Three-dimensional Fire codes

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Abstract. Binary cyclic three–dimensional array codes that can correct three–dimensional bursts (or clusters) of errors are considered. The class of binary cyclic three–dimensional burst–error–correcting array codes, called three–dimensional Fire codes, is constructed. Several important properties such as the burst–error–correcting capability and the positions of the parity–check symbols are presented. Also, encoding and decoding algorithms are given.

Keywords: Burst–error–correcting codes, three–dimensional array codes, cyclic Fire codes, encoding and decoding

1 Introduction

In some memory devices, information is stored in three dimensions. In such cases, errors usually take the form of three–dimensional bursts. For example, correction of three–dimensional error bursts is required in holographic storage. Three–dimensional array codes are very suitable for correcting error bursts in such data structures.

For integers $n_l,\ l = 1, 2, 3$ we consider the linear space $V(n_1, n_2, n_3)$ of all binary three–dimensional $n_1 \times n_2 \times n_3$ arrays. A linear $k-$dimensional $(k \leq n_1 n_2 n_3)$ subspace $C(n_1, n_2, n_3)$ of the space $V(n_1, n_2, n_3)$ is called a linear binary three–dimensional array $[n_1 \times n_2 \times n_3,\ k]$ code of size $n_1 \times n_2 \times n_3$ with k information symbols and $r = n_1 n_2 n_3 - k$ parity-check symbols. Thus, a code word of the binary linear three–dimensional array code $C(n_1, n_2, n_3)$ is a three-dimensional array $c = (c_{i,j,h})$ where $c_{i,j,h} = 0, 1$ for $i = 0, 1, \ldots, n_1 - 1;\ j = 0, 1, \ldots, n_2 - 1;\ h = 0, 1, \ldots, n_3 - 1.$

A linear three–dimensional array code $C(n_1, n_2, n_3)$ is said to be a three–dimensional cyclic array code if and only if any cyclic shift of any codeword $c = (c_{i,j,h})$ is also a codeword of the code $C(n_1, n_2, n_3)$. The cyclic shift of $c = (c_{i,j,h})$ is $n_1 \times n_2 \times n_3$ array $c = (c_{i+l_1,j+l_2,h+l_3})$ where l_1, l_2, l_3 are arbitrary integers and the subscripts m, t, u of $c = (c_{m,t,u})$ are calculated modulo $n_1, n_2, n_3,$ respectively.

A three–dimensional array $e = (e_{i,j,h})$ of size $n_1 \times n_2 \times n_3$ is called a three–dimensional parallelepipedal $b_1 \times b_2 \times b_3$–burst if

1) $e_{u_1+p_1,u_2+p_2,u_3+p_3} = 0$ or 1 for some $0 \leq u_l < n_l - b_l$ and any $0 \leq p_l < b_l,\ l = 1, 2, 3;$

2) $e_{u_1+p_1,u_2,u_3}e_{u_1,u_2+p_2,u_3}e_{u_1,u_2,u_3+p_3} = 1$ for at least one triple p_1, p_2, p_3;

3) the others $e_{i,j,h} = 0$.

Thus, a $b_1 \times b_2 \times b_3$–burst $e = (e_{i,j,h})$ can contain nonzero symbols only within submatrix $e_p = (e_{u_1+p_1,u_2+p_2,u_3+p_3})$, $p_l = 0, 1, \ldots, b_l - 1$, $l = 1, 2, 3$.

The triple u_1, u_2, u_3 and submatrix e_p are called the starting position and the pattern of a three–dimensional parallelepipedal $b_1 \times b_2 \times b_3$–burst e, respectively.

A cyclic shift of a $b_1 \times b_2 \times b_3$–burst is called a cyclic $b_1 \times b_2 \times b_3$–burst. Of course, a $b_1 \times b_2 \times b_3$–burst is a cyclic burst, but the converse does not always hold. The starting position and the pattern of the cyclic burst are not necessarily unique. A necessary and sufficient condition for all cyclic $b_1 \times b_2 \times b_3$–bursts $e_{i,j,h}$ to have unique starting positions and patterns is $n_l \geq 2b_l - 1, l = 1, 2, 3$ (see Lemma 1 in [1]).

