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Abstract. In this paper, we present various results on arcs in projective three-dimensional Hjelmslev spaces over finite chain rings of nilpotency index 2. A table is given containing exact values and bounds for projective arcs in the geometries over the two chain rings with four elements.

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1 Introduction

From the point of view of coding theory, arcs in higher dimensional Hjelmslev spaces are of particular interest. The research in the past ten years is focused mainly on plane projective arcs. In this paper, we present some constructions and upper bounds on arcs in the three-dimensional Hjelmslev spaces over the finite chain rings of nilpotency index 2. As a by-product, we obtain new results on non-projective plane arcs. In order to save space, we do not introduce the basic facts on projective Hjelmslev geometries over finite chain rings. A self-contained introduction to Hjelmslev geometries over finite chain rings is given e.g. in [1].

This note is organized as follows. In section 2, we present several upper bounds on the size of an arc with given parameters in $\mathrm{PHG}(R^t_R)$. Section 3 contains results on arcs with multiple points in projective Hjelmslev planes. In particular, we give a complete solution for the optimal sizes in the two planes over chan rings with four elements. In section 4, building on the results in the previous two sections, we compile a table with exact values and bounds for the largest arcs in the three-dimensional projective Hjelmslev geometries over the chain rings with four elements.

2 Upper bounds on the arc size

Denote by $m_n(R^t_R)$ the largest size of a (k,n)-arc in $\mathrm{PHG}(R^t_R)$.

Theorem 1. *Let $\mathfrak{K}$ be a (k,n)-(multi)arc in* $\mathrm{PHG}(R_R^t)$*, where R is a chain ring with $|R| = q^2$, $R/\operatorname{Rad} R \cong \mathbb{F}_q$, and let x be a point with $\mathfrak{K}(x) = a$. Then*

$$k \leq a + m_{n-a}(R_R^{t-1}).$$

Proof. Fix a hyperplane H with $x \not\frown H$. Define a projection φ from x onto H by

$$\varphi : \begin{cases} \mathcal{P} \setminus \{x\} \to & H, \\ \quad y \to \cup_{L: L \in \mathcal{L}, x \in L} L \cap H, \end{cases}$$

where $\mathcal{P}$ is the set of points of $\mathrm{PHG}(R_R^t)$. If $y \in \mathcal{P}$ is not a neighbour to x then its image is a point; if $y \frown x$ then the image of y is a neighbour class of points in $H \cong \mathrm{PHG}(R_R^{t-1})$. The image of an s-dimensional Hjelmslev subspace through x contains an $(s-1)$-dimensional subspace in H. Conversely, every $(s-1)$-dimensional subspace in H is contained in the projection of some s-dimensional subspace of $\mathrm{PHG}(R_R^t)$ through x.

Define a new arc $\mathfrak{K}^\varphi : H \to \mathbb{Q}$ via

$$\mathfrak{K}^\varphi(z) := \sum_{y: \varphi(y) = z, y \not\frown x} \mathfrak{K}(y) + \frac{1}{q^{t-3}} \sum_{y: \varphi(y) = z, y \frown x} \mathfrak{K}(y).$$

Let H' be a hyperplane in $\mathrm{PHG}(R_R^t)$ containig x, and let F' be the unique hyperplane in $\varphi(H')$. We have

$$\mathfrak{K}^\varphi(F') = \mathfrak{K}(H') - \mathfrak{K}(x).$$

Set $\mathfrak{K}([x]) = b$. Then $\mathfrak{K}^\varphi$ is a $(k - a + q(b-a), n-a)$-arc with rational multiplicities of the points.

