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# Quadratic functions with prescribed spectra 

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#### Abstract

We study quadratic Boolean functions $f$ from $\mathbb{F}_{2^{n}}$ to $\mathbb{F}_{2}$, which are well-known to have plateaued Fourier spectrum $\mathcal{F}_{s, f}$, i.e., their Fourier coefficients are in the set $\left\{0, \pm 2^{(n+s) / 2}\right\}$ for some integer $0 \leq s \leq$ $n-1$. For various types of integers $n$, we determine possible values of $s$, construct $f$ with $\mathcal{F}_{s, f}$ for a prescribed $s$, and present enumeration results in case $n$ is a power of 2 . Our work generalizes some of the earlier results of Khoo et. al. ([5]) on near-bent functions and provides a simple proof of a result of Fitzgerald ([2]) on degenerate quadratic forms.


Keywords: Quadratic Boolean functions, s-plateaued functions, nearbent functions, self-reciprocal polynomials, linear complexity

## 1 Introduction

We study quadratic functions

$$
\begin{equation*}
f(x)=\operatorname{Tr}_{\mathrm{n}}\left(\sum_{i=0}^{\lfloor(n-1) / 2\rfloor} a_{i} x^{2^{i}+1}\right) \tag{1}
\end{equation*}
$$

from $\mathbb{F}_{2^{n}}$ to $\mathbb{F}_{2}$, with coefficients in $\mathbb{F}_{2}$.
It is well known that any quadratic function is plateaued i.e., it has (plateaued) Fourier spectrum

$$
\mathcal{F}_{s, f}
$$

, in other words, its Fourier coefficients lie in $\left\{0, \pm 2^{(n+s) / 2}\right\}$ for some integer $0 \leq s \leq n-1$. In this case we call $f s$-plateaued. 1-plateaued functions have been widely studied, and are called near-bent or semi-bent (when $n$ is odd), see for instance $[1,6]$.

One of the problems, that [5] focuses on, is to characterize integers $n$, for which all $f$ from $\mathbb{F}_{2^{n}}$ to $\mathbb{F}_{2}$ of the form (1) are near-bent.

The more general question we address here is the following: Given an integer $n$, characterize those integers $s$, for which $s$-plateaued functions from $\mathbb{F}_{2^{n}}$ to $\mathbb{F}_{2}$ of the form (1) exist. We obtain the characterization when $n$ is a squarefree integer or is a power of 2 . For these classes of integers $n$, we give methods
for constructing $s$-plateaued functions for all possible $s$. We also enumerate the $s$-plateaued functions in case $n=2^{m}, m \geq 1$.

Using standard Welch-squaring techniques one can see that the integer $s$ is the dimension over $\mathbb{F}_{2}$ of the kernel of the linear transformation defined on $\mathbb{F}_{2^{n}}$ by

$$
L(x)=\sum_{i=0}^{\lfloor(n-1) / 2\rfloor}\left(a_{i} x^{2^{i}}+a_{i}^{2^{n-i}} x^{2^{n-i}}\right),
$$

i.e., $\operatorname{gcd}\left(x^{2^{n}}+x, L(x)\right)$ has degree $2^{s}$. Equivalently $\operatorname{ker}(L)$ has dimension $s$ if and only if the associates $A(x)$ and $x^{n}+1$ of $L(x)$ and $x^{2^{n}}+x$, respectively, satisfy (see [7, p.118])

$$
\operatorname{deg}\left(\operatorname{gcd}\left(A(x), x^{n}+1\right)\right)=s
$$

The associate $A(x)$ corresponding to $f$ in (1) is

$$
\begin{equation*}
A(x)=\sum_{i=0}^{\lfloor(n-1) / 2\rfloor} a_{i} x^{i}+a_{i} x^{n-i}=x^{i_{0}} g(x) \tag{2}
\end{equation*}
$$

where $i_{0}$ is the smallest integer such that $a_{i_{0}} \neq 0$, and $g(x) \in \mathbb{F}_{2}[x]$ is the self-reciprocal polynomial

$$
g(x)=\sum_{i=i_{0}}^{\lfloor(n-1) / 2\rfloor} a_{i}\left(x^{i-i_{0}}+x^{n-i_{0}-i}\right)
$$

