A Non-Standard Semantics for Kahn Networks in Continuous Time

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Abstract

In a seminal article, Kahn has introduced the notion of process network and given a semantics for those using Scott domains whose elements are (possibly infinite) sequences of values. This model has since then become a standard tool for studying distributed asynchronous computations. From the beginning, process networks have been drawn as particular graphs, but this syntax is never formalized. We take the opportunity to clarify it by giving a precise definition of these graphs, that we call *nets*. The resulting category is shown to be a *fixpoint category*, i.e. a cartesian category which is traced wrt the monoidal structure given by the product, and interestingly this structure characterizes the category: we show that it is the free fixpoint category containing a given set of morphisms, thus providing a complete axiomatics that models of process networks should satisfy. We then use these tools to build a model of networks in which data vary over a *continuous* time, in order to elaborate on the idea that process networks should also be able to encompass computational models such as hybrid systems or electric circuits. We relate this model to Kahn's semantics by introducing a third model of networks based on non-standard analysis, whose elements form an *internal complete partial order* for which many properties of standard domains can be reformulated. The use of hyperreals in this model allows it to formally consider the notion of infinitesimal, and thus to make a bridge between discrete and continuous time: time is "discrete", but the duration between two instants is infinitesimal. Finally, we give some examples of uses of the model by describing some networks implementing common constructions in analysis.

Process networks have been introduced by Kahn, together with an associated semantics based on Scott domains, as one of the first model for concurrent and asynchronous computations [19]. These networks are constituted of *processes* (which may be thought as computers on a network or threads on a computer for instance) which perform computations and can exchange information through *channels* acting as unbounded FIFO queues. Finite or infinite sequences of values, that are called *streams*, are communicated on the channels, and the semantics of the whole process network is considered to be the streams that can be exchanged, depending on the data provided by the environment. The set of streams can be structured as a complete partial order, and the semantics of networks is modeled by Scott-continuous functions on this domain: the fact that these functions admit a smallest fixpoint turns out to be crucial in order to model "loops" formed by channels in the network. A series of subsequent works have provided a precise understanding of this fixpoint construction [11, 16].

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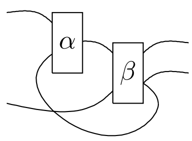
In this model, time is *discrete* in the sense that a countable number of values might be exchanged during an execution: we can consider that each value occurs at a given instant $t \in \mathbb{N}$. In this article, we are interested in understanding how to extend the usual semantics of process networks in order to consider computations in *continuous* time, by replacing $\mathbb{N}$ by $\mathbb{R}^+$ for the domain of time, so as to embrace computational models such as electric circuits or hybrid systems, with which it bears many similarities. However, how would such a semantics relate to the usual discrete semantics of networks? The fundamental intuition in order to relate the two models is the following: if we allow ourselves to consider *infinitesimal* durations $\mathrm{d}t$, then we can think of continuous time as being somehow "discrete", its instants being $0, \mathrm{d}t, 2\,\mathrm{d}t, 3\,\mathrm{d}t, \ldots$ This idea of *infinitesimal time* originates in the works of Bliudze and Krob [7], and was later on developed by Benveniste, Caillaud and Pouzet [6], who formalized it by using tools provided by *non-standard analysis* [27, 12] in order to give a rigorous meaning to infinitesimals. Here, we develop on these ideas by structuring the resulting notion of stream into *internal Scott domains*, which are shown to provide a model of process networks, and explain how the resulting model provides a nice bridge between discrete and continuous time.

For this purpose, we introduce a new model for Kahn process networks. However, what is precisely the syntax for these networks that we want to model? Here, we formalize the definition of the graphs which are often used to informally represent process networks, by a structure that we call *nets*. We show here that the resulting category is a *fixpoint category*, i.e. a cartesian category which is traced [18] wrt the monoidal structure provided by products. Moreover, this structure characterizes the category in the sense that the category of nets is a free category of such kind. This result thus provides a complete description of the axioms that a model of nets should satisfy. We finally use this structure to show that streams in infinitesimal time form such a model. We elaborate here on a series of earlier works. The structure of traced monoidal category for the Kahn model has been introduced by Hildebrandt, Panangaden and Winskel [16] and the construction of nets introduced here is a generalization of the one introduced in [15]. Properties of fixpoint categories and their relationship to fixpoint operators have been studied in details by Hasegawa [14].

We begin by defining nets (Section 1.1), show that they are free fixpoint categories (Section 1.2) and explain that Scott-domain semantics can be given for nets (Section 1.3). We then recall basic constructions and properties of non-standard analysis (Section 2.1), define the notion of internal domain of which infinitesimal-time streams are an instance (Section 2.2) and relate models in infinitesimal and continuous time (Section 2.3). We finally conclude in Section 3.

1 Nets and their semantics



A Kahn process network [19] can be thought as a finite set of boxes, with inputs and outputs, linked through wires, producing outputs depending on their inputs which will be asynchronously transmitted over the wires. Over time, data circulates through the network, producing streams of data. The dataflow semantics of these networks has been well-studied in relation with Scott domains and category theory [25, 16]. However, there is no canonical syntax for them, even though a graphical notation (as pictured on the right) is often used. Since the purpose of this paper is to provide a new semantics for process networks, we take this opportunity to clarify the definition of the syntax, by formalizing the graphical notation and relating it with the categorical structure of the models. The ideas developed here in order to develop an axiomatics for Kahn process networks originate in various previous works in the field. Kahn's original paper [19] mentions results of decidability of the equivalence of graphs representing networks (which are called *schemata*) based on earlier works [9]. Many subsequent articles underline the importance of operations on networks such as sequential composition, parallel composition, tupling (products) and feedback [11, 25], and a traced monoidal category modeling Kahn networks was constructed in [16]. On the categorical side, the "drawings" used here have been formalized as *string diagrams* representing morphisms in monoidal categories [17]. Traced monoidal categories were introduced in [18] and turned out to be a fundamental tool in computer science [1]. Their axiomatics was simplified and studied in the cartesian case [14] and constructions of free traced monoidal categories were provided in [2, 15].

1.1 Nets

A *signature* $\Sigma = (\Sigma, \sigma, \tau)$ consists of a set Σ of *symbols* and two functions $\sigma, \tau : \Sigma \to \mathbb{N}$, which to every symbol α associate its *arity* and *coarity* respectively – we thus sometimes write $\alpha : \sigma(\alpha) \to \tau(\alpha)$: the symbols should be thought as possible building blocks for a process network, with specified number of inputs and outputs. Given such a signature, a net consists of instances of symbols (called operators) whose inputs and outputs are linked together through wires (called ports) defined as follows. Given an integer n, we write $\langle n \rangle$ for the set $\{0, \dots, n-1\}$.

▶ **Definition 1** (Net). A *net* $N = (P, O, \lambda, s, t)$ from m to n, with $m, n \in \mathbb{N}$, consists of

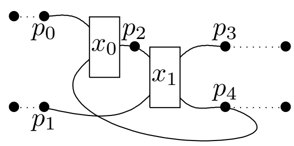
- a finite set P of *ports*,
- a finite set O of *operators*,
- a *labeling function* $\lambda : O \to \Sigma$,
- a *source function* $s : S_N \to P$ and an injective *target function* $t : T_N \to P$, where

$$S_N = \{(x,i) \mid x \in O, i \in \langle \sigma \circ \lambda(x) \rangle\} \uplus \langle n \rangle \qquad T_N = \{(x,i) \mid x \in O, i \in \langle \tau \circ \lambda(x) \rangle\} \uplus \langle m \rangle$$

We sometimes write $N : m \to n$ to indicate that N is a net from m to n.

