Optimal Gathering in Radio Grids with Interference

Jean-Claude Bermond, Joseph Peters

OPTIMAL GATHERING IN RADIO GRIDS WITH INTERFERENCE

JEAN-CLAUDE BERMOND[†] AND JOSEPH G. PETERS[‡]

Abstract. We study the problem of gathering information from the nodes of a radio network into a central node. We model the network of possible transmissions by a graph and consider a binary model of interference in which two transmissions interfere if the distance in the graph from the sender of one transmission to the receiver of the other is d_I or less. A *round* is a set of non-interfering transmissions. In this paper, we determine the exact number of rounds required to gather one piece of information from each node of a square two-dimensional grid into the central node. If $d_I = 2k - 1$ is odd, then the number of rounds is $k(N - 1) - c_k$ where N is the number of nodes and c_k is a constant that depends on k. If $d_I = 2k$ is even, then the number of rounds is $(k + \frac{1}{4})(N - 1) - c'_k$ where c'_k is a constant that depends on k. The even case uses a method based on linear programming duality to prove the lower bound, and sophisticated algorithms using the symmetry of the grid and non-shortest paths to establish the matching upper bound. We then generalize our results to hexagonal grids.

Key words. Radio communication, interference, grids, gathering

AMS subject classifications. 68M12, 94A05, 90B18, 68M10

1. Introduction. In this paper, we study a problem suggested by France Telecom concerning the design of efficient strategies to provide Internet access using wireless devices (see [9]). Typically, several houses in a village need access to a gateway (a satellite antenna) to transmit and receive data over the Internet. To reduce the cost of the transceivers, multi-hop wireless relay routing is used. Information can be transmitted from a node to any node within distance d_T. In this paper, we assume that $d_T = 1$ and we will model the network of possible communications by a symmetric directed communication graph $G = (V, E)$ in which the vertices represent the nodes (wireless devices) of the network and there is a pair of arcs, one arc in each direction, between two vertices if the corresponding nodes can communicate.

However, a transmission can *interfere* with reception at nodes that are close to the transmitter. If two transmissions are mutually non-interfering, we say that they are *compatible*. The goal is to provide efficient access by the users to the gateway within these interference constraints. We will use the term *round* to mean a time slot during which there can be only compatible transmissions or *calls*. We are interested in schedules that minimize the number of rounds (completion time).

Time is slotted and the network is assumed to be synchronous, so a one-hop transmission of one piece of information consumes one time slot (round). These hypotheses are strong and assume a centralized view. However, the values of the completion time that we obtain will give lower bounds for the corresponding real life values. Stated differently, if the value of the completion time is fixed, then our results will give upper bounds on the maximum possible number of users in the network.

In this paper, we will use a binary model of interference based on distance in the communication graph. Let $d(u, v)$ denote the distance (that is, the length of a shortest

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path) between u and v in G. We assume that when a vertex u transmits, all vertices v such that $d(u,v) \leq d_I$ are subject to interference from u's transmission. This model is a simplification of reality in which a node can be subject to interference from all of the other nodes, and models based on signal-to-noise ratio are more accurate. However, our model is more accurate than both the classical half duplex model of wired networks in which a vertex of a communication graph cannot transmit and receive at the same time, and the basic binary model ($d_I = 1$) in which a vertex only experiences interference when one of its neighbours transmits. We assume that all vertices of G have the same interference range d_I; in fact d_I is only an upper bound on the possible range of interference because obstacles can reduce the interference range.

Some authors consider models based on Euclidean distance, but these models do not take into account obstacles. In this paper, we consider square grids as models of urban situations. The distance in a grid is the rectilinear distance between the corresponding nodes in the Euclidean plane. Rectilinear distance is a good approximation to Euclidean distance when d_I is small, and this is usually the case in practice. Later, we will generalize our results to hexagonal grid graphs which provide an even better approximation to Euclidean distance and are a good model of cellular networks.

We study the problem of *gathering* one piece of information from each vertex into a central gateway vertex for transmission over the Internet. The inverse problem of gathering, in which each vertex receives a personalized piece of information from the central vertex, is called *distribution* or *personalized broadcasting*. When the graph is symmetric, the two problems are equivalent; the personalized broadcasting problem can be solved by reversing the order and directions of the transmissions in a gathering protocol. Indeed, if two calls (s, r) and (s', r') are compatible, then $d(s, r') > d_I$ and $d(s', r) > d_I$, so the reverse calls are also compatible. We assume that all pieces of information are of the same size, and that pieces of information cannot be concatenated, so each transmission involves one piece of information, which we call a *message*, and takes one time unit (round). The gathering problem then becomes one of organizing the transmissions into rounds of compatible calls so that the number of rounds is minimized.

A problem that is similar to ours appears in the context of sensor networks. (See [15] for an on-line list of references.) Each device in a sensor network collects data from its immediate environment and the information from all sensors needs to be gathered into a base station. A major goal in sensor network protocols is to minimize energy consumption and most research assumes that data can be combined (or *aggregated*) to reduce transmission costs. In contrast, our goal is to minimize time and we do not allow any combination of data. A model that is closer to ours is considered in [12]. The model includes reachability and interference constraints like our model, but there are a number of differences. The nodes in [12] have directional antennae and no buffering capacity whereas we assume omni-directional transmission and reception and allow buffering of messages. Furthermore, most of the results in [12] use an interference model in which each node can either send or receive a message in each time slot. This can be viewed as $d_I = 0$ in our model. Under their assumptions, the authors give optimal (polynomial-time) gathering protocols for paths and tree networks. Their work has been extended to general graphs with unit-length messages in [13].

Gathering problems like the one that we study in this paper have received much recent attention. A survey can be found in [10]. A protocol for general graphs with an arbitrary amount of information to be transmitted from each vertex is presented in [3]. The protocol is an approximation algorithm with performance ratio at most 4. It is also shown in [3] that there is no fully polynomial time approximation scheme for gathering if $d_I > d_T$, unless $\mathcal{P} = \mathcal{NP}$, and the problem is $\mathcal{NP}$-hard if $d_I = d_T$. If each vertex has exactly one piece of information to transmit, the problem is $\mathcal{NP}$-hard if $d_I > d_T$ [3] and if $d_I = d_T = 1$ [17]. A modified version of the problem in which messages can be released over time is considered in [11] and a 4-approximation algorithm is presented. In [2], general lower bounds and protocols are given for $d_T \geq 1$ for various networks such as trees and stars.

The one-dimensional version of the problem studied in this paper, that is, gathering into a designated vertex of a path, is considered in [1]. The problem is solved when the gateway vertex is at one end of the path and is partly solved when the gateway is in the centre of the path. Optimal protocols have also been designed for trees with $d_I = 1$ in [8]. When no buffering is allowed, the problem has been solved for trees for $d_I = 1$ [5] and for general d_I [4] (where a closed-form expression is given when all vertices have exactly one piece of information to transmit). For square grids with the gateway in the centre, a multiplicative 1.5-approximation algorithm is given in [18] and an additive +1 approximation algorithm is given in [6].

A model with continuous traffic demands and a symmetric interference condition is considered in [16] and systolic algorithms are given. In this model, the problem is to satisfy a flow demand in minimum time. The problem is shown to be related to an optimization problem called the *round weighting problem* and duality is used to find optimal solutions. The problem studied in [16] can be viewed as a relaxation of the problem that we study and we will extend their duality method to prove our lower bounds. Note that the interference condition in [16] is symmetric; two calls interfere if any two vertices, one from each call, are within distance d_I. The results for this continuous model have been used in [14] to obtain results for the grid with the gateway in any position, arbitrary traffic demands, and symmetric interference with $d_I = 1$.

In Section 3, we determine the exact number of rounds to gather one message from each vertex into the central gateway vertex of a square grid with $N = n^2$ vertices and odd interference distance $d_I = 2k-1$. The first few values are $N-1$ (the total number of messages to be gathered) when $d_I = 1$, $2(N-1)-4$ when $d_I = 3$, and $3(N-1)-16$ when $d_I = 5$. In general, the number of rounds is $k(N-1)-c_k$ where c_k is a constant that depends on k. We give a short direct proof of the lower bound. We establish the matching upper bound by providing a protocol and proving that it is correct. In Section 4, we determine the exact number of rounds to gather in a square grid with N vertices and even interference distance $d_I = 2k$. The first few values are $\frac{5}{4}(N-1)-1$ when $d_I = 2$, $\frac{9}{4}(N-1)-6$ when $d_I = 4$, and $\frac{13}{4}(N-1)-20$ when $d_I = 6$. The general pattern is $(k+\frac{1}{4})(N-1)-c'_k$ where c'_k is a constant that depends on k. The bounds for even d_I are considerably more difficult to prove than the bounds for odd d_I. We prove the lower bound by extending a method based on linear programming duality from [16]. The matching upper bound is established by giving a protocol and proving its correctness. In Section 5, we generalize our techniques to hexagonal grids. The next section contains definitions and notation. Early versions of some of the results in this paper were presented in [7].

2. Definitions and Notation. We assume that $G = (V, E)$ is a square grid with $N = n^2$ vertices. We will concentrate on the case when $n = 2p+1$ is odd and the vertices are arranged symmetrically around a central vertex v_0 with p columns of vertices on either side of the vertical axis through v_0 and p rows above and below the horizontal axis through v_0. The vertices of the grid are labelled (x, y) with $-p \leq x \leq p$ and $-p \leq y \leq p$, and the central vertex is $v_0 = (0, 0)$. The vertex (x, y) has four neighbours in G, namely the vertices $(x, y \pm 1)$ and $(x \pm 1, y)$. We will use N_d to denote the number of vertices that are at distance exactly d from v_0. We have that $N_0 = 1$, $N_d = 4d$ for $1 \leq d \leq p$, and $N_d = 4(2p + 1 - d)$ for $p < d \leq 2p$.

We define the rotation ρ to be the one-to-one mapping $\rho((x, y)) = (-y, x)$, which corresponds to a rotation in the plane of $\frac{\pi}{2}$ around the central vertex v_0. Similarly, $\rho^2((x, y)) = (-x, -y)$ corresponds to a rotation of π, and $\rho^3((x, y)) = (y, -x)$ corresponds to a rotation of $\frac{3\pi}{2}$. For a set S of vertices, we define $\rho(S) = \{\rho(v) | v \in S\}$. For an arc $e = (u, v)$, $\rho(e)$ is the arc $(\rho(u), \rho(v))$. Similarly, for a directed path P consisting of the sequence of vertices $v_1, v_2, \ldots, v_h$, we define $\rho(P)$ to be the directed path $\rho(v_1), \rho(v_2), \ldots, \rho(v_h)$.

It will be useful to have names for various regions of the grid. We split the grid into four disjoint regions R_E, R_N, R_W, and R_S. Region R_E consists of the vertices (x, y) with $0 < x \leq p$ and $-x < y \leq x$. The other regions are obtained by rotations, namely $R_N = \rho(R_E)$, $R_W = \rho(R_N) = \rho^2(R_E)$, and $R_S = \rho(R_W) = \rho^2(R_N) = \rho^3(R_E)$.

In a radio network, a transmission is sent to all neighbours of the transmitter (at distance $d_T = 1$ in this paper). However, only one copy of the message needs to reach v_0, so it is only necessary for one of the neighbours to forward the message. Thus, we can consider a transmission to be a call involving a single pair (s, r) where s is the sender and r the receiver of the message, and we can represent calls as arcs (arrows) in our figures. To be successful, a call should not interfere with any other calls that occur during the same time slot. As we said in the introduction, we will use a binary model of interference based on distance in the communication graph. When the distance $d(s_i, r_j)$ between the sender of one call (s_i, r_i) and the receiver of a second call (s_j, r_j) is such that $1 < d(s_i, r_j) \leq d_I$, then the transmission of s_i is too weak to be received by r_j, but it is strong enough to interfere with the reception of call (s_j, r_j) by r_j.