of degree $n-2 i_{0}$.
Note that $\operatorname{gcd}\left(\sum_{i=0}^{\lfloor(n-1) / 2\rfloor} a_{i} x^{i}+a_{i} x^{n-i}, x^{n}+1\right)=\operatorname{gcd}\left(\left(\sum_{i=1}^{\lfloor(n-1) / 2\rfloor} a_{i} x^{i}+\right.\right.$ $\left.\left.a_{i} x^{n-i}\right)+a_{0}\left(x^{n}+1\right), x^{n}+1\right)$, i.e., $a_{0}$ does not effect the value of $s$. Hence we can suppose that the degree of $A(x)$ is at most $n-1$.

We recall that the linear complexity $L(S)$ of an $n$-periodic sequence $S=$ $s_{0}, s_{1}, \ldots$ over $\mathbb{F}_{2}$ is determined by

$$
L(S)=n-\operatorname{deg}\left(\operatorname{gcd}\left(x^{n}+1, S(x)\right)\right)
$$

where $S(x)$ is the generating polynomial of $S$, i.e., the polynomial of degree at most $n-1$ given by $S(x)=s_{0}+s_{1} x+\cdots+s_{n-1} x^{n-1}$. Therefore the calculation of the values in the Fourier spectrum of a quadratic function (1) is equivalent to the determination of the linear complexity of an $n$-periodic sequence with generating polynomial of the form (2). More precisely $s=n-L$ if $L$ is the linear complexity of the corresponding $n$-periodic sequence.

## 2 Main Results

### 2.1 The case $n=2^{m}$

In this subsection we will employ the well known Games-Chan algorithm (see [3]) to enumerate the functions (1) from $\mathbb{F}_{2^{2^{m}}}$ to $\mathbb{F}_{2}$ that yield $s$-plateaued functions. The algorithm also leads to a tool of constructing $s$-plateaued functions for a given $s$.

The following example describes how one can calculate $s$.

Example 1. For $m=4$ consider $f(x)=\operatorname{Tr}_{\mathrm{n}}\left(x^{2}+x^{3}+x^{2^{4}+1}+x^{2^{5}+1}\right)$, then $A(x)=1+x+x^{4}+x^{5}+x^{11}+x^{12}+x^{15}+x^{16}$. For our purpose we consider this polynomial modulo $x^{16}+1$ and put

$$
A(x)=x+x^{4}+x^{5}+x^{11}+x^{12}+x^{15}
$$

and obtain the corresponding 16-periodic binary sequence

$$
\begin{aligned}
& S=(0100110000011001)^{\infty} . \\
& 010011000101 \\
& \begin{array}{r}
00011001 \\
01010101
\end{array} \frac{0101}{0000} \\
& L=8 \quad L=8 \\
& 010 \\
& \begin{array}{l}
01 \\
00
\end{array} \quad L=L+1=10 . \\
& L=8 \quad L=9
\end{aligned}
$$

As the 16 -periodic sequence $S$ corresponding to $A(x)$ has linear complexity $L=$ 10 , the quadratic function $f$ is $s$-plateaued with $s=16-10=6$.

The Games-Chan algorithm motivates the definition of a mapping $\varphi_{m}$ from $\mathbb{F}_{2}^{2^{m}}$ to $\mathbb{F}_{2}^{2^{m-1}}, m \geq 1$, as follows:

$$
\varphi_{m}\left(\left(s_{0}, s_{1}, \ldots, s_{2^{m}-1}\right)\right)=\left(s_{0}+s_{2^{m-1}}, s_{1}+s_{2^{m-1}+1}, \ldots, s_{2^{m-1}-1}+s_{2^{m}-1}\right)
$$

In the following proposition we collect some simple observations for $2^{m}$-periodic sequences corresponding to polynomials $A(x)$ in (2) with $n=2^{m}$. As remarked above we can assume that $a_{0}=0$, thus $\operatorname{deg}(A) \leq n-1$. The strings $\mathbf{s}^{(m)}=$ $s_{0}, s_{1}, \ldots, s_{2^{m}-1}$ of our interest can easily be seen to satisfy $s_{0}=s_{n / 2}=0$, $s_{i}=s_{n-i}, i=1, \ldots, n / 2-1$. We will call a string satisfying these properties antisymmetric. Accordingly, we call the corresponding sequence antisymmetric $2^{m}$-periodic sequence.