The three–dimensional $b_1 \times b_2 \times b_3$–bursts have a parallelepipedal shape. In some papers (see, for example, [2]), it is assumed that bursts can have an arbitrary shape. The approach in these papers is to use interleaving schemes. In this paper we consider parallelepipedal bursts.

A linear three–dimensional array code is a $b_1 \times b_2 \times b_3$–burst–error–correcting code if it has no nonzero codeword which is a $b_1 \times b_2 \times b_3$–burst or a sum of two $b_1 \times b_2 \times b_3$–bursts. A linear three–dimensional cyclic array a $b_1 \times b_2 \times b_3$–burst–error–correcting code corrects cyclic $b_1 \times b_2 \times b_3$–bursts of errors.

The redundancy (the number of parity–check symbols) of a linear three–dimensional array $b_1 \times b_2 \times b_3$–burst–error–correcting code satisfies the inequality

$$r \geq 2b_1b_2b_3. \tag{1}$$

There are some papers, devoted completely or partly three–dimensional array burst–error–correcting codes ([2] — [6]). In [2] the construction of three–dimensional interleaving schemes is given. In [3] a class of linear three–dimensional $b_1 \times b_2 \times b_3$–burst–error–correcting array codes with the redundancy $4b_1b_2b_3$ is given. In [4], [5] the constructions of linear three–dimensional $b_1 \times b_2 \times b_3$–burst–error–correcting array codes with the small excess redundancy are presented. In [6] the existence of almost optimal m–dimensional $(m \geq 3)$ burst–error–correcting array codes is shown.

In this paper we consider binary linear cyclic three–dimensional burst–error–correcting array codes. The class of binary cyclic three–dimensional burst–error–correcting array codes, called three–dimensional Fire codes, is constructed. Several important properties such as the burst–error–correcting capability and the positions of the parity–check symbols are presented. Also, encoding and decoding algorithms are given.

2 Three-Dimensional Cyclic Array Codes and Bursts

An one–dimensional binary cyclic (n, k) code is defined as an ideal of the residue class ring $R_n = F_2[x]/\langle x^n - 1\rangle$ of the polynomial ring $F_2[x]$ over

$F_2 = GF(2)$ with respect to an ideal $\langle x^n - 1\rangle$ [7]. The polynomial representation of codewords is used for description of cyclic two–dimensional array codes [1], [8]—[12]. For the description of cyclic three–dimensional array codes it is also convenient to use a polynomial representation of codewords.

Let $F_2[x, y, z]$ be a polynomial ring in three variables x, y, z over F_2. Let

$$c(x, y, z) = \sum_{i\in I}\sum_{j\in J}\sum_{h\in H} c_{i,j,h} x^i y^j z^h$$

where I, J, H are finite sets of integers.

We will suppose that monomials $c_{i,j,h}x^i y^j z^h$ in $c(x, y, z)$ are lexicographically ordered [16]. Denote by $\deg_x c(x, y, z)$, $\deg_y c(x, y, z)$ and $\deg_z c(x, y, z)$ the maximal degrees of variables x, y and z in monomials $c_{i,j,h}x^i y^j z^h \in c(x, y, z)$, respectively.

If $f_1(x, y, z), \ldots, f_s(x, y, z)$ are polynomials in $F_2[x, y, z]$, then

$$\langle f_1(x, y, z), \ldots, f_s(x, y, z)\rangle = \{ \sum_{l=1}^{s} a_l(x, y, z) f_l(x, y, z) : a_l(x, y, z) \in F_2[x, y, z] \} \tag{2}$$

is the ideal in $F_2[x, y, z]$ and the set $\{f_1(x, y, z), \ldots, f_s(x, y, z)\}$ is the basis of the ideal.

Let

$$R_{n_1,n_2,n_3} = F_2[x, y, z]/\langle x^{n_1} - 1,\ \ y^{n_2} - 1,\ \ z^{n_3} - 1\rangle$$

be a residue class ring of $F_2[x, y, z]$ with respect to an ideal $\langle x^{n_1} - 1, y^{n_2} - 1, z^{n_3} - 1\rangle$. All distinct polynomials $c(x, y, z) \in F_2[x, y, z]$, such that $\deg_x c(x, y, z) < n_1$, $\deg_y c(x, y, z) < n_2$ and $\deg_z c(x, y, z) < n_3$, belong to distinct residue classes of R_{n_1,n_2,n_3}. A residue class of R_{n_1,n_2,n_3} will be represented by such an unique polynomial.

