Unique decomposition for a polynomial of low rank

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UNIQUE DECOMPOSITION FOR A POLYNOMIAL OF LOW RANK

EDOARDO BALLICO, ALESSANDRA BERNARDI

Abstract

Let F be a homogeneous polynomial of degree d in $m+1$ variables defined over an algebraically closed field of characteristic 0 and suppose that F belongs to the s-th secant variety of the d-uple Veronese embedding of $\mathbb{P}^m$ into $\mathbb{P}^{\binom{m+d}{d}-1}$ but that its minimal decomposition as a sum of d-th powers of linear forms requires more than s addenda. We show that if $s \leq d$ then F can be uniquely written as $F = M_1^d + \cdots + M_t^d + Q$, where $M_1, \ldots, M_t$ are linear forms with $t \leq (d-1)/2$, and Q a binary form such that $Q = \sum_{i=1}^q l_i^{d-d_i} m_i$ with l_i's linear forms and m_i's forms of degree d_i such that $\sum(d_i+1) = s-t$.

Introduction

In this paper we will always work with an algebraically closed field K of characteristic 0. Let $X_{m,d} \subset \mathbb{P}^N$, with $m \geq 1$, $d \geq 2$ and $N := \binom{m+d}{m} - 1$, be the classical Veronese variety obtained as the image of the d-uple Veronese embedding $\nu_d : \mathbb{P}^m \to \mathbb{P}^N$. The *$s$-th secant variety* $\sigma_s(X_{m,d})$ of the Veronese variety $X_{m,d}$ is the Zariski closure in $\mathbb{P}^N$ of the union of all linear spans $\langle P_1, \ldots, P_s \rangle$ with $P_1, \ldots, P_s \in X_{m,d}$. For any point $P \in \mathbb{P}^N$, we indicate with $\mathrm{sbr}(P) = s$ the minimum integer s such that $P \in \sigma_s(X_{m,d})$. This integer is called the *symmetric border rank* of P. Since $\mathbb{P}^m \simeq \mathbb{P}(K[x_0, \ldots, x_m]_1) \simeq \mathbb{P}(V^*)$, with V an $(m+1)$-dimensional vector space over K, the generic element belonging to $\sigma_s(X_{m,d})$ is the projective class of a form (a symmetric tensor) of type:

$$F = L_1^d + \cdots + L_r^d, \quad (T = v_1^{\otimes d} + \cdots + v_r^{\otimes d}). \tag{1}$$

The decomposition of a homogeneous polynomial that combines a minimum number of terms and that involves a minimum number of variables is a problem that is having a lot of attentions not only form classical Algebraic Geometry ([1], [7], [5], [6], [9]), but also from applications like Computational Complexity ([8]) and Signal Processing ([10]).

At the Workshop on Tensor Decompositions and Applications (September 13–17, 2010, Monopoli, Bari, Italy), A. Bernardi presented the following result:

([2], Corollary 1) Let $F \in K[x_0, \ldots, x_m]_d$ be such that $\mathrm{sbr}(F) + \mathrm{sr}(F) \leq 2d+1$ and $\mathrm{sbr}(F) < \mathrm{sr}(F)$. Then there are an integer $t \geq 0$, linear forms $L_1, L_2, M_1, \ldots, M_t \in K[x_0, \ldots, x_m]_1$, and a form $Q \in K[L_1, L_2]_d$ such that $F = Q + M_1^d + \cdots + M_t^d$, $t \leq \mathrm{sbr}(F) + \mathrm{sr}(F) - d - 2$, and $\mathrm{sr}(F) = \mathrm{sr}(Q) + t$. Moreover t, $M_1, \ldots, M_t$ and the linear span of L_1, L_2 are uniquely determined by F.

