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Abstract. We study divisibility by eight of exponential sums of several classes of functions over finite fields of characteristic two. For the binary classical Kloosterman sums $K(a)$ over such fields we give a simple recurrent algorithm for finding the largest k, such that 2^k divides the Kloosterman sum $K(a)$. This gives a simple description of zeros of such Kloosterman sums.

Keywords: exponential sum, Kloosterman sum, divisibility by power of two, zero of Kloosterman sum

1. Introduction. Let $\mathbb{F} = \mathbb{F}_{2^m}$ be the field of characteristic 2 of order 2^m, where $m \geq 3$ is an integer. By $\mathbb{F}_2$ denote the field, consisting of two elements. Set

$$e(x) \; = \; (-1)^{Tr(x)}, \quad x \in \mathbb{F},$$

where $Tr(x)$ is the trace function from $\mathbb{F}$ into $\mathbb{F}_2$. For an arbitrary polynomial $f(x)$ over $\mathbb{F}$ define its exponential sum $S(f)$,

$$S(f) \; = \; \sum_{x \in \mathbb{F}} e(f(x)).$$

Recall that under x^{-i} we understand x^{2^m-1-i}, avoiding by this way a division into 0. The sum

$$K(a) \; = \; \sum_{x \in \mathbb{F}} e(x \; + \; a/x), \quad a \in \mathbb{F}^*,$$

is called the *Kloosterman sum*.

Given a polynomial $f(x)$ over $\mathbb{F}$ it is a hard mathematical problem to write out the value of its exponential sum $S(f)$ or the module $|S(f)|$. This problem is interesting from several points of view, for example, for the theory of error-correcting codes, for sequences, for solving of equations over finite fields, for

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cryptography and etc. In some cases the value $S(f)$ is known (see [1], [2], [8] and references there). From this point of view, the divisibility of $S(f)$ is an interesting important open problem. Recently, the divisibility of $K(a)$ has been considered in several papers. In particular, the divisibility of $K(a)$ modulo 24 has been solved in [4], and the divisibility by 16 has been solved in [9].

In the present paper we use an elementary combinatorial approach to study divisibility by eight of exponential sums of polynomials of special type (such as $a(x^d + x^{-d})$, $ax^d + bx^{-1}$, $a(x + x^{-1})^d$, and etc.). This is considered in the first three sections. In the last section we consider the divisibility of Kloosterman sums $K(a)$ into the maximal power of two, which results in a simple algorithm of finding of zeros of Kloosterman sums. Kloosterman zeros play a significant role in construction of highly nonlinear functions that are used in cryptography, in particular, for construction of *bent functions* (see [5], [6]) and *hyperbent functions* (see [5]).

2. Polynomials of type $a(x^d + x^{-d})$. We start by studying the divisibility by 8 of exponential sums of polynomials $f(x)$ of the type

$$f(x) \; = \; a(x^d \; + \; x^{-d}), \quad a \in \mathbb{F}^*,$$

where d is any odd integer, $1 \le d \le 2^m - 3$. All our proofs of divisibility by eight are based on the following quite evident observation.

Proposition 1 . *Let $\boldsymbol{b} = (b_1, \dots, b_k, b_{k+1}, \dots, b_{2k})$ be a binary sequence of even length $2k$, and let $\boldsymbol{b}_{inv} = (b_{2k}, \dots, b_{k+1}, b_k, \dots, b_1)$ be the inverse sequence. Then the Hamming distance*

$$d\,(\boldsymbol{b}, \boldsymbol{b}_{inv}) \equiv \begin{cases} 0 \pmod 4, & \text{if} \quad \mathrm{wt}(\boldsymbol{b}) \quad \text{even}, \\ 2 \pmod 4, & \text{if} \quad \mathrm{wt}(\boldsymbol{b}) \quad \text{odd}, \end{cases}$$

where $\mathrm{wt}(\boldsymbol{b})$ *is the Hamming weight of* $\boldsymbol{b}$ *(i.e. the number of ones in the sequence* $\boldsymbol{b}$*).*

