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Three-Dimensional Fire Codes

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Abstract. Binary cyclic three-dimensional array codes that can correct three-dimensional bursts (or clusters) of errors are considered. The class of binary cyclic three-dimensional burst-error-correcting array codes, called three-dimensional Fire codes, is constructed. Several important properties such as the burst-error-correcting capability and the positions of the parity-check symbols are presented. Also, encoding and decoding algorithms are given.

Keywords: Burst-error-correcting codes, three-dimensional array codes, cyclic Fire codes, encoding and decoding

1 Introduction

In some memory devices, information is stored in three dimensions. In such cases, errors usually take the form of three-dimensional bursts. For example, correction of three-dimensional error bursts is required in holographic storage. Three-dimensional array codes are very suitable for correcting error bursts in such data structures.

For integers n_l , $l = 1, 2, 3$ we consider the linear space $V(n_1, n_2, n_3)$ of all binary three-dimensional $n_1 \times n_2 \times n_3$ arrays. A linear k -dimensional ($k \leq n_1 n_2 n_3$) subspace $C(n_1, n_2, n_3)$ of the space $V(n_1, n_2, n_3)$ is called a linear binary three-dimensional array $[n_1 \times n_2 \times n_3, k]$ code of size $n_1 \times n_2 \times n_3$ with k information symbols and $r = n_1 n_2 n_3 - k$ parity-check symbols. Thus, a code word of the binary linear three-dimensional array code $C(n_1, n_2, n_3)$ is a three-dimensional array $c = (c_{i,j,h})$ where $c_{i,j,h} = 0, 1$ for $i = 0, 1, \dots, n_1 - 1$; $j = 0, 1, \dots, n_2 - 1$; $h = 0, 1, \dots, n_3 - 1$.

A linear three-dimensional array code $C(n_1, n_2, n_3)$ is said to be a three-dimensional cyclic array code if and only if any cyclic shift of any codeword $c = (c_{i,j,h})$ is also a codeword of the code $C(n_1, n_2, n_3)$. The cyclic shift of $c = (c_{i,j,h})$ is $n_1 \times n_2 \times n_3$ array $c = (c_{i+l_1, j+l_2, h+l_3})$ where l_1, l_2, l_3 are arbitrary integers and the subscripts m, t, u of $c = (c_{m,t,u})$ are calculated modulo n_1, n_2, n_3 , respectively.

A three-dimensional array $e = (e_{i,j,h})$ of size $n_1 \times n_2 \times n_3$ is called a three-dimensional parallelepipedal $b_1 \times b_2 \times b_3$ -burst if

1) $e_{u_1+p_1, u_2+p_2, u_3+p_3} = 0$ or 1 for some $0 \leq u_l < n_l - b_l$ and any $0 \leq p_l < b_l$, $l = 1, 2, 3$;

- 2) $e_{u_1+p_1, u_2, u_3} e_{u_1, u_2+p_2, u_3} e_{u_1, u_2, u_3+p_3} = 1$ for at least one triple p_1, p_2, p_3 ;
 3) the others $e_{i,j,h} = 0$.

Thus, a $b_1 \times b_2 \times b_3$ -burst $e = (e_{i,j,h})$ can contain nonzero symbols only within submatrix $e_p = (e_{u_1+p_1, u_2+p_2, u_3+p_3})$, $p_l = 0, 1, \dots, b_l - 1$, $l = 1, 2, 3$.

The triple u_1, u_2, u_3 and submatrix e_p are called the starting position and the pattern of a three-dimensional parallelepipedal $b_1 \times b_2 \times b_3$ -burst e , respectively.

A cyclic shift of a $b_1 \times b_2 \times b_3$ -burst is called a cyclic $b_1 \times b_2 \times b_3$ -burst. Of course, a $b_1 \times b_2 \times b_3$ -burst is a cyclic burst, but the converse does not always hold. The starting position and the pattern of the cyclic burst are not necessarily unique. A necessary and sufficient condition for all cyclic $b_1 \times b_2 \times b_3$ -bursts $e_{i,j,h}$ to have unique starting positions and patterns is $n_l \geq 2b_l - 1$, $l = 1, 2, 3$ (see Lemma 1 in [1]).