▶ **Example 2.** Suppose that Σ contains two symbols α and β whose sources (given by σ) are both 2 and targets (given by τ) are respectively 1 and 2. The net drawn in the introduction of this section can be formalized as a net $N : 2 \to 2$ defined by $P = \{p_0, \dots, p_4\}$, $O = \{x_0, x_1\}$, $\lambda(x_0) = \alpha$, $\lambda(x_1) = \beta$, s and t are defined by

$$s(x_0,0) = p_0 \qquad s(x_0,1) = p_4 \qquad s(x_1,0) = p_2 \qquad s(x_1,1) = p_1 \qquad s(0) = p_3 \qquad s(1) = p_4$$
$$\text{and} \qquad t(x_0,0) = p_2 \qquad t(x_1,0) = p_3 \qquad t(x_1,1) = p_4 \qquad t(0) = p_0 \qquad t(1) = p_1$$



Graphically, this can be pictured as on the right. The bullets on the left and on the right indicate the source and target m and n and the dotted lines represent the induced source and target functions $\langle m \rangle \to P$ and $\langle n \rangle \to P$ respectively. Notice that a port can be used as input for multiple wires as it is the case for the port p_4 in the example. However, t being injective, two wires cannot have the same output port.

▶ **Definition 3.** A *morphism* $\varphi : M \to N$ between two nets $M, N : m \to n$ (with the same source and target) consists of a pair of functions $\varphi_P : P_M \to P_N$ and $\varphi_O : O_M \to O_N$ such that for every operator $x \in O_M$, $\lambda_N(\varphi_O(x)) = \lambda_M(x)$, for every source $(x,i) \in S_M$, $\varphi_P(s_M(x,i)) = s_N(\varphi_O(x), i)$, for every $k \in \langle n \rangle$, $\varphi_P(s_N(k)) = s_M(k)$ and similar equations for target functions. Two nets M and N are isomorphic when there exists an invertible morphism between them, which we write $M \approx N$.

▶ **Definition 4.** In order to define a category whose objects are integers and morphisms are nets (considered up to isomorphism), we introduce the following constructions:

- *Identity.* The identity net $N : n \to n$ is the net such that $P = \langle n \rangle$, $O = \emptyset$ and $s, t : \langle n \rangle \to P$ are both the identity function.
- *Composition.* Given two nets $M : n_1 \to n_2$ and $N : n_2 \to n_3$, their composite $N \circ M : n_1 \to n_3$ is the net defined by $P = P_M \uplus P_N / \sim$ where $\sim$ is the smallest equivalence relation such that $s_M(k) \sim t_N(k)$ for every $k \in \langle n_2 \rangle$, $O = O_M \uplus O_N$, $\lambda = \lambda_M \uplus \lambda_N$, s is defined by $s(x, i) = s_M(x, i)$ if $x \in O_M$, $s(x, i) = s_N(x, i)$ if $x \in O_N$ and $s(k) = s_N(k)$ if $k \in \langle n_3 \rangle$, and t is defined similarly.
- *Tensor.* Given two nets $M : m \to m'$ and $N : n \to n'$, their tensor product net $M \otimes N : m + n \to m' + n'$ is the net which is defined by $P = P_M \uplus P_N$, $O = O_M \uplus O_N$, $\lambda = \lambda_M \uplus \lambda_N$, s is defined by $s(x, i) = s_M(x, i)$ if $x \in O_M$ and $s(x, i) = s_N(x, i)$ if $x \in O_N$, $s(k) = s_M(k)$ if $k \in \langle m' \rangle$ and $s(k) = s_N(k - m')$ if $k \in \langle n' \rangle$, and t is defined similarly.
- *Trace.* Given a net $N : n_1 + n \to n_2 + n$, we define the net $\mathrm{Tr}^n_{n_1,n_2}(N) : n_1 \to n_2$ by $P = P_N / \sim$ where $\sim$ is the smallest equivalence relation such that $s_N(n_2+k) = t_N(n_1+k)$ for every $k \in \langle n \rangle$, $O = O_N$, $\lambda = \lambda_N$, $s = q \circ s_N$ and $t = q \circ t_N$ where $q : P_N \to P$ is the canonical quotient map.

It can easily be shown that the constructions above are compatible with isomorphisms of nets (e.g. if $M \approx M'$ and $N \approx N'$ then $M \otimes N \approx M' \otimes N'$). It thus makes sense to define the following category:

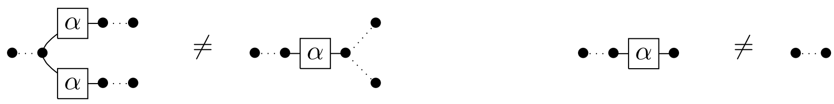
▶ **Definition 5.** We write $\mathbf{Net}_\Sigma$ for the category $\mathbf{Net}_\Sigma$ whose objects are natural integers, morphisms $N : m \to n$ are isomorphism classes of nets, identities and composition are given by the constructions of Definition 4.

▶ **Lemma 6.** *The category $\mathbf{Net}_\Sigma$ is well-defined and is monoidal with the tensor product of Definition 4 and 0 as unit.*

▶ Remark. In order to make a more fine-grained study of the categorical structure of nets, we could have avoided quotienting morphisms by isomorphisms of net and defined a bicategory [5] whose 0-cells are integers, 1-cells are nets and 2-cells are morphisms of nets. We did not do this here to simplify the presentation.

▶ Remark. This construction, as well as the following, can be extended without difficulty to define multisorted nets (i.e. where the various inputs of operators have different types), see [15] for a similar construction. A nice and abstract description of this construction can be carried on using polygraphs [8], in a way similar to [24].

Even though the output of an operator can be duplicated or erased, the category $\mathbf{Net}_\Sigma$ fails to have finite products. This is essentially due to the fact that duplication and erasure are not natural, e.g. the following nets (of type $1 \to 2$ and $1 \to 0$ respectively) are different:



We can however recover products by considering the two inequalities above as rewriting rules on nets from left to right as follows.

- *Sharing.* Given a net $N : m \to n$, suppose that there exists two operators $x, y \in O$ with the same label and the same inputs, i.e. $\lambda(x) = \lambda(y)$ and for every $i \in \langle \sigma \circ \lambda(x) \rangle$, $s(x, i) = s(y, i)$. We define a net $N' : m \to n$ by $P = P_N / \sim$ where $\sim$ is the smallest

equivalence relation such that $t(x,i) \sim t(y,i)$ for every $i \in \langle \tau \circ \lambda(x) \rangle$, $O = O_N / \sim'$ where $\sim'$ is the smallest equivalence relation identifying x and y, and λ, s and t are obtained by quotienting the corresponding functions of N. The net N' is said to be obtained from N by *sharing*.
- *Erasing.* Given a net $N : m \to n$, suppose that there exists an operator $x \in O$ such that for every $i \in \langle \tau \circ \lambda(x) \rangle$, $s^{-1}(t(x,i)) = \emptyset$. Informally, none of the outputs of the operator x is used as an input for some other operator. We write $N' : m \to n$ for the net obtained from N by removing the operator x as well as all the ports $t(x,i)$ for $i \in \langle \tau \circ \lambda(x) \rangle$. The net N' is said to be obtained from N by *erasing*.