In terms of tensors it can be translates as follows:

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([2], Corollary 2) Let $T \in S^dV^*$ be such that $\mathrm{sbr}(T)+\mathrm{sr}(T) \le 2d+1$ and $\mathrm{sbr}(T) < \mathrm{sr}(T)$. Then there are an integer $t \ge 0$, vectors $v_1, v_2, w_1, \dots, w_t \in S^1V^*$, and a symmetric tensor $v \in S^d(\langle v_1, v_2\rangle)$ such that $T = v + w_1^{\otimes d} + \cdots + w_t^{\otimes d}$, $t \le \mathrm{sbr}(T)+\mathrm{sr}(T)-d-2$, and $\mathrm{sr}(T) = \mathrm{sr}(v)+t$. Moreover t, $w_1, \dots, w_t$ and $\langle v_1, v_2\rangle$ are uniquely determined by T.

The natural questions that arose at that workshop from applied people, were about the possible uniqueness of the binary form Q in [2], Corollary 1 (ie. the vector v in [2], Corollary 2) and a bound on the number t of linear forms (ie. rank 1 symmetric tensors). We are finally able to give the most possible complete answer to this question. The main result of this paper is the following.

Theorem 1. *Let $P \in \mathbb{P}^N$ with $N = \binom{m+d}{d}-1$. Suppose that:*

$$\mathrm{sbr}(P) < \mathrm{sr}(P) \text{ and}$$
$$\mathrm{sbr}(P)+\mathrm{sr}(P) \le 2d+1.$$

Let $\mathcal{S} \subset X_{m,d}$ be a 0-dimensional reduced subscheme that realizes the symmetric rank of P, and let $\mathcal{Z} \subset X_{m,d}$ be a 0-dimensional non-reduced subscheme such that $P \in \langle \mathcal{Z} \rangle$ and $\deg \mathcal{Z} \le \mathrm{sbr}(P)$. There is a unique rational normal curve $C_d \subset X_{m,d}$ such that $C_d \cap (\mathcal{S} \cup \mathcal{Z}) \ge d+2$. Then, for all points $P \in \mathbb{P}^N$ as above we have that:

$$\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2, \qquad \mathcal{Z} = \mathcal{Z}_1 \sqcup \mathcal{S}_2,$$

where $\mathcal{S}_1 = \mathcal{S} \cap C_d$, $\mathcal{Z}_1 = \mathcal{Z} \cap C_d$ and $\mathcal{S}_2 = (\mathcal{S} \cap \mathcal{Z}) \setminus \mathcal{S}_1$.
Moreover C_d, $\mathcal{S}_2$ and $\mathcal{Z}$ are unique, $\deg(\mathcal{Z}) = \mathrm{sbr}(P)$, $\deg(\mathcal{Z}_1)+\deg(\mathcal{S}_1) = d+2$, $\mathcal{Z}_1 \cap \mathcal{S}_1 = \emptyset$ and $\mathcal{Z}$ is a the unique zero-dimensional subscheme N of $X_{m,d}$ such that $\deg(N) \le \mathrm{sbr}(P)$ and $P \in \langle N \rangle$.

In the language of polynomials, Theorem 1 can be rephrased as follows.

Corollary 1. *Let $F \in K[x_0, \dots, x_m]_d$ be such that $\mathrm{sbr}(F)+\mathrm{sr}(F) \le 2d+1$ and $\mathrm{sbr}(F) < \mathrm{sr}(F)$. Then there are an integer $0 \le t \le (d-1)/2$, linear forms $L_1, L_2, M_1, \dots, M_t \in K[x_0, \dots, x_m]_1$, and a form $Q \in K[L_1, L_2]_d$ such that $F = Q + M_1^d + \cdots + M_t^d$, $t \le \mathrm{sbr}(F)+\mathrm{sr}(F)-d-2$, and $\mathrm{sr}(F) = \mathrm{sr}(Q)+t$.*
Moreover the line $\langle L_1, L_2\rangle$, the forms $M_1, \dots, M_t$ and Q such that $Q = \sum_{i=1}^q l_i^{d-d_i} m_i$ with l_i's linear forms and m_i's forms of degree d_i such that $\sum(d_i+1) = s-t$, are uniquely determined by F.