The three-dimensional $b_1 \times b_2 \times b_3$ -bursts have a parallelepipedal shape. In some papers (see, for example, [2]), it is assumed that bursts can have an arbitrary shape. The approach in these papers is to use interleaving schemes. In this paper we consider parallelepipedal bursts.

A linear three-dimensional array code is a $b_1 \times b_2 \times b_3$ -burst-error-correcting code if it has no nonzero codeword which is a $b_1 \times b_2 \times b_3$ -burst or a sum of two $b_1 \times b_2 \times b_3$ -bursts. A linear three-dimensional cyclic array a $b_1 \times b_2 \times b_3$ -burst-error-correcting code corrects cyclic $b_1 \times b_2 \times b_3$ -bursts of errors.

The redundancy (the number of parity-check symbols) of a linear three-dimensional array $b_1 \times b_2 \times b_3$ -burst-error-correcting code satisfies the inequality

$$r \geq 2b_1b_2b_3. \quad (1)$$

There are some papers, devoted completely or partly three-dimensional array burst-error-correcting codes ([2] — [6]). In [2] the construction of three-dimensional interleaving schemes is given. In [3] a class of linear three-dimensional $b_1 \times b_2 \times b_3$ -burst-error-correcting array codes with the redundancy $4b_1b_2b_3$ is given. In [4], [5] the constructions of linear three-dimensional $b_1 \times b_2 \times b_3$ -burst-error-correcting array codes with the small excess redundancy are presented. In [6] the existence of almost optimal m -dimensional ($m \geq 3$) burst-error-correcting array codes is shown.

In this paper we consider binary linear cyclic three-dimensional burst-error-correcting array codes. The class of binary cyclic three-dimensional burst-error-correcting array codes, called three-dimensional Fire codes, is constructed. Several important properties such as the burst-error-correcting capability and the positions of the parity-check symbols are presented. Also, encoding and decoding algorithms are given.

2 Three-Dimensional Cyclic Array Codes and Bursts

An one-dimensional binary cyclic (n, k) code is defined as an ideal of the residue class ring $R_n = F_2[x]/\langle x^n - 1 \rangle$ of the polynomial ring $F_2[x]$ over

$F_2 = GF(2)$ with respect to an ideal $\langle x^n - 1 \rangle$ [7]. The polynomial representation of codewords is used for description of cyclic two-dimensional array codes [1], [8]—[12]. For the description of cyclic three-dimensional array codes it is also convenient to use a polynomial representation of codewords.

Let $F_2[x, y, z]$ be a polynomial ring in three variables x, y, z over F_2 . Let

$$c(x, y, z) = \sum_{i \in I} \sum_{j \in J} \sum_{h \in H} c_{i,j,h} x^i y^j z^h$$

where I, J, H are finite sets of integers.

We will suppose that monomials $c_{i,j,h} x^i y^j z^h$ in $c(x, y, z)$ are lexicographically ordered [16]. Denote by $\deg_x c(x, y, z)$, $\deg_y c(x, y, z)$ and $\deg_z c(x, y, z)$ the maximal degrees of variables x, y and z in monomials $c_{i,j,h} x^i y^j z^h \in c(x, y, z)$, respectively.

If $f_1(x, y, z), \dots, f_s(x, y, z)$ are polynomials in $F_2[x, y, z]$, then

$$\langle f_1(x, y, z), \dots, f_s(x, y, z) \rangle = \left\{ \sum_{l=1}^s a_l(x, y, z) f_l(x, y, z) : a_l(x, y, z) \in F_2[x, y, z] \right\} \quad (2)$$

is the ideal in $F_2[x, y, z]$ and the set $\{f_1(x, y, z), \dots, f_s(x, y, z)\}$ is the basis of the ideal.