The map

$$c = (c_{i,j,h}) \longmapsto c(x, y, z) = \sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}\sum_{h=0}^{n_3-1} c_{i,j,h} x^i y^j z^h$$

defines an isomorphism between V_{n_1,n_2,n_3} and R_{n_1,n_2,n_3}. If C_{n_1,n_2,n_3} is a three-dimensional cyclic array code and $c = (c_{i,j,h}) \in C_{n_1,n_2,n_3}$, then the cyclic shift $c' = (c_{i+l_1,j+l_2,h+l_3}) \in C_{n_1,n_2,n_3}$. The cyclic shift $c' = (c_{i+l_1,j+l_2,h+l_3})$ corresponds the polynomial $c'(x, y, z) = x^{l_1} y^{l_2} z^{l_3} c(x, y, z)$. The residue class ring R_{n_1,n_2,n_3} is a $n_1n_2n_3$–dimensional linear space over F_2.

Thus, a linear cyclic three–dimensional array code C_{n_1,n_2,n_3} is an ideal in the residue class ring $R_{n_1,n_2,n_3} = F_2[x, y, z]/\langle x^{n_1} - 1,\ \ y^{n_2} - 1,\ \ z^{n_3} - 1\rangle$.

In the polynomial representation of a linear cyclic three–dimensional array code, a polynomial

$$e(x, y, z) = \sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}\sum_{h=0}^{n_3-1} e_{i,j,h} x^i y^j z^h$$

is a $b_1 \times b_2 \times b_3$–burst if

$$e(x, y, z) = x^{u_1} y^{u_2} z^{u_3} b(x, y, z)$$

for some $0 \leq u_l < n_l - b_l,\ l = 1, 2, 3$ and a polynomial $b(x, y, z)$, such that $\deg_x b(x, y, z) < b_1$, $\deg_y b(x, y, z) < b_2$, $\deg_y b(x, y, z) < b_3$ and $b(x, 0, 0) \neq 0$, $b(0, y, 0) \neq 0$, $b(0, 0, z) \neq 0$.

The polynomial $e'(x, y, z)$ is a cyclic $b_1 \times b_2 \times b_3$–burst if $e'(x, y, z) = x^{v_1} y^{v_2} z^{v_3} e(x, y, z)$ for some $0 \leq v_l < n_l,\ l = 1, 2, 3$ and $e(x, y, z)$ is a $b_1 \times b_2 \times b_3$–burst.

3 Definition of Three-Dimensional Fire Codes

Two approaches are used for description of binary two-dimensional cyclic codes of area $n_1 \times n_2$.

First approach is the concept of common zeros of code polynomials of cyclic two-dimensional codes [10], [11]. The second approach is the representation of a cyclic two-dimensional code of area $n_1 \times n_2$ as an ideal, that is defined by means of a certain basis [12]. In this paper we use the second approach for the definition and constructing cyclic three-dimensional Fire codes.

First, let us recall ordinary Fire codes [13] – [15]. The binary cyclic Fire code is defined as cyclic code generated by the polynomial $g(x) = p(x)(x^c - 1)$ where $p(x)$ is an irreducible polynomial over $GF(2)$ of degree m whose roots have order e and c is a positive integer not divisible by e. The code is of length $n = LCM(c, e)$ and is capable of correcting a burst of length b, if $b \leq (c + 1)/2$ and $b \leq m$. A cyclic Fire code is an ideal in R_n and can be considered as the intersection of ideals $\langle x^c - 1\rangle$ and $\langle p(x)\rangle$.

By analogy with one-dimensional binary cyclic Fire codes we consider three-dimensional cyclic array codes as ideals, which are the intersections of ideals $\langle x^{c_1} - 1,\ y^{c_2} - 1,\ z^{c_3} - 1\rangle$ and $\langle p_1(x),\ p_2(y),\ p_3(z)\rangle$ in the residue class ring $R_{n_1,n_2,n_3} = F_2[x, y, z]/ \langle x^{n_1} - 1,\ y^{n_2} - 1,\ z^{n_3} - 1\rangle$.