Proposition 1. Let $\mathbf{s}^{(m)}=s_{0}, s_{1}, \ldots, s_{2^{m}-1}$ be a string, $m \geq 1$.
(i) An antisymmetric string $\mathbf{s}^{(m)}$ is determined by the bits $s_{1}, \ldots, s_{2^{m-1}-1}$. There are $2^{2^{m-1}-1}$ distinct antisymmetric strings of length $2^{m}$.
(ii) If $\mathbf{s}^{(m)}$ is antisymmetric, then $\varphi_{m}\left(\mathbf{s}^{(m)}\right)$ is also.
(iii) The set of antisymmetric preimages $\varphi_{m}^{-1}\left(\mathbf{s}^{(m-1)}\right)$ of an antisymmetric string $\mathbf{s}^{(m-1)}$ has cardinality $2^{2^{m-2}}$.
(iv) Let $\mathbf{s}^{(m)}$ be an antisymmetric string satisfying $\varphi_{m}\left(\mathbf{s}^{(m)}\right)=0,0, \ldots, 0$. Then either $s_{0}, s_{1}, \ldots, s_{2^{m-1}-1}$ is itself antisymmetric, or the string

$$
s_{0}, s_{1}, \ldots, s_{2^{m-2}}+1, \ldots, s_{2^{m-1}-1}
$$

is antisymmetric.
In the first case $\varphi_{m-1}\left(s_{0}, s_{1}, \ldots, s_{2^{m-1}-1}\right)=t_{0}, t_{1}, \ldots, t_{2^{m-2}-1}$ is antisymmetric, and in the second case the string $t_{0}-1, t_{1}, \ldots, t_{2^{m-2}-1}$ is antisymmetric.

Theorem 1. For $n=2^{m}$, let $\mathcal{N}_{m}(s)$ denote the number of strings

$$
\left(a_{1}, a_{2}, \ldots, a_{(n / 2)-1}\right) \in \mathbb{F}_{2}^{(n / 2)-1}
$$

for which the quadratic function $f$ from $\mathbb{F}_{2^{n}}$ to $\mathbb{F}_{2}$, given by

$$
f(x)=\operatorname{Tr}_{\mathrm{n}}\left(\sum_{i=1}^{(n / 2)-1} a_{i} x^{2^{i}+1}\right)
$$

is s-plateaued. Then

$$
\mathcal{N}_{m}(s)=\left\{\begin{aligned}
2^{2^{m-1}-1-k} & : \quad s=2 k, k=1, \ldots, 2^{m-1}-1, \\
0 & : s=0 \text { or } s \text { odd } .
\end{aligned}\right.
$$

Proof. We use induction. One can easily see that the assertion holds for small $m$. Now suppose that $\mathcal{N}_{m-1}(s)=2^{2^{m-2}-1-k}$ for $s=2 k, k=1, \ldots, 2^{m-2}-1$, i.e., for these values of $s$, there are $2^{2^{m-2}-1-k}$ sequences, which are $2^{m-1}$-periodic with linear complexity $2^{m-1}-s$, corresponding to antisymmetric strings. By Proposition 1 (iii), for each of these stings we have $2^{2^{m-2}}$ antisymmetric preimages giving rise to antisymmetric $2^{m}$-periodic sequences with linear complexity $2^{m}-s$. Proposition 1 (ii) implies that these are all such sequences. Consequently, we get $\mathcal{N}_{m}(s)=2^{2^{m-2}} 2^{2^{m-2}-1-k}=2^{2^{m-1}-1-k}$ when $s=2 k, k=1, \ldots, 2^{m-2}-1$. It remains to show that the formula holds for $s=2 k$ with $k=2^{m-2}, \ldots, 2^{m-1}-1$. We therefore have to enumerate the antisymmetric $2^{m}$-periodic sequences with a given linear complexity $2^{m}-s \leq 2^{m-1}$. First observe that these are the sequences corresponding to antisymmetric strings $\mathbf{s}^{(m)}=s_{0}, \ldots, s_{2^{m}-1}$ such that (a) $\varphi_{m}\left(\mathbf{s}^{(m)}\right)=0,0, \ldots, 0$, or
(b) the sequence corresponding to $s_{0}, \ldots, s_{2^{m-1}-1}$ has linear complexity $2^{m}-s$.