Let

$$R_{n_1, n_2, n_3} = F_2[x, y, z] / \langle x^{n_1} - 1, y^{n_2} - 1, z^{n_3} - 1 \rangle$$

be a residue class ring of $F_2[x, y, z]$ with respect to an ideal $\langle x^{n_1} - 1, y^{n_2} - 1, z^{n_3} - 1 \rangle$. All distinct polynomials $c(x, y, z) \in F_2[x, y, z]$, such that $\deg_x c(x, y, z) < n_1$, $\deg_y c(x, y, z) < n_2$ and $\deg_z c(x, y, z) < n_3$, belong to distinct residue classes of R_{n_1, n_2, n_3} . A residue class of R_{n_1, n_2, n_3} will be represented by such a unique polynomial.

The map

$$c = (c_{i,j,h}) \mapsto c(x, y, z) = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} \sum_{h=0}^{n_3-1} c_{i,j,h} x^i y^j z^h$$

defines an isomorphism between V_{n_1, n_2, n_3} and R_{n_1, n_2, n_3} . If C_{n_1, n_2, n_3} is a three-dimensional cyclic array code and $c = (c_{i,j,h}) \in C_{n_1, n_2, n_3}$, then the cyclic shift $c' = (c_{i+l_1, j+l_2, h+l_3}) \in C_{n_1, n_2, n_3}$. The cyclic shift $c' = (c_{i+l_1, j+l_2, h+l_3})$ corresponds to the polynomial $c'(x, y, z) = x^{l_1} y^{l_2} z^{l_3} c(x, y, z)$. The residue class ring R_{n_1, n_2, n_3} is a $n_1 n_2 n_3$ -dimensional linear space over F_2 .

Thus, a linear cyclic three-dimensional array code C_{n_1, n_2, n_3} is an ideal in the residue class ring $R_{n_1, n_2, n_3} = F_2[x, y, z] / \langle x^{n_1} - 1, y^{n_2} - 1, z^{n_3} - 1 \rangle$.

In the polynomial representation of a linear cyclic three-dimensional array code, a polynomial

$$e(x, y, z) = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} \sum_{h=0}^{n_3-1} e_{i,j,h} x^i y^j z^h$$

is a $b_1 \times b_2 \times b_3$ -burst if

$$e(x, y, z) = x^{u_1} y^{u_2} z^{u_3} b(x, y, z)$$

for some $0 \leq u_l < n_l - b_l$, $l = 1, 2, 3$ and a polynomial $b(x, y, z)$, such that $\deg_x b(x, y, z) < b_1$, $\deg_y b(x, y, z) < b_2$, $\deg_z b(x, y, z) < b_3$ and $b(x, 0, 0) \neq 0$, $b(0, y, 0) \neq 0$, $b(0, 0, z) \neq 0$.

The polynomial $e'(x, y, z)$ is a cyclic $b_1 \times b_2 \times b_3$ -burst if $e'(x, y, z) = x^{v_1} y^{v_2} z^{v_3} e(x, y, z)$ for some $0 \leq v_l < n_l$, $l = 1, 2, 3$ and $e(x, y, z)$ is a $b_1 \times b_2 \times b_3$ -burst.

3 Definition of Three-Dimensional Fire Codes

Two approaches are used for description of binary two-dimensional cyclic codes of area $n_1 \times n_2$.

First approach is the concept of common zeros of code polynomials of cyclic two-dimensional codes [10], [11]. The second approach is the representation of a cyclic two-dimensional code of area $n_1 \times n_2$ as an ideal, that is defined by means of a certain basis [12]. In this paper we use the second approach for the definition and constructing cyclic three-dimensional Fire codes.

First, let us recall ordinary Fire codes [13] – [15]. The binary cyclic Fire code is defined as cyclic code generated by the polynomial $g(x) = p(x)(x^c - 1)$ where $p(x)$ is an irreducible polynomial over $GF(2)$ of degree m whose roots have order e and c is a positive integer not divisible by e . The code is of length $n = LCM(c, e)$ and is capable of correcting a burst of length b , if $b \leq (c + 1)/2$ and $b \leq m$. A cyclic Fire code is an ideal in R_n and can be considered as the intersection of ideals $\langle x^c - 1 \rangle$ and $\langle p(x) \rangle$.

