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Abstract. An r-identifying code in a graph $G = (V, E)$ is a subset $C \subseteq V$ such that for each $u \in V$ the intersection of C and the ball of radius r centered at u is nonempty and unique. Previously, r-identifying codes have been studied in various grids. In particular, it has been shown that there exists a 2-identifying code in the hexagonal grid with density 4/19 and that there are no 2-identifying codes with density smaller than 2/11. Recently, the lower bound has been improved to 1/5 by Martin and Stanton (2010). In this paper, we prove that the 2-identifying code with density 4/19 is optimal, i.e. that there does not exist a 2-identifying code in the hexagonal grid with smaller density.

Keywords: Identifying code; optimal code; hexagonal grid

1 Introduction

Let $G = (V, E)$ be a simple, connected and undirected graph with V as the set of vertices and E as the set of edges. Let u and v be vertices in V. If u and v are adjacent to each other, then the edge between u and v is denoted by $\{u, v\}$ (or in short by uv). The *distance* between u and v is denoted by $d(u, v)$ and is defined as the number of edges in any shortest path between u and v. Let r be a positive integer. We say that u *r-covers* v if the distance $d(u, v)$ is at most r. The *ball of radius r centered at u* is defined as

$$B_r(u) = \{x \in V \mid d(u, x) \leq r\}.$$

A nonempty subset of V is called a *code* in G, and its elements are called *codewords*. Let $C \subseteq V$ be a code and u be a vertex in V. An *I-set* (or an *identifying set*) of the vertex u with respect to the code C is defined as

$$I_r(C; u) = I_r(u) = B_r(u) \cap C.$$

The following definition is due to Karpovsky *et al.* [9].

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Definition 1. *Let r be a positive integer. A code $C \subseteq V$ is said to be r-*identifying *in G if for all $u, v \in V$ $(u \neq v)$ the set $I_r(C; u)$ is nonempty and*

$$I_r(C; u) \neq I_r(C; v).$$

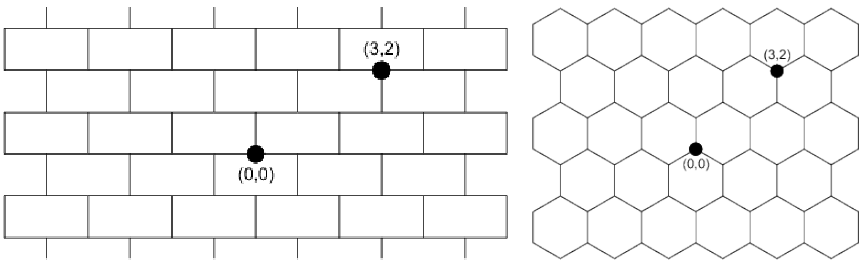
Let X and Y be subsets of V. The *symmetric difference* of X and Y is defined as $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$. We say that the vertices u and v are *r-separated* by a code $C \subseteq V$ (or by a codeword of C) if the symmetric difference $I_r(C; u) \triangle I_r(C; v)$ is nonempty. The definition of r-identifying codes can now be reformulated as follows: $C \subseteq V$ is an r-identifying code in G if and only if for all $u, v \in V$ $(u \neq v)$ the vertex u is r-covered by a codeword of C and

$$I_r(C; u) \triangle I_r(C; v) \neq \emptyset.$$

In this paper, we study identifying codes in the hexagonal grid. We define the hexagonal grid $G_H = (V_H, E_H)$ using the brick wall representation as follows: the set of vertices $V_H = \mathbb{Z}^2$ and the set of edges

$$E_H = \{\{\mathbf{u} = (i, j), \mathbf{v}\} \mid \mathbf{u}, \mathbf{v} \in \mathbb{Z}^2, \mathbf{u} - \mathbf{v} \in \{(0, (-1)^{i+j+1}), (\pm 1, 0)\}\}.$$

This definition is illustrated in Figure 1(a). The hexagonal grid can also be illustrated using the honeycomb representation as in Figure 1(b). In both illustrations, lines represent the edges and intersections of the lines represent the vertices of G_H. The labeling of the vertices in the brick wall representation is self-explanatory. This labeling can also be applied to the honeycomb representation, if we visualize the honeycomb to be obtained from the brick wall by squeezing it from left and right. For an example of the labeling of the vertices, we refer to Figure 1.