Now define

$$\varphi_0 : \begin{cases} \mathcal{P} \setminus \{x\} \to & H, \\ \quad y \to L \cap H, \end{cases}$$

where L is some arbitrarily chosen line in $\mathrm{PHG}(R_R^t)$ that contains x and y. Set

$$\mathfrak{K}^{\varphi_0}(z) := \sum_{y: \varphi_0(y) = z} \mathfrak{K}(y).$$

It is easily verified that $\mathfrak{K}^{\varphi_0}(F') \leq \mathfrak{K}^\varphi(F') \leq n - a$. The arc $\mathfrak{K}^{\varphi_0}$ is integer-valued $(k-a, n-a)$-arc in H hence

$$k - a \leq m_{n-a}(R_R^{t-1}),$$

which was to be proved.

Corollary 1. *Let $\mathfrak{K}$ be a projective (k,n)-arc in* $\mathrm{PHG}(R_R^4)$ *where R is a chain ring with $|R| = q^2$, $R/\operatorname{Rad} R \cong \mathbb{F}_q$. Then*

$$k \leq 1 + m_{n-1}(R_R^3).$$

Corollary 2. *Let $\mathfrak{K}$ be a (k,n)-(multi)arc in* $\mathrm{PHG}(R_R^t)$*, where R is a chain ring with $|R| = q^2$, $R/\operatorname{Rad} R \cong \mathbb{F}_q$, and let x be a point with $\mathfrak{K}(x) = a$, $\mathfrak{K}([x]) = b$. Then there exists an arc in* $\mathrm{PHG}(R_R^{t-1})$ *with parameters $(qk + q^2 b - q(q+1)a, q(n-a))$.*

Proof. The arc $q\mathfrak{K}^{\varphi}$ from the proof of Theorem 1 has the desired parameters.

The next theorem gives better upper bounds in three-dimensional spaces when the values of n are large, i.e. close to the size of a plane.

Theorem 2. *Let $\mathfrak{K}$ be a (k,n)-(multi)arc in $\mathrm{PHG}(R_R^4)$, where R is a chain ring with $|R| = q^2$, $R/\operatorname{Rad} R \cong \mathbb{F}_q$. If there exists a neighbor class of lines $[L]$ with $\mathfrak{K}([L]) = c$ then*

$$k \le q(q+1)(n - \lceil \frac{c}{q} \rceil) + c.$$

Proof. Let $\Pi = \mathrm{PHG}(R_R^4)$ and let $[H]$ be a class of neighbour planes containing $[L]$. Consider the incidence structure Π' having as points the plane segments $H' \cap [x]$, where $H' \in [H]$, $x \in H$, and as planes – the elements of $[H]$. It is known (cf. [1,3]) that Π' is isomorphic to the dual affine space $\overline{AG(3,q)}$ and can be imbedded in $\mathrm{PG}(3,q)$. The class $[L]$ in Π' is a dual affine plane. The number of lines in a dual affine plane $\overline{AG(2,q)}$ is q^2 and every point is on exactly q lines. Therefore there exists a line L' in Π' containing at least $\lceil qc/q^2 \rceil = \lceil c/q \rceil$ points. This line L' consists of the points in a class of neighbor lines in some plane $H' \in [H]$, i.e. $L' = H' \cap [L]$.

Denote by $T_1, \dots, T_q$ the planes in Π containing $H' \cap [L]$. Clearly,

$$\mathfrak{K}(T_i \setminus (H' \cap [L])) \le n - \lceil c/q \rceil,$$

whence

$$\mathfrak{K}([H]) \le q(n - \lceil c/q \rceil) + c.$$

Now consider the factor geometry $\Pi/\frown \cong \mathrm{PG}(3,q)$. Denote by $[H_i]$, $i = 0, \dots, q$, the planes through $[L]$ in $\Pi/\frown$. We have

$$\begin{aligned} k &= \sum_{i=0}^{q} \mathfrak{K}([H_i]) - q \cdot \mathfrak{K}([L]) \\ &\le (q+1)(q(n - \lceil c/q \rceil) + c) - qc \\ &= q(q+1)(n - \lceil c/q \rceil) + c, \end{aligned}$$

which is the desired inequality.