By Proposition 1 (iv), (a) implies that $s_{0}, \ldots, s_{2^{m-1}-1}$ or $s_{0}, s_{1}, \ldots, s_{2^{m-2}}+$
$1, \ldots, s_{2^{m-1}-1}$ is antisymmetric. Moreover it is easily seen that for any such string there is exactly one corresponding antisymmetric string $\mathbf{s}^{(m)}$ for which (a) holds. Having an odd number of 1 's, the $2^{m-2}-1$ strings of the second type yield $2^{m-1}$-periodic sequences with linear complexity $L=2^{m-1}$ (thus $s=2^{m}-L=2^{m-1}$ ). Among the $2^{m-2}-1$ strings of the first type, by our hypothesis, precisely $2^{2^{m-2}-1-\kappa}$ yield $2^{m-1}$-periodic sequences with linear complexity $L=2^{m-1}-2 \kappa, \kappa=1, \ldots, 2^{m-2}-1$. Substituting $\kappa$ by $k-2^{m-2}$ we obtain $2^{2^{m-1}-1-k}$ for the number of antisymmetric $2^{m}$-periodic sequences with linear complexity $L=2^{m}-2 k$ (thus $s=2 k$ ) for $k=2^{m-2}+1, \ldots, 2^{m-1}-1$, hence $\mathcal{N}_{m}(s)=2^{2^{m-2}} 2^{2^{m-2}-1-k}=2^{2^{m-1}-1-k}$ for these values of $k$, and $s=2 k$. Note that from the above arguments we also see that $\mathcal{N}_{m}(s)=0$ when $s=0$ and when $s$ is odd. However one can also see directly that $\mathcal{N}_{m}(0)=0$ since antisymmetric strings contain an even number of 1 's, and the statement for odd $s$ simply follows from the Fourier transform being an integer.
Note that the arguments in the proof also enable the construction of $s$-plateaued quadratic functions from $\mathbb{F}_{2^{2^{m}}}$ to $\mathbb{F}_{2}$ for a prescribed value of $s$.

### 2.2 The case $n=p_{1} p_{2} \cdots p_{r}$

The results in this subsection are obtained with a different approach, namely by analysing the factorization of $x^{n}+1$ into self-reciprocal polynomials. With the observation that $\operatorname{gcd}\left(x^{n}+1, A(x)\right)$ is again self-reciprocal if $A(x)$ is self-reciprocal, one obtains the following general theorem, which is valid for arbitrary integers $n$.

Theorem 2. Let $n$ be arbitrary.
(i) If $n$ is odd, then there exists an s-plateaued function of the form (1) if and only if $s$ is odd and $x^{n}+1$ has a self-reciprocal factor $h(x)$ of degree $s$ (in which case $x^{n}+1$ is always divisible by $x+1$ ).
(ii) If $n$ is even then there exists an s-plateaued function of the form (1) if and only if $s$ is even and $x^{n}+1$ has a self-reciprocal factor $h(x)$ of degree $s$ divisible by $(x+1)^{2}$.

Note that if $n=2^{v} n_{1}, n_{1}$ odd, then $x^{n}+1=\left(x^{n_{1}}+1\right)^{2^{v}}$. Thus it is sufficient to analyse the factorization of $x^{n}+1$ for odd $n$. Here we only consider the case of $n$ being square-free. Our main tool for studying the factorization of $x^{n}+1$ into self-reciprocals is, as expected, the use of cyclotomic cosets modulo $n$ relative to powers of 2 .

We denote the $n$-th cyclotomic polynomial by $\mathcal{Q}_{n}$, and denote the 2 -adic valuation of an integer $k$ by $\nu(k)$, i.e., $2^{\nu(k)}$ is the largest power of 2 which divides $k$. The following lemma describes for which squarefree integers $n=p_{1} p_{2} \cdots p_{r}$ the irreducible factors of $\mathcal{Q}_{n}$ are self-reciprocal. Note that $\mathcal{Q}_{n}$ has $d$ irreducible factors where $d=\operatorname{lcm}\left(d_{1}, \ldots, d_{r}\right)=\operatorname{ord}_{n} 2$.

