being either two bad 8-vertices or a bad 8-vertex and a  $6^-$ -vertex. But these two exceptional cases are impossible by (C5').

Finally, let  $f = uvw$  be a  $(4^+, 4^+, 4^+)$ -face. Every incident vertex gives at least 1 to  $f$  by R3'-4'. Hence  $\omega^*(f) \geq 0$ .

It follows that every vertex and face has a non-negative charge as required. This completes the proof.

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