It is well-known that there exists a spread of q^2+1 lines in the factor geometry $\Pi/\frown$. Hence we have $c \ge \lceil k/(q^2+1) \rceil$. Applying Theorem 2, we obtain the following corollary.

Corollary 3. *Let $\mathfrak{K}$ be a projective (k,n)-arc in $\mathrm{PHG}(R_R^4)$. Then*

$$k \le \max_{1 \le c \le q^4+q^3} \min\{c(q^2+1), q(q+1)(n - \lceil c/q \rceil) + c\}.$$

The following theorem is also useful.

Theorem 3. *Let $\mathfrak{F}$ be an (f,m)-blocking (multi)set in $\mathrm{PHG}(R_R^4)$ with $m \le q$, where R is a chain ring with $|R| = q^2$, $R/\operatorname{Rad} R \cong \mathbb{F}_q$. If there exists a neighbor class with $\mathfrak{F}([x]) = 0$ then $f \ge mq(q+1)$.*

Proof. Consider the projection φ from any point in the empty class onto some plane disjoint from $[x]$. Obviously $\mathfrak{F}^{\varphi}$ is an (f,m)-blocking set in an Hjelmslev plane isomorphic to $\mathrm{PHG}(R_R^3)$. The result follows from the trivial upper bound on the size of a plane blocking set.

3 Arcs with multiple points in $\mathrm{PHG}(R_R^3)$

Extensive tables for the optimal sizes of projective arcs in Hjelmslev planes over the small chain rings are given in [2,3,5]. However, the arcs needed in Corollary 1 are not projective. In this section, we collect results on multiarcs of maximal size in projective Hjelmslev geometries of dimension 3. Let us note that the general bound from [5] applies also for arcs with multiple points. Since this bound is our main tool, we state it explicitly.

Theorem 4. *Let $\mathfrak{K}$ be a (k,n)-arc in $\mathrm{PHG}(R_R^3)$ where $|R| = q^2$, $R/N \cong \mathbb{F}_q$. Suppose there exist a point x with $\mathfrak{K}(x) = a$ and a neighbour class of points $[x]$ with $\mathfrak{K}([x]) = b$. Then*

$$k \leq (n-a)q^2 + (n-b)q + b.$$

For some special values of n, the exact value of $m_n(R_R^3)$ is easily found.

Theorem 5. *Let R be a chain ring with $|R| = q^2$, $R/\operatorname{Rad} R \cong \mathbb{F}_q$. Then*

(a) $m_{sq(q+1)}(R_R^3) = sq^2(q^2+q+1)$;
(b) $m_{sq(q+1)+1}(R_R^3) = sq^2(q^2+q+1)+1$.

Proof. Part (a) follows easily from (b), so we provide a proof for part (b) only.

Arcs with parameters $(sq^2(q^2+q+1)+1, sq(q+1)+1)$ are easily obtained as the sum of s copies of the complete plane plus an arbitrary point. Now we are goimg to prove that we cannot have an arc with $n = sq(q+1)+1$ and size larger than $sq^2(q^2+q+1)$.

Let $\mathfrak{K}$ be an (k,n)-arc with $n = sq(q+1)+1$. Assume $[x]$ is a neighbour class of points with $\mathfrak{K}([x]) = sq^2+\alpha$, $\alpha > 0$. Such a class does exist; otherwise $k \leq sq^2(q^2+q+1)$ and we are done.