(a) Brick wall (b) Honeycomb

Fig. 1. The brick wall and the honeycomb representations illustrated.

To measure the size of an identifying code in the infinite hexagonal grid, we introduce the notion of density. For the formal definition, we first define

$$Q_n = \{(x, y) \in V_H \mid |x| \leq n, |y| \leq n\}.$$

Then the *density* of a code $C \subseteq V_H$ is defined as

$$D(C) = \limsup_{n\to\infty} \frac{|C \cap Q_n|}{|Q_n|}.$$

Naturally, we try to construct identifying codes with as small density as possible. Moreover, we say that an r-identifying code is *optimal*, if there do not exist any r-identifying codes with smaller density.

Previously, r-identifying codes in G_H have been studied in various papers. The first results concerning r-identifying codes in G_H have been presented in the seminal paper [9] in the case $r = 1$. Later these results have been improved by showing that there exists a 1-identifying code with density $3/7$ (see Cohen *et al.* [5]) and that there do not exist 1-identifying codes in G_H with density smaller than $12/29$ (see Cranston and Yu [6]). For general $r \geq 2$, Charon *et al.* [2] showed that each r-identifying code C in G_H has $D(C) \geq 2/(5r + 3)$ if r is even and $D(C) \geq 2/(5r + 2)$ if r is odd. They also presented a construction for each $r \geq 2$ giving an r-identifying code $C \subseteq V_H$ with $D(C) \sim 8/(9r)$.

For small values of r, the previous constructions have been improved in [4] by Charon *et al.* In particular, it is shown that there exists a 2-identifying code in G_H with density $4/19$. In the case $r = 2$, the lower bound $2/11$ from [4, Equation (1)] (and the aforementioned general lower bound) is improved in Martin and Stanton [10] by showing that the density of any 2-identifying code in G_H is at least $1/5$. In this paper, we further improve this lower bound to $4/19$. In other words, we show that the previously presented 2-identifying code with density $4/19$ is optimal.

Notice that, besides the hexagonal grid, identification has also been studied in various other grids such as the square, triangular and king grids. For the results in these grids, we refer to [1, 3, 4, 9, 10]. Moreover, in [7], one can found tables summarizing known result in these grids.

2 Lower bounds using share

Let $G = (V, E)$ be a simple, connected and undirected graph. Assume also that C is a code in G. The following concept of the share of a codeword has been introduced by Slater in [11]. The *share* of a codeword $c \in C$ is defined as

$$s_r(C; c) = s_r(c) = \sum_{u \in B_r(c)} \frac{1}{|I_r(C; u)|}.$$

The notion of share proves to be useful in determining lower bounds of r-identifying codes (as explained in the following).

Assume that $G = (V, E)$ is a finite graph and D is a code in G such that $B_r(u) \cap D$ is nonempty for all $u \in V$. Then it is easy to conclude that $\sum_{c \in D} s_r(D; c) = |V|$. Assume further that $s_r(D; c) \leq \alpha$ for all $c \in D$. Then we have $|V| \leq \alpha |D|$, which immediately implies

$$|D| \geq \frac{1}{\alpha} |V|.$$

Assume then that for any r-identifying code C in G we have $s_r(C;c) \leq \alpha$ for all $c \in C$. By the aforementioned observation, we then obtain the lower bound $|V|/\alpha$ for the size of an r-identifying code in G. In other words, by determining the maximum share for any r-identifying code, we obtain a lower bound for the minimum size of an r-identifying code.

The previous reasoning can also be generalized to the case when an infinite graph is considered. In particular, if for any r-identifying code in G_H we have $s_r(C;c) \leq \alpha$ for all $c \in C$, then it can be shown that the density of an r-identifying code in G_H is at least $1/\alpha$ (compare to Theorem 1). The main idea behind the proof of the lower bound (in Section 3) is based on this observation, although we use a more sophisticated method by showing that for any 2-identifying code the share is on *average* at most 19/4. In Theorem 1, we present a formal proof to verify that this method is indeed valid.