Let β be a primitive element of $\mathbb{F}$. For a given $a \in \mathbb{F}^*$ and an odd integer d, $1 \le d \le 2^m - 3$, define a binary vector of length 2^m:

$$\boldsymbol{T}(a\,x^d) \; = \; (Tr(0), Tr(a), Tr(a\,\beta^d), Tr(a\,\beta^{2d}), ..., Tr(a\,\beta^{(2^m-2)d})).$$

In the similar way define the vector $\boldsymbol{T}\,(a\,x^{-d})$ of the same length. Denote

$$\boldsymbol{T}_1(a\,x^d) \; = \; (Tr(a\,\beta^d), Tr(a\,\beta^{2d}), ..., Tr(a\,\beta^{(2^{m-1}-1)d}))$$

and

$$\boldsymbol{T}_2(a\,x^d) \; = \; (Tr(a\,\beta^{2^{m-1}d}), Tr(a\,\beta^{(2^{m-1}+1)d}), ..., Tr(a\,\beta^{(2^m-2)d})).$$

Similarly define also the vectors $\boldsymbol{T}_1(a\,x^{-d})$ and $\boldsymbol{T}_2(a\,x^{-d})$. Note that the vector $(\boldsymbol{T}_1(a\,x^{-d}), \boldsymbol{T}_2(a\,x^{-d}))$ of length $2^m - 2$ is an inversion of the vector

$$(\boldsymbol{T}_1(a\,x^d), \boldsymbol{T}_2(a\,x^d)).$$

Since

$$\sum_{x\in\mathbb{F}} Tr\,(a\,x^d) \;=\; Tr\left(\sum_{i=0}^{2^m-2} a\,\beta^{i\,d}\right) \;=\; Tr\left(a\,\frac{\beta^{d(2^m-1)}-1}{\beta-1}\right) \;=\; 0,$$

the weight of the vector $(\boldsymbol{T}_1(a\,x^d), \boldsymbol{T}_2(a\,x^d))$ is even, if $Tr(a) = 0$. The same property is valid for the vector $(\boldsymbol{T}_1(a\,x^{-d}), \boldsymbol{T}_2(a\,x^{-d}))$. But if $Tr(a) = 1$, then the corresponding weights are odd. Therefore, according to Proposition 1 we obtain that

$$d\,(\boldsymbol{T}(a\,x^d), \boldsymbol{T}(a\,x^{-d})) \equiv \begin{cases} 0 \pmod 4), & \text{if} \quad Tr\,(a) = 0, \\ 2 \pmod 4), & \text{if} \quad Tr\,(a) = 1. \end{cases}$$

Since

$$S(a(x^d + x^{-d})) \;=\; 2^m \;-\; 2\,d\,(\boldsymbol{T}(a\,x^d), \boldsymbol{T}(a\,x^{-d})),$$

the following result holds.

Statement 1 . *Let $a \in \mathbb{F}^*$, and d be any odd integer, $1 \le d \le 2^m - 3$. Then*

$$S(a\,(x^d \;+\; x^{-d})) \equiv \begin{cases} 0 \pmod 8), & \textit{if} \quad \mathrm{Tr}(a) = 0, \\ 4 \pmod 8), & \textit{if} \quad \mathrm{Tr}(a) = 1. \end{cases}$$

The proof given above is quite elementary. Remark, that this result was known for $d = 1$ and, therefore, for any d which is relatively prime to $2^m - 1$ (see [7], where two proofs of this result are given: the first one based on finding of the number of solutions of some system of equations over $\mathbb{F}$, and the second one, suggested by P. Charpin, based on Melas codes).