Lemma 1. *Let* $A_{g_1,g_2,g_3}(x, y, z) = \langle g_1(x),\ g_2(y),\ g_3(z)\ \rangle$ *be an ideal in* $R_{n_1,n_2,n_3} = F_2[x, y, z]/ \langle x^{n_1} - 1,\ y^{n_2} - 1,\ z^{n_3} - 1.\rangle$ *Then*

(i) $g_1(x)$ *divides* $x^{n_1} - 1$, $g_2(y)$ *divides* $y^{n_2} - 1$ *and* $g_3(z)$ *divides* $z^{n_3} - 1$;

(ii) $a(x, y, z) \in A_{g_1,g_2,g_3}(x, y, z)$, *if and only if*

$$a(x, y, z) = a_1(x, y, z)g_1(x) + a_2(x, y, z)g_2(y) + a_3(x, y, z)g_3(z), \tag{3}$$

where $a_l(x, y, z) \in R_{n_1,n_2,n_3}$;

(iii) for any polynomial $a(x, y, z) \in R_{n_1,n_2,n_2}$ *there exists a unique polynomial* $r(x, y, z) \in R_{n_1,n_2,n_3}$, *such that* $\deg_x r(x, y, z) < \deg_x g_1(x)$, $\deg_y r(x, y, z) < \deg_y g_2(y)$, $\deg_z r(x, y, z) < \deg_z g_3(z)$ *and*

$$a(x, y, z) = a_1(x, y, z)g_1(x) + a_2(x, y, z)g_2(y) + a_3(x, y, z)g_3(z) + r(x, y, z), \tag{4}$$

$a_l(x, y, z) \in R_{n_1,n_2,n_3}$, $l = 1, 2, 3$.

The statement (ii) of Lemma 1 is equivalent to the statement, that the basis $\{g_1(x),\ \ g_2(y),\ \ g_3(z)\}$ of the ideal $A_{g_1,g_2,g_3}(x,y,z)$ is a Gröbner basis [16].

Theorem 1. *Let* $p_l(u)$ *be an irreducible polynomial over* F_2 *of degree* m_l *whose roots have order* e_l, $l = 1,2,3$. *Let* c_l *be a positive integer not divisible by* e_l *and* $n_l = LCM(c_l, e_l)$. *Then the intersection* $C_{n_1,n_2,n_3}(x,y,z)$ *of the ideals*

$$A_{c_1,c_2,c_3}(x,y,z) = \langle\ x^{c_1} - 1,\ y^{c_2} - 1,\ \ z^{c_3} - 1\ \rangle$$

and

$$A_{p_1,p_2,p_3}(x,y,z) = \langle p_1(x),\ p_2(y), p_3(z)\ \rangle$$

in the residue class ring $R_{n_1,n_2,n_3} = F_2[x,y,z]/\langle x^{n_1} - 1,\ \ y^{n_2} - 1,\ \ z^{n_3} - 1\rangle$ *is a linear cyclic three–dimensional array code of size* $n_1 \times n_2 \times n_3$ *correcting* $b_1 \times b_2 \times b_3$*–bursts of errors if* $b_l \le (c_l + 1)/2$ *and* $b_l \le m_l$.

The code $C_{n_1,n_2,n_3}(x,y,z)$ is called a binary three–dimensional Fire code.

Let $m_2 = m_3 = 1$ and $c_2 = c_3 = 1$. Then $C_{n_1,n_2,n_3}(x,y,z)$ is reduced to the binary cyclic one–dimensional Fire code, generated by $p_1(x)(x^{c_1} - 1)$.

4 Encoding Three-Dimensional Fire Codes

Let code $C_{n_1,n_2,n_3}(x,y,z)$ be a binary three–dimensional Fire code and

$$c(x,y,z) = \sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}\sum_{h=0}^{n_3-1} c_{i,j,h}x^iy^jz^h$$

be a codeword.