By analogy with one-dimensional binary cyclic Fire codes we consider three-dimensional cyclic array codes as ideals, which are the intersections of ideals $\langle x^{c_1} - 1, y^{c_2} - 1, z^{c_3} - 1 \rangle$ and $\langle p_1(x), p_2(y), p_3(z) \rangle$ in the residue class ring $R_{n_1, n_2, n_3} = F_2[x, y, z] / \langle x^{n_1} - 1, y^{n_2} - 1, z^{n_3} - 1 \rangle$.

Lemma 1. Let $A_{g_1, g_2, g_3}(x, y, z) = \langle g_1(x), g_2(y), g_3(z) \rangle$ be an ideal in $R_{n_1, n_2, n_3} = F_2[x, y, z] / \langle x^{n_1} - 1, y^{n_2} - 1, z^{n_3} - 1 \rangle$. Then

(i) $g_1(x)$ divides $x^{n_1} - 1$, $g_2(y)$ divides $y^{n_2} - 1$ and $g_3(z)$ divides $z^{n_3} - 1$;

(ii) $a(x, y, z) \in A_{g_1, g_2, g_3}(x, y, z)$, if and only if

$$a(x, y, z) = a_1(x, y, z)g_1(x) + a_2(x, y, z)g_2(y) + a_3(x, y, z)g_3(z), \quad (3)$$

where $a_l(x, y, z) \in R_{n_1, n_2, n_3}$;

(iii) for any polynomial $a(x, y, z) \in R_{n_1, n_2, n_3}$ there exists a unique polynomial $r(x, y, z) \in R_{n_1, n_2, n_3}$, such that $\deg_x r(x, y, z) < \deg_x g_1(x)$, $\deg_y r(x, y, z) < \deg_y g_2(y)$, $\deg_z r(x, y, z) < \deg_z g_3(z)$ and

$$a(x, y, z) = a_1(x, y, z)g_1(x) + a_2(x, y, z)g_2(y) + a_3(x, y, z)g_3(z) + r(x, y, z), \quad (4)$$

$$a_l(x, y, z) \in R_{n_1, n_2, n_3}, \quad l = 1, 2, 3.$$

The statement (ii) of Lemma 1 is equivalent to the statement, that the basis $\{g_1(x), g_2(y), g_3(z)\}$ of the ideal $A_{g_1, g_2, g_3}(x, y, z)$ is a Gröbner basis [16].

Theorem 1. *Let $p_l(u)$ be an irreducible polynomial over F_2 of degree m_l whose roots have order e_l , $l = 1, 2, 3$. Let c_l be a positive integer not divisible by e_l and $n_l = LCM(c_l, e_l)$. Then the intersection $C_{n_1, n_2, n_3}(x, y, z)$ of the ideals*

$$A_{c_1, c_2, c_3}(x, y, z) = \langle x^{c_1} - 1, y^{c_2} - 1, z^{c_3} - 1 \rangle$$

and

$$A_{p_1, p_2, p_3}(x, y, z) = \langle p_1(x), p_2(y), p_3(z) \rangle$$

in the residue class ring $R_{n_1, n_2, n_3} = F_2[x, y, z]/\langle x^{n_1} - 1, y^{n_2} - 1, z^{n_3} - 1 \rangle$ is a linear cyclic three-dimensional array code of size $n_1 \times n_2 \times n_3$ correcting $b_1 \times b_2 \times b_3$ -bursts of errors if $b_l \leq (c_l + 1)/2$ and $b_l \leq m_l$.

The code $C_{n_1, n_2, n_3}(x, y, z)$ is called a binary three-dimensional Fire code.

Let $m_2 = m_3 = 1$ and $c_2 = c_3 = 1$. Then $C_{n_1, n_2, n_3}(x, y, z)$ is reduced to the binary cyclic one-dimensional Fire code, generated by $p_1(x)(x^{c_1} - 1)$.

4 Encoding Three-Dimensional Fire Codes

Let code $C_{n_1, n_2, n_3}(x, y, z)$ be a binary three-dimensional Fire code and

$$c(x, y, z) = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} \sum_{h=0}^{n_3-1} c_{i,j,h} x^i y^j z^h$$

be a codeword.