In the proof of the lower bound, we need to determine upper bounds for shares of codewords. To formally present a way to estimate shares, we first need to introduce some notations. Let $D \subseteq V$ be a code and c be a codeword of D. Consider then the I-sets $I_r(D;u)$ when u goes through all the vertices in $B_r(c)$. (Notice that all of these I-sets do not have to be different.) Denote the different identifying sets by $I_1, I_2, \ldots, I_k$, where k is a positive integer. Furthermore, denote the number of identifying sets equal to I_j by i_j ($j = 1, 2, \ldots, k$). Now we are ready to present the following lemma, which provides a method to estimate the shares of the codewords.

Lemma 1. *Let C be an r-identifying code in G and let D be a nonempty subset of C. For $c \in D$, using the previous notations, we have*

$$s_r(C;c) \leq \sum_{j=1}^{k} \left(\frac{1}{|I_j|} + (i_j - 1)\frac{1}{|I_j| + 1} \right).$$

Proof. Assume that $c \in D$. Then, for each $j = 1, 2, \ldots, k$, define $\mathcal{I}_j = \{u \in B_r(c) \mid I_j = I_r(D;u)\}$. Now it is obvious that for at most one vertex $u \in \mathcal{I}_j$ we have $I_j = I_r(C;u)$ and the other vertices of $\mathcal{I}_j$ are r-covered by at least $|I_j| + 1$ codewords of C. Hence, the claim immediately follows. □

The use of the previous lemma is illustrated in the following example.

Example 1. Let C be a 2-identifying code in the hexagonal grid G_H. Assume further that $D = \{(0,0),(0,1),(1,-1)\} \subseteq C$. Now we have $I_2(D;(-2,1)) = I_2(D;(-1,2)) = I_2(D;(1,2)) = I_2(D;(2,1)) = \{(0,1)\}$, $I_2(D;(-1,1)) = I_2(D;(0,1)) = I_2(D;(1,1)) = I_2(D;(-1,0)) = \{(0,0),(0,1)\}$ and $I_2(D;(0,0)) = I_2(D;(1,0)) = \{(0,1),(0,0),(1,-1)\}$. Thus, by Lemma 1, we obtain that

$$s_2(C;(0,1)) \leq \left(1 + 3 \cdot \frac{1}{2}\right) + \left(\frac{1}{2} + 3 \cdot \frac{1}{3}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) = \frac{55}{12}.$$

Furthermore, since we have $I_2(D;(-1,-1)) = I_2(D;(1,-1)) = I_2(D;(2,0)) = \{(0,0),(1,-1)\}$, we obtain (again by Lemma 1) that

$$s_2(C;(0,0)) \leq 1 + 2 \cdot \frac{1}{2} + 6 \cdot \frac{1}{3} + \frac{1}{4} = \frac{17}{4}.$$

3 The proof of the lower bound

From now on, we assume that C is a 2-identifying code in G_H. In this section, we first show that on average $s_2(\mathbf{c}) \leq 19/4$ for all $\mathbf{c} \in C$. Then, in Theorem 1, we finally prove that the density $D(C) \geq 4/19$.

The averaging process is done by introducing a *shifting scheme* designed to even out the shares among the codewords of C. Notice that the shifting scheme can also be understood as a discharging method. The *rules* of the shifting scheme are illustrated in Figure 2. Translations, rotations and reflections (over the line passing vertically through $\mathbf{u}$) can be applied to each rule in such a way that the structure of the underlying graph G_H is preserved. In the rules, share is shifted as follows:

- In the rules 1, 2, 4 and 7, we shift 1/4 units of share from $\mathbf{u}$ to $\mathbf{v}$.
- In the rule 3, we shift 1/6 and 1/12 units of share from $\mathbf{u}$ to $\mathbf{v}$ and $\mathbf{v}'$, respectively.
- In the rule 5, we shift 1/6 units of share from $\mathbf{u}$ to $\mathbf{v}$.
- In the rules 6, 8, 9 and 10, we shift 1/12 units of share from $\mathbf{u}$ to $\mathbf{v}$.