We say that N *se-rewrites* to N' when N' can be obtained from N by sharing or erasing. The *se-equivalence* is the smallest equivalence relation containing the se-rewriting relation.

▶ **Proposition 1.** *The category* $\mathbf{sNet}_\Sigma$ *obtained from* $\mathbf{Net}_\Sigma$ *by quotienting morphisms by se-equivalence has finite products, given on objects by the tensor product of* $\mathbf{Net}_\Sigma$*.*

Proof. The terminal object is 0 and the product of two objects m and n is $m+n$ with the projection of m defined as the net $N : m+n \to m$ such that $P = \langle m+n \rangle$, $O = \emptyset$, $s : \langle m \rangle \to P$ is the canonical injection and $t : \langle m+n \rangle \to P$ is the identity (and the projection on n is defined similarly). All required axioms are easily verified. ◀

It can be shown that the se-rewriting rules form a terminating (they decrease the number of operators) and confluent rewriting system. The normal forms are nets which do not contain two operators with the same label and input ports, and do not contain operators such that none of the outputs are inputs for some other operator. A direct alternative description of nets modulo se-equivalence, called shared nets, can thus be defined as follows.

▶ **Definition 7.** A *shared net* $N = (P, O, s, t)$ from m to n consists of
- a finite set P of *ports*,
- a finite set O of *operators* which are pairs $(\alpha, (s_i)_{i \in \langle \sigma(\alpha) \rangle})$ where $\alpha \in \Sigma$ is a symbol and $(s_i)_{i \in \langle \sigma(\alpha) \rangle}$ is a family of ports called the *sources* of the operators,
- a *source function* $s : \langle n \rangle \to P$,
- an injective *target function* $t : T_N \to P$, where $T_N = \{(x,i) \mid x \in O, i \in \langle \tau \circ \lambda(x) \rangle\} \uplus \langle m \rangle$,

such that for every operator $x \in O$, $s^{-1}(t(T_x)) \neq \emptyset$ where $T_x = \{(x,i) \mid i \in \langle \tau \circ \lambda(x) \rangle\}$.

▶ **Proposition 2.** *A category whose objects are integers and morphisms are shared nets modulo (suitably defined) isomorphism can be defined in a similar way as previously, and this category can be shown to be equivalent to* $\mathbf{sNet}_\Sigma$ *through product-preserving functors.*

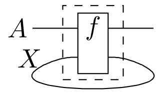
Proof. The canonical forms of nets wrt se-rewriting are in bijection with shared nets. ◀

Next section justifies why the category $\mathbf{sNet}$ provides a convincing definition of the networks.

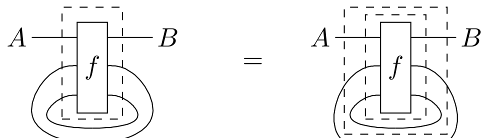
1.2 Nets as free fixpoint categories

We now study the structure of the category $\mathbf{sNet}_\Sigma$ in order to define a proper denotational model this category. Recall that a (strict) *monoidal category* $(\mathcal{C}, \otimes, I)$ consists of a category $\mathcal{C}$ together with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called *tensor*, and an object I, called *unit*, such that the tensor is strictly associative and admits units as neutral elements. A (strict) *symmetric monoidal category* consists of a monoidal category $(\mathcal{C}, \otimes, I)$ equipped with a natural family $\gamma_{A,B} : A \otimes B \to B \otimes A$ of isomorphisms satisfying $\gamma_{B,A} \circ \gamma_{A,B} = \mathrm{id}_{A \otimes B}$ as well as other coherence axioms, see [22] for details. Any category $\mathcal{C}$ with finite products admits a structure of symmetric monoidal category with the cartesian product $\times$ as tensor and the terminal object 1 as unit, and this structure can be chosen to be strict in the case of $\mathbf{sNet}_\Sigma$ (thus we only consider strict monoidal categories in the following for simplicity). A natural notion of "feedback" was formalized in monoidal categories by Joyal, Street and Verity [18] as follows:

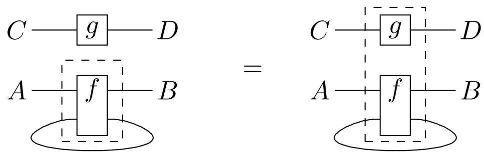
▶ **Definition 8** (Trace). A *trace* on a symmetric monoidal category $\mathcal{C}$ consists of a *natural* family of functions $\mathrm{Tr}^X_{A,B} : \mathcal{C}(A \otimes X, B \otimes X) \to \mathcal{C}(A, B)$. Given a morphism $f : A \otimes X \to B \otimes X$, the morphism $\mathrm{Tr}^X_{A,B}(f) : A \to B$ is often drawn as on the right. A trace should satisfy the following axioms.



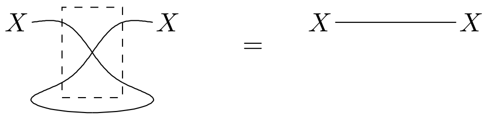
1. *Vanishing*: for every $f : A \otimes X \otimes Y \to B \otimes X \otimes Y$, $\mathrm{Tr}^{X \otimes Y}_{A,B}(f) = \mathrm{Tr}^X_{A,B}(\mathrm{Tr}^Y_{A \otimes X, B \otimes X}(f))$



2. *Superposing*:
for every $f : A \otimes X \to B \otimes X$ and $g : C \to D$, $g \otimes \mathrm{Tr}^X_{A,B}(f) = \mathrm{Tr}^X_{C \otimes A, D \otimes B}(g \otimes f)$



3. *Yanking*: for every object X, $\mathrm{Tr}^X_{X,X}(\gamma_{X,X}) = \mathrm{id}_X$



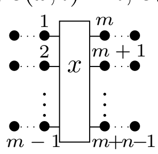
▶ **Proposition 3.** *The construction of Definition 4 induces a trace on* $\mathbf{Net}_\Sigma$ *and on* $\mathbf{sNet}_\Sigma$.

The category $\mathbf{sNet}_\sigma$ is a traced cartesian category that we call a *fixpoint category* in the following. Interestingly, it is actually characterized by this structure in the sense that it is the free fixpoint category containing a Σ-object.

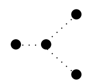
▶ **Definition 9** (Σ-object). Given a signature Σ, a Σ-*object* in a monoidal category $\mathcal{C}$ consists of an object A together with a morphism $f_\alpha : \otimes^{\sigma(\alpha)} A \to \otimes^{\tau(\alpha)} A$ for every symbol $\alpha \in \Sigma$, called the *interpretation* of α, where $\otimes^n A$ denotes the tensor product of n copies of the object A. A Σ-*morphism* between two Σ-objects (A, f_α) and (B, g_α) consists of a morphism $h : A \to B$ such that for every $\alpha \in \Sigma$, $(\otimes^{\tau(\alpha)} h) \circ f_\alpha = g_\alpha \circ (\otimes^{\sigma(\alpha)} h)$.