It is easy to see that the sum $S(a(x^d + x^{-d}))$ depends on the value of the greatest common divisor $h = (d, 2^m - 1)$ of numbers d and $2^m - 1$. In particular, h divides $S(a(x^d + x^{-d})) - 1$. Indeed, denoting $2^m - 1 = hu$, we obtain

$$\begin{aligned}
S(a(x^d + x^{-d})) &= 1 \;+\; \sum_{i=0}^{2^m-2} e(a\,(\beta^{i\,d} \;+\; \beta^{-i\,d})) \\
&= 1 \;+\; \sum_{i=0}^{2^m-2} e(a\,\beta^{i\,d})\,e(a\,\beta^{-i\,d}) \\
&= 1 \;+\; \sum_{j=0}^{u-1}\sum_{s=0}^{h-1} e(a\,(\beta^{(j+s\,u)\,d})e(a\,\beta^{-(j+s\,u)\,d}) \\
&= 1 \;+\; \sum_{j=0}^{u-1} e(a\,\beta^{j\,d})\,e(a\,\beta^{-j\,d}) \sum_{s=0}^{h-1} e(a\,\beta^{s\,u\,d})e(a\,\beta^{-s\,u\,d}) \\
&= 1 \;+\; h\sum_{j=0}^{u-1} e(a\,\beta^{j\,d})\,e(a\,\beta^{-j\,d}) \\
&= 1 \;+\; h\sum_{j=0}^{u-1} e(a\,(\beta^{j\,d} \;+\; \beta^{-j\,d})).
\end{aligned}$$

From the last expression and Statement 1 we have the following

Statement 2 . *Let d be any odd integer, $1 \le d \le 2^m - 3$, and $h = (d, 2^m - 1)$. For any elements $a_1, a_2 \in \mathbb{F}^*$ the following congruence is valid:*

$$S(a_1\,(x^d + x^{-d})) - S(a_2\,(x^d + x^{-d})) \equiv \begin{cases} 0 & (\bmod\ 4\,h), \ \ \textit{if} \ \ \mathrm{Tr}(a_1) \ne \mathrm{Tr}(a_2), \\ 0 & (\bmod\ 8\,h), \ \ \textit{if} \ \ \mathrm{Tr}(a_1) = \mathrm{Tr}(a_2), . \end{cases}$$

From the proof of Statement 1 also follows clearly the following more general

Statement 3 . *Let $a_1, \dots, a_t \in \mathbb{F}^*$ and $d_1, \dots, d_t$ be any odd numbers, $1 \le d_i \le 2^m - 3$, $i = 1, \dots, t$. Then*

$$S\left(\sum_{i=1}^{t} a_i\,(x^{d_i} \ + \ x^{-d_i})\right) \equiv \begin{cases} 0 & (\bmod\ 8), \ \ \textit{if} \ \ \mathrm{Tr}\,(\sum_{i=1}^{t} a_i) = 0, \\ 4 & (\bmod\ 8), \ \ \textit{if} \ \ \mathrm{Tr}\,(\sum_{i=1}^{t} a_i) = 1. \end{cases}$$

3. Polynomials of type $a(x + x^{-1})^d$. Now consider polynomials $f(x) = a(x + x^{-1})^d$, where

$$d \ = \ \sum_{j=0}^{\ell} 2^{d_j}$$

is any odd number, $3 \le d \le 2^m - 3$,

$$m > d_\ell > d_{\ell-1} > \ldots > d_0 = 0$$

and $\ell \ge 1$. Clearly

$$\begin{aligned} f(x) \ &= \ a\,\left(x \ + \ x^{-1}\right)^{\sum_{j=0}^{\ell} 2^{d_j}} \\ &= \ a \prod_{j=0}^{\ell} \left(x^{2^{d_j}} + x^{-2^{d_j}}\right) \\ &= \ \sum_{k} a\,(x^k + x^{-k}), \end{aligned}$$

where summing is taken over all k of the type

$$k \ = \ 2^{d_\ell} \pm 2^{d_{\ell-1}} \pm \ldots \pm 1$$

(i.e. the value k takes 2^ℓ different values). Since $Tr(\sum_k a) = 0$ (indeed, the sum consists of even number of monoms), then according to Statement 3 (taking into account, that $a_i = a$ for all $i = 1, \dots, 2^\ell$) we obtain the following result.