Let $P_1 = \{i, j, h : 0 \leq i < c_1, 0 \leq j < c_2, 0 \leq h < c_3\}$; $P_2 = \{i, j, h : c_1 \leq i < c_1 + m_1, 0 \leq j < m_2, 0 \leq h < m_3\}$; $P_3 = \{i, j, h : 0 \leq i < m_1, c_2 \leq j < c_2 + m_2, 0 \leq h < m_3\}$; $P_4 = \{i, j, h : 0 \leq i < m_1, 0 \leq j < m_2, 0 \leq h < c_3 + m_3\}$.

We put coefficients $c_{i,j,h}$ of the code polynomial $c(x, y, z)$ to be information symbols if triple $i, j, h \notin P_l$ and parity-check symbols if triple $i, j, h \in P_l$, $l = 1, 2, 3, 4$.

Let

$$c_0(x, y, z) = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} \sum_{h=0}^{n_3-1} c_{i,j,h}^{(0)} x^i y^j z^h \quad (5)$$

be a polynomial whose coefficients $c_{i,j,h}^{(0)}$, $i, j, h \notin P_l$ are information symbols of the code polynomial and other coefficients are $c_{i,j,h}^{(0)} = 0$, $i, j, h \in P_l$, $l = 1, 2, 3, 4$.

Dividing the polynomial $c_0(x, y, z)$ by the basis $s = \{x^{c_1} - 1, y^{c_2} - 1, z^{c_3} - 1\}$ of the ideal $A_{c_1, c_2, c_3}(x, y, z)$, we obtain

$$c_0(x, y, z) = a_1(x, y, z)(x^{c_1} - 1) + a_2(x, y, z)(y^{c_2} - 1) + a_3(x, y, z)(z^{c_3} - 1) + r_0(x, y, z), \quad (6)$$

where $\deg_x r_0(x, y, z) < c_1$, $\deg_y r_0(x, y, z) < c_2$ and $\deg_z r_0(x, y, z) < c_3$.

Further, dividing the polynomials $a_l(x, y, z)$, $l = 1, 2, 3$ by the basis $\{p_1(x), p_2(y), p_3(z)\}$ of the ideal $A_{p_1, p_2, p_3}(x, y, z)$, we obtain

$$a_l(x, y, z) = a_1^{(l)}(x, y, z)p_1(x) + a_2^{(l)}(x, y, z)p_2(y) + a_3^{(l)}(x, y, z)p_3(z) + r_l(x, y, z), \quad (7)$$

where $\deg_x r_l(x, y, z) < m_1$, $\deg_y r_l(x, y, z) < m_2$ and $\deg_z r_l(x, y, z) < m_3$.

Combining (6) and (7), we get

$$\begin{aligned} & c_0(x, y, z) - r_1(x, y, z)x^{c_1} - r_2(x, y, z)y^{c_2} - r_3(x, y, z)z^{c_3} - r_0(x, y, z) + \\ & r_1(x, y, z) + r_2(x, y, z) + r_3(x, y, z) = (a_1(x, y, z) - r_1(x, y, z))(x^{c_1} - 1) + \\ & (a_2(x, y, z) - r_2(x, y, z))(y^{c_2} - 1) + (a_3(x, y, z) - r_3(x, y, z))(z^{c_3} - 1) \end{aligned} \quad (8)$$

Since the right-hand side of (8) is a codeword of the code $C_{n_1, n_2, n_3}(x, y, z)$, the left-hand side of (8) is also a codeword of the code. Thus, we get the codeword

$$c(x, y, z) = c_0(x, y, z) + c_1(x, y, z) \quad (9)$$

where the coefficients of the polynomial $c_0(x, y, z)$ are the information symbols and the coefficients of the polynomial

$$\begin{aligned} c_1(x, y, z) = & r_1(x, y, z) + r_2(x, y, z) + r_3(x, y, z) - r_0(x, y, z) - \\ & r_1(x, y, z)x^{c_1} - r_2(x, y, z)y^{c_2} - r_3(x, y, z)z^{c_3} \end{aligned} \quad (10)$$

are the parity-check symbols of the codeword $c(x, y, z)$.