An analogous corollary can be stated for symmetric tensors.

Corollary 2. *Let $T \in S^dV^*$ be such that $\mathrm{sbr}(T)+\mathrm{sr}(T) \le 2d+1$ and $\mathrm{sbr}(T) < \mathrm{sr}(T)$. Then there are an integer $0 \le t \le (d-1)/2$, vectors $v_1, v_2, w_1, \dots, w_t \in S^1V^*$, and a symmetric tensor $v \in S^d(\langle v_1, v_2\rangle)$ such that $T = v + w_1^{\otimes d} + \cdots + w_t^{\otimes d}$, $t \le \mathrm{sbr}(T)+\mathrm{sr}(T)-d-2$, and $\mathrm{sr}(T) = \mathrm{sr}(v)+t$. Moreover the line $\langle v_1, v_2\rangle$, the vectors $v_1, \dots, v_t$ and the tensor v such that $v = \sum_{i=1}^q u_i^{\otimes(d-d_i)} \otimes z_i$ with $u_i \in \langle v_1, v_2\rangle$ and $z_i \in S^{d_i}(\langle v_1, v_2\rangle)$ such that $\sum(d_i+1) = s-t$, are uniquely determined by T.*

Moreover in Theorem 2 and in Corollary 4, by introducing the notion of linearly general position of a scheme (Definition 1), we can also extend to a geometric description the condition for the uniqueness of the scheme $\mathcal{Z}$ of Theorem 1. We can rephrase their contents in therms of homogeneous polynomials and symmetric tensors in the following Corollary.

Corollary 3. *Fix integers $m \ge 2$ and $d \ge 4$. Fix F an homogeneous polynomial in $m+1$ variables of degree d ($T \in S^dV$ respectively) such that $\mathrm{sbr}(F) \le d$ ($\mathrm{sbr}(T) \le d$). Let $Z \subset \mathbb{P}^m$ be* *any smoothable zero-dimensional scheme such that* $\nu_d(Z)$ *computes* $\mathrm{sbr}(F)$ *(*$\mathrm{sbr}(T)$*). Assume that* Z *is in linearly general position. Then* Z *is the unique scheme computing* $\mathrm{sbr}(P)$ *(*$\mathrm{sbr}(F)$*).*

1. Proofs

The existence of such a scheme $\mathcal{Z}$ was known from [3] and [4] (see Remark 1 of [2]).

Lemma 1. *Fix integers* $m \ge 2$ *and* $d \ge 2$*, a line* $\ell \subset \mathbb{P}^m$ *and any finite set* $E \subset \mathbb{P}^m \setminus \ell$ *such that* $\sharp(E) \le d$*. Then* $\dim(\langle \nu_d(E) \rangle) = \sharp(E) - 1$ *and* $\langle \nu_d(\ell) \rangle \cap \langle \nu_d(E) \rangle = \emptyset$*.*

Proof. Since $h^0(\ell \cup E, \mathcal{O}_{\ell \cup E}(d)) = d + 1 + \sharp(E)$, to get both statements it is sufficient to prove $h^1(\mathcal{I}_{\ell \cup E}(d)) = 0$. Let $H \subset \mathbb{P}^m$ be a general hyperplane containing ℓ. Since E is finite and H is general, we have $H \cap E = \emptyset$. Hence the residual exact sequence of the scheme $\ell \cup E$ with respect to the hyperplane H is the following exact sequence on $\mathbb{P}^m$:

$$0 \to \mathcal{I}_E(d-1) \to \mathcal{I}_{\ell \cup E}(d) \to \mathcal{I}_{\ell,H}(d) \to 0 \tag{2}$$

Since $h^1(\mathcal{I}_E(d-1)) = h^1(H, \mathcal{I}_{\ell,H}(d)) = 0$, we get the lemma. □