Statement 4 . *Let $a \in \mathbb{F}^*$ and d be any odd number, $3 \le d \le 2^m - 3$. Then*

$$S\left(a\,(x \ + \ x^{-1})^d\right) \equiv 0 \quad (\bmod\ 8).$$

From Statement 3 also follows more general result.

Statement 5 . *Let $a_1, \dots, a_t \in \mathbb{F}^*$, $d_1, \dots, d_t$ be any odd numbers, $3 \le d_i \le 2^m - 3$. Then*

$$S\left(\sum_{j=1}^{t} a_j\,(x \;+\; x^{-1})^{d_j}\right) \equiv 0 \pmod 8.$$

Remark 1 . *Clearly Statement 5 is still satisfied, if we change $x + x^{-1}$ by $x^r + x^{-r}$, where r is any integer, $3 \le r \le 2^m - 3$.*

4. Polynomials of type $a\,x^d + b\,x^{-1}$. Consider now polynomials $f(x) = a\,x^d + b\,x^{-1}$, where $a, b \in \mathbb{F}$, $a \ne 0$, and d is any odd number, $1 \le d \le 2^m - 3$. First prove the following statement.

Proposition 2 . *Let $c \in \mathbb{F}^*$. Then*

$$S(c\,x^d + x^{-1}) \;+\; S(c\,x^d) \;=\; S(c\,(x + x^{-1})^d).$$

Proof. The following chain of equalities takes place:

$$\begin{aligned}
& S(c\,x^d + x^{-1}) \;+\; S(c\,x^d) \\
= & \sum_{Tr(x^{-1})=0} e(cx^d + x^{-1}) \;+\; \sum_{Tr(x^{-1})=1} e(cx^d + x^{-1}) \\
+ & \sum_{Tr(x^{-1})=0} e(cx^d) \;+\; \sum_{Tr(x^{-1})=1} e(cx^d) \\
= & \sum_{Tr(x^{-1})=0} e(cx^d + x^{-1}) \;+\; \sum_{Tr(x^{-1})=0} e(cx^d) \\
= & \; 2 \times \sum_{x\in\mathbb{F}:\; Tr(x^{-1})=0} e(cx^d).
\end{aligned}$$

Since the equation

$$y \;+\; \frac{1}{y} \;=\; x, \quad x \in \mathbb{F} \tag{1}$$

has two distinct zeros y_1, y_2 in the field $\mathbb{F}$, if and only if $Tr(x^{-1}) = 0$ (see [8]) and the number of the elements $x \in \mathbb{F}$, such that $Tr(x^{-1}) = 0$, is equal to 2^{m-1}, then the solutions y_1, y_2 of the equation (1), for all such x, run over the all field $\mathbb{F}$. Therefore,

$$2 \times \sum_{x\in\mathbb{F}:\; Tr(x^{-1})=0} e\,(\,c\,x^d) \;=\; \sum_{y\in\mathbb{F}} e\,(c\,(y + \frac{1}{y})^d) \;=\; S\,(c\,(x + x^{-1})^d). \qquad \triangle$$

From Proposition 2 and Statement 4 the following statement follows.

Statement 6 . *Let $a, b \in \mathbb{F}^*$ and d be any odd integer, $3 \le d \le 2^m - 3$. Then*

$$S\,(a\,x^d \;+\; b\,x^{-1}) \;\equiv\; 0 \pmod 8,$$

if and only if

$$S\,(a\,x^d) \equiv 0 \pmod 8.$$

Proof. Since

$$S\,(a\,x^d \;+\; b\,x^{-1}) \;=\; S\,(c\,x^d \;+\; x^{-1}).$$

where $c = a\,b^d$, and $S\,(c\,x^d) = S\,(a\,x^d)$, then, according to Proposition 2,

$$S\,(a\,x^d \;+\; b\,x^{-1}) \;=\; S\,(c\,(x \;+\; x^{-1})^d) \;-\; S\,(a\,x^d).$$

Now to complete the proof it is enough to use Statement 4. □

Recall that if d and $2^m - 1$ are relatively prime, then $S\,(a\,x^d) = 0$ for any element a from $\mathbb{F}$.