5 Decoding Three-Dimensional Fire Codes

Let

$$c'(x, y, z) = c(x, y, z) + e(x, y, z), \quad (11)$$

where $c(x, y, z)$ is a codeword of the three-dimensional Fire code and $e(x, y, z)$ is $b_1 \times b_2 \times b_3$ -burst of errors.

The syndrome of $c'(x, y, z)$ is

$$S_{c'} = S_e = (r_c(x, y, z), r_p(x, y, z)), \quad (12)$$

where the polynomials $r_c(x, y, z)$ and $r_p(x, y, z)$ are the remainders of dividing $c'(x, y, z)$ by the basis $\{x^{c_1} - 1, y^{c_2} - 1, z^{c_3} - 1\}$ of the ideal $A_{c_1, c_2, c_3}(x, y, z)$ and the basis $\{p_1(x), p_2(y), p_3(z)\}$ of the ideal $A_{p_1, p_2, p_3}(x, y, z)$, respectively.

Thus,

$$c'(x, y, z) = a_1(x, y, z)(x^{c_1} - 1) + a_2(x, y, z)(y^{c_2} - 1) + a_3(x, y, z)(z^{c_3} - 1) + r_c(x, y, z), \quad (13)$$

and

$$c'(x, y, z) = d_1(x, y, z)p_1(x) + d_2(x, y, z)p_2(y) + d_3(x, y, z)p_3(z) + r_p(x, y, z). \quad (14)$$

where $\deg_x r_c(x, y, z) < c_1$, $\deg_y r_c(x, y, z) < c_2$, $\deg_z r_c(x, y, z) < c_3$ and $\deg_x r_p(x, y, z) < m_1$, $\deg_y r_p(x, y, z) < m_2$, $\deg_z r_p(x, y, z) < m_3$.

A decoding algorithm for three-dimensional Fire Codes can be described as follows.

1. Compute the syndrome $S_{c'} = S_e = (r_c(x, y, z), r_p(x, y, z))$ of the received word $c'(x, y, z)$.

2. If $S_{c'} = (0, 0)$, there are no errors in $c'(x, y, z)$ and $c'(x, y, z)$ is a codeword.

3. If $\deg_x r_c(x, y, z) < b_1$, $\deg_y r_c(x, y, z) < b_2$, $\deg_z r_c(x, y, z) < b_3$, $r_c(x, 0, 0) \neq 0$, $r_c(0, y, 0) \neq 0$, $r_c(0, 0, z) \neq 0$ and $r_c(x, y, z) = r_p(x, y, z)$, then there is the burst error $e(x, y, z) = r_c(x, y, z)$ in $c'(x, y, z)$ and $c(x, y, z) = c'(x, y, z) - e(x, y, z)$ is a codeword.

4. For $i \in I$, $j \in J$, $h \in H$ find the remainders $r_c^{(i,j,h)}(x, y, z)$ and $r_p^{(i,j,h)}(x, y, z)$ of dividing the polynomial $x^i y^j z^h c'(x, y, z)$ by the basis $\{x^{c_1} - 1, y^{c_2} - 1, z^{c_3} - 1\}$ of the ideal $A_{c_1, c_2, c_3}(x, y, z)$ and the basis $\{p_1(x), p_2(y), p_3(z)\}$ of the ideal $A_{p_1, p_2, p_3}(x, y, z)$, respectively.

If $\deg_x r_c^{(i,j,h)}(x, y, z) < b_1$, $\deg_y r_c^{(i,j,h)}(x, y, z) < b_2$, $\deg_z r_c^{(i,j,h)}(x, y, z) < b_3$, $r_c^{(i,j,h)}(x, 0, 0) \neq 0$, $r_c^{(i,j,h)}(0, y, 0) \neq 0$, $r_c^{(i,j,h)}(0, 0, z) \neq 0$ and $r_c^{(i,j,h)}(x, y, z) = r_p^{(i,j,h)}(x, y, z)$, then $e(x, y, z) = x^{n_1-i} y^{n_2-j} z^{n_3-h} r_c^{(i,j,h)}(x, y, z)$ and $c(x, y, z) = c'(x, y, z) - e(x, y, z)$.

5. In all other cases, the received word $c'(x, y, z)$ has uncorrectable errors.

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