Time integration of nonsmooth mechanical systems with unilateral contact. Conservation and stability of position and velocity constraints in discrete time

Vincent Acary, Olivier Bonnefon

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Vincent Acary* and Olivier Bonnefon*
*INRIA, Grenoble Rhne–Alpes, BIPOP team–project, France

Summary. This work addresses the problem of the numerical time–integration of nonsmooth mechanical systems subjected to unilateral contacts and impacts. The considered systems may be the standard multi–body systems or the space–discretized continuous systems obtained by using FEM approach. Up to now, two main numerical schemes are available to perform this task: the Moreau–Jean scheme which solves the constraints at the velocity level together with a Newton impact law and the Schatzman–Paoli scheme which directly considers the constraints at the position level. In both schemes, the position and velocity constraints are not both satisfied in discrete time. The aim of this work is to propose a new scheme inspired by the GGL approach in DAE that solves, in discrete time, the constraints on both position and velocity levels. The stability and the local order of the scheme will be discussed. Some comparisons with recent works on the adaption of Newmark's scheme will be presented.

Nonsmooth mechanical systems with unilateral contact

This work addresses the problem of the numerical time–integration of nonsmooth mechanical systems subjected to unilateral contact. The considered systems may be the standard multi–body systems or the space–discretized continuous systems obtained by using FEM approach. These systems can be written in a pure Lagrangian setting as

$$\begin{cases} M(q(t))\dot{v}(t) = F(t, q(t), v(t)) + G(q(t))\lambda(t), \\ \dot{q}(t) = v(t), \\ y(t) = g(q(t)), \end{cases} \tag{1}$$

where

- $q \in \mathbb{R}^n$ is the generalized coordinates vector which collects the position and rotation parameters for multi–body systems or the node unknowns for FEM applications,
- $v \in \mathbb{R}^n$ is the generalized velocities vector directly given by the time derivative of q,
- $M(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix and $F(t, q, v) \in \mathbb{R}^n$ collects the internal forces and applied forces,
- the function $g(q)$ and the associated vector $y \in \mathbb{R}^m$ is used to define constraints or joints in the dynamical system, and $G^T(q) = \nabla_q^T g(q)$ is the Jacobian matrix of g with respect to q,
- $\lambda \in \mathbb{R}^m$ is the Lagrange multiplier vector associated with the constraints.

For the sake of simplicity, we restrict our presentation to holonomous perfect unilateral constraints given by the complementarity relations also known as the Signorini condition

$$0 \leq y(t) \perp \lambda(t) \geq 0. \tag{2}$$

For the finite freedom mechanical systems, an impact law must be added to complete the system. The most simple impact law will be considered in this work given by Newton's impact law

$$\dot{y}^+(t) = -e\dot{y}^-(t), \tag{3}$$

where e is the coefficient of restitution.

Background on the time–integration methods

Leaving aside the time–integration methods based on an accurate event detection procedure (event–tracking schemes or event–driven schemes [2, Chap. 8]), two main numerical schemes are available to date for integrating nonsmooth mechanical systems which are sound from the mathematical analysis point of view and which take advantage of a strong practical experience: the Moreau–Jean scheme and the Schatzman–Paoli scheme. The Moreau–Jean scheme [9, 5] is based on the Moreau sweeping process which enables to write the unilateral constraints at the velocity level including Newton's impact law,

$$\begin{cases} M(q(t))dv = F(t, q(t), v^+(t))dt + G(q(t))di, \\ \dot{q}(t) = v^+(t), \\ y(t) = g(q(t)), \\ \text{ if } y(t) \leq 0, \text{ then } 0 \leq \dot{y}^+(t) + e\dot{y}^-(t) \perp di \geq 0. \end{cases} \tag{4}$$

where dt is the Lebesgue measure, dv is a differential measure associated with v and di is an impulse reaction measure. The velocity level formulation can be viewed as an index–reduction procedure in the differential Algebraic Equations (DAE) theory. The numerical scheme which solves (4) enforces in discrete time the Newton law at each time step. On the contrary, a drift of the constraints in y is generally observed. The Schatzman–Paoli scheme [12, 10, 11] deals directly with the unilateral constraint in discrete time and incorporates the Newton impact law such that the law is satisfied in two or three time steps. In this scheme, the position constraints is satisfied in discrete time but not the impact law.

The proposed approach

In this work, we propose a scheme which both satisfies in discrete time the position constraints and the velocity constraints, i.e., the impact law. This scheme is an adaption of the Moreau–Jean scheme inspired by the Gear–Gupta–Leimkuhler (GGL) method. Let us start by considering the following "augmented" system

$$\begin{cases} M(q(t))dv = F(t, q(t), v^+(t))dt + G(q(t))di, \\ \dot{q}(t) = v^+(t) + G(q(t))\mu(t), \\ y(t) = g(q(t)), \\ \text{ if } y(t) \leq 0, \text{ then } 0 \leq \dot{y}^+(t) + e\dot{y}^-(t) \perp di \geq 0. \\ 0 \leq y(t) \perp \mu(t) \geq 0. \end{cases} \tag{5}$$

where $\mu(t)$ is a new multiplier which corresponds to the redundant constraints $y(t) \geq 0$. Thanks to Moreau's viability lemma, we expect that the multiplier is identically zero and that the solution of (5) is equivalent to the solution of (4). The proposed time–stepping scheme reads as

$$\begin{cases} M(q_{k+\theta})(v_{k+1} - v_k) - hF_{k+\theta} = G(q_{k+\theta})P_{k+1}, \\ q_{k+1} = q_k + hv_{k+\theta} + G(q_{k+\theta})\tau_{k+1}, \\ U_{k+1} = G^T(q_{k+\theta})\, v_{k+1}, \\ y_{k+1} = g(q_{k+1}), \\ \text{if } \tilde{y}_{k+1} \leq 0, \text{ then } 0 \leq U_{k+1} + eU_k \perp P_{k+1} \geq 0. \\ 0 \leq y_{k+1} \perp \tau_{k+1} \geq 0. \end{cases} \tag{6}$$

where h is the time step, $\tilde{y}_{k+1}$ is an explicit forecast of the constraint and the notation $x_{k+\alpha}$ stands for $(1-\alpha)x_{k+1} + \alpha x_k$. It is shown that the discrete multiplier τ_{k+1} is of order $\mathcal{O}(h)$ extending results of [1] and allows us to satisfy the constraints at the position level, that is $g(q_{k+1})$. The choice of the explicit forecast $\tilde{y}_{k+1}$ will be discussed in details as it plays a leading part in the global behavior of the scheme. Furthermore, it is shown that satisfying the impact law in discrete time leads to good stability properties. In particular, spurious oscillations of the velocity and other artifacts on stresses are eliminated over permanent contact periods [7, 4]. The link between this approach and the introduction of some persistence contact conditions as in [5, 3] will be also discussed. Finally, we will try to make the difference and outline the similarities with the numerous variants of the Newmark scheme dedicated to computational contact mechanics [6, 8, 4].

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