Proof of Theorem 1. All the statements are contained in [2], Theorem 1, except the uniqueness of $\mathcal{Z}$, that $\deg(\mathcal{Z}_1) + \deg(\mathcal{S}_1) = d + 2$ and that $\mathcal{Z}_1 \cap \mathcal{S}_1 = \emptyset$. Let $\ell \subset \mathbb{P}^m$ be the line such that $\nu_d(\ell) = C_d$. Take $Z, S, Z_i, S_i \subset \mathbb{P}^m$, $i = 1, 2$, such that $\nu_d(Z) = \mathcal{Z}$, $\nu_d(S) = \mathcal{S}$, $\nu_d(Z_i) = \mathcal{Z}_i$, and $\nu_d(S_i) = \mathcal{S}_i$. Assume the existence of another subscheme $\mathcal{Z}' \subset X_{m,d}$ such that $P \in \langle \nu_d(\mathcal{Z}') \rangle$ and $\deg(\mathcal{Z}') \le \mathrm{sbr}(P)$. Set $\mathcal{Z}_1' := \mathcal{Z}' \cap C_d$. The proof of [2], Theorem 1, gives $\mathcal{Z}' = \mathcal{Z}_1' \sqcup S_2$. Since C_d is a smooth curve, $\mathcal{Z}_1 \cup \mathcal{Z}_1' \subset C_d$, $S_2 \cap \ell = \emptyset$, and $\mathcal{Z} \cup \mathcal{Z}' = (\mathcal{Z}_1 \cup \mathcal{Z}_1') \sqcup S_2$, the schemes $\mathcal{Z}$ and $\mathcal{Z}'$ are curvilinear. Hence all subschemes of $\mathcal{Z}$ and $\mathcal{Z}'$ are smoothable. Hence any subscheme of either $\mathcal{Z}$ or $\mathcal{Z}'$ may be used to compute the border rank of some point of $\mathbb{P}^N$. Since $\deg(\ell \cap (Z \cup S)) \ge d + 2$, $\nu_d((Z \cup S) \cap \ell)$ spans $\langle C_d \rangle$. Lemma 1 implies $\langle C_d \rangle \cap \langle \mathcal{S}_2 \rangle = \emptyset$. Since $P \in \langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle$ and $\sharp(S) = \mathrm{sr}(P)$, we have $P \notin \langle \mathcal{A} \rangle$ for any $\mathcal{A} \subsetneq \mathcal{S}$. Therefore we get that $\langle \{P\} \cup \mathcal{S}_2 \rangle \cap \langle \mathcal{S}_1 \rangle$ is a unique point. Call P_1 this point. Similarly, $\langle \mathcal{Z}_1 \rangle \cap \langle \mathcal{S}_2 \rangle$ is a unique point and we call it P_2. Since $\langle C_d \rangle \cap \langle \mathcal{S}_2 \rangle = \emptyset$, the set $\langle C_d \rangle \cap \langle \{P\} \cup \mathcal{S}_2 \rangle$ is at most one point. Since $P_i \in \langle C_d \rangle \cap \langle \{P\} \cup \mathcal{S}_2 \rangle$, $i = 1, 2$, we have $P_1 = P_2$ and $\{P_1\} = \langle C_d \rangle \cap \langle \{P\} \cup \mathcal{S}_2 \rangle$. Since $P_1 = P_2$ and $P_1 \in \langle \mathcal{S}_1 \rangle$ and $P_2 \in \langle \mathcal{Z}_1' \rangle$, we have $P_1 \in \langle \mathcal{Z}_1' \rangle \cap \langle \mathcal{S}_1 \rangle$. Take any $E \subseteq \mathcal{Z}_1$ such that $P_1 \in \langle E \rangle$. Since $P \in \langle \{P_1\} \cup \mathcal{S}_2 \rangle \subseteq \langle E \cup \mathcal{S}_2 \rangle$ and $P \notin \langle \mathcal{U} \rangle$ for any $\mathcal{U} \subsetneq \mathcal{Z}$, we get $E \cup \mathcal{S}_2 = \mathcal{Z}$. Hence $E = \mathcal{Z}_1$. Therefore $\mathcal{Z}_1$ computes $\mathrm{sbr}(P_1)$ with respect to C_d. Similarly, $\mathcal{Z}_1'$ computes $\mathrm{sr}(P_2)$ with respect to the same rational normal curve C_d. Since $P_1 = P_2$, we have $\mathcal{Z}_1' = \mathcal{Z}_1$ (as for all curves we get the uniqueness of $\mathcal{Z}_1$ for $\mathrm{sbr}(P_1) \le \lfloor (d+2)/2 \rfloor$). Since $\mathrm{sbr}(P_1) \ne \mathrm{sr}(P_1)$, a theorem of Sylvester gives $\mathrm{sbr}(P_1) + \mathrm{sr}(P_1) = d + 2$, i.e. $\deg(\mathcal{Z}_1) + \deg(\mathcal{S}_1) = d + 2$.