Several examples of interference are shown in Figure 1. In the figure, the calls (s_1, r_1) and (s_3, r_3) are compatible when $d_I = 3$ and so are the calls (s_3, r_3) and (s_4, r_4). All other pairs of calls are incompatible. For example, the call (s_1, r_1) does not interfere with reception at r_2, but (s_2, r_2) interferes with reception at r_1, so these calls are incompatible.

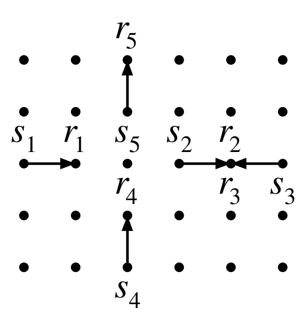

FIG. 1. *Examples of interference for $d_I = 3$.*

For both odd $d_I = 2k - 1$ and even $d_I = 2k$, the *interference zone* consists of the vertices (x, y) at distance at most k from v_0, that is $|x| + |y| \leq k$. The interference zones are shown as shaded areas in Figure 2. For even $d_I = 2k$, the vertices at distance

$k+1$ from v_0 define the *partial interference boundary* which is shown as a dashed box in Figure 2(b).

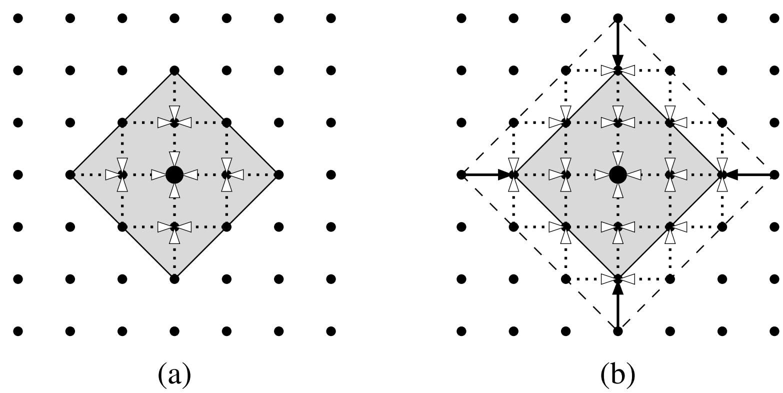

FIG. 2. *Interference zones for (a) $d_I = 3$, (b) $d_I = 4$.*

Figure 2 shows some of the possible calls around the central vertex v_0, which is represented by a large circle. In Figure 2(a), $d_I = 3$ is odd. None of the calls shown in the shaded *interference zone* are compatible with each other, so at most one of these calls can be done at any given time. The situation is more complicated when d_I is even. In Figure 2(b), $d_I = 4$. All of the calls shown in the shaded interference zone interfere with each other as in the odd case, but the ways in which information can enter vertices on the boundary of the interference zone are more restricted. The largest subset of compatible calls from vertices on the partial interference boundary to vertices on the boundary of the interference zone is the subset of four calls shown with solid arrows. All other such calls can only be done two or three at a time.

3. Square Grids - Odd Interference Distance. In this section we assume a square grid with $N = n^2$ vertices, $n = 2p+1$, odd interference distance $d_I = 2k-1$, $k \geq 1$, and $p \geq k$.

THEOREM 1. *Suppose that $n = 2p+1$ and $d_I = 2k-1$ are odd and $p \geq k$. Then the number of rounds needed to gather in a square grid with $N = n^2$ vertices is at least $k(N-1) - c_k$, where $c_k = \frac{2k(k+1)(k-1)}{3}$.*

Proof. The message of each vertex at distance $i > k$ from v_0 must use k calls inside the interference zone, all of them pairwise interfering, to reach v_0. The message of each vertex at distance $i \leq k$ from v_0 must use i calls inside the interference zone. So, the total number of rounds is at least $\sum_{i=1}^{k} iN_i + k(N - \sum_{i=0}^{k} N_i) = k(N-1) - \sum_{i=1}^{k}(k-i)N_i$. Noting that $N_i = 4i$ for $1 \leq i \leq k$, we get $c_k = \sum_{i=1}^{k}(k-i)N_i = 4k\sum_{i=1}^{k} i - 4\sum_{i=1}^{k} i^2 = \frac{2k(k+1)(k-1)}{3}$. □

Now we describe a protocol that achieves the bound of Theorem 1. The general idea is to organize the calls into stages of $4k$ rounds. We say that a vertex is *active* if it has messages that need to be sent or forwarded to v_0. Otherwise it is called *dormant*. In each stage, we select four active vertices that are outside the interference zone and arranged symmetrically around v_0 and four directed paths (*dipaths* connecting the selected vertices to v_0. Messages are forwarded along the four dipaths for $4k$ rounds. At the end of the stage, the four selected vertices become dormant, all other vertices on the four dipaths have sent one message and received another, and v_0 has received four more messages. The dipaths are chosen in such a way that the calls in each round are compatible. We iterate this procedure until the only remaining active vertices are inside the interference zone around v_0. Sequential calls inside the interference zone are then used to move the remaining messages into v_0.

Figure 3 shows two examples of stages, one with solid arrows and the other with dotted arrows. The labels indicate the rounds during which the calls are made. Stages are executed sequentially, so that at any given time, only one set of dipaths is being used. It is not hard to verify that the calls on the solid dipaths are compatible in each round. Similarly, the calls of the dotted dipaths are compatible. In general, dipaths in region R_E go towards the positive x axis and then along the axis to v_0. The dipaths in the three other quadrants are obtained by rotations.

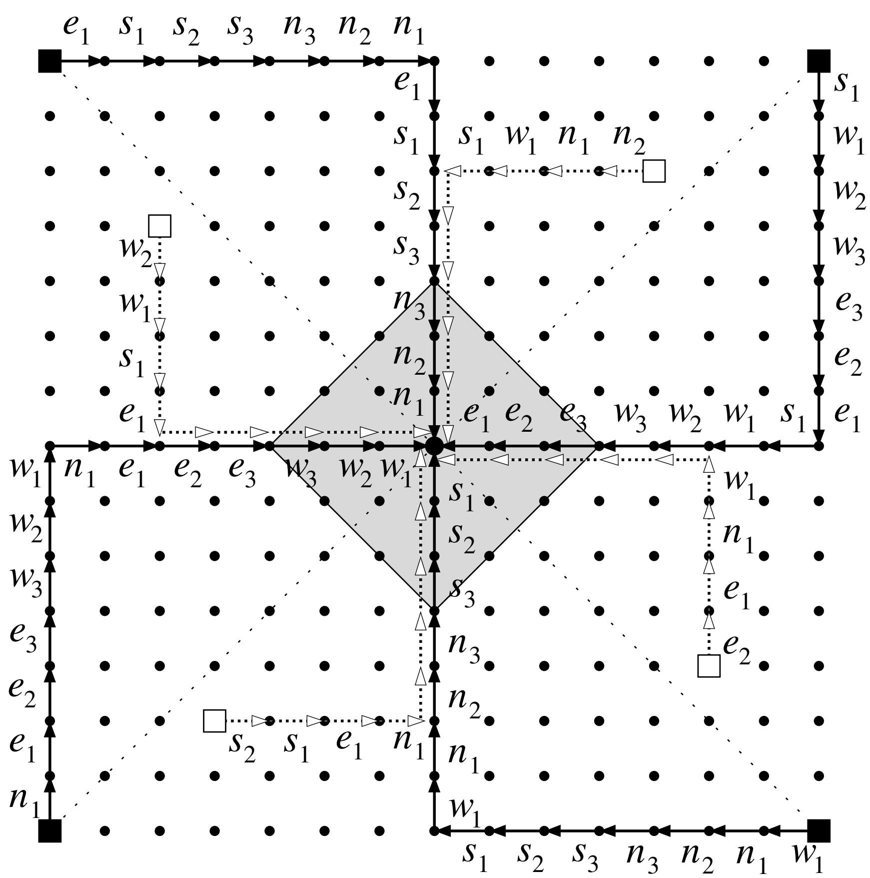

FIG. 3. *Gathering stages for $d_I = 5$.*

THEOREM 2. *Suppose that $n = 2p + 1$ and $d_I = 2k - 1$ are odd and $p \geq k$. Then gathering in a square grid with $N = n^2$ vertices can be completed in $k(N - 1) - c_k$ rounds, where $c_k = \frac{2k(k+1)(k-1)}{3}$ and this is optimal.*

Proof. We define the dipaths to be used precisely by constructing a sequence of directed trees called *gathering trees* containing active vertices. Initially, all vertices are active and are included in the gathering tree. The initial tree consists of the arcs directed towards v_0 along the four axes and the arcs directed towards v_0 along the perpendicular lines inside each of the four regions. For region R_E, the tree contains the horizontal arcs $((x, 0), (x + 1, 0))$, $0 \leq x < p$, the vertical arcs $((x, y), (x, y + 1))$,

$1 \le x \le p$, $0 \le y < x$, and the vertical arcs $((x, y), (x, y+1))$, $2 \le x \le p$, $-x+1 \le y \le -1$. The arcs in the other regions are obtained by rotations. Figure 4 shows a gathering tree for $p = 6$ (and $n = 2p + 1 = 13$). All arcs in the tree are directed towards v_0, but the arrowheads are omitted from Figure 4 (and some later figures) to simplify the diagram.

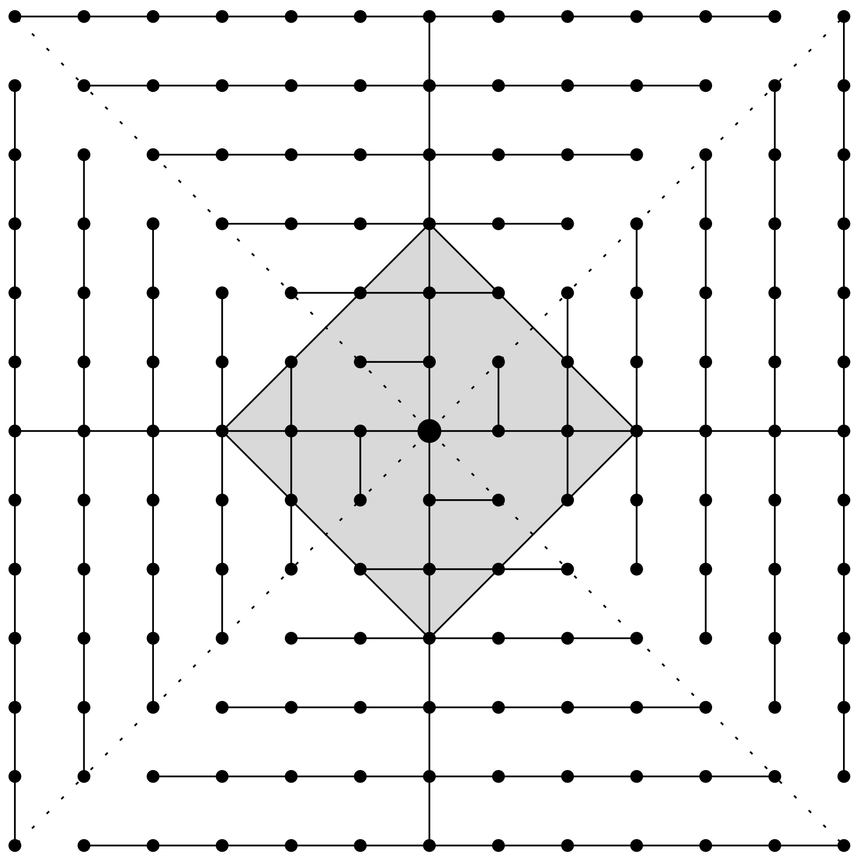

FIG. 4. *Gathering tree for odd d_I.*

In each stage, we select a leaf $v = (x, y)$ of the current gathering tree in the region R_E and outside the interference zone and its three rotated images $\rho(v)$, $\rho^2(v)$, and $\rho^3(v)$. The calls are done for $4k$ rounds along the four dipaths $P(v)$, $\rho(P(v))$, $\rho^2(P(v))$, and $\rho^3(P(v)))$ where $P(v)$ is the dipath in the gathering tree from v to v_0. After the stage, the four selected leaves become dormant and are deleted from the gathering tree. The other vertices on the dipaths (except v_0) will have sent one message and received one message. So, all vertices remaining in the gathering tree will be active and will have exactly one message. After $\frac{1}{4}(N - \sum_{i=0}^{k} N_i)$ stages of $4k$ rounds, all of the vertices outside of the interference zone will be dormant. It then takes $\sum_{i=1}^{k} iN_i$ sequential calls inside the interference zone to move the remaining messages into v_0. This establishes the upper bound $\sum_{i=1}^{k} iN_i + k(N - \sum_{i=0}^{k} N_i)$ on the number of rounds which matches the lower bound of Theorem 1.

