Timestepping Schemes Based On Time Discontinuous Galerkin Methods

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The contribution deals with timestepping schemes for nonsmooth dynamical systems. Traditionally, these schemes are locally of integration order one, both in smooth and nonsmooth periods. This is inefficient for applications with few events like circuit breakers, valve trains or slider-crank mechanisms. To improve the behavior during smooth episodes, we start activities twofold. First, we include the classic schemes in time discontinuous Galerkin methods. Second, we split smooth and nonsmooth force propagation. The correct mathematical setting is established with mollifier functions, Clenshaw-Curtis quadrature rules and appropriate impact representation. The result is a Petrov-Galerkin distributional differential inclusion. It defines two Runge-Kutta collocation families and enables higher integration order during smooth transition phases. As the framework contains the classic Moreau-Jean timestepping schemes for constant ansatz and test functions on velocity level, it can be considered as a consistent enhancement. An experimental convergence analysis with the bouncing ball example illustrates the capabilities.

1 Introduction

The *bouncing ball* is a typical *nonsmooth dynamical system* in the field of mechanics. The most important realisation is the occurrence of a *velocity jump* due to the *impact*. The function describing the *state* of position and velocity contains smooth and nonsmooth propagation episodes. Using a description based on classic function derivatives, one has to distinguish these two ranges. However, with appropriate mathematical objects, i.e. measures, it is possible to enter the *modern theory* of nonsmooth dynamical systems. The interpretation in the sense of distributions offers the connection to Galerkin schemes known from the numerical treatment of PDEs.

Problem 1 (Distribution differential inclusion) *Solve*

$$q(0) := q_0 \in \mathbb{R} \,, \tag{1}$$

$$v(0) := v_0 \in \mathbb{R} \,, \tag{2}$$

$$< \dot{q},\, \varphi_p >_{\mathcal{D}^*,\mathcal{D}} =< v,\, \varphi_p >_{\mathcal{D}^*,\mathcal{D}} \,, \quad \forall \varphi_p \in \mathcal{D}(I) \,, \tag{3}$$

$$< D^1 v,\, \varphi_v >_{\mathcal{D}^*,\mathcal{D}} =< m^{-1} f,\, \varphi_v >_{\mathcal{D}^*,\mathcal{D}} + < m^{-1} \mathrm{d}i,\, \varphi_v >_{\mathcal{D}^*,\mathcal{D}} \,, \quad \forall \varphi_v \in \mathcal{D}(I) \,, \tag{4}$$

$$\textit{contact and impact relations}\,(q, v, \mathrm{d}i, t) \in \mathcal{N} \,. \tag{5}$$

With inverse mass m^{-1}, external force f as well as time t, position q, velocity v and reaction $\mathrm{d}i$, a uniform descripton is found. The space of all real-valued, $\mathcal{C}^\infty$-functions with compact support is denoted by $\mathcal{D}(I)$, the primal-dual pairing by $< *, * >$ and the distributional derivative of v by $D^1 v$.

Classical timestepping schemes discretise the equations of motion including the constraints with integration order one. This avoids *event* detections, i.e. impact detections: a large number of contact transitions can be handled with increased computational efficiency when the influence of particular events is not as important as the mean. *Event-driven schemes* resolve the exact contact transition times. Between the events, the motion of the system is computed by a classic integration method for *differential algebraic equations* (DAE). This is very accurate but the detection of events can be time consuming and is not possible for Zeno phenomena. If an underlying mathematical model exhibits only few events, event-driven schemes are our methods of choice.