Lemma 1. Let $n=p_{1} p_{2} \cdots p_{r}, d_{i}=\operatorname{ord}_{p_{i}} 2$ and $d=\operatorname{ord}_{n} 2$. Suppose the irreducible factors of $\mathcal{Q}_{n}$ are $f_{1}, \ldots, f_{\varphi(n) / d}$. Then
(i) The polynomials $f_{1}, \ldots, f_{\varphi(n) / d}$ are self-reciprocal if and only if $\nu\left(d_{1}\right)=$ $\nu\left(d_{2}\right)=\cdots=\nu\left(d_{r}\right)>0$. In particular, if $n$ is a prime $p$, then $f_{1}, \ldots, f_{(p-1) / d}$ are self-reciprocal if and only if $d$ is even.
(ii) If $\nu\left(d_{i}\right) \neq \nu\left(d_{j}\right)$ for some $1 \leq i, j \leq \varphi(n) / d$, then none of the polynomials $f_{t}$, $1 \leq t \leq \varphi(n) / d$, is self-reciprocal, and for each $t, 1 \leq t \leq \varphi(n) / d$, there exists a unique $t^{\prime} \neq t, 1 \leq t^{\prime} \leq \varphi(n) / d$ such that the product $f_{t} f_{t^{\prime}}$ is self-reciprocal.

Idea of Proof. First observe that the irreducible factors of $\mathcal{Q}_{n}$ are self-reciprocal if every cyclotomic coset modulo $n$ relative to powers of 2 containing the element $a$ also contains the element $-a$. Therefore an irreducible factor of $\mathcal{Q}_{n}$ is selfreciprocal if the cyclotomic coset of 1 also contains -1 , i.e., $2^{k} \equiv-1 \bmod n$ for some integer $k$. This is equivalent to $2^{k} \equiv-1 \bmod p_{1}, i \leq i \leq r$, which holds if and only if $d_{i}$ divides $2 k$ but not $k$ for each $i$. This leads to the condition $\nu\left(d_{1}\right)=\nu\left(d_{2}\right)=\cdots=\nu\left(d_{r}\right)>0$.
By Lemma 1 and [7, Exercise 3.15] the polynomial $x^{n}+1$ factors into selfreciprocal irreducible polynomials if and only $\nu\left(d_{1}\right)=\nu\left(d_{2}\right)=\cdots=\nu\left(d_{r}\right)>0$.

Example 2. I. $n=5 \cdot 13=65$, then $d_{1}=4, d_{2}=12$, hence $\nu\left(d_{1}\right)=\nu\left(d_{2}\right)$. Consequently $x^{65}+1$ factors into self-reciprocal irreducible polynomials.
II. $n=3 \cdot 5 \cdot 7=105$, then $d_{1}=2, d_{2}=4, d_{3}=3$ and $\nu\left(d_{1}\right) \neq \nu\left(d_{2}\right)$. Hence not all the irreducible factors of $x^{105}+1$ are self-reciprocal.

A simple consequence of the above lemma is also the main result of [5]: All functions of the form (1) from $\mathbb{F}_{2^{n}}$ to $\mathbb{F}_{2}$ are 1-plateaued (near-bent) only when $n=p$ is a prime, satisfying $p \equiv 3 \bmod 4$ with $\operatorname{ord}_{p} 2=(p-1) / 2$, or 2 is a primitive root modulo $p$. Note that the self-reciprocal factors of $x^{n}+1$ are exactly $x+1$ and $1+x \cdots+x^{n-1}$ only for such $n$.