▶ **Theorem 10.** *The category* $\mathbf{sNet}_\Sigma$ *is the free category containing a* Σ*-object in the sense that for every fixpoint category* $\mathcal{C}$*, the category* $\mathbf{Mod}(\Sigma, \mathcal{C})$ *of* Σ*-objects and* Σ*-morphisms is equivalent to the category* $\mathbf{Fix}(\mathbf{sNet}_\Sigma, \mathcal{C})$ *whose objects are fixpoint functors (preserving cartesian product and trace) and morphisms are monoidal natural transformations.*

Proof. The category $\mathbf{sNet}_\Sigma$ contains a Σ-object whose underlying object is 1 and the interpretation of a symbol α with $\sigma(\alpha) = m$ and $\tau(\alpha) = n$ is the net $N : m \to n$ such that $P = \langle m + n \rangle$, $O = \{x\}$, $\lambda(x) = \alpha$, $s(x, i) = i$, $s(k) = m + k$, $t(x, i) = m + i$, $t(k) = k$:



A construction of the free traced symmetric monoidal category containing a Σ-object was provided in [2] and reformulated in [15] using a variant of the nets defined here, that we call *traced nets*. It is easy to see that we recover traced nets by restricting $\mathbf{sNet}_\Sigma$ to the nets such that the source function s is a bijective function. We thus have to show that our category of nets is the free category over the category of traced nets. Recall that a cocommutative comonoid (M, δ, ε) in a symmetric monoidal category consists of an object M together with two morphisms $\delta : M \to M \otimes M$ (called *duplication*) and $\varepsilon : M \to I$ (called *erasure*), which are such that $(\delta \otimes \mathrm{id}_I) \circ \delta = (\mathrm{id}_I \otimes \delta) \circ \delta$, $(\varepsilon \otimes \mathrm{id}_I) \circ \delta = \delta = (\mathrm{id}_I \otimes \varepsilon) \circ \delta$ and $\gamma_{M,M} \circ \delta = \delta$. Now, it has been shown [8] that the category whose objects are integers and whose morphisms $f : m \to n$ are functions $f : \langle m \rangle \to \langle n \rangle$ is the free monoidal symmetric monoidal category containing a commutative monoid, and that the free cartesian category over a symmetric monoidal category is obtained by freely adding a natural structure of cocommutative comonoids over each object: precisely, this means that each object M is equipped with a structure $(M, \delta_M, \varepsilon_M)$ of cocommutative comonoid and these are natural in the sense that for every morphism $f : M \to N$, $\delta_N \circ f = (f \otimes f) \circ \delta_M$ and $\varepsilon_N \circ f = \varepsilon_M$. From this it can be deduced that going from nets with bijective s to nets with any function as s, and quotienting by se-equivalence, amounts to construct the free cartesian category over the category of traced nets. Namely, allowing any function equips the object 1 with a structure of comonoid with the duplication δ_1 being the net $N_{\delta_1} : 1 \to 2$ such that $P = \{p\}$, $O = \emptyset$, $s(k) = p$ and $t(k) = p$ and the duplication ε_1 being the net $N_{\varepsilon_1} : 1 \to 0$ such that $P = \{p\}$, $O = \emptyset$ and $t(k) = p$:

$N_{\delta_1} \quad =$  $\qquad N_{\varepsilon_1} \quad =$

(and every object can be equipped with a structure of cocommutative comonoid in a similar way). Quotienting by se-equivalence amounts to impose that the structures of cocommutative comonoid on the objects are natural. ◀

1.3 Models of nets

The properties of fixpoint categories have been studied in details by Hasegawa and Hyland [14]. In particular, they have shown that a cartesian category $\mathcal{C}$ is traced if and only if it contains a fixpoint operator satisfying suitable axioms (these are sometimes called *Conway fixpoint operators*). For instance, the category of Scott domains recalled below admits such a fixpoint and is therefore a fixpoint category thus providing a natural semantics for nets.

A *directed complete partial order* (or *dcpo*) is a poset $(D, \leqslant)$ such that every directed subset X has a supremum $\bigvee X$ and a *complete partial order* (or *cpo*) is a dcpo with a least element $\bot$ [3, 4, 10]. A function $f : A \to B$ between two dcpo is *Scott-continuous* when it preserves supremums. By the Kleene fixpoint theorem, every Scott-continuous function $f : X \to X$ admits a least fixpoint $\mathrm{fix}_X(f)$ defined by $\mathrm{fix}_X(f) = \bigvee_{n \in \mathbb{N}} f^n(\bot_X)$, where f^n denotes the composite of n instances of f. Suppose given a function $f : A \times X \to B \times X$. We write $\pi_B : B \times X \to B$ and $\pi_X : B \times X \to X$ for the canonical projections. Given $a \in A$, we write $f_a = x \mapsto f(a, x)$ for the partial application of f to a. A trace can defined on f by

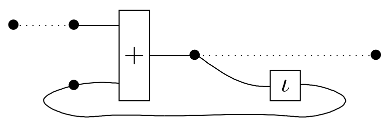
$$\mathrm{Tr}^X_{A,B}(f) \quad = \quad a \quad \mapsto \quad \pi_B(\mathrm{fix}_{B \times X}(f_a \circ \pi_X)) \tag{1}$$

and this function can be shown to be Scott-continuous.

▶ **Proposition 4.** *The category* **Cpo** *of cpo and Scott-continuous functions is a fixpoint category with* (1) *as trace.*

By Theorem 10, any Σ-object in **Cpo** canonically induces a functor $F : \mathbf{Net}_\Sigma \to \mathbf{Cpo}$ which associates to every net its *domain semantics*: once we have interpreted each symbol as a Scott-continuous function (by fixing a Σ-object), the interpretation of each network is also fixed. In particular, when the Σ-object is the domain R^∞ of R-valued streams (for some set R), we recover the usual Kahn semantics [19] of nets: R^∞ is the domain whose elements are finite or infinite sequences (called *streams*) of elements of R, ordered by inclusion. The intuition here is that time is discrete (because we only consider the times where some information is passed on a wire as *instants*) and the elements of the domain are the sequences of values passed on wires at the various instants.

▶ **Example 11.** Consider a signature Σ containing two symbols $+ : 2 \to 1$ and $\iota : 1 \to 1$. We consider the Σ-object $\mathbb{R}^\infty$ in **Cpo** such that the interpretation of $+$ is the Scott-continuous function $\mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ such that the image of two streams of same length is their pointwise addition and the interpretation of ι is the function $\mathbb{R}^\infty \to \mathbb{R}^\infty$ which prepends a 0 to streams. The interpretation of the net on the right is the function which returns the stream whose n-th value is the sum of the $n+1$ first values of the stream.