We also have the following modifications to the previous rules:

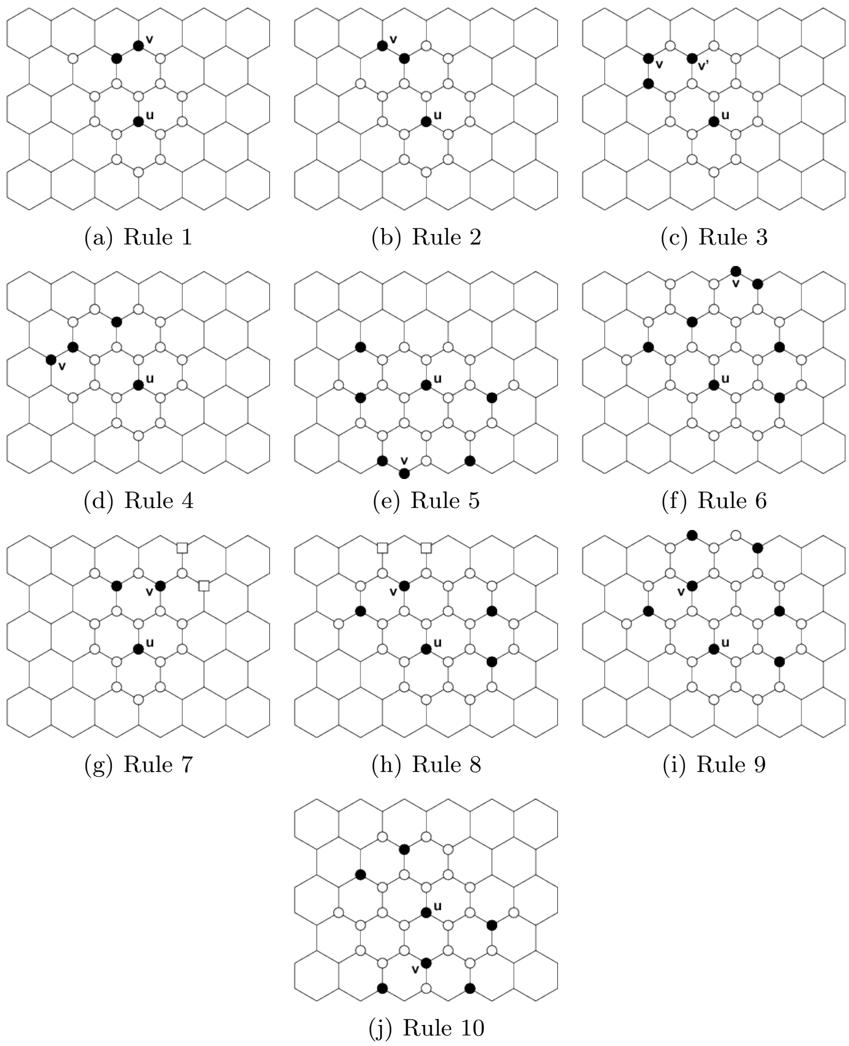
- If in the rules 1, 2 and 7 we have $\mathbf{u} + (0, -1) \in C$, then we only shift 1/12 units of share from $\mathbf{u}$ to $\mathbf{v}$ and denote these new rules (respectively) by 1.1, 2.1 and 7.1. Moreover, in the rule 1.1, we shift 1/12 units of share to $\mathbf{v}$ whether $(-3, 2)$ belongs to C or not.
- If in the rule 1 we have $\mathbf{u} + (-3, 2) \in C$, then we shift 1/4 units of share from $\mathbf{u}$ to $\mathbf{u} + (-1, 2)$ (no share is shifted to $\mathbf{v}$) and denote this new rule by 1.2.
- If in the rule 2 we have $\mathbf{u} + (-3, 1) \in C$, then we shift 1/6 units of share from $\mathbf{u}$ to $\mathbf{v}$ and denote this new rule by 2.2.
- If in the rule 2 we have $\mathbf{u} + (1, 2) \in C$, then we shift 1/12 units of share from $\mathbf{u}$ to $\mathbf{v}$ and denote this new rule by 2.3.

The modified share of a codeword $\mathbf{c} \in C$, which is obtained after the shifting scheme is applied, is denoted by $\bar{s}_2(\mathbf{c})$. In what follows, we show that $\bar{s}_2(\mathbf{c}) \leq 19/4$ for all $\mathbf{c} \in C$.