Let $P_1 = \{i,j,h : 0 \le i < c_1,\ 0 \le j < c_2,\ 0 \le h < c_3\}$; $P_2 = \{i,j,h : c_1 \le i < c_1 + m_1,\ 0 \le j < m_2,\ 0 \le h < m_3\}$; $P_3 = \{i,j,h : 0 \le i < m_1,\ c_2 \le j < c_2 + m_2,\ 0 \le h < m_3\}$; $P_4 = \{i,j,h : o \le i < m_1,\ 0 \le j < m_2,\ 0 \le h < c_3 + m_3\}$.

We put coefficients $c_{i,j,h}$ of the code polynomial $c(x,y,z)$ to be information symbols if triple $i,j,h \notin P_l$ and parity–check symbols if triple $i,j,h \in P_l$, $l = 1,2,3,4$.

Let

$$c_0(x,y,z) = \sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}\sum_{h=0}^{n_3-1} c^{(0)}_{i,j,h}x^iy^jz^h \tag{5}$$

be a polynomial whose coefficients $c^{(0)}_{i,j,h}$, $i,j,h \notin P_l$ are information symbols of the code polynomial and other coefficients are $c^{(0)}_{i,j,h} = 0$, $i,j,h \in P_l$, $l = 1,2,3,4$.

Dividing the polynomial $c_0(x,y,z)$ by the basis s $\{\ x^{c_1}-1,\ y^{c_2}-1,\ \ z^{c_3}-1\ \}$ of the ideal $A_{c_1,c_2,c_3}(x,y,z)$, we obtain

$$c_0(x,y,z) = a_1(x,y,z)(x^{c_1}-1)+a_2(x,y,z)(y^{c_2}-1)+a_3(x,y,z)(z^{c_3}-1)+r_0(x,y,z), \tag{6}$$

where $\deg_x r_0(x,y,z) < c_1$, $\deg_y r_0(x,y,z) < c_2$ and $\deg_z r_0(x,y,z) < c_3$.

Further, dividing the polynomials $a_l(x,y,z)$, $l = 1,2,3$ by the basis $\{p_1(x), p_2(y), p_3(z)\}$ of the ideal $A_{p_1,p_2,p_3}(x,y,z)$, we obtain

$$a_l(x,y,z) = a_1^{(l)}(x,y,z)p_1(x) + a_2^{(l)}(x,y,z)p_2(y) + a_3^{(l)}(x,y,z)p_3(z) + r_l(x,y,z), \tag{7}$$

where $\deg_x r_l(x,y,z) < m_1$, $\deg_y r_l(x,y,z) < m_2$ and $\deg_z r_l(x,y,z) < m_3$.

Combining (6) and (7), we get

$$\begin{aligned} c_0(x,y,z) - r_1(x,y,z)x^{c_1} - r_2(x,y,z)y^{c_2} - r_3(x,y,z)z^{c_3} - r_0(x,y,z) + \\ r_1(x,y,z) + r_2(x,y,z) + r_3(x,y,z) = (a_1(x,y,z) - r_1(x,y,z))(x^{c_1} - 1) + \\ (a_2(x,y,z) - r_2(x,y,z))(y^{c_2} - 1) + (a_3(x,y,z) - r_3(x,y,z))(z^{c_3} - 1) \end{aligned} \tag{8}$$

Since the right–hand side of (8) is a codeword of the code $C_{n_1,n_2,n_3}(x,y,z)$, the left–hand side of (8) is also a codeword of the code. Thus, we get the codeword

$$c(x,y,z) = c_0(x,y,z) + c_1(x,y,z) \tag{9}$$

where the coefficients of the polynomial $c_0(x,y,z)$ are the information symbols and the coefficients of the polynomial

$$\begin{aligned} c_1(x,y,z) = r_1(x,y,z) + r_2(x,y,z) + r_3(x,y,z) - r_0(x,y,z) - \\ r_1(x,y,z)x^{c_1} - r_2(x,y,z)y^{c_2} - r_3(x,y,z)z^{c_3} \end{aligned} \tag{10}$$

are the parity–check symbols of the codeword $c(x,y,z)$.