An element of the Kahn domain can be considered as a partial function $s : \mathbb{N} \to R$ whose domain of definition is an initial segment of $\mathbb{N}$ (the integers in $\mathbb{N}$ represent the instants of the time). Our goal here is to consider a semantics where *time is continuous*, i.e. streams are generalized to partial functions $s : \mathbb{R}^+ \to R$ defined on an initial segment of $\mathbb{R}^+$ and to relate it to the Kahn semantics. In order to build bridge between the two models, the intuition here is that continuous time can be considered as "discrete" if we allow ourselves to consider *infinitesimals*: the time in $\mathbb{R}^+$ can namely be thought as a sequence of instants $0, \mathrm{d}t, 2\,\mathrm{d}t, 3\,\mathrm{d}t, \ldots$ where $\mathrm{d}t$ is an infinitesimal, thus enabling us to extend techniques developed for Kahn networks to continuous time semantics. Moreover, many operations of common use in analysis can be simply formulated by an appropriate net with the continuous time semantics. For instance, the derivative of a function whose definition can be formulated as $f'(t) = (f(t) - f(t - \mathrm{d}t)) / \,\mathrm{d}t$ can be implemented by a net of the form (4) which directly translates to nets the above formula. The rest of the paper is devoted to explaining and formalizing these ideas by using of non-standard analysis which allows us to rigorously make sense of the notion of infinitesimal.

2 A non-standard semantics for Kahn networks in continuous time

2.1 Hyperreals

We give here a brief introduction to non-standard analysis and refer the reader to textbooks [12, 27, 26] for details. The motivation underlying the construction of hyperreals is to extend the field $\mathbb{R}$ of real numbers into a field $^*\mathbb{R}$ in which one can give a meaning to the notions of infinitesimal and infinite numbers. Basically, hyperreal numbers are defined as countable sequences $(x_i)_{i\in\mathbb{N}}$ of real numbers, the sequences converging towards 0 representing infinitesimals. Any real x can be seen as the hyperreal which is the constant sequence whose elements are equal to x, moreover the usual operations are extended pointwise to hyperreals, e.g. the multiplication is defined by $(x_i) \times (y_i) = (x_i \times y_i)$. In order for suitable axioms to be satisfied (for instance every non-null hyperreal should have an inverse) one has to consider equivalence classes of such sequences; in particular, any two sequences which only differ on a finite number of values should be equivalent. The starting point of non-standard analysis is the fact that a suitable equivalence relation can be defined from an ultrafilter:

▶ **Definition 12** (Ultrafilter). A *filter* on a set I is a collection $\mathcal{F}$ of sets which is closed under intersection and under supersets (i.e. if $U \subseteq V \subseteq I$ and $U \in \mathcal{F}$ then $V \in \mathcal{F}$). A filter is *proper* when $\emptyset \notin \mathcal{F}$. An *ultrafilter* on I is a proper filter such that for every $U \subseteq I$, either $U \in \mathcal{F}$ or $I \setminus U \in \mathcal{F}$. An ultrafilter $\mathcal{F}$ is *principal* when there exists $i \in I$ such that $\mathcal{F} = \{U \subseteq I \mid i \in U\}$, and *non-principal* otherwise.

Assuming Zorn's lemma (which is equivalent to the axiom of choice), it can be shown that

▶ **Proposition 5.** *Any infinite set I has a non-principal ultrafilter on it.*

We now fix such an ultrafilter $\mathcal{F}$ on $\mathbb{N}$ whose elements are called *large sets*. The fact that $\mathcal{F}$ is non-principal implies that it does not contain any finite subset of $\mathbb{N}$: the construction of the ultrafilter can thus be thought as a way of constructing a set, starting from all cofinite sets, and coherently adding either I or its complement for every set $I \subseteq \mathbb{N}$ which is neither finite nor cofinite. We define an equivalence relation $\equiv$ on countable sequences of reals by $(x_i) \equiv (y_i)$ when $\{i \in \mathbb{N} \mid x_i = y_i\} \in \mathcal{F}$ and denote by $\langle x_i \rangle$ the equivalence class of (x_i).

▶ **Definition 13** (Hyperreals). The set ${}^*\mathbb{R}$ of *hyperreals* is the set of equivalence classes (wrt $\equiv$) of countable sequences of reals.

The set ${}^*\mathbb{N}$ of *hyperintegers* is defined similarly and there is a canonical inclusion ${}^*\mathbb{N} \subseteq {}^*\mathbb{R}$.

Any countable sequence (x_i) of reals induces an hyperreal $\langle x_i \rangle$, and in particular a real r can be seen as an hyperreal ${}^*r = \langle r \rangle$ by considering the equivalence class of the constant sequence whose elements are equal to r (we sometimes leave this conversion implicit). Similarly, a countable sequence (X_i) of subsets of $\mathbb{R}$ induces a set $\langle X_i \rangle$ of hyperreals defined as the set of $\langle x_i \rangle \in {}^*\mathbb{R}$ such that $\{i \in \mathbb{N} \mid x_i \in X_i\} \in \mathcal{F}$. A subset of ${}^*\mathbb{R}$ is an *internal set* if it can be obtained this way, in particular any set $X \subseteq \mathbb{R}$ induces an internal set ${}^*X = \langle X \rangle$, associated to the constant sequence (for instance $\langle \mathbb{R} \rangle = {}^*\mathbb{R}$). Similarly, a countable sequence of functions $(f_i : A_i \to B_i)$, where the A_i and B_i are subsets of $\mathbb{R}$, extends to a function $\langle f_i \rangle : \langle A_i \rangle \to \langle B_i \rangle$, defined on $\langle x_i \rangle \in \langle A_i \rangle$ by $\langle f_i \rangle(\langle x_i \rangle) = \langle \overline{f_i}(x_i) \rangle$ where $\overline{f_i}(x_i) = f_i(x_i)$ if $x_i \in A_i$ and $\overline{f_i}(x_i) = 0$ otherwise. Such a function is called an *internal function*. Any real-valued function $f : A \to B$ may be seen as an internal function ${}^*f = \langle f \rangle : \langle A \rangle \to \langle B \rangle$. The notion of *internal relation* is defined similarly.

▶ Remark. As we explain in Section 2.1.1, it is important to keep in mind that not every set $X \subseteq {}^*\mathbb{R}$ (or function, or relation) is internal.

Notice that in the above definition of an internal function, we have used 0 as "default value" for the functions f_i on the elements $x_i \notin A_i$. This could be avoided by choosing a suitable representative in the equivalence class $\langle x_i \rangle$:

▶ **Lemma 14.** *Given an element x of an internal set $\langle A_i \rangle$, there exists a sequence (y_i), such that $y_i \in A_i$ for every index i, satisfying $\langle y_i \rangle = x$.*

In the way described above, all the usual operations on reals extend to hyperreals (and similarly for hyperintegers). For instance, the absolute value of an hyperreal $\mathtt{x} = \langle x_i \rangle$ is defined by $|\mathtt{x}| = \langle |x_i| \rangle$. An hyperreal $\mathtt{x}$ of ${}^*\mathbb{R}$ is *infinitesimal* whenever $|\mathtt{x}| < r$ for every real $r > 0$, and *unlimited* if $r < |\mathtt{x}|$ for every real $r \in \mathbb{R}$. Given a hyperreal $\mathtt{x}$ which is not unlimited, there exists a unique real y such that $\mathtt{x} - y$ is infinitesimal: this real is called the *standard part* of $\mathtt{x}$ and denoted by $\mathrm{st}(\mathtt{x})$. We define an equivalence relation $\approx$ on hyperreals by $\mathtt{x} \approx \mathtt{y}$ whenever $\mathrm{st}(\mathtt{x} - \mathtt{y}) = 0$.