Now we specify the rounds precisely for the stage when $v = (x, y)$ and its rotated images are the selected leaves. First, suppose that $y \ge 0$. (The case $y < 0$ is similar and is discussed later.) The dipath $P((x, y))$ consists of the y vertical arcs $((x, z), (x, z-1))$ for $y \geq z > 0$, followed by the x horizontal arcs $((t, 0), (t-1, 0))$ for $x \geq t > 0$. Each arc will be used by exactly one call during the stage and the call will be made during a round that depends on the distance of the arc from v_0. We label the $4k$ rounds of each stage with the labels e_i, n_i, w_i, s_i, $1 \leq i \leq k$. We specify the labels for $P((x, y))$ in the opposite direction to the dipath, that is, starting at v_0 and working towards (x, y). The first $2k+1$ labels are $e_1, e_2, \ldots, e_k, w_k, w_{k-1}, \ldots, w_1, s_1$. If the dipath has more than $2k+1$ arcs, then the pattern is repeated until all arcs from v_0 to (x, y) are labelled. According to this labelling, a call (s, r) on P that satisfies $d(s, v_0) = d$ is labelled e_i if $d \equiv i \pmod{2k+1}$ and $1 \leq i \leq k$, w_{2k+1-i} if $d \equiv i \pmod{2k+1}$ and $k+1 \leq i \leq 2k$, and s_1 if $d \equiv 0 \pmod{2k+1}$.

To specify the labels for the three rotated dipaths, we associate a one-to-one mapping ω with the rotation ρ. The mapping ω acts on the labels of the arcs as follows: $\omega(e_i) = n_i$, $\omega(n_i) = w_i$, $\omega(w_i) = s_i$, and $\omega(s_i) = e_i$. So, if arc e in $P(v)$ is labelled l, then arc $\rho(e)$ in the rotated dipath $\rho(P(v))$ is labelled $\omega(l)$. For example, the arcs of $\rho(P(x, y))$ starting at v_0 are labelled with the repeating pattern $n_1, n_2, \ldots, n_k, s_k, s_{k-1}, \ldots, s_1, e_1$. Figure 3 shows the dipaths (solid arrows) and labels for $v = (x, y) = (7, 7)$, $k = 3$, and $d_I = 2k - 1 = 5$.

To finish the proof, we have to show that there is no interference among the $4(x+y)$ calls in the stage. Two calls can only interfere if they have the same label (i.e., they are made in the same round). Suppose that two calls (s, r) and (s', r') have the same label. We will show that our labelling scheme ensures that $d(s, s') \geq 2k+1$ so the two calls are compatible.

Case 1: the two calls are on the same dipath. Since the dipath is a shortest path and the repeated sequence of labels has length $2k+1$, the distance between s and s' is $2k+1$.

Case 2: (s, r) is on the dipath P and (s', r') is on $\rho^2(P)$. If $d(s, v_0) \geq 2k+1$ or $d(s', v_0) \geq 2k+1$, then $d(s, s') \geq 2k+1$ and the calls are compatible, so the only possibility for conflicts is when $d(s, v_0) \leq 2k$ and $d(s', v_0) \leq 2k$. If both calls are labelled e_i, then $d(s, v_0) = i$, $d(s', v_0) = 2k+1-i$, and $d(s, s') = 2k+1$. If both calls are labelled w_i, then $d(s, v_0) = 2k+1-i$, $d(s', v_0) = i$, and $d(s, s') = 2k+1$. The proof for the pair of dipaths $\rho(P)$ and $\rho^3(P)$ is similar.

Case 3: (s, r) is on P and (s', r') is on $\rho(P)$. (The proofs for other pairs of dipaths that differ by a rotation of $\frac{\pi}{2}$ are similar.) If $x \leq k$, then $d(s, v_0) \leq 2k$ (because $-x < y \leq x$ in region R_E), and there are no common labels on the two dipaths. Otherwise the only possible common labels are s_1 and e_1.

Subcase 3(a): $k+1 \leq x \leq 2k$. The dipaths are of length at most $4k$ and there is at most one call labelled s_1 on P and at most one call labelled $\omega(s_1) = e_1$ on $\rho(P)$. If there is a call (s, r) labelled s_1 on P, then the coordinates of s are $x_s = x$ and $y_s = 2k+1-x$, while the only call (s', r') labelled s_1 on $\rho(P)$ has $x_{s'} = -(2k-x)$ and $y_{s'} = x$. Therefore, $d(s, s') = x + (2k-x) + x - (2k+1-x) = 2x - 1 \geq 2k+1$, as $x \geq k+1$. If there is a call (s', r') labelled e_1 on $\rho(P)$, then the coordinates of s' are $y_{s'} = x$ and $x_{s'} = -(2k+1-x)$, and $d(s', v_0) = 2k+1$, so the call (s, r) labelled e_1 with $s = (1, 0)$ has $d(s', s) = 2k+2$. If there is a second call (s'', r'') labelled e_1 on P, then its coordinates are $x_{s''} = x$ and $y_{s''} = 2k+2-x$, and $d(s'', s') = x + (2k+1-x) + x - (2k+2-x) = 2x - 1 \geq 2k+1$.

Subcase 3(b): $x \geq 2k+1$. The sending vertices of all arcs labelled s_1 on P are at distance at least $2k+1$ from all vertices of $\rho(P)$, so there are no conflicts. Similarly, the senders of all arcs labelled e_1 on $\rho(P)$ are at distance at least $2k+1$ from P.

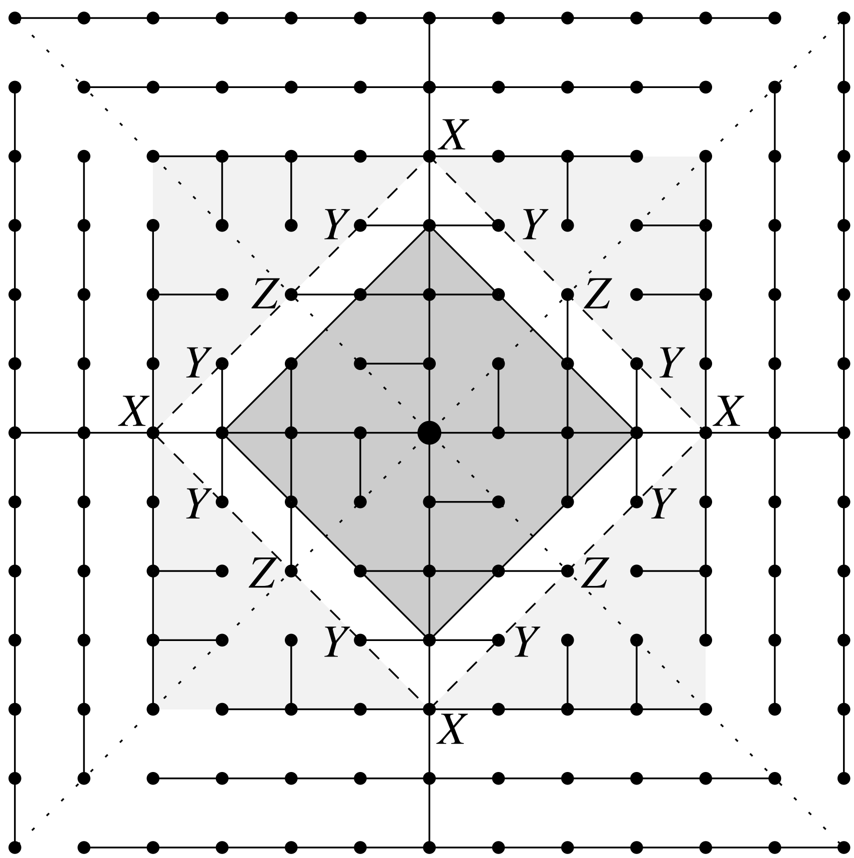

FIG. 5. *Gathering tree for $d_I = 6$.*

The proof for the case $y < 0$ is similar to the case $y \geq 0$. The only difference is that the label s_1 is replaced by n_1 in the pattern of $2k+1$ labels for dipath P, and corresponding changes are made in the rotated dipaths. Figure 3 shows the dipaths (dotted arrows) and labels for $v = (x, y) = (5, -4)$, $k = 3$, and $d_I = 2k - 1 = 5$. ∎

4. Square Grids - Even Interference Distance. In this section, we assume a square grid with $N = n^2$ vertices, $n = 2p + 1$, even interference distance $d_I = 2k$, $k \geq 1$, and $p \geq k + 1$. Both the protocol and the proof of the lower bound for even d_I are more complicated than for odd d_I because the interference pattern is more complicated. Some of the differences can be seen by comparing Figures 4 and 5. When $d_I = 2k-1$ is odd, as it is in Figure 4, we only have to distinguish between calls inside the shaded *interference zone* bounded by vertices (x, y) at distance k from v_0 and calls outside the interference zone. When $d_I = 2k$ is even, there are four zones as shown in Figure 5. (Note that the paths in the gathering tree in Figure 5 are directed towards v_0 but the arrowheads are omitted to simplify the diagram.) The behaviour inside the darkly-shaded interference zone and in the area outside of the square bounded by vertices with $|x| \geq k + 1$ and $|y| \geq k + 1$ is the same as when d_I is odd. The interference patterns for calls originating on the *partial interference boundary* defined by vertices at distance $k+1$ from v_0 (shown as a dashed box in Figure 5) are different and affect both the lower bound and the protocol. Calls originating outside the partial interference boundary but inside the square with $|x| \geq k+1$ and $|y| \geq k+1$ (the lightly shaded area of Figure 5) do not affect the lower bound, but the gathering tree must be modified to avoid interference. The labels X, Y, and Z in Figure 5 will be explained later.