Lemma 2. *Let $\mathbf{c} \in C$ be a codeword such that $\mathbf{c}$ is adjacent to another codeword and share is shifted to $\mathbf{c}$ according to the previous rules. Then we have $\bar{s}_2(\mathbf{c}) \leq 19/4$.*

Proof. Since $\mathbf{c}$ is adjacent to another codeword, it is immediate that $\mathbf{c}$ can receive share only according to the rules 1–6 and their modifications. The proof of the lemma is now divided into three cases depending on the number of codewords adjacent to $\mathbf{c}$.

Assume first that $\mathbf{c}$ is adjacent to exactly one codeword. Without loss of generality, we may assume that $\mathbf{c} = (0, 0)$ and the adjacent codeword is $(0, 1)$. (Notice that $(-1, 0) \notin C$ and $(1, 0) \notin C$.) Now the only possibilities for $\mathbf{c}$



(a) Rule 1 (b) Rule 2 (c) Rule 3

(d) Rule 4 (e) Rule 5 (f) Rule 6

(g) Rule 7 (h) Rule 8 (i) Rule 9

(j) Rule 10

Fig. 2. The rules of the shifting scheme illustrated. The black dots represent codewords and the white dots represent non-codewords. In the rules 7 and 8, at least one of the vertices marked with a white square is a codeword.

to receive share is from the vertices $(-3,0)$ or $(3,0)$ (the rule 1.2) and from the vertices that belong to $S_1 = \{(5,1),(4,1),(3,1),(3,2),(2,2)\}$ and $S_2 = \{(-5,1),(-4,1),(-3,1),(-3,2),(-2,2)\}$. It is straightforward to verify that the codewords in each of the sets S_1 and S_2 can shift at most $1/4$ units of share to $\mathbf{c}$. (Notice that only the rules 2.1 and 5, or the rules 2.1 and 6 can be used simultaneously.)

Assume that the rule 1.2 is used. (Clearly, this rule can be used only once.) Without loss of generality, we may assume that $(-3,0) \in C$ and $(1,-1) \in C$. Therefore, choosing $D = \{\mathbf{c},(0,1),(-3,0),(1,-1)\}$ in Lemma 1, we obtain that $s_2(\mathbf{c}) \leq 15/4$. Thus, since the codewords in each of the sets S_1 and S_2 can shift at most $1/4$ units of share to $\mathbf{c}$, we have $\bar{s}_2(\mathbf{c}) \leq s_2(\mathbf{c}) + 3 \cdot 1/4 \leq 9/2$. Assume then that the rule 1.2 cannot be applied to $\mathbf{c}$. Since $\mathbf{c}$ and $(0,1)$ are 2-separated by C, there exists at least one codeword in the symmetric difference $B_2(\mathbf{c}) \triangle B_2(0,1)$. Thus, without loss of generality, we may assume that $(1,-1) \in C$, $(1,2) \in C$, $(2,0) \in C$ or $(2,1) \in C$.

Assume first that $(1,-1) \in C$. Now, by Example 1, we know that $s_2(\mathbf{c}) \leq 17/4$. Therefore, we have $\bar{s}_2(\mathbf{c}) \leq s_2(\mathbf{c}) + 2 \cdot 1/4 \leq 19/4$. Assume then that $(1,2) \in C$. It is straightforward to verify that share can be shifted to $\mathbf{c}$ only from $(-3,1)$, $(-2,2)$, $(-3,2)$, $(-4,1)$ and $(-5,1)$ according to the rules 1.1, 2.3, 3, 5 and 6, respectively. Moreover, it is easy to see that at most one of these rules can be used (and only once). Thus, the codeword $\mathbf{c}$ receives at most $1/6$ units of share. Hence, we have $\bar{s}_2(\mathbf{c}) \leq s_2(\mathbf{c}) + 1/6 \leq 19/4$ since $s_2(\mathbf{c}) \leq 55/12$ by Example 1.