▶ Remark. The existence of a standard part might be surprising at first: for instance, given the sequence x_i such that $x_i = 0$ if i is even and $x_i = 1$ otherwise, what should be the standard part of $\langle x_i \rangle$? The result is given by the chosen ultrafilter $\mathcal{F}$: if the set I of even integers is in $\mathcal{F}$ then $\mathrm{st}(\langle x_i \rangle) = 0$, otherwise the set $\mathbb{N} \setminus I$ of odd integers is in $\mathcal{F}$ and $\mathrm{st}(\langle x_i \rangle) = 1$.

▶ Remark. The method used to construct $^*\mathbb{R}$ is an instance of a very general construction of ultraproducts in model theory, which can be used to define a non-standard model $^*\mathbb{X}$ from any model $\mathbb{X}$ [21, 27, 12]. In particular, given sets $\mathbb{X}$ and $\mathbb{Y}$, this construction applied to the set $\mathbb{Y}^{\mathbb{X}}$ of functions from $\mathbb{X}$ to $\mathbb{Y}$ constructs the set $^*(\mathbb{Y}^{\mathbb{X}})$ of internal functions from $^*\mathbb{X}$ to $^*\mathbb{Y}$.

2.1.1 The transfer principle

A fundamental tool in non-standard analysis is the *transfer principle*, which follows from Łoś theorem [21]. Informally, this principle can be formulated as follows

▶ **Proposition 6** (Transfer principle). *A first-order formula φ is satisfied in $\mathbb{R}$ if and only if it is satisfied in $^*\mathbb{R}$, if we assume that all the sets, functions and relations involved in the formula are internal.*

A similar theorem can be formulated for $^*\mathbb{N}$. Many constructions of standard analysis can thus be transferred to non-standard analysis. For instance, the sets $^*\mathbb{N}$ and $^*\mathbb{R}$ are, respectively, a ring and a field and both are totally ordered.

▶ Remark. The assumption that we consider only internal objects is very important. For instance the formula $((\forall x \in A.\ x \in \mathbb{N}) \wedge (\exists x \in \mathbb{N}.\ x \in A)) \Rightarrow (\exists x \in A. \forall y \in A.\ x \leqslant y)$ is true in $\mathbb{N}$: every non-empty subset A of $\mathbb{N}$ admits a smallest element. From this, we can deduce by transfer that every non-empty *internal* subset of $^*\mathbb{N}$ admits a smallest element. However, this property does not hold for every subset of $^*\mathbb{N}$: for instance, $^*\mathbb{N} \setminus \mathbb{N}$ does not have a smallest element since it can be shown to be closed under predecessor, and is thus not internal. Likewise a subset of $^*\mathbb{N}$ (resp. $^*\mathbb{R}$) is internal if and only if it is finite.

2.1.2 Non-standard analysis

One of the most interesting property of hyperreals is that it allows one to rigorously consider infinitesimals and thus formalize in an elegant way many of the tools in common use in standard analysis. We give below some of these reformulations which will be of use afterward.

▶ **Proposition 7** (Continuity). *A function $f : \mathbb{R} \to \mathbb{R}$ is* continuous *at x when for every $\mathrm{y} \in {}^*\mathbb{R}$ such that $\mathrm{y} \approx x$, we have $^*f(\mathrm{y}) \approx f(x)$. Otherwise said, f is continuous at x when for every infinitesimal $\delta \approx 0$, there exists an infinitesimal $\varepsilon \approx 0$ such that $^*f(x+\delta) = f(x)+\varepsilon$.*

▶ **Proposition 8** (Differentiation). *A function $f : \mathbb{R} \to \mathbb{R}$ admits $y \in \mathbb{R}$ as* derivative *at $x \in \mathbb{R}$ when for every non-null infinitesimal $\delta \approx 0$, we have $({}^*f(x+\delta) - f(x))/\delta \approx y$. Furthermore, if f is continuously differentiable on $\mathbb{R}$, then for any two non-null distinct infinitesimals δ and ε, and for any $x \in \mathbb{R}$, we have*

$$f'(x) \quad = \quad \mathrm{st}\left(\frac{{}^*f(x+\delta) - {}^*f(x+\varepsilon)}{\delta - \varepsilon}\right)$$

Given a continuous function $f : \mathbb{R} \to \mathbb{R}$, its integral on an interval $[a, b]$ is defined by

$$\int_a^b f(x)\,\mathrm{d}x \quad = \quad \lim_{n\to\infty}\left(\sum_{k=0}^{n-1} f\left(a + \frac{k}{n}(b-a)\right)\frac{1}{n}\right)$$

Notice that each sum makes sense because it is finite since it is indexed over the finite set $\{k \in \mathbb{N} \mid 0 \leqslant k < n\}$. This notion of finite set can be generalized to internal sets as follows: an internal set $A = \langle A_i \rangle$ is *hyperfinite* if almost all the A_i are finite, i.e. $\{i \in \mathbb{N} \mid A_i \text{ is finite}\} \in \mathcal{F}$. By an argument similar to Lemma 14, we can always suppose that all the A_i are finite by choosing a suitable sequence of finite sets B_i such that $\langle B_i \rangle = \langle A_i \rangle$. Given such an internal set and an internal function $\langle f_i \rangle$, we define $\sum_{\langle x_i \rangle \in \langle A_i \rangle} \langle f_i \rangle(\langle x_i \rangle) = \langle \sum_{x_i \in A_i} f_i(x_i) \rangle$.

▶ **Proposition 9** (Integration). *Given a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous on an interval $[a, b]$, excepting possibly a finite number of points of discontinuity, we have*

$$\int_a^b f(x)\,\mathrm{d}x \quad = \quad \mathrm{st}\left(\sum_{x \in N} {}^*f(a + x(b-a))\delta \right) \tag{2}$$

where $\delta = 1/\mathtt{n}$ for some unlimited $\mathtt{n} = \langle n_i \rangle \in {}^*\mathbb{N}$ and N is the hyperfinite set $N = \langle N_i \rangle \subseteq {}^*\mathbb{R}$ with $N_i = \{k_i/n_i \in \mathbb{R} \mid k_i \in \mathbb{N}, 0 \leqslant k_i < n_i\}$. In particular, the result does not depend on the choice of the unlimited hyperinteger $\mathtt{n} \in {}^*\mathbb{N}$.

The notion defined above corresponds to the Riemann integral. More refined notions (such as the Lebesgue integral) can also be adapted to the non-standard setting.

2.2 Internal domains

In this section, we introduce the notion of internal domain, which we use to define a non-standard denotational semantics for process networks. Given a totally ordered set T and a set R, we write $R^{\leqslant T}$ for the set of partial functions $s : T \to R$ defined on an initial segment of T, called the *domain of definition* of s. The elements of $R^{\leqslant T}$ are called *streams*: the set T can be thought as *time* and the elements of R as the possible *values* of a stream over time. Every such set can be equipped with a partial order $\sqsubseteq$ such that, given $r, s \in R^{\leqslant T}$, we have $f \sqsubseteq g$ whenever the definition domain of r is included in the definition domain of s and r and s coincide on the domain of definition of r.