In the previous section, we gave a short direct proof of a lower bound when d_I is odd. We have not found a convincing direct proof of a lower bound when d_I is even because of the large number of cases that must be argued. We will use a different method based on linear programming duality to prove a lower bound when d_I is even. Our method is based on a proof technique that was introduced in [16] to solve bandwidth allocation problems in radio networks with continuous traffic demands. The continuous gathering problem in [16] is a special case that can be formulated as a linear programming problem. The solution of the linear programming problem gives an upper bound on the gathering time. The solution of the dual linear programming problem gives a lower bound on the time to gather information into the central vertex v_0. Our problem is different in that each vertex only sends one piece of information to v_0 and we seek an integral solution that minimizes the number of rounds. However, we can extend the technique of [16] to provide tight lower bounds for our problem.

A feasible solution for our gathering problem in a grid $G = (V, E)$ consists of a set of dipaths to v_0, one dipath $P(v)$ from each $v \in V$, $v \neq v_0$, and an ordered sequence of rounds that specifies the calls. For each dipath $P(v)$, the sequence of rounds must contain a subsequence that includes the arcs of $P(v)$ in the order that they occur on $P(v)$. This is necessary to allow the message of v to reach v_0. We want to find an optimal feasible solution that minimizes the total number of rounds.

Let $\mathscr{R} = \{R_1, R_2, \ldots, R_r\}$ be the set of all possible different rounds, where a round is any set of compatible calls in $G = (V, E)$. Note that $\mathscr{R}$ can be exponential in size. A gathering protocol uses a sequence of rounds from $\mathscr{R}$. Typically, a protocol will use only a small subset of $\mathscr{R}$ and may use some R_i more than once. Let $\mathscr{P} = \{P(v) | v \in V, v \neq v_0\}$ be a set of dipaths and let $T_{\mathscr{P}}$ be the minimum number of rounds to complete gathering using $\mathscr{P}$. We want to determine the minimum time T over all possible sets of dipaths $\mathscr{P}$, that is $T = \min_{\mathscr{P}} T_{\mathscr{P}}$.

To obtain a lower bound, it suffices to consider a relaxed version of the problem in which we concentrate on the structure of the rounds and ignore their order in the sequence. In particular, the number of rounds containing each arc e must be at least as large as the number of dipaths containing e. This condition is necessary so that all messages that need to traverse arc e can do so.

Let $\pi_{\mathscr{P}}(e)$ denote the number of dipaths of a set $\mathscr{P}$ that contain arc $e \in E$. A feasible solution of the relaxed problem for a given set $\mathscr{P}$ is a set of integers $W_{\mathscr{P}} = \{w_i | 1 \leq i \leq r\}$, where w_i is the number of times that round R_i is used in the solution. Let $\mathscr{R}_e = \{i | e \in R_i\}$. Then the number of times that an arc $e \in E$ is used in the solution is $\eta_W(e) = \sum_{i \in \mathscr{R}_e} w_i$. We want to find a solution with the minimum total number of rounds such that the number of rounds containing each arc e is at least as large as the number of dipaths containing e. In order to use linear programming duality, we need to further relax our problem to allow non-integer solutions $W_{\mathscr{P}}$. With this further relaxation, we can now state the relaxed problem for a given set of dipaths $\mathscr{P}$ as:

$$\text{Minimize } S_{\mathscr{P}} = \sum_{i=1}^{r} w_i \text{ subject to } (\forall e \in E)\ \eta_W(e) \geq \pi_{\mathscr{P}}(e).$$

We can express this problem in terms of matrices as:

$$\text{Minimize } S_{\mathscr{P}} = \mathbf{1} \cdot W^T \text{ subject to } R \cdot W^T \geq \Pi_{\mathscr{P}}^T,$$

where W^T is the column vector $[w_1, w_2, \ldots, w_r]^T$, $\Pi_{\mathscr{P}}^T$ is the column vector $[\pi_{\mathscr{P}}(e_1), \pi_{\mathscr{P}}(e_2), \ldots, \pi_{\mathscr{P}}(e_{|E|})]^T$, $\mathbf{1}$ is the vector $[1, 1, \ldots, 1]$ of length r, and R is the binary matrix with $|E|$ rows corresponding to the arcs of G, r columns corresponding to the rounds of $\mathscr{R}$, and a 1 in row j and column i if arc e_j is used in R_i.

The dual problem has the form:

$$\text{Maximize } S_{\mathscr{P}}^D = \Pi_{\mathscr{P}} \cdot \Lambda^T \text{ subject to } R^T \cdot \Lambda^T \leq \mathbf{1}^T,$$

where R^T is the transpose of matrix R, $\mathbf{1}^T$ is the column vector $[1, 1, \ldots, 1]^T$ of length r, and the solution $\Lambda = [\lambda(e_1), \ldots, \lambda(e_{|E|})]$ is a vector of weights on the arcs of E with $0 \leq \lambda(e) \leq 1$, $\forall e \in E$. The weight $\lambda(e)$ can be viewed as the cost (fraction of a round) to move a message across the arc e in a dipath. The dual problem can also be expressed as:

$$\text{Maximize } S_{\mathscr{P}}^D = \sum_e \pi_{\mathscr{P}}(e)\lambda(e) \text{ subject to}$$

$$(\forall R_i \in \mathscr{R}) \sum_{e \in R_i} \lambda(e) \leq 1. \qquad (*)$$

By linear programming duality, we have $S_{\mathscr{P}}^D = S_{\mathscr{P}}$. Furthermore, $S_{\mathscr{P}} \leq T_{\mathscr{P}}$ because $S_{\mathscr{P}}$ is an upper bound for a relaxed version of our gathering problem. Therefore, a lower bound on $S_{\mathscr{P}}^D$ for all feasible sets $\mathscr{P}$ is a lower bound on the time $T = \min_{\mathscr{P}} T_{\mathscr{P}}$ for our gathering problem.

Let $\tau_{\mathscr{P}}(v) = \sum_{e \in P(v)} \lambda(e)$. Then $S_{\mathscr{P}}^D = \sum_e \pi_{\mathscr{P}}(e)\lambda(e) = \sum_v \tau_{\mathscr{P}}(v)$. Intuitively, $\tau_{\mathscr{P}}(v)$ is the cost (in rounds) to move a message from v to v_0 along the dipath $P(v) \in \mathscr{P}$ and $\tau_{\min}(v) = \min_{\mathscr{P}} \tau_{\mathscr{P}}(v)$ is the minimum cost to move a message from v to v_0 along any dipath. For any set of values $\{\lambda(e_1), \lambda(e_2), \ldots, \lambda(e_{|E|})\}$ satisfying constraint $(*)$, we have

$$T \geq \sum_v \tau_{\min}(v). \qquad (**)$$

The lower bound method works for both even and odd d_I. The application of the method is considerably simpler for odd d_I than for even d_I, and is also easier to explain because we can appeal to the direct proof of Theorem 1 for intuition. So, we will give a second proof of Theorem 1 to illustrate the application of the method. Then we will use it to prove a lower bound for the more complicated even case.

Let us apply this method for odd $d_I = 2k - 1$. Choose $\lambda(e) = 1$ for each arc $e = (s, r)$ inside the interference zone with $1 \leq d(s, v_0) \leq k$ and $d(r, v_0) = d(s, v_0) - 1$, and choose $\lambda(e) = 0$ for all other arcs. Since all calls inside the interference zone interfere with each other, at most one arc with $\lambda(e) = 1$ can be used in a round, and constraint $(*)$ is satisfied. Now, for a vertex v inside the interference zone with $d(v, s_0) = i \leq k$, any dipath from v to v_0 uses at least i arcs with $\lambda(e) = 1$ and so $\tau_{\min}(v) \geq i$. For a vertex v with $d(v, v_0) \geq k$, any dipath from v to v_0 uses at least k arcs with $\lambda(e) = 1$ and so $\tau_{\min}(v) \geq k$. Therefore, using $(**)$, we have

$T \geq \sum_v \tau_{\min}(v) \geq \sum_{i=1}^{k} iN_i + k(N - \sum_{i=0}^{k} N_i)$ which matches the lower bound of Theorem 1.

Before we apply the method for even d_I, we need to distinguish among three types of vertices on the partial interference boundary (i.e., at distance $k+1$ from v_0). These three types of vertices are labelled X, Y, and Z in Figures 5 and 6. There are four vertices of type X: $v = (k+1, 0)$, and $\rho(v), \rho^2(v)$, and $\rho^3(v)$. For $k \geq 2$, there are eight vertices of type Y: $v = (k, 1)$, $v' = (k, -1)$, and their rotated images. If $k = 1$, there are only four vertices of type Y: $(1, 1)$ and its three rotated images. If $k > 2$, then all of the $4k - 8$ other vertices on the partial interference boundary are of type Z. Now, for even $d_I = 2k \geq 2$, we get the following lower bound:

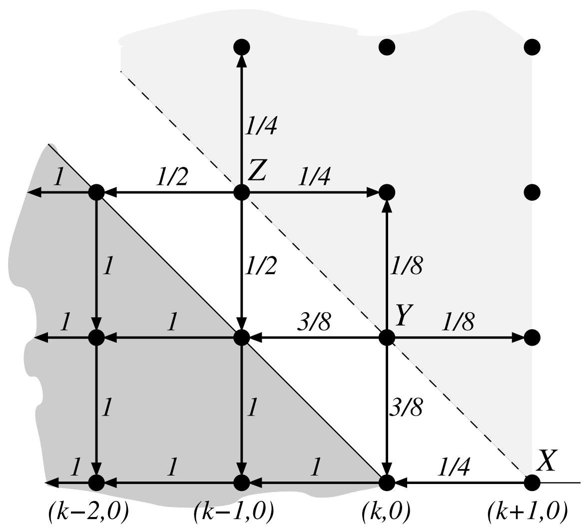

FIG. 6. *Weights for even interference distance lower bound.*

THEOREM 3. *Suppose that $n = 2p+1$ is odd, $d_I = 2k \geq 2$ is even, and $p \geq k+1$. Then the number of rounds needed to gather in a square grid with $N = n^2$ vertices is at least $(k + \frac{1}{4})(N-1) - c'_k$, where $c'_k = \frac{k(k+1)(4k-1)}{6} - \min\{1, k-1\}$.*

Proof. The lower bound follows with the following choices for $\lambda(e)$ (see Figures 5 and 6). Choose $\lambda(e) = 1$ for each arc e inside the interference zone that is directed towards v_0 (i.e., $e = (s, r)$ with $1 \leq d(s, v_0) \leq k$ and $d(r, v_0) = d(s, v_0) - 1$). For each of the four arcs (s, r) directed towards v_0 with sender s of type X (and $d(r, v_0) = k$), choose $\lambda(e) = \frac{1}{4}$. For the arcs with sender of type Y, choose $\lambda(e) = \frac{3}{8}$ if the arc is directed towards v_0 (i.e., $d(r, v_0) = k$), and $\lambda(e) = \frac{1}{8}$ if the arc is directed away from v_0 ($d(r, v_0) = k+2$). Finally, if an arc has a sender of type Z, choose $\lambda(e) = \frac{1}{2}$ if the arc is directed towards v_0 ($d(r, v_0) = k$), and $\lambda(e) = \frac{1}{4}$ if the arc is directed away from v_0 ($d(r, v_0) = k+2$). All other arcs have $\lambda(e) = 0$.

CLAIM 4. *Constraint $(*)$ is satisfied for these values of $\lambda(e)$.*

Proof of the claim. Every arc inside the interference zone with $\lambda(e) = 1$ is directed towards v_0 and any call that uses such an arc conflicts with any call from a sender on or inside the partial interference boundary. So, a round can use at most one arc with $\lambda(e) = 1$ and if it uses one such arc then it uses no other arc e' with $\lambda(e') > 0$. Thus, constraint $(*)$ is satisfied for such a round.