5 Decoding Three-Dimensional Fire Codes

Let

$$c'(x,y,z) = c(x,y,z) + e(x,y,z), \tag{11}$$

where $c(x,y,z)$ is a codeword of the three-dimensional Fire code and $e(x,y,z)$ is $b_1 \times b_2 \times b_3$–burst of errors.

The syndrome of $c'(x,y,z)$ is

$$S_{c'} = S_e = (r_c(x,y,z),\ r_p(x,y,z)), \tag{12}$$

where the polynomials $r_c(x,y,z)$ and $r_p(x,y,z)$ are the remainders of dividing $c'(x,y,z)$ by the basis $\{\ x^{c_1}-1,\ y^{c_2}-1,\ \ z^{c_3}-1\ \}$ of the ideal $A_{c_1,c_2,c_3}(x,y,z)$ and the basis $\{p_1(x), p_2(y), p_3(z)\}$ of the ideal $A_{p_1,p_2,p_3}(x,y,z)$, respectively.

Thus,

$$c'(x,y,z) = a_1(x,y,z)(x^{c_1}-1)+a_2(x,y,z)(y^{c_2}-1)+a_3(x,y,z)(z^{c_3}-1)+r_c(x,y,z), \tag{13}$$

and

$$c'(x,y,z) = d_1(x,y,z)p_1(x)+d_2(x,y,z)p_2(y)+d_3(x,y,z)p_3(z)+r_p(x,y,z). \tag{14}$$

where $\deg_x r_c(x,y,z) < c_1$, $\deg_y r_c(x,y,z) < c_2$, $\deg_z r_c(x,y,z) < c_3$ and $\deg_x r_p(x,y,z) < m_1$, $\deg_y r_p(x,y,z) < m_2$, $\deg_z r_p(x,y,z) < m_3$.

A decoding algorithm for three-dimensional Fire Codes can be described as follows.

1. Compute the syndrome $S_{c'} = S_e = (r_c(x,y,z),\ r_p(x,y,z))$ of the received word $c'(x,y,z)$.

2. If $S_{c'} = (0,0)$, there are no errors in $c'(x,y,z)$ and $c'(x,y,z)$ is a codeword.

3. If $\deg_x r_c(x,y,z) < b_1$, $\deg_y r_c(x,y,z) < b_2$, $\deg_z r_c(x,y,z) < b_3$, $r_c(x,0,0) \neq 0$, $r_c(0,y,0) \neq 0$, $r_c(0,0,z) \neq 0$ and $r_c(x,y,z) = r_p(x,y,z)$, then there is the burst error $e(x,y,z) = r_c(x,y,z)$ in $c'(x,y,z)$ and $c(x,y,z) = c'(x,y,z) - e(x,y,z)$ is a codeword.

4. For $i \in I$, $j \in J$, $h \in H$ find the the remainders $r_c^{(i,j,h)}(x,y,z)$ and $r_c^{(i,j,h)}(x,y,z)$ of dividing the polynomial $x^i y^j z^h c'(x,y,z)$ by the basis $\{\ x^{c_1} - 1,\ y^{c_2} - 1,\ \ z^{c_3} - 1\ \}$ of the ideal $A_{c_1,c_2,c_3}(x,y,z)$ and the basis $\{p_1(x),\ p_2(y), p_3(z)\}$ of the ideal $A_{p_1,p_2,p_3}(x,y,z)$, respectively.

If $\deg_x r_c^{(i,j,h)}(x,y,z) < b_1$, $\deg_y r_c^{(i,j,h)}(x,y,z) < b_2$, $\deg_z r_c^{(i,j,h)}(x,y,z) < b_3$, $r_c^{(i,j,h)}(x,0,0) \neq 0$, $r_c^{(i,j,h)}(0,y,0) \neq 0$, $r_c^{(i,j,h)}(0,0,z) \neq 0$ and $r_c^{(i,j,h)}(x,y,z) = r_p^{(i,j,h)}(x,y,z)$, then $e(x,y,z) = x^{n_1-i} y^{n_2-j} z^{n_3-h} r_c^{(i,j,h)}(x,y,z)$ and $c(x,y,z) = c'(x,y,z) - e(x,y,z)$.

5. In all other cases, the received word $c'(x,y,z)$ has uncorrectable errors.

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