▶ **Proposition 10.** *The poset $(R^{\leqslant T}, \sqsubseteq)$ is a cpo with smallest element $\bot$ being the function nowhere defined.*

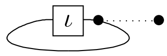
▶ **Example 15.** The Kahn domain described in Section 1.3 is $R^{\leqslant \mathbb{N}}$.

Every function $f : R \to R$ lifts to a function $\tilde{f} : R^{\leqslant T} \to R^{\leqslant T}$ such that, given $s \in R^{\leqslant T}$, the domain of definition of $\tilde{f}(s)$ is the same as the domain of definition of s and the image of s is defined by $\tilde{f}(s)(t) = f \circ s(t)$. The function $\tilde{f}$ is called the *lifting* of f to $R^{\leqslant T}$. It is easy to show that every such lifting is Scott-continuous.

In the following, we will be interested in modeling nets operating in a time which varies continuously. We thus introduce the following domain in order to model the data flowing through the wires:

▶ **Definition 16** (Continuous-time domain). The *continuous-time domain* is the complete partial order $CT = \mathbb{R}^{\leqslant \mathbb{R}^+}$. The *continuous-time domain of continuous functions* CCT is the subdomain of CT whose elements are continuous partial functions $\mathbb{R}^+ \to \mathbb{R}$.

As explained in the introduction, the purpose of this paper is to explain how to implement Scott-continuous functions over this domain using Kahn networks by formalizing the following intuition using non-standard semantics: continuous time can be considered as “discrete” where the duration between two instants is infinitesimal. A natural candidate for this would be the domain ${}^*\mathbb{R}^{\leqslant {}^*\mathbb{N}}$. Namely, in the view of Proposition 9, we would like to relate a stream $s \in \mathbb{R}^{\leqslant \mathbb{R}^+}$ with the stream $\overline{s} \in {}^*\mathbb{R}^{\leqslant {}^*\mathbb{N}}$ defined on $\mathtt{t} \in {}^*\mathbb{N}$ by $\overline{s}(\mathtt{t}) = {}^*s(\mathtt{t}\delta)$, from some infinitesimal $\delta \in {}^*\mathbb{R}$. It turns out that the fixpoints computed in ${}^*\mathbb{R}^{\leqslant {}^*\mathbb{N}}$ are not satisfactory. For instance, consider the net on the right such that the interpretation of the operator ι is the function $\iota : {}^*\mathbb{R}^{\leqslant {}^*\mathbb{N}} \to {}^*\mathbb{R}^{\leqslant {}^*\mathbb{N}}$ such that the image of a stream r is the stream s defined by $s(0) = 0$ and for any non-null hyperinteger $\mathtt{t}$, $s(\mathtt{t}) = r(\mathtt{t} - 1)$. We expect the interpretation of this net to be the constant function equal to 0. However, this is not the case: the semantics s of this net is given by the fixpoint $s = \mathrm{fix}(\iota) = \bigvee_{k \in \mathbb{N}} \iota^k(\bot)$ of ι. Given $k \in \mathbb{N}$, the domain of definition of the stream $\iota^k(\bot)$ is the set $\{\mathtt{p} \in {}^*\mathbb{N} \mid 0 \leqslant \mathtt{p} < k\}$. Therefore, given an unlimited $\mathtt{n} \in {}^*\mathbb{N}$, $\iota^k(\bot)(\mathtt{n})$ is undefined for every $k \in \mathbb{N}$ and thus $\mathrm{fix}(\iota)(\mathtt{n})$ is undefined. Intuitively, the induction on $k \in \mathbb{N}$ defining the smallest fixpoint is not powerful enough to reach all elements of ${}^*\mathbb{N}$. The cpo ${}^*\mathbb{R}^{\leqslant {}^*\mathbb{N}}$ is thus not the appropriate domain, however we explain below that internal domains are a more suitable notion, because they support an induction principle on ${}^*\mathbb{N}$.



(3)

▶ **Definition 17** (Internal cpo). An *internal cpo* $(D, \leqslant)$ in a non-standard model consists of an internal set $D = \langle D_i \rangle$ and an internal relation $\leqslant = \langle \leqslant_i \rangle$ such that for every integer i, $(D_i, \leqslant_i)$ is a cpo. Similarly, an *internal Scott-continuous function* $f : D \to E$ between two internal cpo $D = \langle D_i \rangle$ and $E = \langle E_i \rangle$ consists of an internal function $\langle f_i : D_i \to E_i \rangle$ such that all the f_i are continuous. We write **ICpo** for the category of internal cpo and internal Scott-continuous functions.

▶ Remark. Notice that such an internal cpo $(D, \leqslant)$ is not necessarily a cpo: only internal directed subsets are required to have a supremum. For instance suppose fixed an unlimited hyperinteger $\mathtt{n} \in {}^*\mathbb{N}$. The set $D = \{\mathtt{k} \in {}^*\mathbb{N} \mid \mathtt{k} \leqslant \mathtt{n}\}$ equipped with the usual total order is an internal cpo, but not a cpo because the (non-internal) subset $\mathbb{N} \subseteq D$ is directed and does not have a supremum.

▶ **Proposition 11.** *Any internal Scott-continuous function $f : D \to D$, where D is an internal cpo, admits a least fixpoint $\mathrm{fix}(f)$ which satisfies $\mathrm{fix}(f) = \bigvee \{f^{\mathtt{n}}(\bot) \mid \mathtt{n} \in {}^*\mathbb{N}\}$. Here, if $f = \langle f_i \rangle$ and $\mathtt{n} = \langle n_i \rangle$, $f^{\mathtt{n}}$ is defined as $\langle f_i^{n_i} \rangle$.*

The axioms of fixpoint categories can be formulated in first-order logic. Using the transfer principle (Proposition 6), it can be shown that the fact that **Cpo** is a fixpoint category implies that

▶ **Proposition 12.** *The category* **ICpo** *is a fixpoint category.*

In the category **ICpo**, we will be particularly interested in the following domain:

▶ **Definition 18** (Infinitesimal-time domain). The *infinitesimal-time domain* is the internal complete partial order $IT = {}^*(\mathbb{R}^{\leqslant \mathbb{N}})$.

As explained in the remark in the end of Section 2.1, the elements of $IT = {}^*(\mathbb{R}^{\leqslant \mathbb{N}})$ are the internal partial functions from ${}^*\mathbb{N}$ to ${}^*\mathbb{R}$. The order $\sqsubseteq$ on this domain is such that for any $r, s \in IT$, we have $r \sqsubseteq s$ whenever the domain of definition of r is included in the domain of definition of s and r and s coincide on the domain of definition of r.

2.3 Comparing continuous time and infinitesimal time

In this section, we explain how the semantics of nets in IT can "simulate" operations in CT. We now suppose fixed an infinitesimal δ called *sampling period*. We define a function

$S : CT \to IT$, called *sampling*, which to every stream $s \in CT$ associates the stream $S(s) = \mathtt{k} \mapsto {}^*s(\mathtt{k}\delta)$, and a function $T : IT \to CT$, called *standardization*, which to every stream s associates $T(s) = x \mapsto \mathrm{st}\,(s(\lfloor {}^*x/\delta \rfloor))$, where $\lfloor - \rfloor : {}^*\mathbb{R} \to {}^*\mathbb{N}$ denotes the floor function, and is defined on the biggest initial segment of $\mathbb{R}^+$ for which this definition makes sense. These functions enable us to show that CCT (the domain of *continuous* streams) is a retract of IT. We discuss afterward the possible extensions of this result to elements of CT.