The only arcs outside the interference zone with $\lambda(e) > 0$ are those with senders on the partial interference boundary. All such arcs have $\lambda(e) \leq \frac{1}{2}$, so any round using at most two such arcs satisfies constraint $(*)$. If a round uses three or more such arcs, we can assume, without loss of generality, that one arc has a sender in the upper right quadrant, that is $s = (x, k+1-x)$ with $1 \leq x \leq k+1$. The other cases are obtained by rotations.

We now compute the distance between s and another sender s' according to which group s' belongs:

- *Group 1:* If $s' = (x', k+1-x')$ with $0 \leq x' \leq k+1$, then $d(s, s') = 2|x - x'|$.
- *Group 2:* If $s' = (x', -(k+1-x'))$ with $1 \leq x' \leq k$, then $d(s, s') = |x - x'| + 2k + 2 - x - x'$.
- *Group 3:* If $s' = (-x', k+1-x')$ with $1 \leq x' \leq k$, then $d(s, s') = x + x' + |x - x'|$.
- *Group 4:* If $s' = (-x', -(k+1-x'))$ with $0 \leq x' \leq k+1$, then $d(s, s') = 2k+2$.

Case 1: s is of type Z, so $2 \leq x \leq k-1$.

If $d(s, s') < 2k$ then any call with sender s' interferes with every call with sender s. So the only other possible senders on the partial interference boundary are s' in Group 2 with $x' = 1$, that is $s' = (1, -k)$, or s' in Group 3 with $x' = k$, that is $s' = (-k, 1)$, or s' in Group 4. Note that if $s' = (1, -k)$ or $s' = (-k, 1)$, then (s, r) must be directed away from v_0 to avoid interference with (s', r'), so $\lambda(s, r) = \frac{1}{4}$. A round containing (s, r) contains at most two other arcs (s', r') and (s'', r'') as follows:

- (s', r') with $s' = (1, -k)$ (so $\lambda(s, r) = \frac{1}{4}$) and (s'', r'') with $s'' = (-k, 1)$ or $s'' = (-(k+1), 0)$ or $s'' = (-k, -1)$: in this case, $\sum \lambda(e) \leq \frac{1}{4} + \frac{3}{8} + \frac{3}{8} = 1$.
- (s', r') with $s' = (-k, 1)$ (so $\lambda(s, r) = \frac{1}{4}$) and (s'', r'') with $s'' = (-1, -k)$ or $s'' = (1, -k)$ or $s'' = (0, -(k+1))$: in this case, $\sum \lambda(e) \leq \frac{1}{4} + \frac{3}{8} + \frac{3}{8} = 1$.
- $s' = (-(k+1), 0)$ and $s'' = (0, -(k+1))$ are the two senders of type X in Group 4: in this case, $\sum \lambda(e) \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$.

In summary, if s is of type Z, constraint $(*)$ is always satisfied.

Case 2: All the arcs in the round have senders of types X and Y.

We can have at most one such arc per region, so there are at most four such arcs in a round. If all arcs satisfy $\lambda(e) \leq \frac{1}{4}$, we are done. So, it remains to deal with the case where at least one arc has $\lambda(s, r) = \frac{3}{8}$ which implies that s is of type Y and (s, r) is directed towards v_0. Without loss of generality, suppose that $s = (k, 1)$. Then $s' = (1, -k)$, $s' = (1, k)$, and $s' = (0, k+1)$ cannot be senders because they interfere with (s, r).

Suppose that there is a sender s' in region R_N. Then $s' = (-1, k)$, and (s', r') is directed away from v_0, so $\lambda(s', r') = \frac{1}{8}$. Furthermore, the only other possible senders are $s'' = (-k, -1)$, $s'' = (-1, -k)$, and $s'' = (0, -(k+1))$, and at most one such arc can be included without causing interference, so $\sum \lambda(e) \leq \frac{3}{8} + \frac{1}{8} + \frac{3}{8} < 1$.

It remains to consider the case of two arcs of type X or Y, one in region R_S and one in region R_W. If the arc in region R_S is $s' = (0, -(k+1))$, then $\lambda(s', r') = \frac{1}{4}$ and $\sum \lambda(e) \leq \frac{3}{8} + \frac{1}{4} + \frac{3}{8} = 1$. If it is $s' = (-1, -k)$, then $s'' = (-k, 1)$ and one of the arcs (s', r') and (s'', r'') must be directed away from v_0 to avoid interference between them, so $\sum \lambda(e) \leq \frac{3}{8} + \frac{3}{8} + \frac{1}{8} < 1$.

So, constraint $(*)$ is satisfied in all cases and the claim is proved.

To finish the proof of Theorem 3, it suffices to compute a lower bound on $\sum_v \tau_{\min}(v)$ where $\tau_{\min}(v)$ is the minimum cost $\sum_{e \in P(v)} \lambda(e)$ to move a message from v to v_0 along any dipath $P(v)$. If a vertex v is inside the interference zone, then $d(v, v_0) = i \leq k$, and any dipath from v to v_0 uses at least i arcs with $\lambda(e) = 1$, so $\tau_{\min}(v) \geq i$. For a vertex v that is on or outside the partial interference boundary, $d(v, v_0) \geq k+1$, and any dipath from v to v_0 uses at least k arcs inside the interference zone with $\lambda(e) = 1$ plus at least one additional arc from the partial interference boundary to the boundary of the interference zone (i.e., to a vertex at distance k from v_0). If v is outside the partial interference boundary or is of type X, then the additional arc e has $\lambda(e) \geq \frac{1}{4}$, and $\tau_{\min}(v) \geq k + \frac{1}{4}$. If v is a type Y vertex, then either it uses an arc e towards v_0 with $\lambda(e) = \frac{3}{8}$ or it uses an arc away from v_0 with $\lambda(e) = \frac{1}{8}$ plus another arc e' with $\lambda(e') \geq \frac{1}{4}$ to get to the boundary of the interference zone. In both cases, $\tau_{\min}(v) \geq k+\frac{3}{8}$ for a vertex of type Y. If v is a type Z vertex, then either it uses an arc e towards v_0 with $\lambda(e) = \frac{1}{2}$ or it uses an arc away from v_0 with $\lambda(e) = \frac{1}{4}$ plus another arc e' with $\lambda(e') \geq \frac{1}{4}$ to get to the boundary of the interference zone. In both cases, $\tau_{\min}(v) \geq k+\frac{1}{2}$ for a vertex of type Z. Summing over all vertices and using $(**)$, we get the lower bound $T \geq \sum_v \tau_{\min}(v) \geq \sum_{i=1}^k iN_i + (k + \frac{1}{4})(N - \sum_{i=0}^k N_i) + \frac{1}{8}|Y| + \frac{1}{4}|Z|$, where $|Y|$ and $|Z|$ are the numbers of vertices of types Y and Z, respectively. If $k \geq 2$, then $\frac{1}{8}|Y| + \frac{1}{4}|Z| = \frac{1}{8} \times 8 + \frac{1}{4} \times (4k - 8) = k - 1$. If $k = 1$, then there are only four vertices of type Y and no vertices of type Z, so $\frac{1}{8}|Y| + \frac{1}{4}|Z| = \frac{1}{8} \times 4 + 0 = \frac{1}{2}$. Since the number of rounds is an integer, the lower bound when $k = 1$ is $\sum_{i=1}^k iN_i + (k + \frac{1}{4})(N - \sum_{i=0}^k N_i) + 1$. Putting the two bounds together gives a lower bound of $\sum_{i=1}^k iN_i + (k + \frac{1}{4})(N - \sum_{i=0}^k N_i) + \min\{1, k - 1\}$. Noting that $N_i = 4i$ for $1 \leq i \leq k$, we get a lower bound of $(k + \frac{1}{4})(N - 1) - c'_k$ where $c'_k = \sum_{i=1}^k (k+\frac{1}{4}-i)N_i - \min\{1, k-1\} = (4k+1)\sum_{i=1}^k i - 4\sum_{i=1}^k i^2 - \min\{1, k-1\} = \frac{k(k+1)(4k-1)}{6} - \min\{1, k - 1\}$. □

THEOREM 5. *Suppose that $n = 2p + 1$ is odd, $d_I = 2k \geq 2$ is even, and $p \geq k + 1$. Then gathering in a square grid with $N = n^2$ vertices can be completed in $(k + \frac{1}{4})(N - 1) - c'_k$ rounds, where $c'_k = \frac{k(k+1)(4k-1)}{6} - \min\{1, k - 1\}$ and this is optimal.*

Proof. The protocol for d_I even is similar to the odd case but there are several differences. Firstly, an extra round labelled α is needed in each stage for the four arcs directed towards v_0 from senders of type X. This is the only set of four compatible calls that can be used to transmit simultaneously to vertices on the boundary of the interference zone (see the examples in Figure 7). Secondly, we have to use dipaths that contain arcs that are directed away from the central vertex in some areas of the grid. We also have to deal with the $4k + 4$ vertices on the partial interference boundary (dashed box) as special cases.

(a) For all of the vertices outside the partial interference boundary, each stage consists of $4k+1$ rounds labelled e_i, n_i, w_i, s_i with $1 \leq i \leq k$, and α. Similarly to the odd case, four leaves of the gathering tree will become dormant at the end of each stage. Except for the addition of the rounds labelled α, the gathering tree, the dipaths, and the labellings are the same as in the odd case for the vertices that satisfy $|x| \geq k + 1$ and $|y| \geq k + 1$ (i.e., vertices no closer to v_0 than the boundary of the light grey square). The labels for the dipath $P(x, y)$, starting from v_0 and working in the opposite direction to the dipath towards (x, y), use the repeating pattern of $2k + 2$ labels: $e_1, e_2, \ldots, e_k, \alpha, w_k, w_{k-1}, \ldots, w_1, s_1$. Figure 7 shows the dipaths (solid arrows) and labels for $v = (7, 7)$, $k = 3$, and $d_I = 2k = 6$. When $y < 0$, the label s_1 is replaced by n_1 as it is in the odd case. Figure 7 shows the dipaths (dotted arrows) and labels for $v' = (5, -4)$, $k = 3$, and $d_I = 2k = 6$.