Corollary 1 . *Let $a, b \in \mathbb{F}^*$ and d be any odd integer, such that $3 \leq d \leq 2^m - 3$. If d and $2^m - 1$ are mutually prime, then*

$$S\,(a\,x^d \;+\; b\,x^{-1}) \;\equiv\; 0 \pmod 8.$$

Corollary 2 . *Since $S\,(a\,x^3) \equiv\; 0 \pmod 8$ for even $m \geq 6$ (see [2]), then*

$$S\,(a\,x^3 \;+\; b\,x^{-1}) \equiv\; 0 \pmod 8, \quad a, b \in \mathbb{F}^*,$$

for even $m \geq 6$.

This result was known (see [3]).

5. Kloosterman sums. Now consider Kloosterman sums $K(a)$. Here we study the divisibility of such sums by the maximal possible number of the type 2^k (i.e. 2^k divides $K(a)$, but 2^{k+1} does not divide $K(a)$).

We are going to formulate a simple recurrent algorithm for finding the largest k, such that 2^k divides the Kloosterman sum $K(a)$. Before it, recall Lemma 7.4 in [10].

Lemma A [10]. *Let polynomials $g_i = g_i(x)$ over $\mathbb{F}$ be defined by the following recurrent construction:*

$$\begin{array}{lcll} g_0 & = & x, & \\ g_1 & = & x \;+\; b_1, & \textit{where } b_1^4 = a^2, \\ \cdots & \cdots & & \cdots \end{array}$$

$$g_i \;=\; g_{i-1}^2 + b_i x \prod_{j=1}^{i-1} (g_j)^2, \quad \textit{where } b_i^{2^{i+1}} = a^2 \textit{ for } i \geq 2. \tag{2}$$

Then the all zeros of the polynomial g_{k-1} gives the x-th coordinates of the points of order 2^k of the elliptic curve $E(a)$ over $\mathbb{F}$, defined by the following equation:

$$y^2 \;+\; x\,y \;=\; x^3 \;+\; a^2. \tag{3}$$

Now recall the following result due to Lisonek P. (see reference in [9]), which we formulate only for $p = 2$.

Theorem B [9]). *Let $a \in \mathbb{F}^*$, and let $0 \le k \le m$. Then $2^k | K(a)$ if and only if there exists a point of order 2^k on $E(a)$, where the curve $E(a)$ is defined by (3).*

For a given element $a \in \mathbb{F}^*$, define now the sequence $x_1, x_2, \dots, x_k$ of elements of the field $\mathbb{F}$ by the following recurrent expression:

$$\left.\begin{aligned} x_1 &= 0, \\ x_{i+1}^2 + \sqrt{x_i}\, x_{i+1} + a &= 0, \quad i = 1, \dots, k-1. \end{aligned}\right\} \tag{4}$$

The following one of the main results follows from Lemma A and Theorem B.

Theorem 1 . *Let a be any element of $\mathbb{F}^*$ and let a sequence of elements $x_1, x_2, \dots, x_k$ be constructed in accordance with recurrent relation (4). Let k be the smallest natural number, such that $\mathrm{Tr}(x_k) = 1$. Then the Kloosterman sum $K(a)$ is divisible by 2^k and is not divisible by 2^{k+1}.*

It is interesting to find a direct proof of Theorem 1, which does not use a transition to the number of rational points of the elliptic curve $E(a)$.