In order to determine the possible values of $s$ that a function of the form (1) has Fourier spectrum $\mathcal{F}_{s, f}$, we consider all cyclotomic polynomials $\mathcal{Q}_{m}, m \mid n$ and apply Lemma 1 accordingly:

Example 2.II. continued: $n=3 \cdot 5 \cdot 7=105, d_{1}=2, d_{2}=4, d_{3}=3$. $\varphi(n)=48, d=\operatorname{gcd}(2,4,3)=12$, and $\nu\left(d_{1}\right) \neq \nu\left(d_{2}\right)$. Hence $x^{105}+1$ has 2 self-reciprocal factors of degree 24 .
$\varphi(35)=24, \operatorname{gcd}\left(d_{2}, d_{3}\right)=\operatorname{gcd}(4,3)=12$, and $\nu(4) \neq \nu(3)$, which yields 2 cyclotomic classes of cardinality 12 , and hence one self-reciprocal factor of degree 24. $\varphi(21)=12, \operatorname{gcd}(2,3)=6$. There are 2 cyclotomic classes of cardinality 6 corresponding to one self-reciprocal factor of degree 12.
$\varphi(15)=8, \operatorname{gcd}(2,4)=4, \nu(2) \neq \nu(4)$. There are 2 cyclotomic classes of cardinality 4 , giving one self-reciprocal factor of degree 8 .
Similarly it is easy to see that $x^{105}+1$ has one self-reciprocal factor of degree 6 , and two irreducible self-reciprocal factors; one of degree 4 , and one of degree 2 .

Therefore $s$ can be any integer less than 105 of the form $s=24 k_{1}+12 k_{2}+$ $8 k_{3}+6 k_{4}+4 k_{5}+2 k_{6}+1,0 \leq k_{1} \leq 3$ and $0 \leq k_{i} \leq 1$ for $2 \leq i \leq 6$.

We list the possible values of $s$ in two special cases:
Corollary 1. Let $n$ be an odd prime with ord $_{n} 2=d$.
(i) If $d$ is even, then there exists an s-plateaued function of the form (1) from $\mathbb{F}_{2^{n}}$ to $\mathbb{F}_{2}$ if and only if $s=k d+1$ for some $0 \leq k \leq(n-1) / d-1$.
(ii) If $d$ is odd, then there exists an s-plateaued function of the form (1) from $\mathbb{F}_{2^{n}}$ to $\mathbb{F}_{2}$ if and only if $s=2 k d+1$ for some $0 \leq k \leq(n-1) /(2 d)-1$.

Corollary 2. Let $n=p q$ for two odd primes $p$ and $q$ and let $\operatorname{ord}_{p} 2=d_{p}$, ord $_{q} 2=d_{q}$. The integers $s$ for which there exists an $s$-plateaued function of the form (1) from $\mathbb{F}_{2}^{n}$ to $\mathbb{F}_{2}$ are given as follows: $s<n$ and

1. if $\nu\left(d_{p}\right)=\nu\left(d_{q}\right)>0$, then $s=k_{1} \operatorname{lcm}\left(d_{p}, d_{q}\right)+k_{2} d_{p}+k_{3} d_{q}, 0 \leq k_{1} \leq$ $(p-1)(q-1) / \operatorname{lcm}\left(d_{p}, d_{q}\right), 0 \leq k_{2} \leq(p-1) / d_{p}, 0 \leq k_{3} \leq(q-1) / d_{q}$;
2. if $\nu\left(d_{p}\right)>0, \nu\left(d_{q}\right)>0$ and $\nu\left(d_{p}\right) \neq \nu\left(d_{q}\right)>0$, then $s=2 k_{1} \operatorname{lcm}\left(d_{p}, d_{q}\right)+$ $k_{2} d_{p}+k_{3} d_{q}, 0 \leq k_{1} \leq(p-1)(q-1) /\left(2 \operatorname{lcm}\left(d_{p}, d_{q}\right)\right), 0 \leq k_{2} \leq(p-1) / d_{p}$, $0 \leq k_{3} \leq(q-1) / d_{q} ;$
3. if $\nu\left(d_{p}\right)>0, \nu\left(d_{q}\right)=0$, then $s=2 k_{1} \operatorname{lcm}\left(d_{p}, d_{q}\right)+k_{2} d_{p}+2 k_{3} d_{q}, 0 \leq k_{1} \leq$ $(p-1)(q-1) /\left(2 \operatorname{lcm}\left(d_{p}, d_{q}\right)\right), 0 \leq k_{2} \leq(p-1) / d_{p}, 0 \leq k_{3} \leq(q-1) /\left(2 d_{q}\right) ;$
4. $\nu\left(d_{p}\right)=\nu\left(d_{q}\right)=0$, then $s=2 k_{1} \operatorname{lcm}\left(d_{p}, d_{q}\right)+2 k_{2} d_{p}+2 k_{3} d_{q}, 0 \leq k_{1} \leq$ $(p-1)(q-1) /\left(2 \operatorname{lcm}\left(d_{p}, d_{q}\right)\right), 0 \leq k_{2} \leq(p-1) /\left(2 d_{p}\right), 0 \leq k_{3} \leq(q-1) /\left(2 d_{q}\right)$.