(b) For the vertices strictly inside the light grey square, but outside the partial interference boundary, (i.e., $|x| \leq k$, $|y| \leq k$, and $|x|+|y| > k+1$), each stage consists of $4k+1$ rounds, but the gathering tree differs from the odd case. For

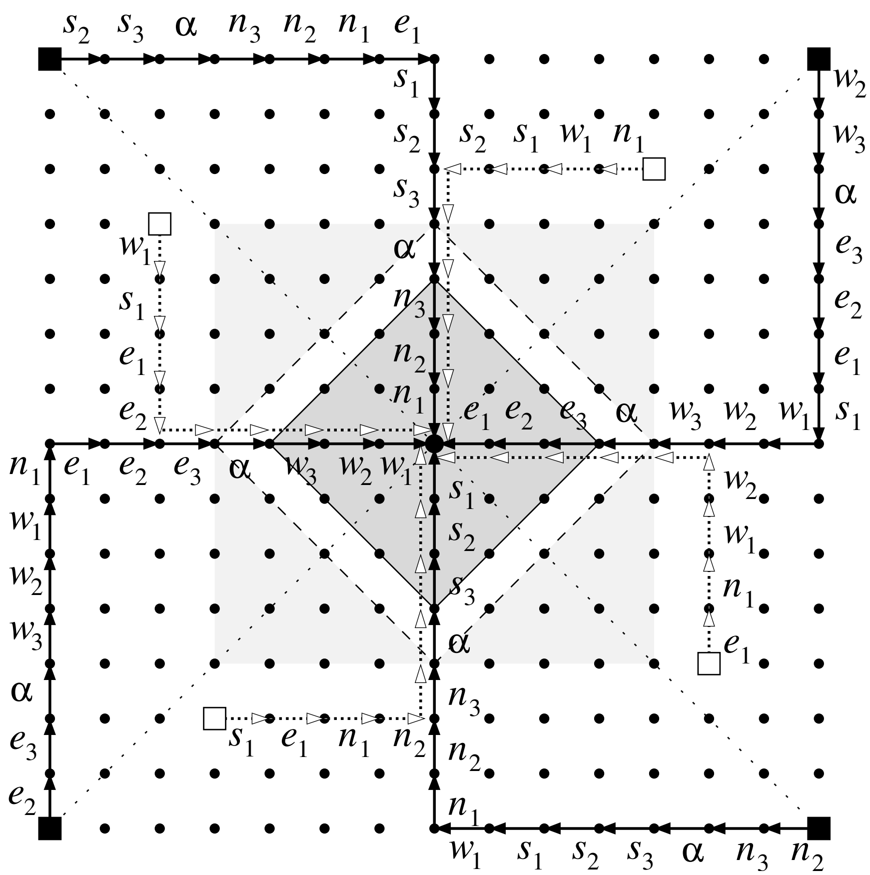

FIG. 7. *Gathering stages for vertices outside the light grey square and $d_I = 6$.*

region R_E, the tree contains horizontal arcs directed towards the vertical line $x = k+1$. More precisely, for a vertex (x, y) in region R_E with $y > 1$, $P(x, y)$ consists of the $k+1-x$ horizontal arcs $((i, y), (i+1, y))$ for $x \leq i \leq k$ followed by the y vertical arcs $((k+1, j), (k+1, j-1))$ for $y \geq j > 0$, and finally the $k+1$ horizontal arcs $((i, 0), (i-1, 0))$, $1 \leq i \leq k+1$. The length of $P(x, y)$ is $(k+1-x)+y+(k+1) \leq 2k+2$. The dipaths for vertices (x, y) with $y < -1$ are similar except the middle set of y vertical arcs is $((k+1, j), (k+1, j+1))$, $y \leq j < 0$. Note that the first calls move information away from v_0, which is necessary to avoid interference.

The labels for the dipath $P(x, y)$, starting from v_0 and working in the opposite direction to the dipath towards (x, y), use the repeating pattern of labels: $e_1, e_2, \ldots, e_k, \alpha, s_k, s_{k-1}, \ldots, s_{k+1-y}, w_{k+2-y}, \ldots, w_{2k+2-y-x}$. According to this labelling, a call of the form $((i, 0), (i-1, 0))$ is labelled e_i, a call of the form $((k+1, j), (k+1, j-1))$ is labelled s_{k+1-j}, and a call of the form $((i, y), (i+1, y))$ is labelled $w_{2k+2-y-i}$. The pattern is similar for vertices (x, y) with $y < 0$ except the labels $s_k, s_{k-1}, \ldots, s_{k+1-y}$ are replaced by $n_k, n_{k-1}, \ldots, n_{k+1-y}$.

The labels for the three rotated dipaths, $\rho(P), \rho^2(P), \rho^3(P)$, are obtained using the same mapping ω that was used for d_I odd: if arc e in $P(x, y)$ is labelled l, then arc $\rho(e)$ in the rotated dipath $\rho(P)$ is labelled $\omega(l)$. Figure 8 shows the dipaths and labels for $v = (x, y) = (3, 3)$, $k = 4$, and $d_I = 2k = 8$.

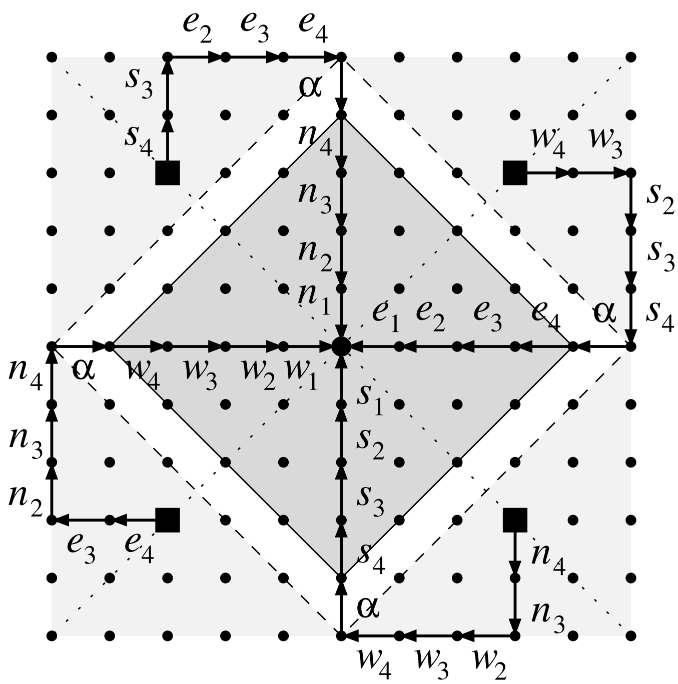

FIG. 8. *Gathering stage close to the interference zone for* $d_I = 8$.

It remains to prove that any pair of calls that have the same label (so they are made in the same round) are compatible. If the label is α, there is no interference as the four calls labelled α are compatible. Now consider a call labelled e_i (the proofs for n_i, w_i, and s_i follow by applying ρ and ω). Three such calls are possible: (s, r) on P with $s = (i, 0)$ and $r = (i-1, 0)$, (s', r') on $\rho(P)$ with $s' = (i-k-1, k+1)$ and $r' = (i-k, k+1)$, and (s'', r'') on $\rho^2(P)$ with $s'' = (y+i-2k-2, -y)$ and $r'' = (y+i-2k-3, -y)$. We have $d(s, r') = d(s', r) = i+k-i+k+1 = 2k+1$, $d(s'', r) = d(s, r'') - 2 = 2k+2-y-i+i-1+y = 2k+1$, and $d(s', r'') = d(s'', r') = 2k+2-y-i+i-k+k+1+y = 2k+3$. In all of these cases the calls are compatible.

(c) Finally, we have to deal with calls sent from vertices on the partial interference boundary. First, assume that $k \geq 2$. We use four special rounds for the twelve vertices of types X and Y. The first round consists of the three calls $((k+1, 0), (k, 0))$, $((-1, k), (0, k))$, and $((-1, -k), (0, -k))$, and the other three special rounds consist of calls obtained by rotations. After each special round, the messages of three vertices have arrived at the boundary of the interference zone and we use $3k$ rounds to move them to v_0. This gives a total of $12(k + \frac{1}{4}) + 1$ rounds for these twelve vertices. Note that the special rounds exactly satisfy the lower bound constraint: $\sum \lambda(e) = \frac{3}{8} + \frac{3}{8} + \frac{1}{4} = 1$. If $k > 2$, then there are $4k - 8$ vertices of type Z on the partial interference boundary, and their messages are sent two at a time to the boundary of the interference zone during special rounds. Any vertex (x, y) of type Z in region R_E or region R_N sends its message to its neighbour in the gathering tree and

$\rho^2(x,y) = (-x,-y)$ sends its message in the same round. For example, (x,y) with $x > 1$ in region R_E uses the call $((x,y),(x,y-1))$ and $(-x,-y)$ uses the call $\rho^2((x,y),(x,y-1)) = ((-x,-y),(-x,-y+1))$. Then $2k$ rounds are needed to move the two messages to v_0. (See Figure 5.) The total number of rounds for the $4k-8$ vertices of type Z is $(k+\frac{1}{2})(4k-8) = (k+\frac{1}{4})(4k-8) + k-2$ rounds. Note that the special rounds exactly satisfy the lower bound constraint: $\sum \lambda(e) = \frac{1}{2} + \frac{1}{2} = 1$.

Altogether we need $\sum_{i=1}^{k} iN_i$ rounds to move the messages of the vertices inside the interference zone to v_0, $(k+\frac{1}{4})N_{k+1} + k - 1$ rounds for the vertices on the partial interference boundary, and $(k+\frac{1}{4})(N - \sum_{i=0}^{k+1} N_i)$ rounds for the vertices outside of the partial interference boundary for a total of $\sum_{i=1}^{k} iN_i + (k+\frac{1}{4})(N - \sum_{i=0}^{k} N_i) + k - 1$.

If $k = 1$, then there are only four vertices of type Y (and none of type Z), so the special rounds for vertices of types X and Y are different. Three special rounds are needed for these eight vertices because $\sum \tau_{\min}(v) \geq 4 \times \frac{3}{8} + 4 \times \frac{1}{4} = 2.5$. For example, the four messages of the type X vertices can be sent to the boundary of the interference zone in one round, the messages of the type Y vertices can be sent two at a time in two rounds, and then $8k = 8$ rounds are needed to move the messages to v_0. The total number of rounds for vertices on the partial interference boundary is therefore $(k+\frac{1}{4})N_{k+1} + 1$ instead of $(k+\frac{1}{4})N_{k+1} + k - 1$. Putting the two bounds together gives an upper bound of $\sum_{i=1}^{k} iN_i + (k+\frac{1}{4})(N - \sum_{i=0}^{k} N_i) + \min\{1, k-1\}$ which matches the lower bound of Theorem 3. ∎

Remark. Our results and proofs for square grids are also valid for grids with different shapes with the condition that when a vertex v has a message to send, then the vertices $\rho(v), \rho^2(v)$, and $\rho^3(v)$ must also have messages to send. For example, the bounds and protocols are the same for the diamond-shaped grid consisting of the $N = 2d^2 + 2d + 1$ vertices at distance at most d from v_0.

5. Hexagonal Grids. The hexagonal grid is similar to the grid except each vertex has degree six and it contains six axes denoted A, B, C, D, E, and F. In this section, we use ρ to denote a rotation of $\frac{\pi}{3}$, so $B = \rho(A)$, $C = \rho(B) = \rho^2(A)$, and so on. Analogously to the grid, we define regions R_A, R_B, R_C, R_D, R_E, and R_F. R_A is the region centred around the A axis and between the dotted lines in Figures 9 and 10. Its positive part is above the A axis and its negative part is below. R_B is the region obtained by rotating region R_A: $R_B = \rho(R_A)$. Similarly, $R_C = \rho(R_B)$, and so on.

We define the interference zone to be the set of vertices at distance at most k from the central vertex v_0. For even $d_I = 2k$, the vertices at distance $k+1$ from v_0 define the partial interference boundary and are of two types. The six type X vertices ($X_A, X_B, X_C, X_D, X_E, X_F$ in Figure 10) are the vertices at distance $k+1$ from v_0 on the axes. All other vertices at distance $k+1$ from v_0 are of type Z. The number of vertices at distance exactly d from v_0 is $N_d = 6d$ for $1 \leq d \leq k$, and $N_0 = 1$.

Similarly to the square grids, the results and proofs for hexagonal grids in this section are valid with the condition that when a vertex v has a message to send, then the five vertices obtained by rotations must also have messages to send. For example, the bounds and protocols apply to the hexagon-shaped grid consisting of the $N = 3d^2 + 3d + 1$ vertices at distance at most d from v_0.