Now in each parallel class of $[x]$ (which has the structure of $\mathrm{AG}(2,q)$) we have a line segment of multiplicity at least $sq + \lceil \alpha/q \rceil$. For each line L containing this line segment, we have

$$\mathfrak{K}(L \setminus [x]) \leq sq(q+1)+1-sq-\lceil \alpha/q \rceil = sq^2+1-\lceil \alpha/q \rceil.$$

This implies

$$\begin{aligned} k &\leq sq^2+\alpha+q(q+1)(sq^2+1-\lceil \alpha/q \rceil) \\ &= sq^2(q^2+q+1)+q^2(1-\lceil \alpha/q \rceil)+q(1-\lceil \alpha/q \rceil)+\alpha. \end{aligned}$$

If $\alpha > q$, then $q^2(1-\lceil \alpha/q \rceil)+q(1-\lceil \alpha/q \rceil)+\alpha < 0$. So let $1 \leq \alpha \leq q$. Clearly, there exists a point in $[x]$ of multiplicity $s+\beta$, $\beta \geq 1$. At least one of the line segments through this point has multiplicity more than $sq+2 = sq+1+\lceil \alpha/q \rceil$. Otherwise, we would have

$$\mathfrak{K}([x]) \leq s+\beta+(q+1)(sq+1-s-\beta) = sq^2+(q+1)(1-\beta) \leq sq^2,$$

a contradiction. Now

$$\begin{aligned} k &\leq sq^2+\alpha+q^2(sq^2+1-\lceil\alpha/q\rceil)+q(sq^2-\lceil\alpha/q\rceil \\ &= sq^2(q^2+q+1)+q^2(1-\lceil\alpha/q\rceil)-q\lceil\alpha/q\rceil)+\alpha. \end{aligned}$$

We have that for $1 \leq \alpha \leq q$,

$$q^2(1-\lceil\alpha/q\rceil)-q\lceil\alpha/q\rceil)+\alpha =)-q\lceil\alpha/q\rceil)+\alpha \leq 1,$$

whith equality for $\alpha = 1$. This proves the theorem.

The next theorem settles the problem of finding the maximal sizes of multiarcs for the two rings with four elements.

Theorem 6. *Let R be a chain ring with $|R| = 4$, $R/\operatorname{Rad} R \cong \mathbb{F}_2$. Then*

(a) $m_{6t}(R_R^3) = 28t$,
(b) $m_{6t+1}(R_R^3) = 28t+1$,
(c) $m_{6t+2}(R_R^3) = \begin{cases} 28t+7 & \text{if } R = \mathbb{Z}_4; \\ 28t+6 & \text{if } R = \mathbb{F}_2[X]/(X^2) \end{cases}$,
(d) $m_{6t+3}(R_R^3) = 28t+10$,
(e) $m_{6t+4}(R_R^3) = 28t+16$,
(f) $m_{6t+5}(R_R^3) = 28t+22$,

where $t = 0,1,2,\ldots$.

Proof. Clearly, (a) and (b) are settled by Theorem 5.

(c) Let $n = 6t+2$. Arcs of cardinality $28t+7$ (resp. $28t+6$) are obtained as a sum of T copies of the complete plane and a $(7,2)$-arc (resp. $(6,2)$-arc). Now assume there exists a (k,n)-arc with $k = 28t+8$. Then there exists a class of points $[x]$ with $\mathfrak{K}([x]) = 4t+2$ and a point in it with $\mathfrak{K}(x) \geq t+1$. By Theorem 4 we get

$$\begin{aligned} k &\leq 4\cdot(6t+2-a)+2\cdot(6t+2-b)+b \\ &= 36t+12-4a-b \\ &\leq 28t+6, \end{aligned}$$

which is a contradiction.

Now assume that $\mathfrak{K}$ is a (k,n)-arc with $k = 28t+7$. By the above argument, all classes of points ahve multiplicity $4t+1$ and every point has multiplicity at most $t+1$. Moreover, the four points in each neighbour class of points must have multiplicities $t+1,t,t,t$ since otherwise we get a contradiction by a counting argument. For instance, if the multiplicities of the points in a neighbour class are $t+1,t+1,t+1,t-2$ then there is a line segment of multiplicity $2t+2$ in each direction and

$$k \leq 4t+1+6\cdot(6t+2-2t-2) = 28t+1.$$

The other possibility of points of multiplicities $t+1,t+1,t,t-1$ is ruled out by the same argument.