Now the methods of constructing $s$-plateaued functions of the form (1) with prescribed $s$ are obvious:

1. Among the self-reciprocal factors of $x^{n}+1$ select some, whose degrees add up to $s$ and form their product $h(x)$. We remark that $s$ must be odd if $n$ is odd, thus $x+1$ must divide $h(x)$. If $n$ is even then also $s$ must be even, hence $h(x)$ will always be divisible by $(x+1)^{g}$ for some even integer $g \geq 2$.
2. Multiply $h(x)$ with a self-reciprocal polynomial of even degree, which is relatively prime to $\left(x^{n}+1\right) / h(x)$. The resulting product $g(x)$ must be of degree at most $n-1$.
3. Multiply $g(x)$ with $x^{i_{0}}$, where $i_{0}$ is the unique integer such that $A(x)=x^{i_{0}} g(x)$ is of the form (2). Note that $a_{0}=0$ for any $A(x)$, obtained this way.
4. The polynomial $f(x)$ of the form (1) corresponding to a $A(x)$ is then $s$ plateaued. Note that $a_{0}$ can be chosen as 0 or 1 .

The following example leads to an easy proof of a result of [2].

Example 3. Construction of $s$-plateaued functions with maximal possible value for $s$ :

As $n+s$ must be even, the maximal possible value for $s$ is $s=n-2$. We have to choose a self-reciprocal divisor $h(x)$ of $x^{n}+1$ of degree $n-2$. The only possible choices for $h(x)$ are
(i) $h(x)=\left(x^{n}+1\right) /(x+1)^{2}$.
(ii) $h(x)=\left(x^{n}+1\right) /\left(x^{2}+x+1\right)$.

Now (i) implies that $n$ is even and since then $(x+1)^{2}$ must divide $h(x)$, we need $4 \mid n$, and (ii) implies that $3 \mid n$. The step 2 in the above procedure can not be carried out, thus $g(x)=h(x)$, and $i_{0}=1$. We then get

$$
\begin{aligned}
& A(x)=x h(x)=x+x^{3}+x^{5}+\cdots+x^{n-1} \quad \text { in case (i), and } \\
& A(x)=x h(x)=x+x^{2}+x^{4}+x^{5}+x^{7}+x^{8}+\cdots+x^{n-2}+x^{n-1} \quad \text { in case (ii). }
\end{aligned}
$$

The following corollary easily follows from the argument used in the above example.
Corollary 3. The quadratic function $f$ of the form (1) is $(n-2)$-plateaued if and only if
(i) $4 \mid n$ and $f(x)=\operatorname{Tr}_{\mathrm{n}}\left(\varepsilon x^{2}+x^{2+1}+x^{2^{3}+1}+x^{2^{5}+1}+\cdots+x^{2^{n / 2-1}+1}\right), \varepsilon \in$ $\{0,1\}$, or
(ii) $3 \mid n$ and $f(x)=\operatorname{Tr}_{\mathrm{n}}\left(\varepsilon x^{2}+\sum_{i=1, i \neq 0 \bmod 3}^{\lfloor n-1 / 2\rfloor} x^{2^{i}+1}\right), \varepsilon \in\{0,1\}$.

Compare our Corollary 3 with Theorem 2.4 in [2].

## 3 Conclusion

We enumerate quadratic $s$-plateaued functions from $\mathbb{F}_{2^{2 m}}$ to $\mathbb{F}_{2}$, given by (1). For squarefree integers $n=p_{1} p_{2} \cdots p_{r}$ we characterize the integers $s$, for which $s$-plateaued functions from $\mathbb{F}_{2^{n}}$ to $\mathbb{F}_{2}$ of the form (1) exist. Methods for constructing such functions are also described. Our results generalize earlier work on the case $s=1$, see $[4,5]$.

## References

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