Theorem 6. *Suppose that $d_I = 2k-1$ is odd and $N \geq 3k^2 + 3k + 1$. Then the number of rounds needed to gather in a hexagonal grid with N vertices is $k(N-1) - h_k$, where $h_k = k(k+1)(k-1)$.*

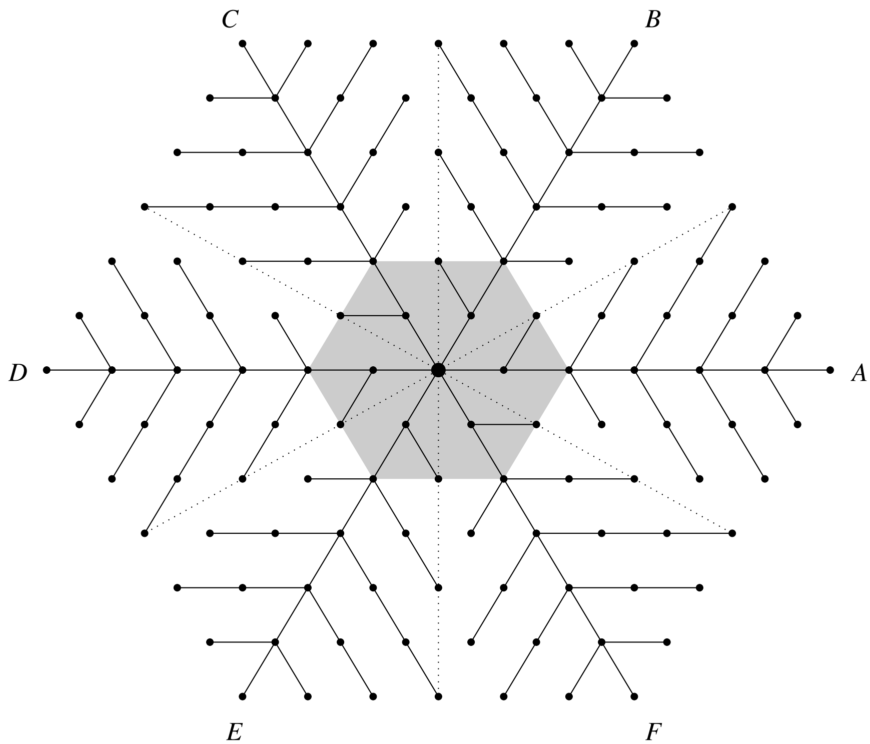

FIG. 9. *Hexagonal gathering tree for odd d_I.*

Proof. The proof is similar to the proof for the grid. We use the dual method to prove the lower bound. We choose $\lambda(e) = 1$ for each arc e that is inside the interference zone and directed towards v_0 and $\lambda(e) = 0$ otherwise. Constraint $(*)$ is satisfied as a round contains at most one arc in the interference zone. For any vertex v at distance i, $\tau_{\min}(v) = \min\{i, k\}$ because a shortest path uses $\min\{i, k\}$ arcs in the interference zone. The total number of rounds is at least $\sum_{i=1}^{k} iN_i + k(N - \sum_{i=0}^{k} N_i)$ and using $N_d = 6d$, $1 \le d \le k$ gives the bound in the statement of the theorem.

For the upper bound, we use the gathering tree shown in Figure 9. Let v be a vertex in the positive part of region R_A outside of the interference zone. We send the message of v along the dipath P containing arcs parallel to the B axis and then arcs on the A axis. We label the $6k$ rounds of each stage with labels $a_i, b_i, c_i, d_i, e_i, f_i$, $1 \le i \le k$. The labels for the dipath P between v and v_0, starting at v_0 (i.e., in reverse order of their occurrence on P), are a repetition of the sequence of $2k+1$ labels: $a_1, a_2, \ldots, a_k, e_k, e_{k-1}, \ldots, e_1, c_1$. We define the dipaths for the regions R_B, R_C, R_D, R_E, R_F by rotations to be $\rho(P), \rho^2(P), \rho^3(P), \rho^4(P), \rho^5(P)$, respectively. If arc e is labelled ℓ, we label arc $\rho(e)$ with label $\omega(\ell)$, where ω is the one-to-one mapping of labels such that $\omega(a_i) = b_i$, $\omega^2(a_i) = c_i$, $\omega^3(a_i) = d_i$, $\omega^4(a_i) = e_i$, $\omega^5(a_i) = f_i$. One can check that two arcs with the same label are non-interfering. The proof is easier than for the grid as P and $\rho(P)$ use different labels, so an arc labelled a_i in P, $i \ge 2$ only appears in $\rho^2(P)$, and the distance between senders is at least $2k+1$. An arc labelled a_1 in P can appear in both $\rho^2(P)$ and $\rho^4(P)$, but all of the senders are at distance at least $2k+1$ from each other.

The proof for the negative part of region R_A is similar except that the dipath

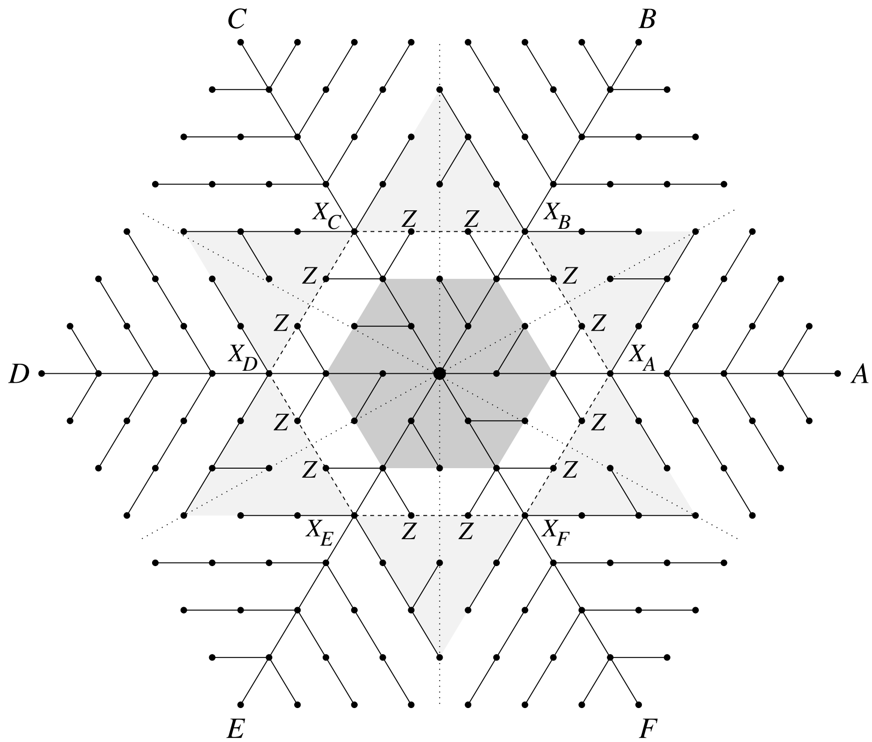

FIG. 10. *Hexagonal gathering tree for $d_I = 4$.*

uses arcs parallel to the F axis and then on the A axis, and the labels are repetitions of the sequence $a_1, a_2, \ldots, a_k, c_k, c_{k-1}, \ldots, c_1, e_1$. □

THEOREM 7. *Suppose that $d_I = 2k$ is even and $N \geq 3(2k+2)^2 + 3(2k+2)k + 1$. Then the number of rounds needed to gather in a hexagonal grid with N vertices is $(k + \frac{1}{3})(N-1) - h'_k$, where $h'_k = k^2(k+1) - k$.*

Proof.

The lower bound is proved using the following choices for $\lambda(e)$. Let $\lambda(e) = 1$ for each arc e inside the interference zone that is directed towards v_0. For each of the six arcs (s, r) directed towards v_0 with sender s of type X (and $d(r, v_0) = k$), choose $\lambda(e) = \frac{1}{3}$. Finally, if an arc has a sender of type Z, choose $\lambda(e) = \frac{1}{2}$ if the arc is directed towards v_0 ($d(r, v_0) = k$) and $\lambda(e) = \frac{1}{6}$ if the arc is directed away from v_0 ($d(r, v_0) = k + 2$). All other arcs have $\lambda(e) = 0$.

A proof similar to the proof for the grid can be used to verify that constraint $(*)$ is satisfied for these values of $\lambda(e)$. The non-trivial cases are when a sender is on the partial interference boundary. If the sender is of type Z and between the A and B axes, then a compatible receiver in the interference zone can only be on the boundary of the interference zone between the D and E axes. So, a round can contain at most two arcs with $\lambda(e) = \frac{1}{2}$. If a round contains one arc e with $\lambda(e) = \frac{1}{2}$, then at most two arcs directed away from v_0 with $\lambda(e) = \frac{1}{6}$ are compatible with it. Finally, if a sender of type X transmits to a vertex closer to v_0 (so the arc has weight $\frac{1}{3}$), then there can be at most one more arc with weight $\frac{1}{2}$ or two more arcs with weight $\frac{1}{3}$. As an example of the latter case, if X_A, X_B, and X_C all transmit towards v_0 simultaneously, then $\sum \lambda(e) = 1$.

To finish the proof of the lower bound, it suffices to compute $\sum_v \tau_{\min}(v)$. If a vertex v is inside the interference zone, then $d(v, v_0) = i \leq k$, and so $\tau_{\min}(v) \geq i$. If v is outside the partial interference boundary or is of type X, then any dipath from v to v_0 uses at least k arcs inside the interference zone with $\lambda(e) = 1$ plus at least one additional arc with $\lambda(e) \geq \frac{1}{3}$, so $\tau_{\min}(v) \geq k + \frac{1}{3}$. If v is of type Z, then any dipath from v to v_0 uses at least k arcs inside the interference zone with $\lambda(e) = 1$ and either it uses an arc e towards v_0 with $\lambda(e) = \frac{1}{2}$ or it uses an arc away from v_0 with $\lambda(e) = \frac{1}{6}$ plus another arc e' with $\lambda(e') \geq \frac{1}{3}$ to get to the boundary of the interference zone. In both cases, $\tau_{\min}(v) \geq k + \frac{1}{2}$ for a vertex of type Z. Summing up, we get $\sum_{i=1}^{k} iN_i + (k + \frac{1}{3})(N - \sum_{i=0}^{k} N_i) + \frac{1}{6}|Z|$ rounds. Since $|Z| = 6k$, and $N_i = 6i$, $1 \leq i \leq k$, we obtain the lower bound in the statement of the theorem.

The proof of the upper bound is also similar to the proof for the grid. We will use the gathering tree shown in Figure 10 with $6k + 2$ labels: $a_i, b_i, c_i, d_i, e_i, f_i$, $1 \leq i \leq k$ as in the case of odd d_I, and two extra labels α and β. The mapping ω associated with ρ extends the mapping for odd d_I with the addition of $\omega(\alpha) = \beta$ and $\omega(\beta) = \alpha$. The messages of vertices inside the interference zone are sent along shortest paths. A vertex at distance $i \leq k$ from v_0 uses i rounds matching the lower bound. The vertices of type X send their messages three at a time towards the interference zone during a round labelled α (for X_A, X_C, and X_E) or β (for X_B, X_D, and X_F), and then each message needs k more rounds inside the interference zone to reach v_0. The vertices of type Z transmit their messages two at a time towards the interference zone during a round labelled α. More precisely, a vertex v of type Z in region R_A uses a shortest path with labels (in reverse order of their occurrence on the dipath starting at v_0) $a_1, a_2, \ldots, a_k, \alpha$, and simultaneously the symmetric vertex $\rho^3(v)$ in region R_D uses a shortest path with labels (starting at v_0) $d_1, d_2, \ldots, d_k, \alpha$. The dipaths for type Z vertices in other regions are obtained by rotations and most of the labels are obtained using the mapping ω. The exception is that the first arc of each dipath is labelled α (i.e., label β is not used). So, the cost for vertices of type Z matches the lower bound of $k + \frac{1}{2}$.