Now we have seven points of multplicity $t+1$ and obviously no three of them are collinear. Hence they form a $(7,2)$-arc which is known to exist when $R=\mathbb{Z}_4$ and not to exist in case of $R=\mathbb{F}_2[X]/(X^2)$.

(d) Obviously, we can construct $(28t+10,6t+3)$-arcs as a sum of t copies of the plane and a $(10,3)$-arc. Assume there is a (k,n)-arc $\mathfrak{K}$ with $k=28t+11$ and $n=6t+3$. Then there exists a class $[x]$ with $\mathfrak{K}([x])\geq 4t+2$ and a point in it, x say, with $\mathfrak{K}(x)\geq 4t+1$. By Theorem 4, $k\leq 28t+12$. If $k=28t+11$ or $28t+12$ then $\mathfrak{K}([x])\leq 4t+3$ for every class $[x]$ and $\mathfrak{K}(y)\leq t+1$ for every point y in such a class.

Classes of multiplicity $4t+3$ are easily ruled out since they must consist of three $(t+1)$-points and one t-point, and must have segments of multiplicity $2t+2$ in every direction. By a similar argument, a class of multiplicity $4t+2$ consists of two $(t+1)$-points and two t-points. Now the $(t+1)$-points form a $(\kappa,3)$-arc with $\kappa=11$ or 12, which is impossible.

The proofs of (e) and (f) use similar arguments.

4 Arcs in three-dimensional Hjelmslev spaces

In this section, we present a table with exact values and bounds on $m_n(R_R^4)$, $n\leq 28$, where R is any of the two chain rings with four elements. Let us note that for $n=3,4,5$ the exact values are computed in [4]. In the remaining cases the bounds are obtained in the way described below.

$n=6$: The upper bound is obtained by Corollary 2; an $(18,6)$-arc is obtained as the union of three $(6,2)$-arcs in three planes (non-neighbors) with a common empty line.

$n=7$: The upper bound follows by Corollary 1; if $R=\mathbb{Z}_4$ a $(21,7)$ is obtained as the sum of three plane $(7,2)$-arcs; if $R=\mathbb{F}_2[X]/(X^2)$ a $(19,7)$ is obtained as the sum of two plane $(6,2)$-arcs and a plane $(7,3)$-arc.

$n=8$: The upper bound is obtained by Corollary 1; for $R=\mathbb{Z}_4$ a $(23,8)$-arc is obtained as the union of two $(8,3)$-arcs and a $(7,2)$-arc in three planes with a common line; for $R=\mathbb{F}_2[X]/(X^2)$ replace the $(7,2)$-arc by a $(6,2)$-arc.

$n=9$: The upper bound follows by Corollary 1; a $(27,9)$-arc is obtained as the union of three $(9,3)$-arcs in three planes (non-neighbors) with a common empty line.

$n=10$: The upper bound follows by Corollary 1; for $R=\mathbb{Z}_4$ a $(30,10)$-arc is obtained as the union of three $(10,3)$-arcs; for $R=\mathbb{F}_2[X]/(X^2)$ a $(38,10)$-arc is obtained as the union of one $(10,4)$-arc and two $(9,3)$-arcs.

$n=11$: The upper bound follows by Corollary 1; for $R=\mathbb{Z}_4$ a $(32,11)$-arc is obtained as the union of two $(11,4)$-arcs and a $(10,3)$-arc in three planes with a common line; for $R=\mathbb{F}_2[X]/(X^2)$ replace the $(10,3)$-arc by a $(9,3)$-arc.

$n=12$: The upper bound follows by Corollary 1; a $(36,12)$-arc is obtained as the union of three $(12,4)$-arcs in three planes (non-neighbors) with a common empty line.

$n=13$: The upper bound follows by Corollary 1; for the construction delete an affine $(16,3)$-blocking set from $\mathrm{AHG}(R_R^4)$.