2 Time Discontinuous Galerkin Methods

To consistently improve the behavior during smooth episodes, we embed the classical timestepping schemes in *time discontinuous Galerkin (TDG) methods*. We start from Problem 1 and assume that ansatz and test functions might have jumps across discretisation intervals, and that they are of higher order inside discretisation intervals. The first assumption leads to the expression of discontinuous Galerkin methods. The second claim states that there is not an instantaneous influence of the analytic nonsmooth dynamics on the numerical solution in-between an interval: the exact time of discontinuity is not resolved. We choose appropriate ansatz and test functions and decide about the location of discontinuities of velocity and interaction impulses at the left or right

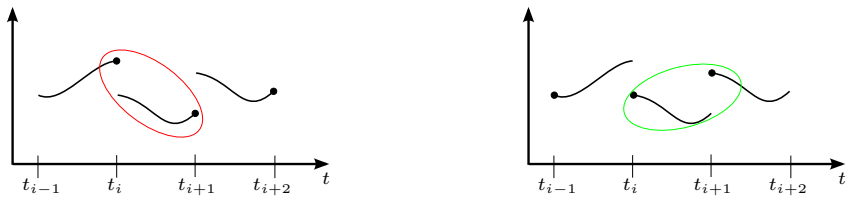

Figure 1: Piecewise smooth propagation with left and right discontinuity.

interval border. Piecewise constant functions and right interaction impulse discontinuity yield classic explicit (right velocity discontinuity) and implicit (left velocity discontinuity) timestepping schemes.

For each interval $[t_i, t_{i+1}]$, we use splitting techniques

$$\mathrm{d}i := r\mathrm{d}t + p_{i+1}\delta_{t_{i+1}}$$

to take care of contact reactions r separated from the single impact reaction p_{i+1} always evaluated at t_{i+1} with the Dirac measure $\delta_{t_{i+1}}$ because of stability reasons. Hence, contacts benefit from higher order ansatz functions and impacts yield local integration order one. The result are two Runge-Kutta collocation families for either left or right velocity discontinuity being represented by trapezoidal rules for linear velocity ansatz functions.

Theorem 2 (Order of local error) *Using Clenshaw-Curtis quadrature rules, the order of the local error is defined by the order of the underlying polynomial quadrature rule in sufficiently smooth intervals.*

3 Numerical Experiments

We analyse the convergence of D^{1-} timestepping scheme (right velocity discontinuity, linear velocity ansatz function) and Moreau-Jean timestepping scheme (constant velocity ansatz function) with the bouncing ball example.

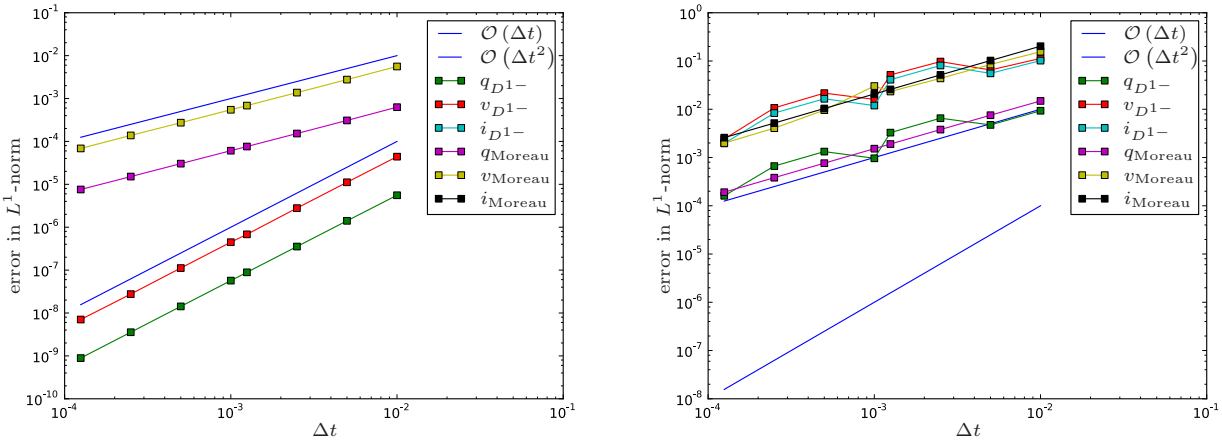

Figure 2: Free flight confirms theory and finite accumulation of impacts yields first order convergence.

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