We need to match the lower bound of $k + \frac{1}{3}$ for all other vertices. The protocol is straightforward for most of the vertices outside the partial interference boundary, but it is quite complicated for the vertices in the light grey triangles of Figure 10. Our discussion will focus on vertices inside the triangle bounded by the line segment joining X_A and X_B, the line segment parallel to the B axis starting from X_A in the direction away from v_0, and the line segment parallel to the A axis starting from X_B. The dipaths for vertices in the other light grey triangles are obtained by rotations and the labels are obtained using ω. Figure 11 shows a detailed view.

Consider a vertex v in the light grey triangle in the positive part of region R_A. Figure 11 shows an example. Using the same idea as for the grid with even d_I, the first arcs of $P(v)$ move information away from v_0 to avoid interference. The natural approach would be to use $P(v)$ and the five dipaths obtained from it by rotations during a stage of $6k + 2$ rounds to deliver six messages to v_0. Unfortunately, this will not avoid all interference. Instead, we consider two consecutive stages with a total of $12k + 4$ rounds to deliver twelve messages to v_0 along twelve dipaths: $P(v)$ and $P(f(v))$ for a vertex $f(v)$ to be defined below, and the ten dipaths obtained from $P(v)$ and $P(f(v))$ by rotations.

The dipath $P(v)$ for a vertex v in the light grey triangle in the positive part of region R_A consists of three parts:

- $\ell_1 > 0$ arcs from v to the boundary of the light grey triangle in the di-

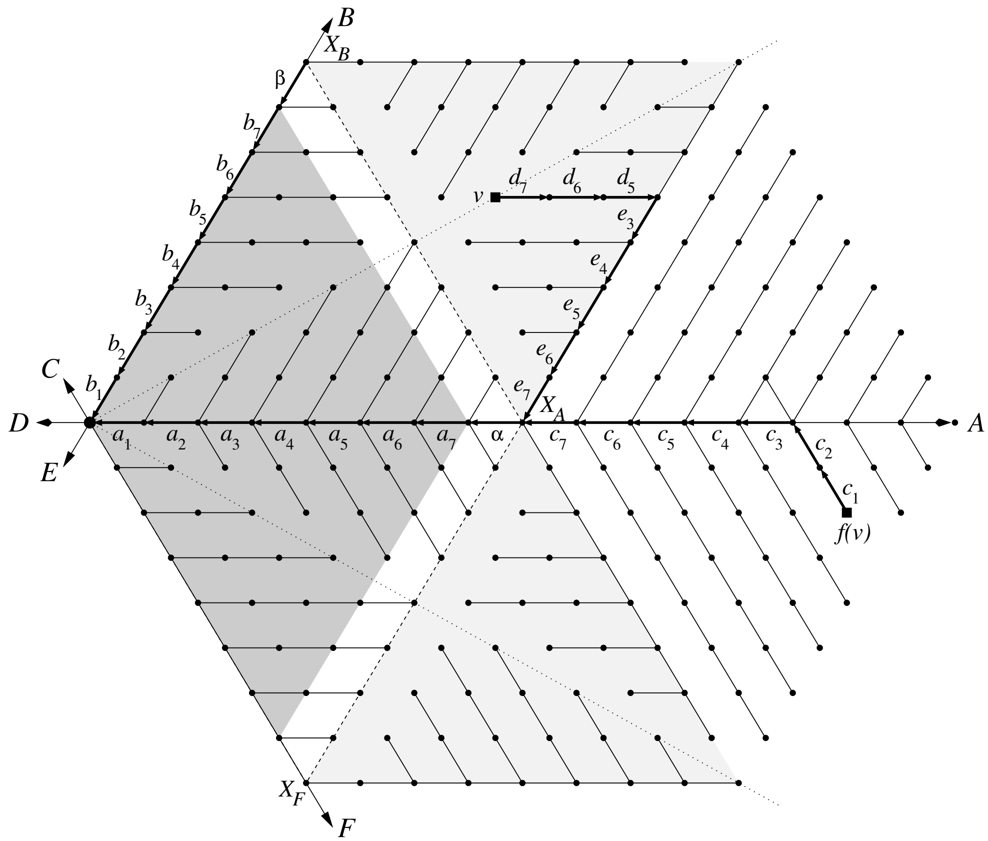

FIG. 11. *Gathering stage for hexagonal grid with $d_I = 14$.*

rection parallel to the A axis and away from v_0. The arcs are labelled $d_k, d_{k-1}, \ldots, d_{k+1-\ell_1}$.
- $\ell_2 \geq 2$ arcs to X_A along the boundary of the triangle in the direction parallel to the B axis. The arcs are labelled $e_{k+1-\ell_2}, \ldots, e_{k-1}, e_k$. Note that $\ell_1 + \ell_2 \leq k+1$ by the definition of the grey triangle.
- $k+1$ arcs along the A axis from X_A to v_0 with labels $\alpha, a_k, a_{k-1}, \ldots, a_1$.

The values of ℓ_1 and ℓ_2 are determined by the location of v. Figure 11 shows an example with $k = 7$, $d_I = 2k = 14$, $\ell_1 = 3$, and $\ell_2 = 5$.

The vertex $f(v)$ is defined by specifying the dipath $P(f(v))$ starting from X_A and working in the direction away from v_0 towards $f(v)$. The dipath consists of two parts:
- ℓ_2 arcs from X_A along the A axis in the direction away from v_0 labelled $c_k, c_{k-1}, \ldots, c_{k+1-\ell_2}$.
- $\ell_1 - 1$ arcs in the direction away from v_0 and parallel to the F axis labelled $c_{k-\ell_2}, \ldots, c_{k-\ell_2-\ell_1+2}$. For the last label, $k - \ell_2 - \ell_1 + 2 \geq 1$ because $\ell_1 + \ell_2 \leq k+1$.

Note that $f(v)$ is in the negative part of region R_A and not in a light grey triangle. Furthermore, for any two vertices v and v', $v \neq v'$ implies that $f(v) \neq f(v')$. Also note that our definition of $f(v)$ requires that v is a leaf when $f(v)$ is a leaf. (See Figure 11.)

Finally, let $P(X_B)$ be the dipath going from X_B along the B axis to v_0 with labels $\beta, b_k, b_{k-1}, \ldots, b_1$.

We divide the $12k + 4$ rounds into two stages of $6k + 2$ rounds. During the first stage, we use the nine dipaths $P(v)$, $P(f(v))$, $P(X_B)$ and their rotated images $\rho^2(P(v))$, $\rho^2(P(f(v)))$, $\rho^2(P(X_B))$, $\rho^4(P(v))$, $\rho^4(P(f(v)))$, $\rho^4(P(X_B))$ labelled using the mapping ω. One can check that no two arcs with the same label interfere. Furthermore, at the end of this stage, six messages have been received by v_0, and all of the vertices have one message except the six leaves v, $\rho^2(v)$, $\rho^4(v)$, $f(v)$, $\rho^2(f(v))$, $\rho^4(f(v))$ which have no messages, the three vertices X_B, $X_D = \rho^2(X_B)$, $X_F = \rho^4(X_B)$ which also have no messages, and the three vertices X_A, $X_C = \rho^2(X_A)$, $X_E = \rho^4(X_A)$ which now have two messages. In the second stage, we use the nine dipaths obtained by rotations from the nine dipaths of the first stage. At the end of the second stage, v_0 will have received six new messages (so twelve messages at the end of the two stages), and all of the vertices will have exactly one message except the twelve leaves v, $f(v)$, and the ten vertices obtained from v and $f(v)$ by rotations, which will have no messages and will become dormant. Indeed, X_A, X_C, and X_E send one message and receive none during the second stage, and X_B, X_D, and X_F send two messages and receive one. The rounds labelled α and β are done last to ensure that X_B, X_D, and X_F receive a message before they have to send it.

Finally, let v be in the positive part of region R_A and not inside a light grey triangle. When v becomes a leaf in the gathering tree and we decide to send its message, we first check whether v is a vertex of type $f(u)$ for some u inside the light grey triangle in the negative part of region R_A. If it is, then we send its message and the message of the corresponding u as described above. Otherwise, we use a stage of $6k + 2$ rounds to send the messages of the six leaves v and $\rho^j(v)$, $1 \le j \le 5$. The dipath $P(v)$ consists of the arcs parallel to the B axis from v to the A axis followed by the arcs along the A axis to v_0. The labels for $P(v)$ starting from v_0 and working towards v use the repeating pattern of $2k + 2$ labels $a_1, a_2, \ldots, a_k, \alpha, e_k, \ldots, e_1, c_1$. The labels for the five rotated dipaths $\rho^j(P(v))$, $1 \le j \le 5$, are obtained using the mapping $\omega(l)$.

The dipaths for vertices in the negative part of region R_A and their rotated images are similar to the dipaths for vertices in the positive part. □

6. Conclusions. In this paper, we determined the exact number of rounds to gather one message from each vertex into a central gateway vertex of a square grid with $N = n^2$ vertices in a wireless radio network with interference constraints. The proof of the lower bound for the case of odd interference distance is straightforward. The matching upper bound is established by specifying an algorithm and proving its correctness. The proofs for the case of even interference distance are considerably more difficult. To prove the lower bound, we developed a new technique based on a relaxation of the problem and linear programming duality. The matching upper bound is proved with a sophisticated algorithm that uses the symmetry of the grid and non-shortest paths.

In a square grid with $N = n^2$, v_0 will be slightly off-centre if n is even. Minor modifications of the techniques described in this paper will work for n even, but it might not be possible to obtain matching upper and lower bounds due to the asymmetry. Similarly, if the grid is not square, then the techniques described in this paper will work as long as the grid is large enough to completely contain the interference zone and other regions that required special attention.rected

We generalized our results to hexagonal grids and again obtained matching lower and upper bounds. Hexagonal tilings of the plane are commonly used to assign frequencies in cell phone networks because hexagons are good approximations to circles, and graph distance in hexagonal grids is a good approximation to Euclidean distance in the plane.

There are several possible generalizations of our work including the following:

- We have assumed that the gateway vertex is in the centre of a symmetrical square grid or hexagonal grid. Experience with the one-dimensional version of the problem [1] suggests that moving the gateway to a different location will make the problem more difficult.
- In practice, the communication graph is unlikely to be a perfect grid graph. It is more likely to be missing some vertices and edges. The techniques in this paper can provide bounds for such graphs, but the algorithms will require a different approach. An interesting problem for general communication graphs would be to identify the best location for the gateway vertex. Another generalization would be to allow multiple gateway vertices. In practice, this would likely involve the use of multiple communication frequencies. (We only used one frequency in this paper.)
- We have assumed that each vertex has one message to send. Our proofs can be easily extended if the number of messages outside the interference zone is balanced so that each vertex and its rotated images have exactly the same number of messages to transmit (which could be zero). In the bounds, N will be the total number of messages instead of the number of vertices, and the constant c_k (c'_k, h_k, h'_k) will be different.
- An interesting variant would be to accommodate different levels of service; different customers could have different contracts with the service provider and would send and receive information at different rates.
- We have assumed that $d_T = 1$. This is a realistic assumption when the cost of the devices sold to consumers is to be minimized because inexpensive devices will have less sophisticated capabilities to handle interference. However, $d_T > 1$ merits further study. Some work in this direction appears in [2].

We believe that our new technique for proving lower bounds based on the relaxation of problem constraints and linear programming duality has significant potential for application to other problems.

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