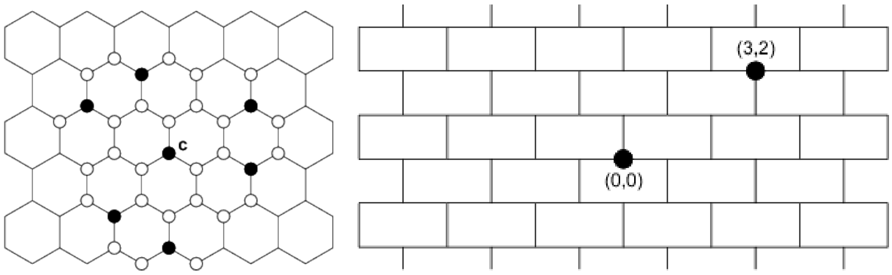
Assume now that $(2,0) \in C$. Then it is easy to conclude that the vertices $(5,1)$, $(4,1)$ and $(3,1)$ cannot shift share to $\mathbf{c}$. Hence, only either $(3,2)$ according to the rule 3 or $(2,2)$ according to the rule 2.2 (but not both) is capable of shifting share to $\mathbf{c}$. In both cases, $\mathbf{c}$ receives at most $1/6$ units of share. Choosing $D = \{\mathbf{c},(0,1),(2,0)\}$ in Lemma 1, we obtain that $s_2(\mathbf{c}) \leq 13/3$. Therefore, since at most $1/4$ units of share is received from S_2, we have $\bar{s}_2(\mathbf{c}) \leq s_2(\mathbf{c}) + 1/4 + 1/6 \leq 19/4$. Finally, assume that $(2,1) \in C$. By Lemma 1, we have $s_2(\mathbf{c}) \leq 9/2$. Since now share can be shifted to $\mathbf{c}$ only from S_2, we obtain that $\bar{s}_2(\mathbf{c}) \leq s_2(\mathbf{c}) + 1/4 \leq 19/4$.

The proofs of the other two cases, where $\mathbf{c}$ is adjacent to two or three codewords, are similar to (and even easier than) the one of the first case. Hence, these proofs are omitted here. However, for the proofs, the interested reader is referred to [8]. □

The proof of the following lemma is analogous to the previous proof and it is omitted in this presentation. However, the complete proof of the lemma can be found from [8].

Lemma 3 ([8]). *Let $\mathbf{c} \in C$ be a codeword such that $\mathbf{c}$ is not adjacent to another codeword and share is shifted to $\mathbf{c}$ according to the previous rules. Then we have $\bar{s}_2(\mathbf{c}) \leq 19/4$.*

Lemma 4. *Let $\mathbf{c} \in C$ be a codeword such that no share is shifted to $\mathbf{c}$ according to the previous rules. Then we have $\bar{s}_2(\mathbf{c}) \leq 19/4$.*



(a) (b)

Fig. 3. Two cases of the proof of Lemma 4 illustrated.

Proof. Without loss of generality, we may assume that $\mathbf{c} = (0, 0)$. Assume first that $|I_2(\mathbf{c})| \geq 2$. If now $\mathbf{c}$ is adjacent to another codeword, then $\bar{s}_2(\mathbf{c}) \leq 19/4$ (by Lemma 1). Hence, we may assume that $(-1, 0), (1, 0), (0, 1) \notin C$ and $(2, 0) \in C$. Since the vertices $\mathbf{c}$ and $(1, 0)$ are 2-separated by C, there is at least one codeword in the symmetric difference $B_2(\mathbf{c}) \triangle B_2(1, 0)$. Therefore, by Lemma 1, we have $\bar{s}_2(\mathbf{c}) \leq s_2(\mathbf{c}) \leq 19/4$ (in all possible cases).

From now on, we may assume that $I_2(\mathbf{c}) = \{\mathbf{c}\}$. Furthermore, suppose first that $(-2, 1), (2, 1), (0, -1) \notin C$. Since $\mathbf{c}$, $(-1, 0)$, $(1, 0)$ and $(0, 1)$ are 2-separated by C, each of the sets $\{(-1, 2), (1, 2)\}$, $\{(3, 0), (2, -1)\}$ and $\{(-3, 0), (-2, -1)\}$ contains at least one codeword. Therefore, by Lemma 1, we have $s_2(\mathbf{c}) \leq 1 + 6 \cdot 1/2 + 3 \cdot 1/3 = 5$.