Definition 1. A scheme $Z \subset \mathbb{P}^m$ is said to be in *linearly general position* if for every linear subspace $R \subsetneq \mathbb{P}^m$ we have $\deg(R \cap Z) \le \dim(R) + 1$.

Notice that the next theorem is false if either $d = 2$ or $m = 1$.

Theorem 2. *Fix integers* $m \ge 2$ *and* $d \ge 4$*. Fix* $P \in \mathbb{P}^N$ *Let* $Z \subset \mathbb{P}^m$ *be any smoothable zero-dimensional scheme such that* $P \in \langle \nu_d(Z) \rangle$ *and* $P \notin \langle \nu_d(Z') \rangle$*. Assume* $\deg(Z) \le d$ *and that* Z *is in linearly general position. Then* Z *is the unique scheme* $Z' \subset \mathbb{P}^m$ *such that* $\deg(Z') \le d$ *and* $P \in \langle \nu_d(Z') \rangle$*. Moreover* $\nu_d(Z)$ *computes* $\mathrm{sbr}(P)$*.*

Proof. The existence of a scheme computing $\mathrm{sbr}(P)$ follows from =[2], Remark ++= and the assumption " $\mathrm{sbr}(P) \le d$ ". Fix any scheme $Z' \subset \mathbb{P}^m$ such that $Z' \ne Z$, $\deg(Z') \le d$, $P \in \langle \nu_d(Z') \rangle$, and $P \notin \langle \nu_d(Z'') \rangle$ for any $Z'' \subsetneq Z'$. Assume $Z' \ne Z$. Since $\deg(Z \cup Z') \le 2d + 1$ and $h^1(\mathbb{P}^m, \mathcal{I}_{Z \cup Z'}(d)) > 0$ ([2], Lemma 1), there is a line $D \subset \mathbb{P}^m$ such that $\deg(D \cap (Z \cup Z')) \ge d + 2$. Since Z is in linearly general position and $m \ge 2$, we have $\deg(Z \cap D) \le 2$. Hence $\deg(Z' \cap D) \ge d$.

Hence $\deg(Z') = d$. Since $\deg(Z') = d$, we get $Z' \subset D$. Hence $P \in \langle \nu_d(D) \rangle$. Hence $\mathrm{sr}(P) = d$. As for all curves we get $\mathrm{sbr}(P) \le \lfloor (d+2)/2 \rfloor$. Since $\deg(Z') = d$, we assumed $\deg(Z') \le \mathrm{sbr}(P)$, contradicting the assumption $d \ge 4$. $\square$

Corollary 4. *Fix integers $m \ge 2$ and $d \ge 4$. Fix $P \in \mathbb{P}^N$ such that $\mathrm{sbr}(P) \le d$. Let $Z \subset \mathbb{P}^m$ be any smoothable zero-dimensional scheme such that $\nu_d(Z)$ computes $\mathrm{sbr}(P)$. Assume that Z is in linearly general position. Then Z is the unique scheme computing $\mathrm{sbr}(P)$.*

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