$n=14$: The upper bound follows by Corollary 1; for the construction delete an affine $(12,2)$-blocking set from $\mathrm{AHG}(R_R^4)$.

$n=15$: The upper bound follows by Corollary 1; for the construction delete an affine $(8,1)$-blocking set from $\mathrm{AHG}(R_R^4)$.

$n = 16$: The upper bound follows by Corollary 1; $\mathrm{AHG}(R_R^4)$ is a $(64,16)$-arc.

$n = 17$: The upper bound follows by Corollary 3; for the constructions, add one point to a $(64,16)$-arc.

$n = 18$: The upper bound follows by Corollary 3; for the constructions, add two points to a $(64,16)$-arc.

$n = 19$: The upper bound follows by Corollary 3; a $(76,19)$-arc is obtained by removing an affine $(16,3)$-blocking set (four skew lines) from $AHG(R_R^3)$.

$n = 20$: The upper bound follows by Corollary 3; an $(80,20)$-arc is obtained by removing an affine $(12,2)$-blocking set (three skew lines) from $AHG(R_R^3)$.

$n = 21$: The upper bound follows by Corollary 3; an $(84,21)$-arc is obtained by removing an affine $(8,1)$-blocking set (two skew lines) from $AHG(R_R^3)$.

$n = 22$: The upper bound follows by Corollary 3; a $(92,22)$-arc is obtained by removing an Hjelmslev plane from $\mathrm{PHG}(R_R^4)$.

$n = 23$: The upper bound follows by Corollary 3; a $(93,23)$-arc is obtaind by adding a point to the $(92,22)$-arc, constructed above.

$n = 24$: The upper bound follows by Corollary 3; in the case $R = \mathbb{F}[X]/(X^2)$ we get a $(99,24)$-arc by removing from $\mathrm{PHG}(R_R^4)$ a copy of $\mathrm{PG}(3,2)$ and a line disjoint from the subgeometry; in the case $R = \mathbb{Z}_4$ we can get a $(96,24)$-arc by removing four disjoint lines.

$n = 25$: The upper bound follows from Theorem 3; in the case $R = \mathbb{F}[X]/(X^2)$ we can get a $(105,25)$-arc by removing a copy of $\mathrm{PG}(3,2)$ from $\mathrm{PHG}(R_R^4)$; in the case $R = \mathbb{Z}_4$ we can get only a $(102,25)$-arc by removing three disjoint lines.

$n = 26$: The upper bound follows from Theorem 3; a $(108,26)$-arc is obtained by removing two disjoint lines from $\mathrm{PHG}(R_R^4)$.

$n = 27$: The upper bound follows from Theorem 3; a $(114,27)$-arc is obtained by removing a line from $\mathrm{PHG}(R_R^4)$.

$n = 28$: A $(120,28)$-arc is the Hjelmslev geometry $\mathrm{PHG}(R_R^4)$.

Table 1. Values of $m_n(R_R^4)$ for Hjelmslev spaces over rings with $|R| = 4$

n/R	$\mathbb{Z}_4$	$\mathbb{F}_2[X]/(X^2)$	n/R	$\mathbb{Z}_4$	$\mathbb{F}_2[X]/(X^2)$
3	8	6	16	64 – 67	64 – 67
4	10	11	17	65 – 70	65 – 70
5	16	16	18	66 – 76	66 – 76
6	18 – 21	18 – 21	19	76 – 80	76 – 80
7	21 – 27	19 – 27	20	80 – 84	80 – 84
8	23 – 30	22 – 30	21	84 – 90	84 – 90
9	27 – 36	27 – 35	22	92	92
10	30 – 39	28 – 39	23	93 – 98	93 – 98
11	32 – 45	31 – 45	24	96 – 100	96 – 100
12	36 – 50	36 – 50	25	102 – 105	105
13	48 – 54	48 – 54	26	108	108
14	52 – 58	52 – 58	27	114	114
15	56 – 62	56 – 62	28	120	120

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