Consider then the set $\{(-1, 2), (1, 2)\}$ that contains at least one codeword as stated above. Assume first that both $(-1, 2) \in C$ and $(1, 2) \in C$. If the vertex $(0, 2)$ also belongs to C, then we have $\bar{s}_2(\mathbf{c}) \leq s_2(\mathbf{c}) \leq 14/3$ by Lemma 1 (and we are done). Hence, suppose that $(0, 2) \notin C$. If now $(-2, 2) \in C$ or $(2, 2) \in C$, then $s_2(\mathbf{c}) \leq 29/6$ and the rule 2.3 can be applied to $\mathbf{c}$. Therefore, we have $\bar{s}_2(\mathbf{c}) \leq s_2(\mathbf{c}) - 1/12 \leq 19/4$. Thus, we may assume that $(-2, 2) \notin C$ and $(2, 2) \notin C$. Since the vertices $(-1, 2)$ and $(1, 2)$ are 2-separated by C, at least one of the vertices $(-3, 2)$, $(-2, 3)$, $(2, 3)$ and $(3, 2)$ belongs to C. Hence, we can shift at least $1/4$ units of share from $\mathbf{c}$ according to the rule 7. Therefore, we have $\bar{s}_2(\mathbf{c}) \leq s_2(\mathbf{c}) - 1/4 \leq 19/4$.

By the considerations above, we may without loss of generality assume that $(-1, 2) \in C$ and $(1, 2) \notin C$. If $(0, 2) \in C$, then $1/4$ units of share can be shifted from $\mathbf{c}$ according to the rule 1 or 1.2, and we are done. Thus, suppose that $(0, 2) \notin C$. Assume then that $(-2, 2) \in C$. If the rule 2 applies to $\mathbf{c}$, then we are immediately done ($1/4$ units of share is shifted from $\mathbf{c}$). On the other hand, if $(-3, 1) \in C$, then by Lemma 1 $s_2(\mathbf{c}) \leq 59/12$ and we are again done since at least $1/6$ units of share is shifted from $\mathbf{c}$ according to the rule 2.2. Hence, assume that $(-2, 2) \notin C$. Since $(0, 1)$ and $(-1, 1)$ are 2-separated by C, the vertex $(-3, 1)$ belongs to C. If now either $(-3, 2) \in C$, or $(-4, 1) \in C$, $(-3, 2) \notin C$ and $(-3, 0) \notin C$, then (respectively) either the rule 3 or the rule 4 can be used and we are done. Furthermore, if $(-4, 1) \in C$, $(-3, 2) \notin C$ and $(-3, 0) \in C$, then instead of the set $\{(-1,2),(1,2)\}$ we can consider the set $\{(-3,0),(-2,-1)\}$. Using similar arguments as above, we obtain that $\bar{s}_2(\mathbf{c}) \leq 19/4$ also in this case. Thus, we may assume that $(-4,1) \notin C$ and $(-3,2) \notin C$.

The previous reasoning also applies when we consider the sets $\{(3,0),(2,-1)\}$ and $\{(-3,0),(-2,-1)\}$ instead of $\{(-1,2),(1,2)\}$. This leads straightforwardly to the observation that we have only two different neighbourhoods of $\mathbf{c}$ (up to rotations and reflections). These neighbourhoods are illustrated in Figure 3.

Consider first the case in Figure 3(a). In what follows, we show that $\mathbf{c}$ shifts 1/12 units of share to $(-1,2)$ or $(1,3)$, or that we originally have $s_2(\mathbf{c}) \leq 5 - 1/12 = 59/12$. This observation can then be generalized to the (other two) symmetrical cases implying $\bar{s}_2(\mathbf{c}) \leq 5 - 3 \cdot 1/12 = 19/4$. If $(2,2) \in C$, then we have $s_2(\mathbf{c}) \leq 29/6 \leq 59/12$. Assume then that $(2,2) \notin C$. If we can shift 1/12 units of share from $\mathbf{c}$ to $(-1,2)$ according to the rule 8, then we are immediately done. Hence, we may assume that $(-2,3) \notin C$ and $(0,3) \notin C$. Since $(-1,2)$ and $(1,2)$ are 2-separated by C, the vertex $(2,3)$ belongs to C. Furthermore, since $(-1,2)$ and $(0,2)$ are 2-separated by C, at least one of the vertices $(-1,3)$ and $(1,3)$ is a codeword. Therefore, at least 1/12 units of share can shifted from $\mathbf{c}$ according to the rule 6 or 9.