▶ **Proposition 13.** *The restriction of the composite $T \circ S$ to CCT is the identity.*

Proof. Suppose given a stream $s \in CCT$. For any $x \in \mathbb{R}^+$, the fact that s is continuous at x implies, by Proposition 7, that for every $\mathtt{k} \in {}^*\mathbb{N}$ such that $\mathtt{k}\delta \approx x$, we have $S(s)(\mathtt{k}\delta) \approx s(x)$. From this we deduce that $T(S(s))(x) = s(x)$. ◀

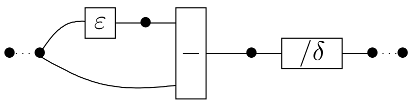
▶ Remark. The function $T \circ S$ is generally *not* the identity on CT. For instance, suppose that $\delta = 1/\mathtt{n}$, where $\mathtt{n} \in {}^*\mathbb{N}$ is unlimited, and consider the stream $s \in CT$ whose value is 0 everywhere except at $\sqrt{2}$ where $s(\sqrt{2}) = 1$. Using the transfer principle, it is easy to show that for every $\mathtt{k} \in {}^*\mathbb{N}$, we have $\mathtt{k}/\mathtt{n} \neq \sqrt{2}$. From this we can deduce that $T \circ S(s)$ is the constant stream equal to 0.

In order to make a more convincing case of the interest of the domain IT as a model of nets and study further its relationship with CT, we give below some examples of nets interpreted in IT which implement common constructions in analysis, and relate them to the corresponding constructions in CT through S and T. For concision, we do not detail the easy verification that the interpretations of operators are internal Scott-continuous functions.

As a first simple example, consider the net (3). From the characterization of the fixpoint of internal Scott-continuous functions given by Proposition 11, it is easy to check that its semantics s in the domain IT is the constant function (defined everywhere) as expected: if we write $c_0 : \mathbb{R}^+ \to \mathbb{R}$ for the constant function equal to 0, we have $s = S(c_0)$ and $c_0 = T(s)$.

2.3.1 Differentiation

The *differentiation* is the following net where "ε" is interpreted as the function which drops the first element of a stream (i.e. $\varepsilon(s)(\mathtt{k}) = s(\mathtt{k}+1)$), "$-$" is interpreted as the pointwise difference (i.e. $(-(s,t))(\mathtt{k}) = s(\mathtt{k}) - t(\mathtt{k})$), and "$/\delta$" is interpreted as the pointwise division by δ (i.e. $(s/\delta)(\mathtt{k}) = s(\mathtt{k})/\delta$).



(4)

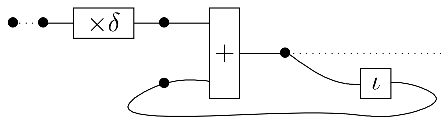
We write $\varphi : IT \to IT$ for the semantics of the net. By Proposition 8, we have immediately:

▶ **Proposition 14.** *For any continuously differentiable function $s : \mathbb{R}^+ \to \mathbb{R}$ in CCT, $T(\varphi(S(s))) = s'$.*

Proof. Given $\mathtt{k} \in {}^*\mathbb{N}$ such that $\mathrm{st}(\mathtt{k}\delta) \in \mathbb{R}^+$, $\varphi(S(s))(\mathtt{k}) = \frac{S(s)(\mathtt{k}+1) - S(s)(\mathtt{k})}{\delta} \approx s'(\mathrm{st}(\mathtt{k}\delta))$. The second step is proved by Proposition 8. Therefore, $T(\varphi(S(s))) = s'$. ◀

2.3.2 Integration

The *integration* is the following net where "$\times\delta$" is the pointwise multiplication of a stream by δ, "$+$" is the pointwise addition of streams, and ι prepends 0 to a stream (see Section 2.2).



(5)

We write $\varphi : IT \to IT$ for the semantics of the net. By Proposition 9, we have immediately:

▶ **Proposition 15.** *For any function $s : \mathbb{R}^+ \to \mathbb{R}$ in CCT, $T(\varphi(S(s))) = x \mapsto \int_0^x s(t)\,\mathrm{d}t$.*

Proof. The semantics φ of the net is computed by a fixpoint as explained in Section 1, defined by $\varphi(S(s))(0) = \delta S(s)(0)$ and $\varphi(S(s))(\mathtt{n}+1) = \varphi(S(s))(\mathtt{n}) + \delta S(s)(\mathtt{n}+1)$. Therefore we have $\varphi(S(s))(\mathtt{n}) = \delta \sum_{\mathtt{k}=0}^{\mathtt{n}} S(s)(\mathtt{k})$. Finally, by Proposition 9, it can be shown that if $\mathrm{st}(\mathtt{x}\delta) = x \in \mathbb{R}^+$, then $\delta \sum_{\mathtt{k}=0}^{\mathtt{n}} S(s)(\mathtt{k}) \approx \int_0^x s(t)\,\mathrm{d}t$ and thus $T(\varphi(S(s)))(x) = \int_0^x s(t)\,\mathrm{d}t$. ◀

This construction can be generalized in order to describe solvers for ordinary differential equations [7, 6]. It should be noticed that the above propositions show that the choice of the infinitesimal sampling period δ is essentially irrelevant.

Most of the preceding results can be adapted to the case where the streams considered are only piecewise continuous, with a finite number of discontinuities. In particular, for any such stream s we have $T \circ S(s) \hat{=} s$ where $\hat{=}$ denotes the equality almost everywhere (this weakening of equality is necessary because of phenomena such as the one described in the remark following Proposition 13). However, the formalization of this is obscured by the fact that piecewise continuous functions, with a finite number of discontinuities, do not form a cpo because the supremum in CT of a directed set of such functions might have an infinite number of points of discontinuity: this is sometimes referred to as the *Zeno phenomenon* in the study of hybrid systems.

3 Conclusion and future works

We have defined nets which provide a formal syntax for process networks, studied the categorical structure of their models, and constructed the infinitesimal-time model as an internal cpo. The fascinating links between denotational semantics of concurrent systems and non-standard analysis have started to be explored only recently and many structures are still yet to be clarified. As explained above, the study of the infinitesimal-time domain has to be refined in order to cope with streams which are not necessarily continuous, and thus to properly model full-fledged hybrid systems. In particular, Proposition 13 fails to be true if we replace CCT by CT: we plan to investigate generalizations of this property where S and T form an adjunction. We are also investigating possible adaptations of nets in the case where we consider a synchronous semantics (in which there is a notion of simultaneity of events). In this setting, the usual delay operator can elegantly be modeled in feedback categories [20] (which are traced monoidal categories with the yanking axiom removed) and we plan to study nets for those categories. Finally, we envisage many connections with other areas of denotational semantics. For instance, the trace semantics of Kahn networks is closely related to game semantics [23] and it is thus natural to wonder if non-standard analysis can provide insights about a possible definition of a "continuous game semantics" in the spirit of the geometric models for concurrent programs [13].

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