Corollary 3 *[7]. Since $x_2 = \sqrt{a}$, then $K(a)$ is divisible by 4, but not divisible by 8, if $Tr(a) = 1$.*

Corollary 4 *[7], [9]. Let $Tr(a) = 0$. Then a can be presented as $a = z^8 + z^{16}$. In this case $x_3 = z^6 + z^8$. If $Tr(z) \ne Tr(z^3)$, then $K(a)$ is divisible by 8, but not divisible by 16.*

Corollary 5 *[9]. Under conditions of the Corollary 4 above, if $Tr(z) = Tr(z^3)$, then $K(a)$ is divisible by 16.*

Recall (see [5] and references there) that the function $f_{a,r}(x)$,

$$f_{a,r}(x) = Tr\left(a\, x^{r(2^{m/2}-1)}\right), \quad a \in \mathbb{F}^*,$$

for even m and natural r, such that r and $2^{m/2}+1$ are mutually prime, is bent, if and only if $K(a) = 0$, and the function $f_a(x)$

$$f_a(x) = Tr\left(a\, x^{2^m-1}\right), \quad a \in \mathbb{F}^*,$$

is hyperbent, if and only if $K(a) = 0$.

Theorem 1 implies the following simple necessary and sufficient condition for an element a to be a zero of the Kloosterman sum $K(a)$, i.e. in order to have $K(a) = 0$.

Theorem 2 . *Let a be any element of $\mathbb{F}^*$ and let a sequence $u_1, u_2, \dots, u_m$ of elements from $\mathbb{F}$ be defined in accordance with the following recurrent relation:*

$$u_{i+1} = u_i^2 + \frac{a^2}{u_i^2},$$

where $u_1 \in \mathbb{F}^*$ *is any element of* $\mathbb{F}^*$*, such that*

$$\mathrm{Tr}(u_1) \ = \ 1 \quad \textit{and} \quad \mathrm{Tr}(u_1 + \frac{a}{u_1}) \ = \ 0.$$

Then $K(a) = 0$*, if and only if* $u_m = 0$*, and the all* $m-1$ *elements* $u_1, \dots u_{m-1}$ *are nonzero. Furthermore, if the first zero element in the sequence* $u_1, u_2, \dots, u_m$ *appears on the kth place, where* $k < m$*, then* k *is the largest integer, such that* 2^k *divides* $K(a)$*.*

References

1. Bassalygo L.A., & Zinoviev V.A., *Polynomials of special form over finite fields with given values of exponential sum*, Matematicheskie Zametki, 2007, vol. 82, pp. 16 - 25.
2. L. Carlitz, *Explicit evaluation of certain exponential sums*, Math. Scand., 1979, vol. 44, pp. 5-16.
3. P. Charpin, T. Helleseth & V.A. Zinoviev, *On cosets of weight* 4 *of binary primitive BCH codes of length* 2^m *(m even) with minimum distance* 8 *and exponential sums*, SIAM J. of Discrete Math., 2008, vol. 23, No. 1, p. 59 - 78.
4. P. Charpin, T. Helleseth & V.A. Zinoviev, *On divisibility properties of classical binary Kloosterman sums*, Discrete Mathematics, 2009, vol. 309, no. 12, pp. 3975-3984.
5. P. Charpin & G. Gong, *Hyperbent functions, Kloosterman sums, and Dickson polynomials*, IEEE Transactions on Information Theory, 2008, vol. 54, no. 9, pp. 4230-4238.
6. J.F. Dillon, Elementary Hadamard difference sets. In Proc. *Sixth Southeastern Conference on Combinatorics, Graph theory, and Computing.* Congressus Numerantium, No. XIV, Utilitas Math., Winnipeg, Man., 1975, pp. 237-249.
7. T. Helleseth & V.A. Zinoviev, *On* Z_4*-Linear Goethals Codes and Kloosterman Sums*, Designs, Codes and Cryptography, 1999 vol. 17, No. 1-3, pp. 246-262.
8. R. Lidl & H. Niederreiter, *"Finite Fields", Encyclopedia of Mathematics and Its Applications*, Reading, MA: Addison Wesley, 1983, vol. 20.
9. P. Lisonek, M. Moisio, *On zeros of Kloosterman sums*, Designs, Codes and Cryptography, 2011, to appear.
10. A. Menezes, *Elliptic Curve Public Key Cryptosystems*, Kluwer Academic Publishers, Boston-Dordrecht-London, 1993.