Consider then the case in Figure 3(b). Let us now show that $\mathbf{c}$ shifts at least 1/6 units of share to $(-1,-2)$ or $(1,-2)$, or that we originally have $s_2(\mathbf{c}) \leq 5 - 1/6 = 29/6$. This result together with the observation above, which states that $\mathbf{c}$ shifts 1/12 units of share to $(-1,2)$ or $(1,3)$, or that we originally have $s_2(\mathbf{c}) \leq 5 - 1/12 = 59/12$, implies that $\bar{s}_2(\mathbf{c}) \leq 5 - 1/6 - 1/12 = 19/4$. If now $(0,-2) \in C$, then we obtain by Lemma 1 that $s_2(\mathbf{c}) \leq 29/6$. Hence, we may assume that $(0,-2) \notin C$. Since $\mathbf{c}$, $(-1,-1)$ and $(1,-1)$ are 2-separated by C, the vertices $(-2,-2)$ and $(2,-2)$ belong to C. Furthermore, since $(0,-1)$ is 2-covered by a codeword of C, we have $(-1,-2) \in C$ or $(1,-2) \in C$. Therefore, we can shift at least 1/6 units of share to $(-1,-2)$ or $(1,-2)$ according to the rule 5. In conclusion, if $\mathbf{c}$ is a codeword such that $I_2(\mathbf{c}) = \{\mathbf{c}\}$ and $(-2,1),(2,1),(0,-1) \notin C$, then we have $\bar{s}_2(\mathbf{c}) \leq 19/4$.

Assume then that $\mathbf{c}$ is a codeword such that $I_2(\mathbf{c}) = \{\mathbf{c}\}$, $(-2,1) \notin C$, $(2,1) \notin C$ and $(0,-1) \in C$. The proof of this case is similar to the previous one. Hence, the proof of this case is omitted here. For the complete proof, we again refer to [8]. Notice that in this case the modified rules 1.1, 2.1 and 7.1 are used.

Finally, assume that $\mathbf{c}$ is a codeword such that $I_2(\mathbf{c}) = \{\mathbf{c}\}$, and that at least two of the vertices $(-2,1)$, $(2,1)$ and $(0,-1)$ belong to C. Then, by Lemma 1, we have $s_2(\mathbf{c}) \leq 14/3$. This observation completes the proof of the lemma. □

In the previous lemmas, we have shown that $\bar{s}_2(\mathbf{c}) \leq 19/4$ for any $\mathbf{c} \in C$. Now we are ready to prove the main theorem of the paper.

Theorem 1. *If C is a 2-identifying code in the hexagonal grid G_H, then the density*

$$D(C) \geq \frac{4}{19}.$$

Proof. Assume that C is a 2-identifying code in G_H. Since each vertex $\mathbf{u} \in Q_{n-2}$ with $|I_2(\mathbf{u})| = i$ contributes the summand $1/i$ to $s_2(\mathbf{c})$ for each of the i codewords

$\mathbf{c} \in B_2(\mathbf{u})$, we have

$$\sum_{\mathbf{c} \in C \cap Q_n} s_2(\mathbf{c}) \geq |Q_{n-2}|. \tag{1}$$

Furthermore, we have

$$\sum_{\mathbf{c} \in C \cap Q_n} s_2(\mathbf{c}) \leq \sum_{\mathbf{c} \in C \cap Q_n} \bar{s}_2(\mathbf{c}) + \frac{19}{4} |Q_{n+6} \setminus Q_n|, \tag{2}$$

because shifting shares inside Q_n does not affect the sum and each codeword in $Q_{n+6} \setminus Q_n$ can receive at most 19/4 units of share (by Lemma 2 and 3). Notice also that codewords in Q_n cannot shift share to codewords outside Q_{n+6}. Therefore, combining the equations (1) and (2) with the fact that $\bar{s}_2(\mathbf{c}) \leq 19/4$ for any $\mathbf{c} \in C$, we obtain

$$\frac{|C \cap Q_n|}{|Q_n|} \geq \frac{4}{19} \cdot \frac{|Q_{n-2}|}{|Q_n|} - \frac{|Q_{n+6} \setminus Q_n|}{|Q_n|}.$$

Since $|Q_k| = (2k+1)^2$ for any positive integer k, it is straightforward to conclude from the previous inequality that the density $D(C) \geq 4/19$. □

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